Imaginary Geometry and the Gaussian Free Field

Jason Miller and Scott Sheffield

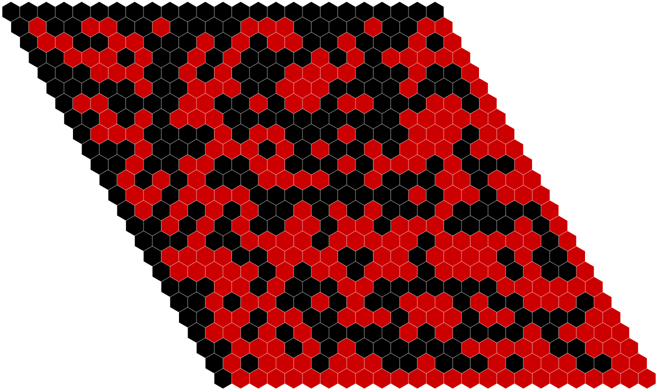
Massachusetts Institute of Technology

May 23, 2013

Part I: SLE

SLE_κ

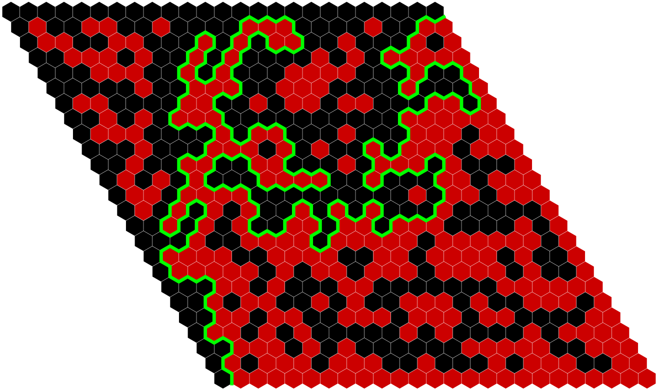
- **Random, fractal curve** in $D \subseteq \mathbf{C}$ simply connected
- Introduced by Oded Schramm as a candidate for the **scaling limit** of loop erased random walk and the interfaces in critical percolation.



(Percolation simulations due to Sam Watson)

SLE_κ

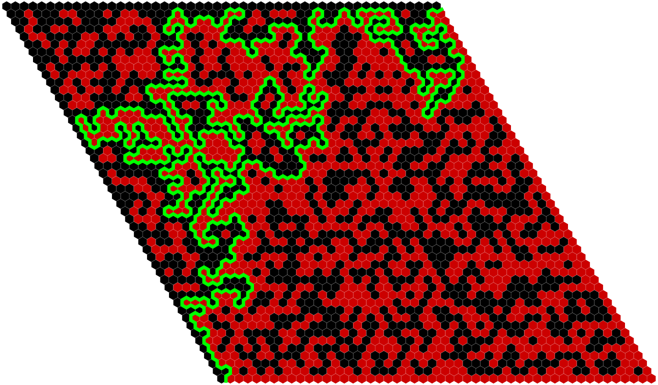
- **Random, fractal curve** in $D \subseteq \mathbf{C}$ simply connected
- Introduced by Oded Schramm as a candidate for the **scaling limit** of loop erased random walk and the interfaces in critical percolation.



(Percolation simulations due to Sam Watson)

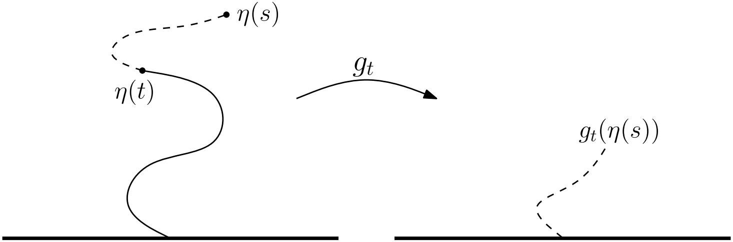
SLE_κ

- **Random, fractal curve** in $D \subseteq \mathbf{C}$ simply connected
- Introduced by Oded Schramm as a candidate for the **scaling limit** of loop erased random walk and the interfaces in critical percolation.



(Percolation simulations due to Sam Watson)

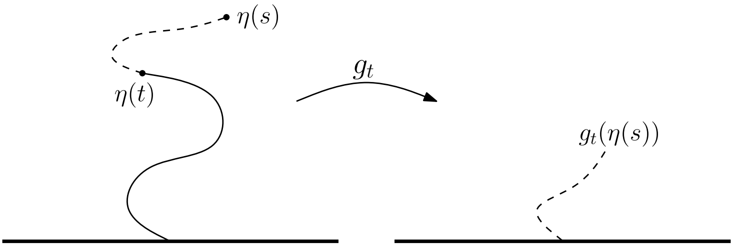
SLE$_\kappa$



Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\bigstar)$$

SLE_κ

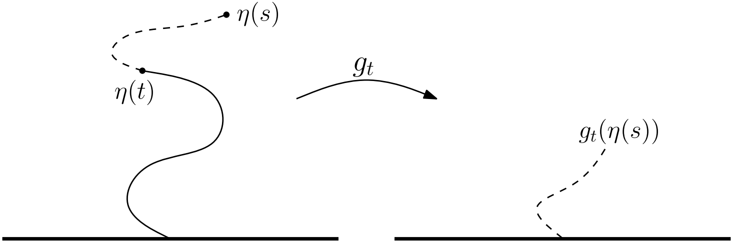


Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\star)$$

SLE_κ in H: The random curve associated with ($\star$) with $W_t = \sqrt{\kappa} B_t$, B a standard Brownian motion.

SLE_κ



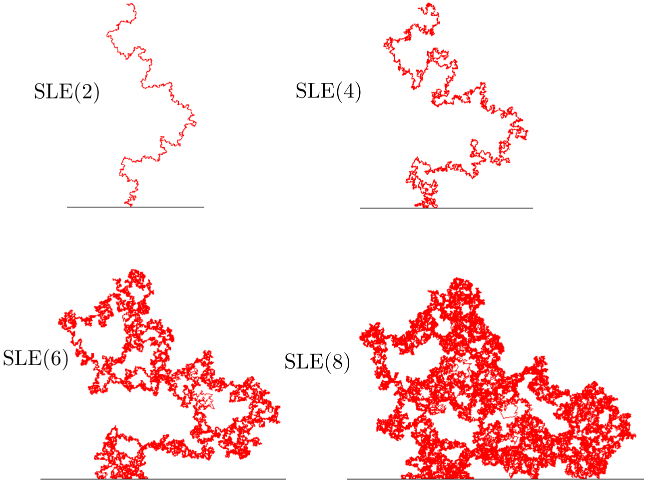
Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\bigstar)$$

SLE_κ **in H**: The random curve associated with ($\bigstar$) with $W_t = \sqrt{\kappa} B_t$, B a standard Brownian motion. Other domains: apply conformal mapping.

SLE_κ

SLE_κ is a continuous path ($\kappa \neq 8$, Rohde-Schramm $\kappa = 8$, Lawler, Schramm, Werner),
Dimension: $1 + \frac{\kappa}{8}$ (Beffara), Simple if $\kappa \in (0, 4]$, self-touching if $\kappa > 4$, space-filling if $\kappa \geq 8$.



(Simulations due to Tom Kennedy)

Sample path properties of the SLE trace

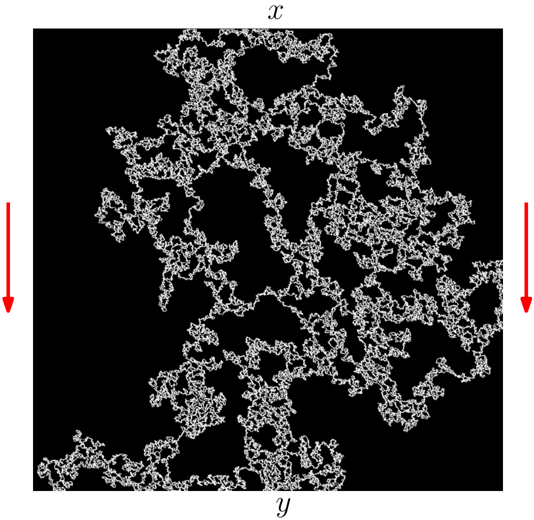
- Continuity; Hölder exponent
- Dimensions
- Multi-fractal spectra
- "Natural parameterization"
- Duality (path decompositions)
- CLE (loop version of SLE)
- SLE on multiply connected domains
- Reversibility

Reversibility of SLE_κ

An SLE_κ process η from x to y. **Question:** If we run η in the **reverse** direction (i.e., from y to x), is it still an SLE_κ?

Reversibility of SLE_κ

An SLE_κ process η from x to y. **Question:** If we run η in the **reverse** direction (i.e., from y to x), is it still an SLE_κ? **Answer:** Yes for $\kappa \leq 8$.



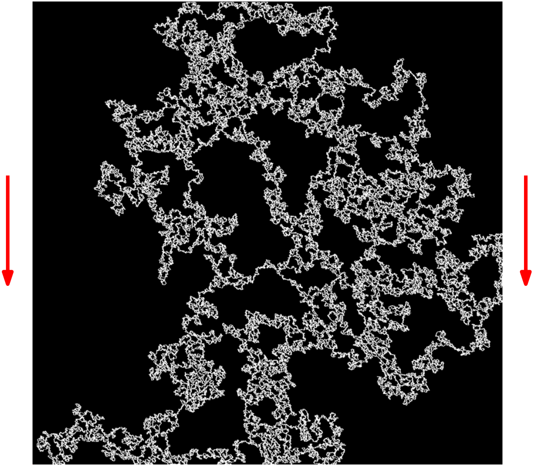
A related question

An SLE_κ process η from x to y. **Question:** If we observe the **range** of η, can we determine the path?

A related question

An SLE_κ process η from x to y. **Question:** If we observe the **range** of η, can we determine the path? **Answer:** No for $\kappa > 4$.

x

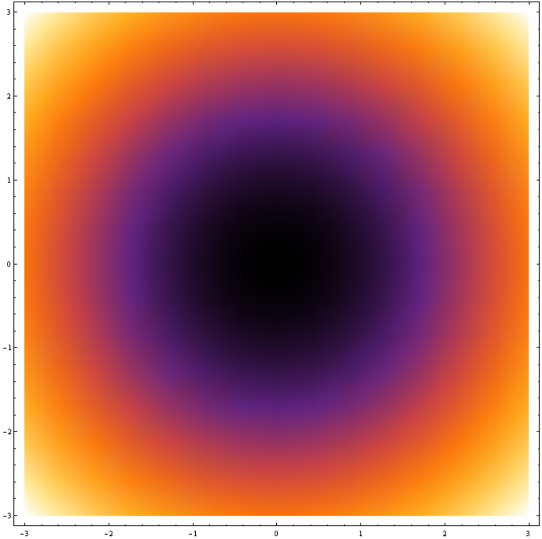


y

Part II:
SLE and the Gaussian Free Field

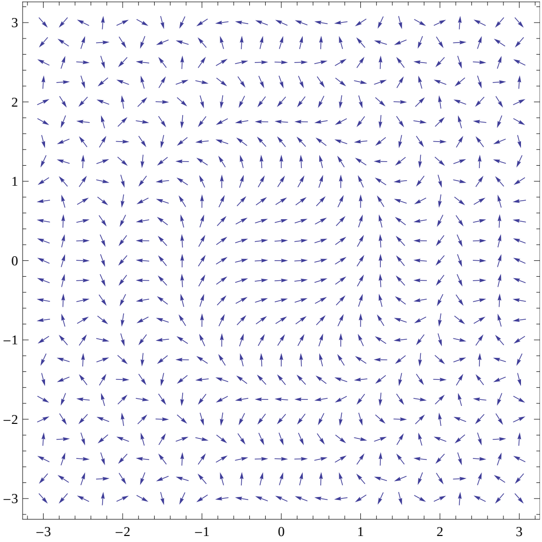
Rays of a Smooth Function

- h smooth $[h(x, y) = x^2 + y^2]$



Rays of a Smooth Function

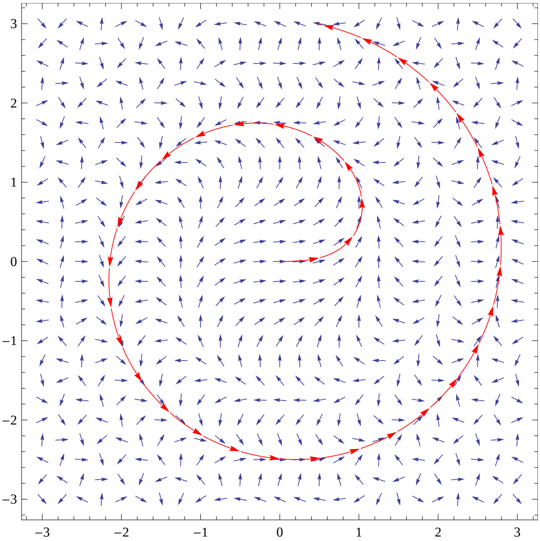
- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$



Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **ray** of h is a **flow line** of e^{ih} , i.e. a solution to

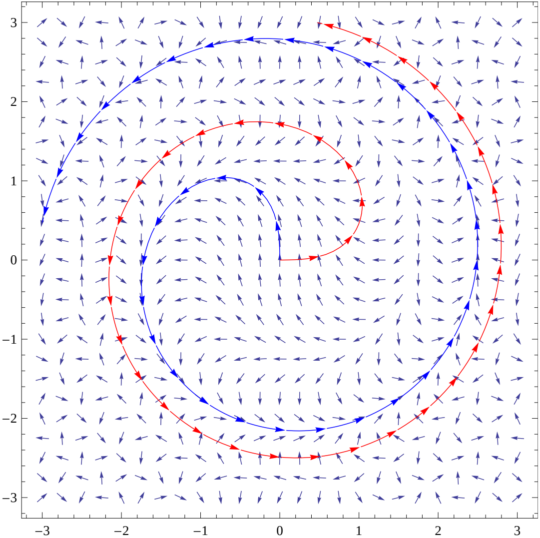
$$\frac{d}{dt}\eta(t) = e^{ih(\eta(t))}$$



Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **θ-angle ray** of h is a **flow line** of $e^{i(h+\theta)}$, i.e. a solution to

$$\frac{d}{dt}\eta(t) = e^{i(h(\eta(t))+\theta)}$$

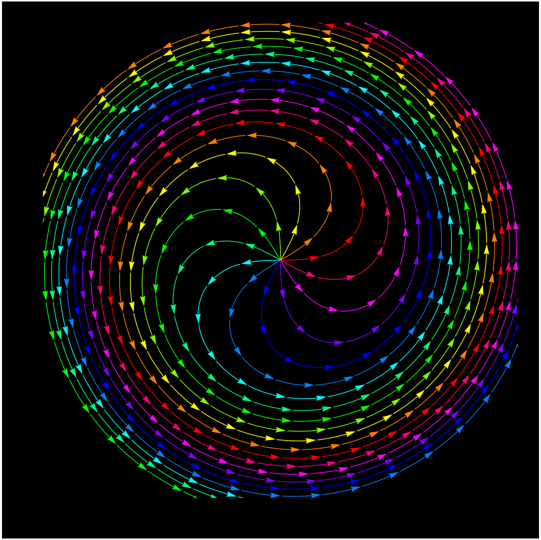


Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **θ-angle ray** of h is a **flow line** of $e^{i(h+\theta)}$, i.e. a solution to

$$\frac{d}{dt}\eta(t) = e^{i(h(\eta(t))+\theta)}$$

- The rays of h vary smoothly and monotonically with θ and are non-intersecting.

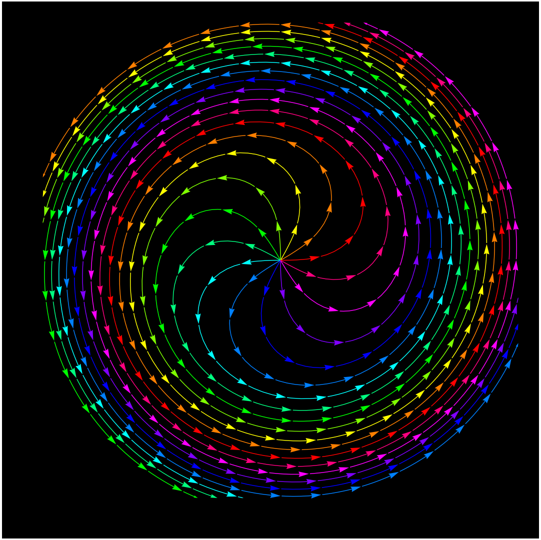


Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **θ-angle ray** of h is a **flow line** of $e^{i(h+\theta)}$, i.e. a solution to

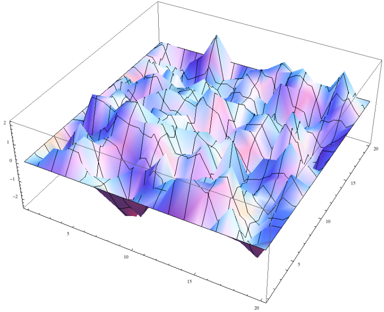
$$\frac{d}{dt}\eta(t) = e^{i(h(\eta(t))+\theta)}$$

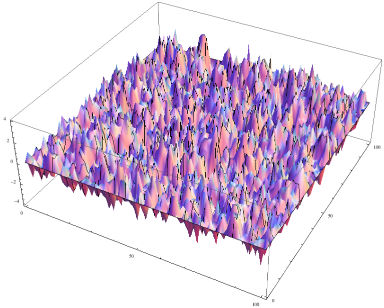
- The rays of h vary smoothly and monotonically with θ and are non-intersecting.
- **Plan:**
  - *Describe the interaction of the rays of the GFF*
  - *Explain how this can be used to study* SLE



The Gaussian Free Field

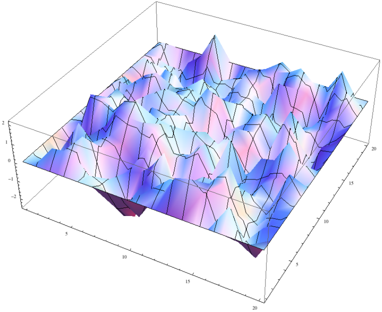
- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.

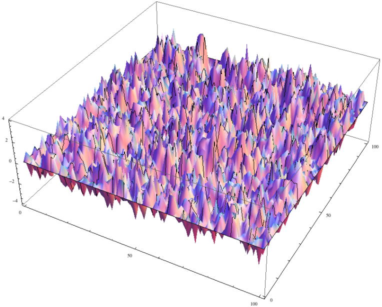




The Gaussian Free Field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Gaussian measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ where
  - **Covariance**: Green's function for discrete Δ
  - **Mean Height**: discrete harmonic ext. of ψ

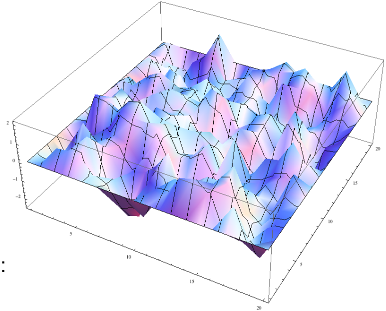


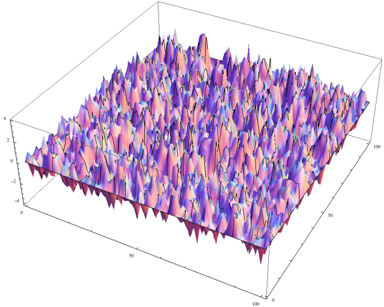


The Gaussian Free Field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Gaussian measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ where
  - **Covariance**: Green's function for discrete Δ
  - **Mean Height**: discrete harmonic ext. of ψ
- Density with respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$





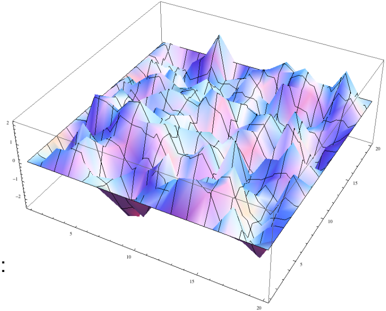
The Gaussian Free Field

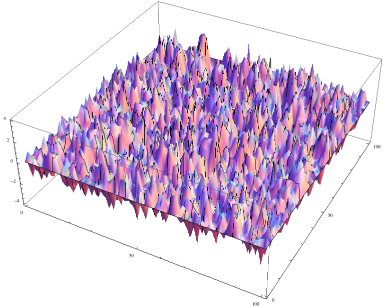
- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Gaussian measure on functions $h \colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ where
  - **Covariance**: Green's function for discrete Δ
  - **Mean Height**: discrete harmonic ext. of ψ
- Density with respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

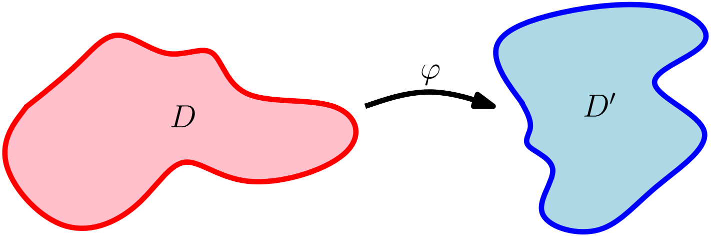
$$(f, g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) dx.$$





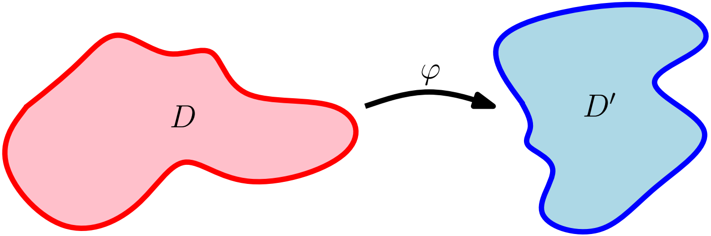
Properties of the GFF

Conformal invariance $D, D' \subseteq \mathbf{C}$ domains, $\varphi \colon D \to D'$ conformal transformation. If h is a GFF on D then $h \circ \varphi^{-1}$ is a GFF on D'.

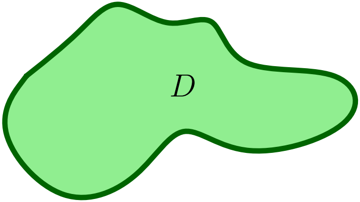


Properties of the GFF

Conformal invariance $D, D' \subseteq \mathbf{C}$ domains, $\varphi \colon D \to D'$ conformal transformation. If h is a GFF on D then $h \circ \varphi^{-1}$ is a GFF on D'.

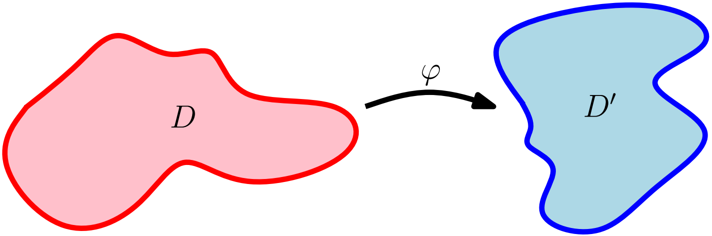


Markov property $D \subseteq \mathbf{C}$ domain, h a GFF on D.

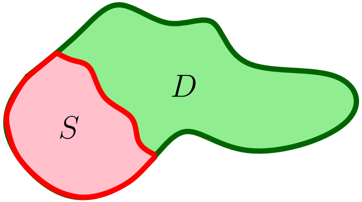


Properties of the GFF

Conformal invariance $D, D' \subseteq \mathbf{C}$ domains, $\varphi\colon D \to D'$ conformal transformation. If h is a GFF on D then $h \circ \varphi^{-1}$ is a GFF on D'.

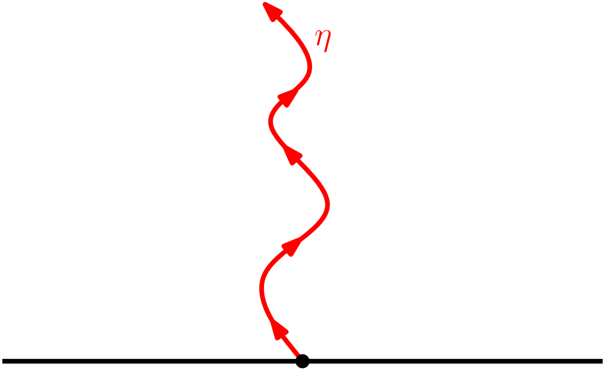


Markov property $D \subseteq \mathbf{C}$ domain, h a GFF on D. Fix $S \subseteq D$. Conditional law of $h|_{D \setminus S}$ given $h|_S$ is a GFF on $D \setminus S$ plus the harmonic extension of $h|_{\partial(D \setminus S)}$ to $D \setminus S$.



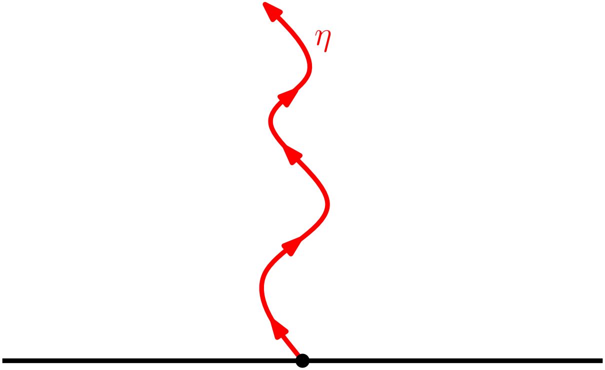
Constructing the rays of the GFF

- Suppose h is smooth on $\mathbf{H}$ and η is a ray of h with $\eta(0) = 0$, i.e. $\frac{d}{dt}\eta(t) = e^{ih(\eta(t))}$



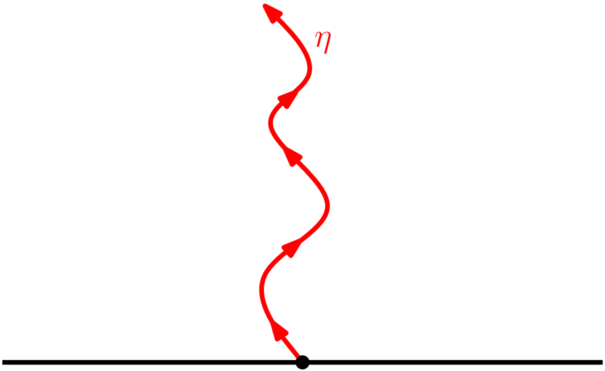
Constructing the rays of the GFF

- Suppose h is smooth on $\mathbf{H}$ and η is a ray of h with $\eta(0) = 0$, i.e. $\frac{d}{dt}\eta(t) = e^{ih(\eta(t))}$
- η **determines** the values of h along η and
- η is **independent** of the values of h off of η



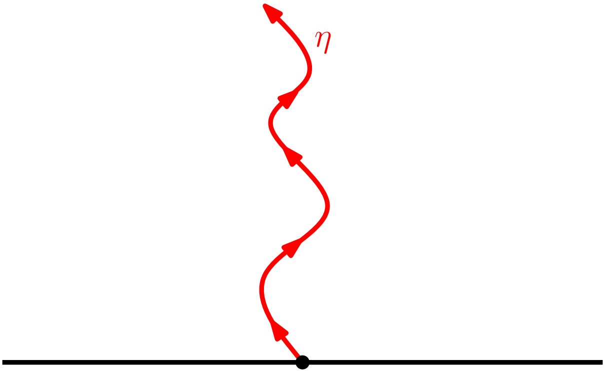
Constructing the rays of the GFF

- Sample a (simple) SLE_κ process η in $\mathbf{H}$ from 0 to ∞ (i.e. $\kappa \in (0,4)$)



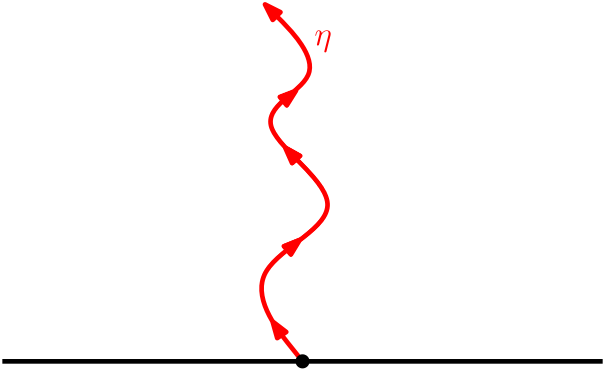
Constructing the rays of the GFF

- Sample a (simple) SLE_κ process η in **H** from 0 to ∞ (i.e. $\kappa \in (0,4)$)
- Given η, let h be a GFF on $\mathbf{H} \setminus \eta$ with boundary data on η given by what it **would** be if η **were** a flow line of $e^{ih/\chi}$ where $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$



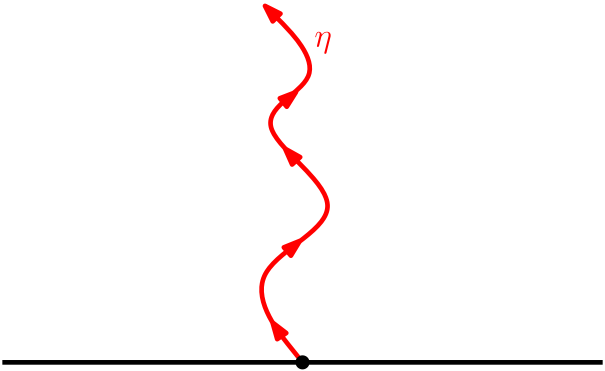
Constructing the rays of the GFF

- Sample a (simple) SLE_κ process η in **H** from 0 to ∞ (i.e. $\kappa \in (0,4)$)
- Given η, let h be a GFF on $\mathbf{H} \setminus \eta$ with boundary data on η given by what it **would** be if η **were** a flow line of $e^{ih/\chi}$ where $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$
- **Theorem** (Sheffield) The marginal law of h is that of a GFF on all of **H**



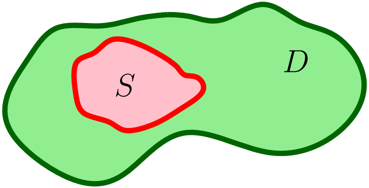
Constructing the rays of the GFF

- Sample a (simple) SLE_κ process η in **H** from 0 to ∞ (i.e. $\kappa \in (0,4)$)
- Given η, let h be a GFF on $\mathbf{H} \setminus \eta$ with boundary data on η given by what it **would** be if η **were** a flow line of $e^{ih/\chi}$ where $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$
- **Theorem** (Sheffield) The marginal law of h is that of a GFF on all of **H**
- **Theorem** (Dubédat) η is almost surely determined by h



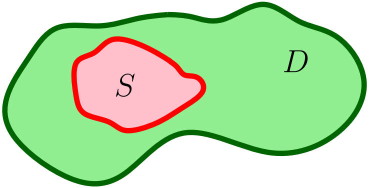
Multiple rays of the GFF

A random closed set S coupled with a GFF h on a domain D is said to be **local** for h if the conditional law of $h|_{D \setminus S}$ given S is that of the sum of a zero-boundary GFF on $D \setminus S$ and an independent harmonic function.



Multiple rays of the GFF

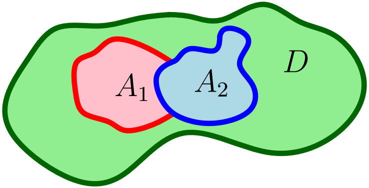
A random closed set S coupled with a GFF h on a domain D is said to be **local** for h if the conditional law of $h|_{D \setminus S}$ given S is that of the sum of a zero-boundary GFF on $D \setminus S$ and an independent harmonic function.



Examples Deterministic closed sets. An SLE path coupled with h as a flow line drawn up to a stopping time.

Multiple rays of the GFF

A random closed set S coupled with a GFF h on a domain D is said to be **local** for h if the conditional law of $h|_{D \setminus S}$ given S is that of the sum of a zero-boundary GFF on $D \setminus S$ and an independent harmonic function.

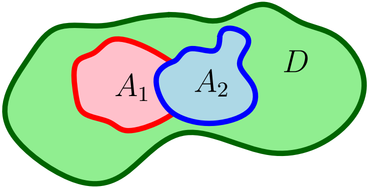


Examples Deterministic closed sets. An SLE path coupled with h as a flow line drawn up to a stopping time.

Fact (Schramm, Sheffield) If A_1, A_2 are local sets for a GFF h which are conditionally independent given h, then $A_1 \cup A_2$ is a local set for h.

Multiple rays of the GFF

A random closed set S coupled with a GFF h on a domain D is said to be **local** for h if the conditional law of $h|_{D \setminus S}$ given S is that of the sum of a zero-boundary GFF on $D \setminus S$ and an independent harmonic function.

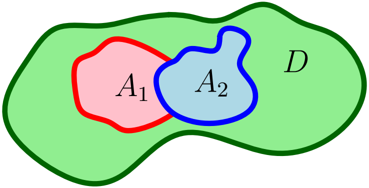


Examples Deterministic closed sets. An SLE path coupled with h as a flow line drawn up to a stopping time.

Fact (Schramm, Sheffield) If A_1, A_2 are local sets for a GFF h which are conditionally independent given h, then $A_1 \cup A_2$ is a local set for h. Moreover, the boundary behavior for the conditional law of h given $A_1 \cup A_2$ agrees with that of h given A_1 near points points $z \in A_1$ with $\text{dist}(z, A_2) > 0$ and vice-versa.

Multiple rays of the GFF

A random closed set S coupled with a GFF h on a domain D is said to be **local** for h if the conditional law of $h|_{D \setminus S}$ given S is that of the sum of a zero-boundary GFF on $D \setminus S$ and an independent harmonic function.

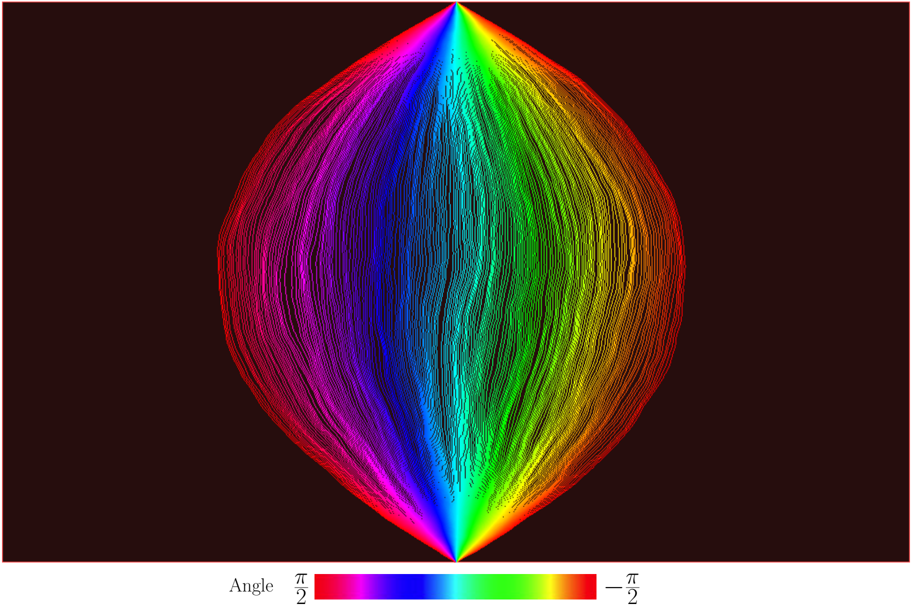


Examples Deterministic closed sets. An SLE path coupled with h as a flow line drawn up to a stopping time.

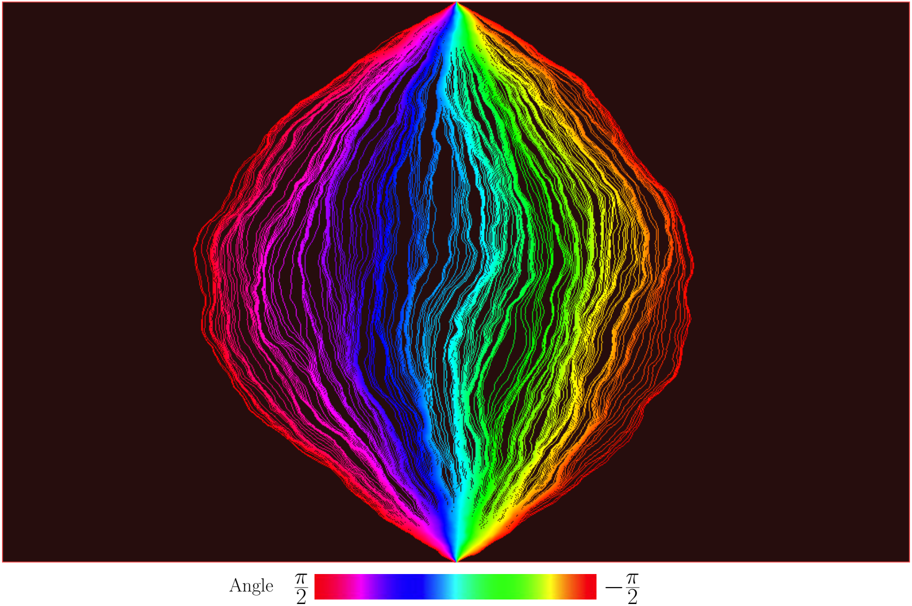
Fact (Schramm, Sheffield) If A_1, A_2 are local sets for a GFF h which are conditionally independent given h, then $A_1 \cup A_2$ is a local set for h. Moreover, the boundary behavior for the conditional law of h given $A_1 \cup A_2$ agrees with that of h given A_1 near points points $z \in A_1$ with $\text{dist}(z, A_2) > 0$ and vice-versa.

Leads to couplings with many SLE paths. Intersection points of paths cause trouble.

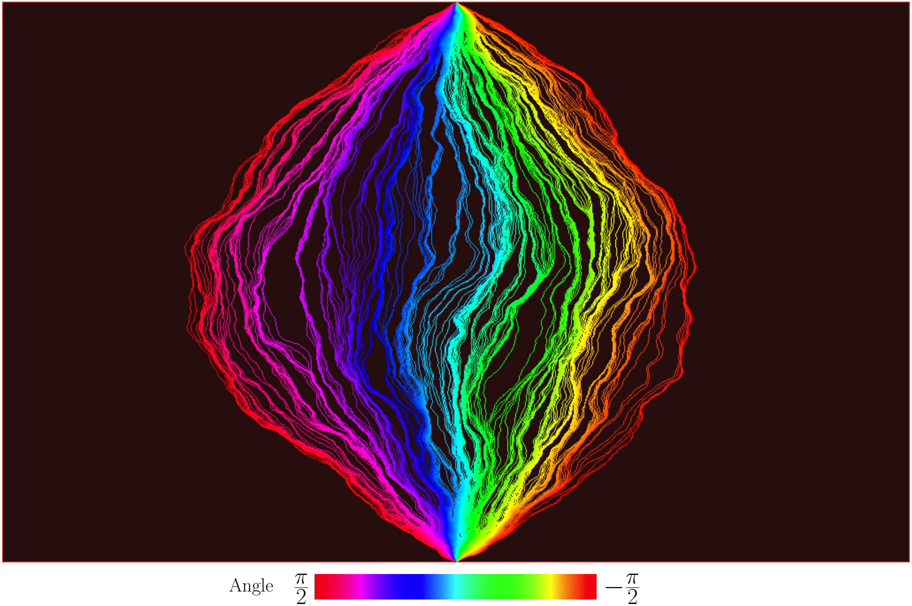
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 31.97$ $[\kappa = 1/256]$



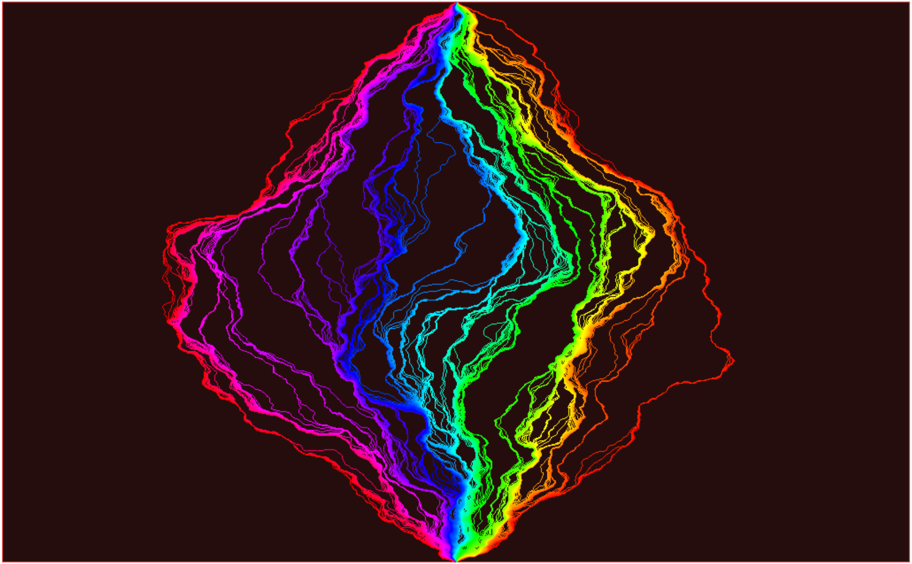
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 11.23$ $[\kappa = 1/32]$



Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 7.88$ [$\kappa = 1/16$]

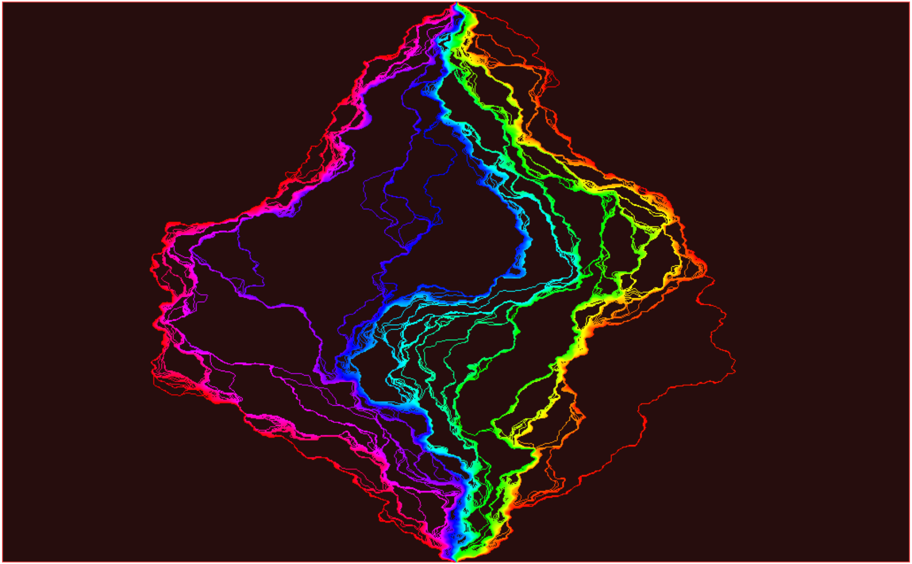


Rays of $e^{ih/\chi}$, h GFF, $\chi = 3.75$ $[\kappa = 1/4]$

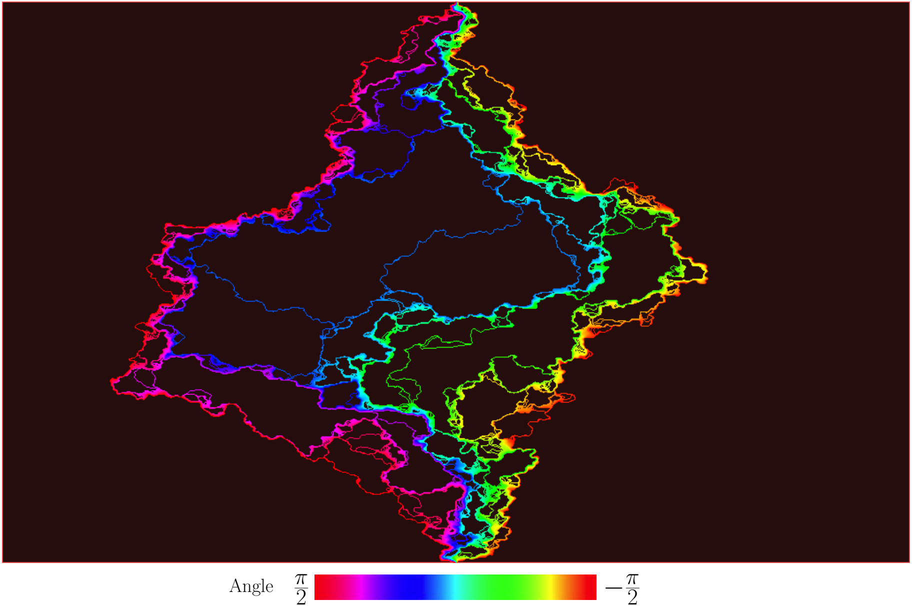


Angle $\frac{\pi}{2}$ $-\frac{\pi}{2}$

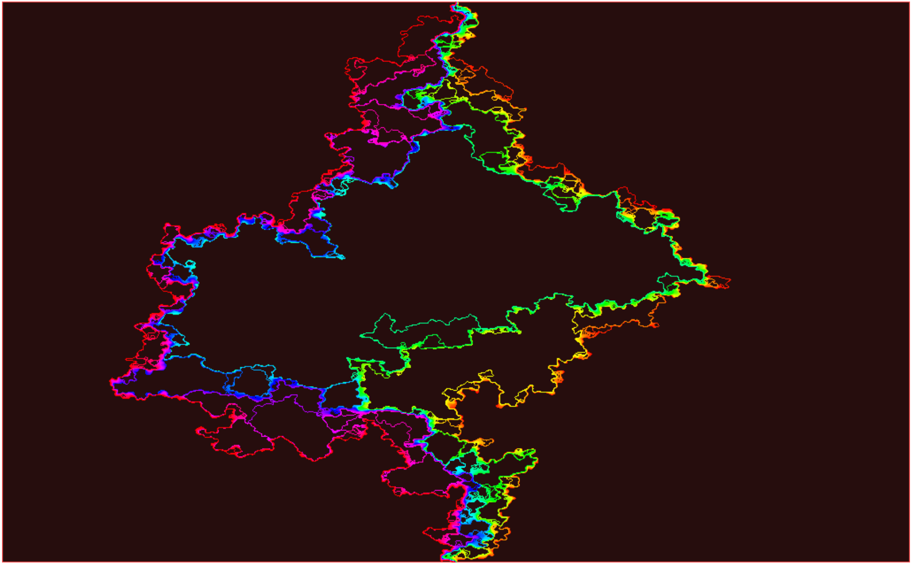
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$



Rays of $e^{ih/\chi}$, h GFF, $\chi = 1.5$ [$\kappa = 1$]

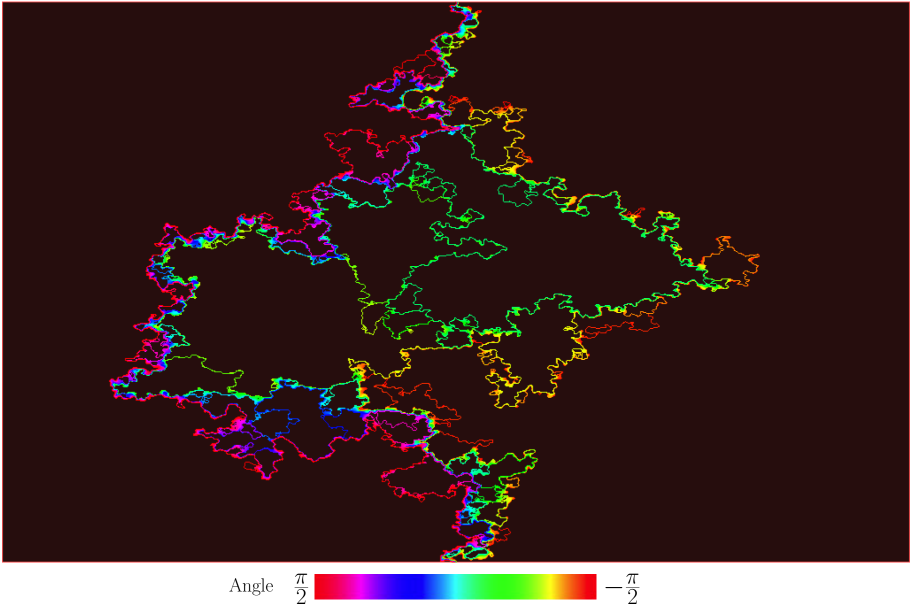


Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 1.02$ $[\kappa = 3/2]$

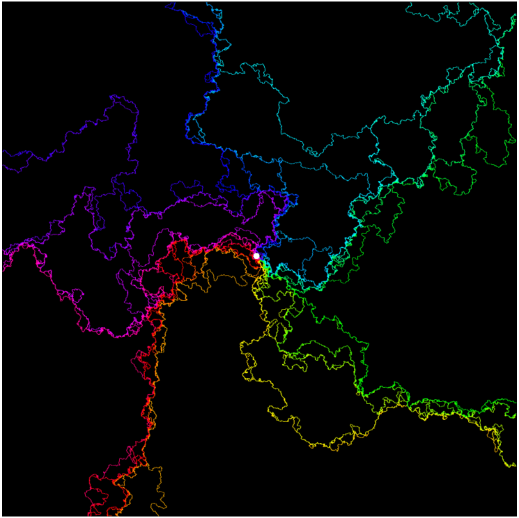


Angle $\frac{\pi}{2}$ $-\frac{\pi}{2}$

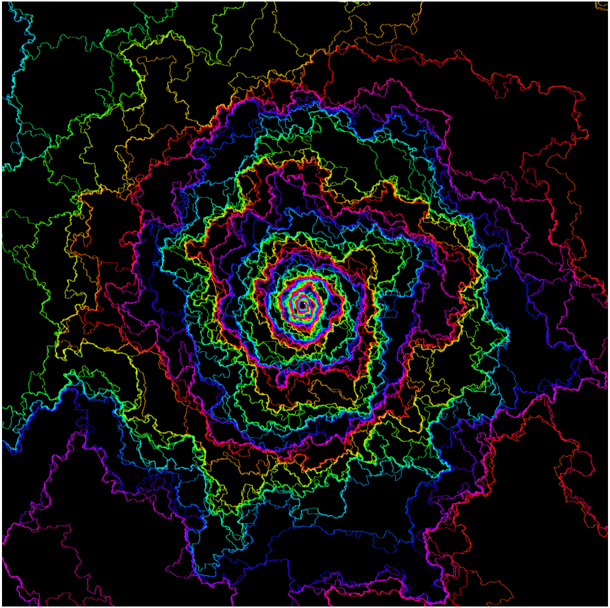
Rays of $e^{ih/\chi}$, h GFF, $\chi = 0.71$ [$\kappa = 2$]



Rays of $e^{ih/\chi}$ starting from an interior point

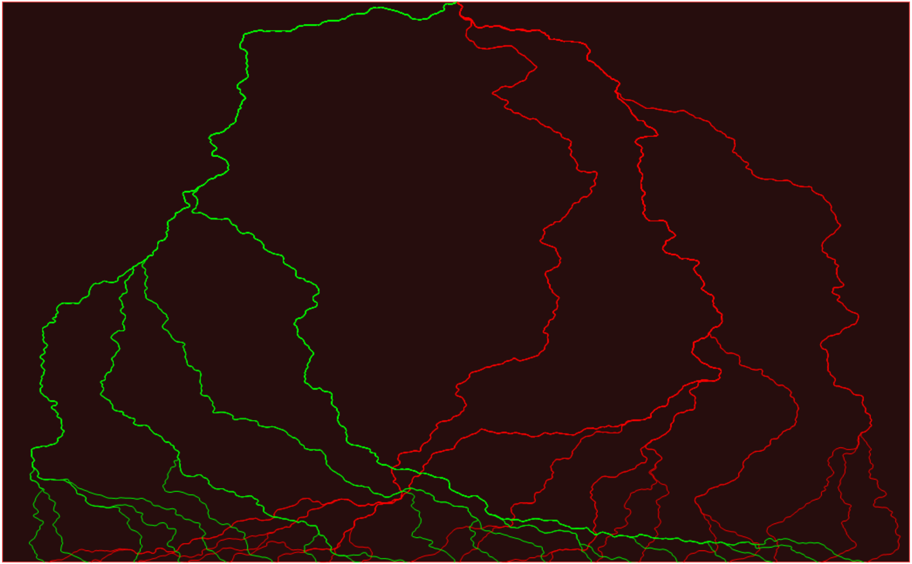


Rays of $e^{i\widetilde{h}/\chi}$, $\widetilde{h} = h + \beta \log|\cdot|$, h GFF, $\beta < 0$

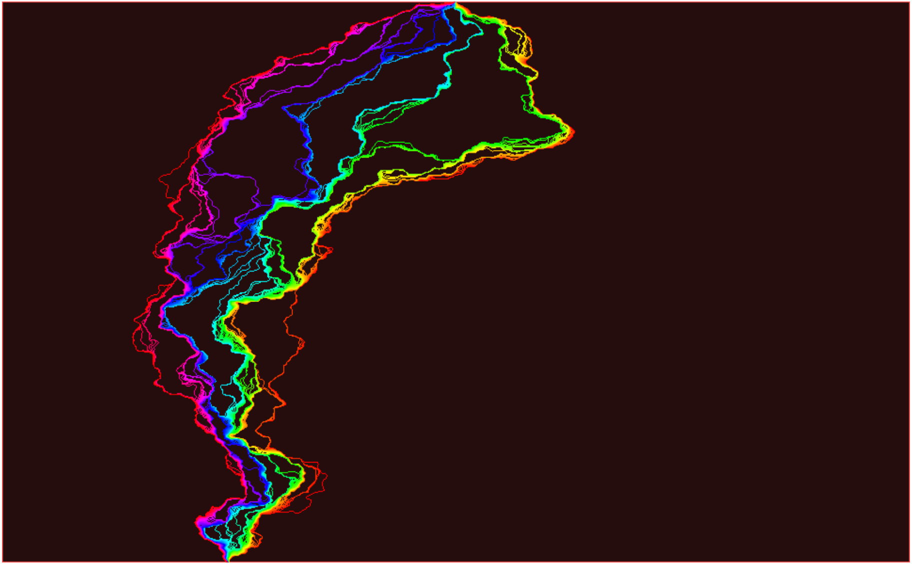


Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$

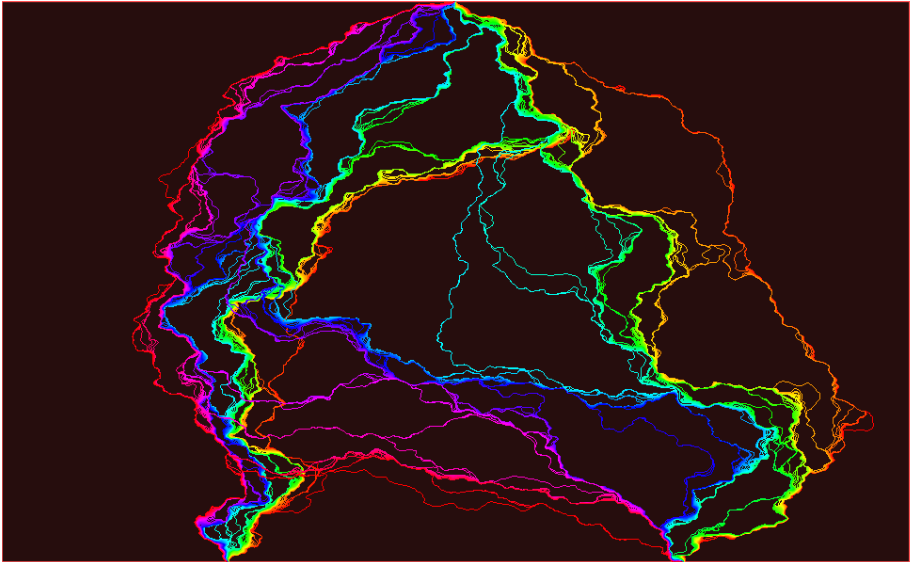
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$



Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$



Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$



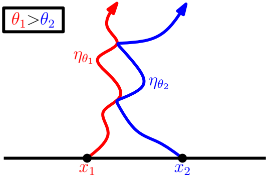
Background

Existence and uniqueness of couplings (η, h) of a GFF h and $\eta \sim \mathrm{SLE}_\kappa$ are studied in the works of Sheffield, Schramm-Sheffield, Dubédat, and Izyurov-Kytöla

New Contributions:

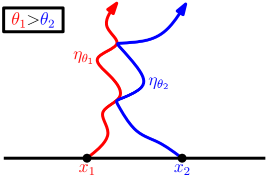
- Developed a robust theory of flow line interaction to make the phenomena observed in the simulations precise
- General forms of SLE duality — the SLE light cone
- $\mathrm{SLE}_\kappa(\underline{\rho})$ processes are continuous (even when they hit the boundary)
  - Important variant of SLE_κ
  - The drift for the driving function includes a linear combination of the Loewner evolution of a collection "force points"
- Double and cut point dimension of SLE
- Reversibility results

Flow line interaction



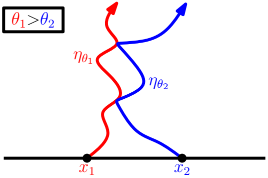
- h GFF on $\mathbf{H}$, $x_1, x_2 \in \partial \mathbf{H}$ with $x_1 \leq x_2$.
- $\eta_{\theta_1}, \eta_{\theta_2}$ flow line of h with angles θ_1, θ_2 starting from x_1, x_2

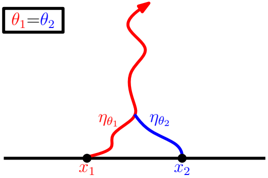
Flow line interaction



- h GFF on $\mathbf{H}$, $x_1, x_2 \in \partial \mathbf{H}$ with $x_1 \leq x_2$.
- $\eta_{\theta_1}, \eta_{\theta_2}$ flow line of h with angles θ_1, θ_2 starting from x_1, x_2
- **Theorem** (M., Sheffield)
  - If $\theta_1 > \theta_2$, then η_{θ_1} stays to the left of η_{θ_2}

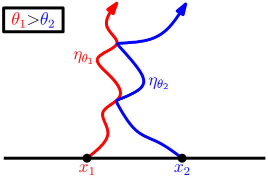
Flow line interaction





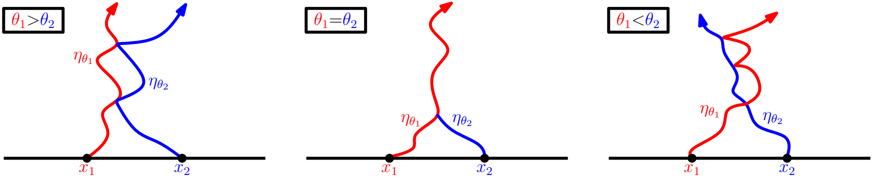
- h GFF on $\mathbf{H}$, $x_1, x_2 \in \partial \mathbf{H}$ with $x_1 \leq x_2$.
- $\eta_{\theta_1}, \eta_{\theta_2}$ flow line of h with angles θ_1, θ_2 starting from x_1, x_2
- **Theorem** (M., Sheffield)
  - If $\theta_1 > \theta_2$, then η_{θ_1} stays to the left of η_{θ_2}
  - If $\theta_1 = \theta_2$, then η_{θ_1} merges with η_{θ_2} upon intersecting

Flow line interaction



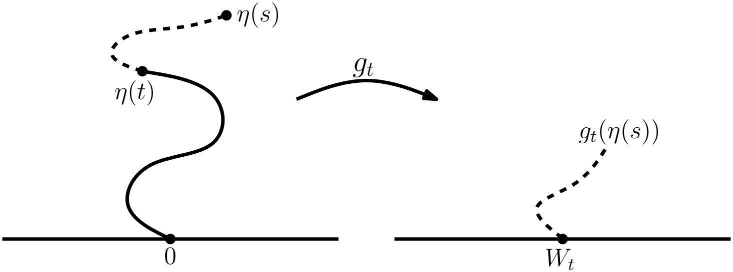
- h GFF on $\mathbf{H}$, $x_1, x_2 \in \partial \mathbf{H}$ with $x_1 \leq x_2$.
- $\eta_{\theta_1}, \eta_{\theta_2}$ flow line of h with angles θ_1, θ_2 starting from x_1, x_2
- **Theorem** (M., Sheffield)
  - If $\theta_1 > \theta_2$, then η_{θ_1} stays to the left of η_{θ_2}
  - If $\theta_1 = \theta_2$, then η_{θ_1} merges with η_{θ_2} upon intersecting
  - If $\theta_1 < \theta_2$, then η_{θ_1} crosses η_{θ_2} upon intersecting

Flow line interaction



- h GFF on $\mathbf{H}$, $x_1, x_2 \in \partial \mathbf{H}$ with $x_1 \leq x_2$.
- $\eta_{\theta_1}, \eta_{\theta_2}$ flow line of h with angles θ_1, θ_2 starting from x_1, x_2
- **Theorem** (M., Sheffield)
  - If $\theta_1 > \theta_2$, then η_{θ_1} stays to the left of η_{θ_2}
  - If $\theta_1 = \theta_2$, then η_{θ_1} merges with η_{θ_2} upon intersecting
  - If $\theta_1 < \theta_2$, then η_{θ_1} crosses η_{θ_2} upon intersecting
- In each case, the conditional law of η_{θ_1} given η_{θ_2} is an $\mathrm{SLE}_\kappa(\underline{\rho})$ process

$\mathrm{SLE}_\kappa(\rho)$

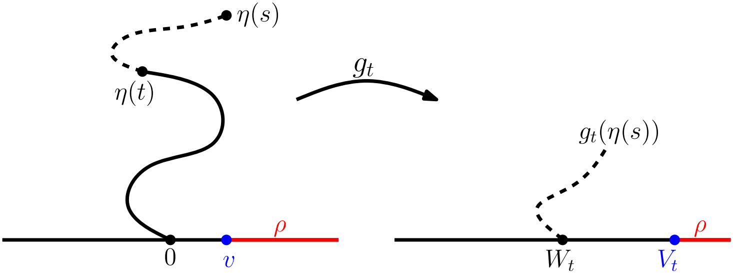


Loewner ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \qquad \text{where} \qquad g_0(z) = z.$$

SLE_κ**:** $W_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion

$\mathrm{SLE}_\kappa(\rho)$



Loewner ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \qquad \text{where} \qquad g_0(z) = z.$$

SLE_κ**:** $W_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion

$\mathrm{SLE}_\kappa(\rho)$**:** Force point $\mathbf{v} > 0$ and a weight $\rho \in \mathbf{R}$.

$$dW_t = \sqrt{\kappa} dB_t + \frac{\rho}{W_t - \mathbf{V_t}} dt \quad \text{where} \quad d\mathbf{V_t} = \frac{2}{\mathbf{V_t} - W_t} dt, \quad \mathbf{V_0} = \mathbf{v}.$$

Continuity $\mathrm{SLE}_\kappa(\underline{\rho})$

SLE_κ is continuous ($\kappa \neq 8$: Rohde-Schramm, $\kappa = 8$: Lawler-Schramm-Werner)

Continuity $\mathrm{SLE}_\kappa(\underline{\rho})$

SLE_κ is continuous ($\kappa \neq 8$: Rohde-Schramm, $\kappa = 8$: Lawler-Schramm-Werner)

Theorem (M., Sheffield)

$\mathrm{SLE}_\kappa(\underline{\rho})$ *processes are continuous.*

Continuity $\mathrm{SLE}_\kappa(\underline{\rho})$

SLE_κ is continuous ($\kappa \neq 8$: Rohde-Schramm, $\kappa = 8$: Lawler-Schramm-Werner)

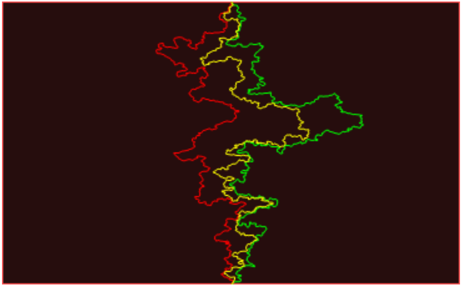
Theorem (M., Sheffield)

$\mathrm{SLE}_\kappa(\underline{\rho})$ *processes are continuous.*

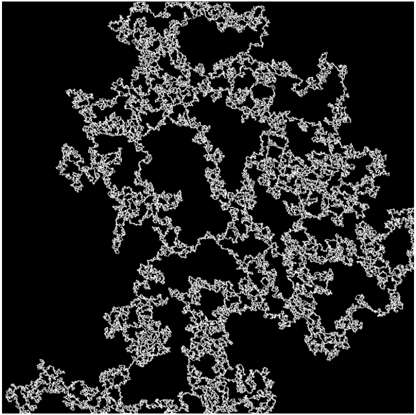
Proof: (for $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$)
Let η_θ be the flow line with angle θ and fix $\theta_1 > 0 > \theta_2$.

Fact: $\eta_{\theta_1}, \eta_{\theta_2}, \eta_0$ are continuous if they do not hit the boundary.
Reason: Their law is mutually absolutely continuous with respect to SLE_κ.

The law of η_0 given η_{θ_1} and η_{θ_2} is an $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ process. □



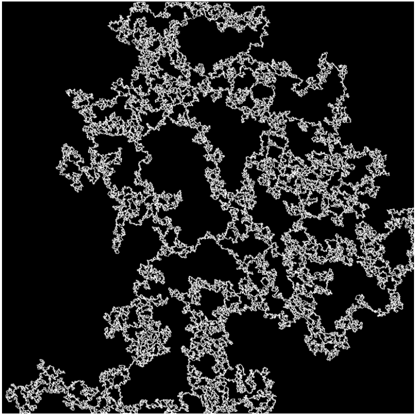
SLE Duality



Fix $\kappa \in (0,4)$. The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_κ process.

- Predicted by Duplantier

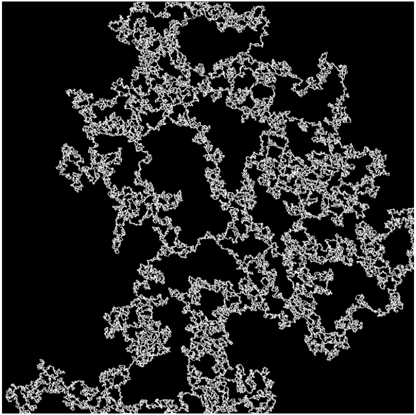
SLE Duality



Fix $\kappa \in (0,4)$. The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_κ process.

- Predicted by Duplantier
- Natural for certain values of κ, i.e. $\kappa = 2$ (LERW) and $16/\kappa = 8$ (UST)

SLE Duality

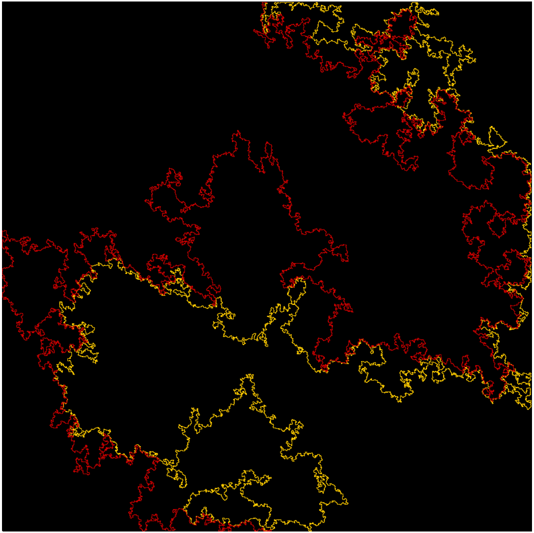


Fix $\kappa \in (0, 4)$. The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_κ process.

- Predicted by Duplantier
- Natural for certain values of κ, i.e. $\kappa = 2$ (LERW) and $16/\kappa = 8$ (UST)
- Proved in various forms by Zhan and Dubédat

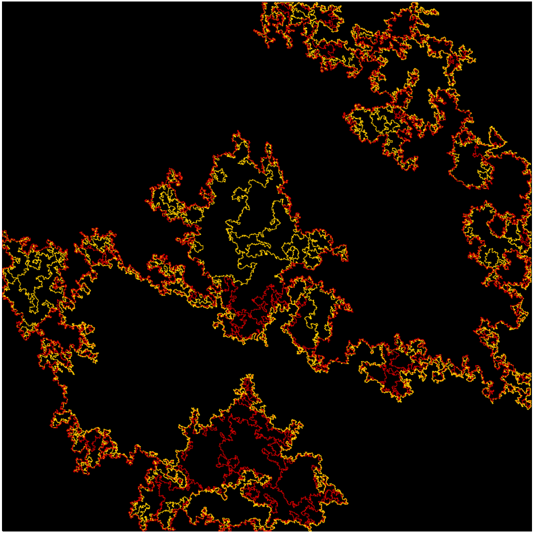
Duality in the GFF: the SLE Light Cone

Flow lines with fixed angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.



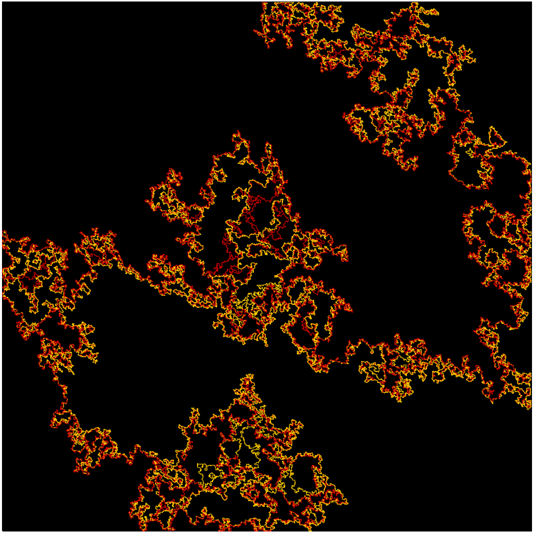
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; one direction change.



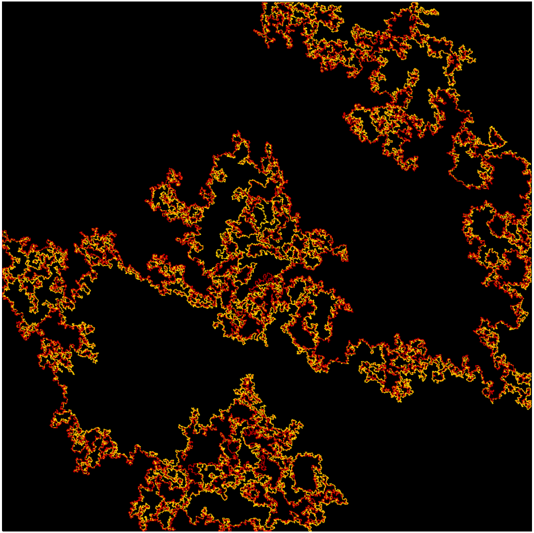
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; two direction changes.



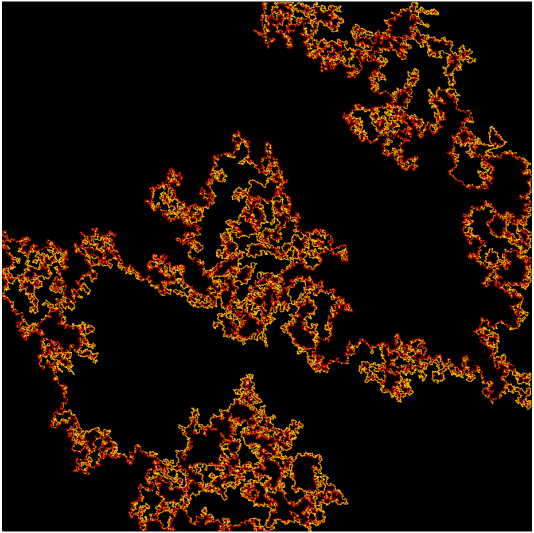
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; three direction changes.



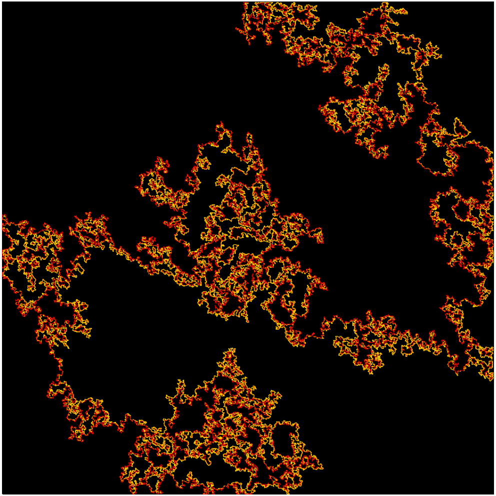
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; four direction changes.

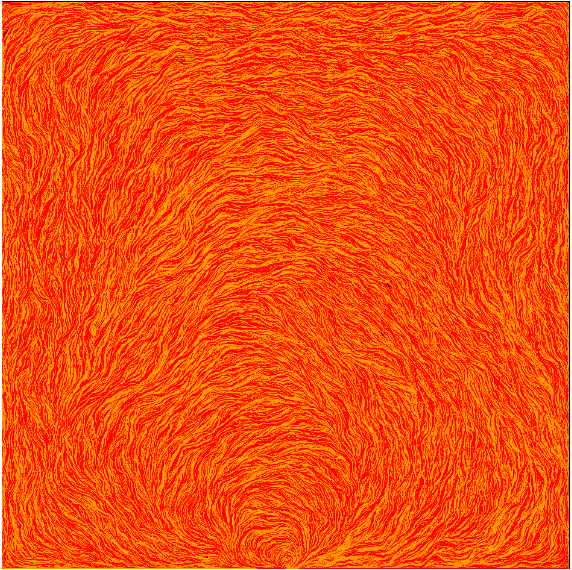


Duality in the GFF: the SLE Light Cone

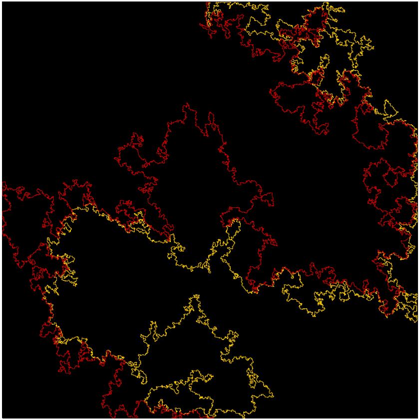
Theorem (M., Sheffield): The set of all points accessible by SLE_κ flow lines ($\kappa \in (0,4)$) with angles restricted in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is an $\mathrm{SLE}_{16/\kappa}$ process.



SLE_{128} Light Cone

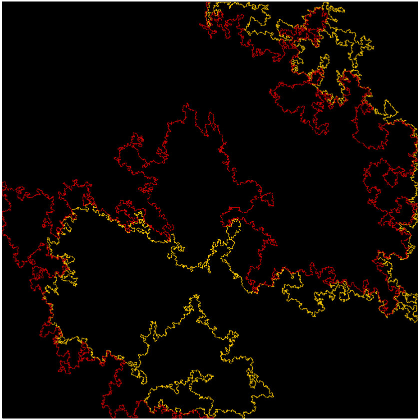


Dimensions

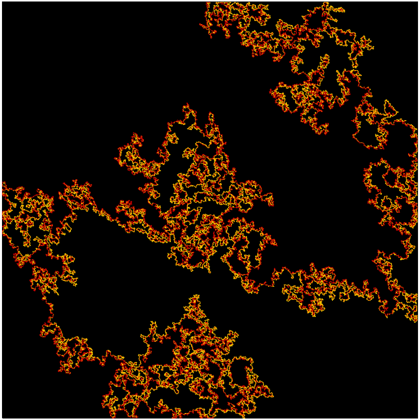


Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$

Dimensions



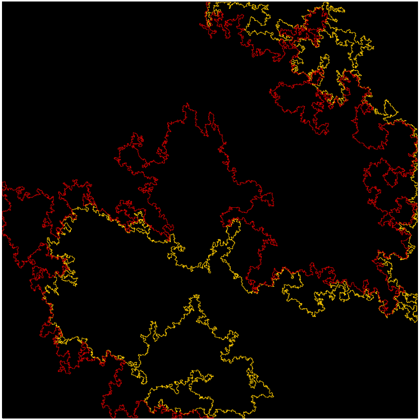
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$



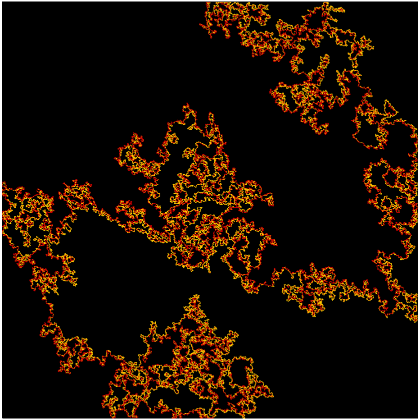
$\mathrm{SLE}_{\kappa'}$ for $\kappa' = 16/\kappa$

Dimensions

Theorem (M., Wu) The cut point dimension of $\mathrm{SLE}_{\kappa'}$ for $\kappa' \in (4,8)$ is $3 - \frac{3\kappa'}{8}$



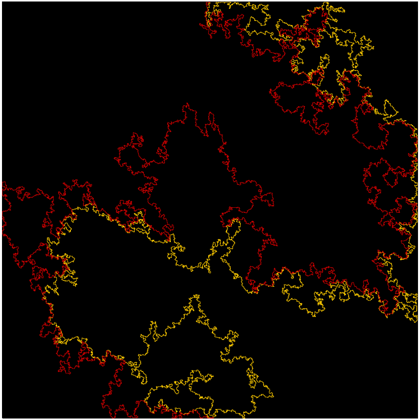
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$



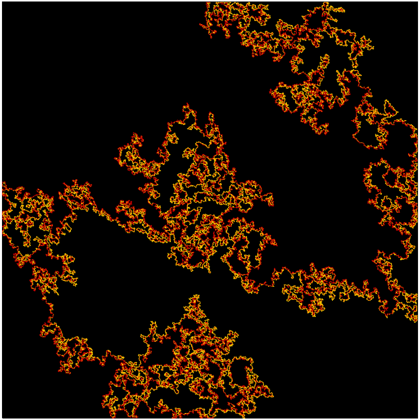
$\mathrm{SLE}_{\kappa'}$ for $\kappa' = 16/\kappa$

Dimensions

Theorem (M., Wu) The cut point dimension of $\mathrm{SLE}_{\kappa'}$ for $\kappa' \in (4,8)$ is $3 - \frac{3\kappa'}{8}$
Corollary of a result which gives the dimension of the intersection of flow lines with different angles.



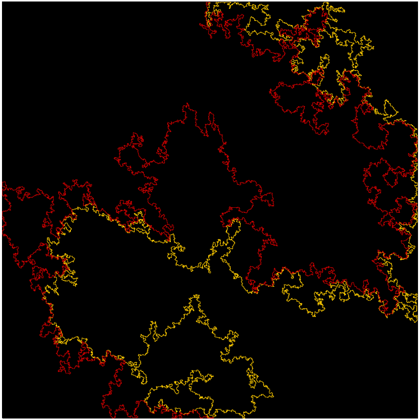
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$



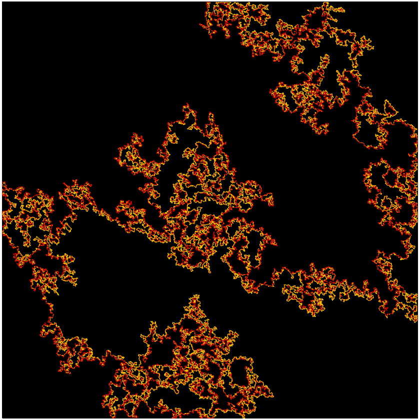
$\mathrm{SLE}_{\kappa'}$ for $\kappa' = 16/\kappa$

Dimensions

Theorem (M., Wu) The cut point dimension of $\mathrm{SLE}_{\kappa'}$ for $\kappa' \in (4,8)$ is $3 - \frac{3\kappa'}{8}$
Corollary of a result which gives the dimension of the intersection of flow lines with different angles. Also gives the double point dimension is $2 - (12-\kappa')(4+\kappa')/(8\kappa')$.



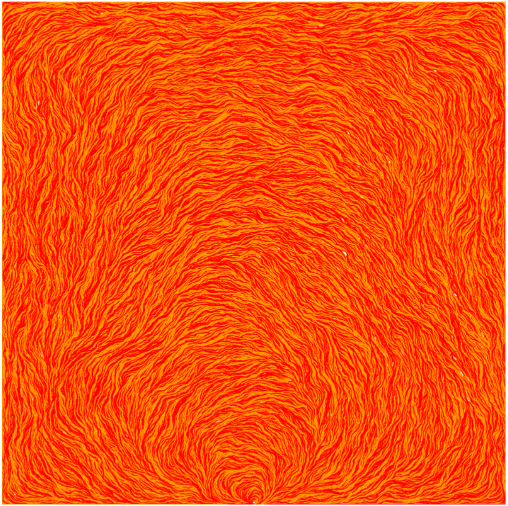
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$



$\mathrm{SLE}_{\kappa'}$ for $\kappa' = 16/\kappa$

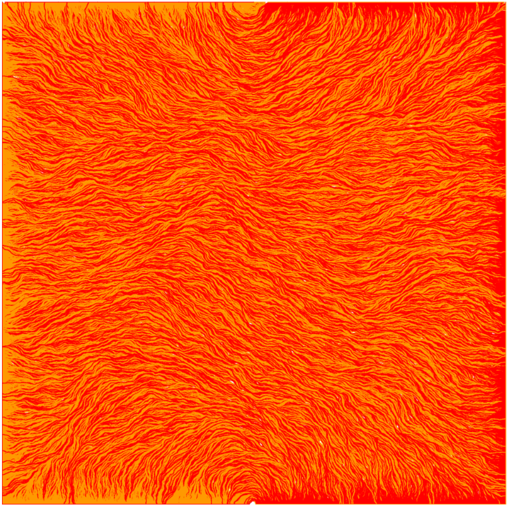
Reversibility of SLE_κ for $\kappa \geq 8$

The time-reversal of an SLE_κ is not an SLE_κ for $\kappa > 8$ (Rohde-Schramm).



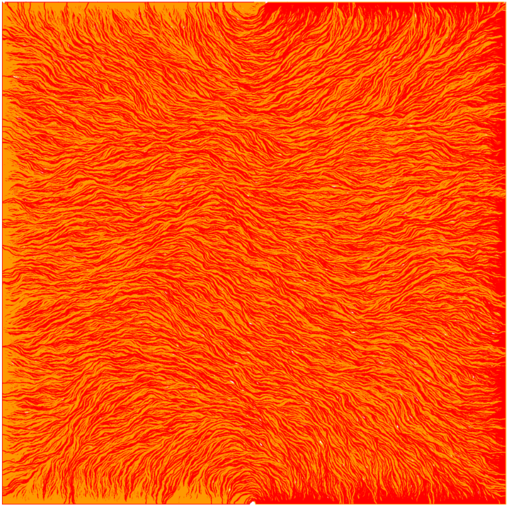
Reversibility of SLE_κ for $\kappa \geq 8$

Theorem (M., Sheffield) $\mathrm{SLE}_\kappa(\frac{\kappa}{4}-2;\frac{\kappa}{4}-2)$ is reversible for $\kappa \geq 8$.



Reversibility of SLE_κ for $\kappa \geq 8$

Theorem (M., Sheffield) $\mathrm{SLE}_\kappa(\frac{\kappa}{4} - 2; \frac{\kappa}{4} - 2)$ is reversible for $\kappa \geq 8$. The time-reversal of ordinary SLE_κ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$.



The SLE_κ fan is "reversible"