AREA DISTORTION, QUASICIRCLES AND HARMONIC MEASURE

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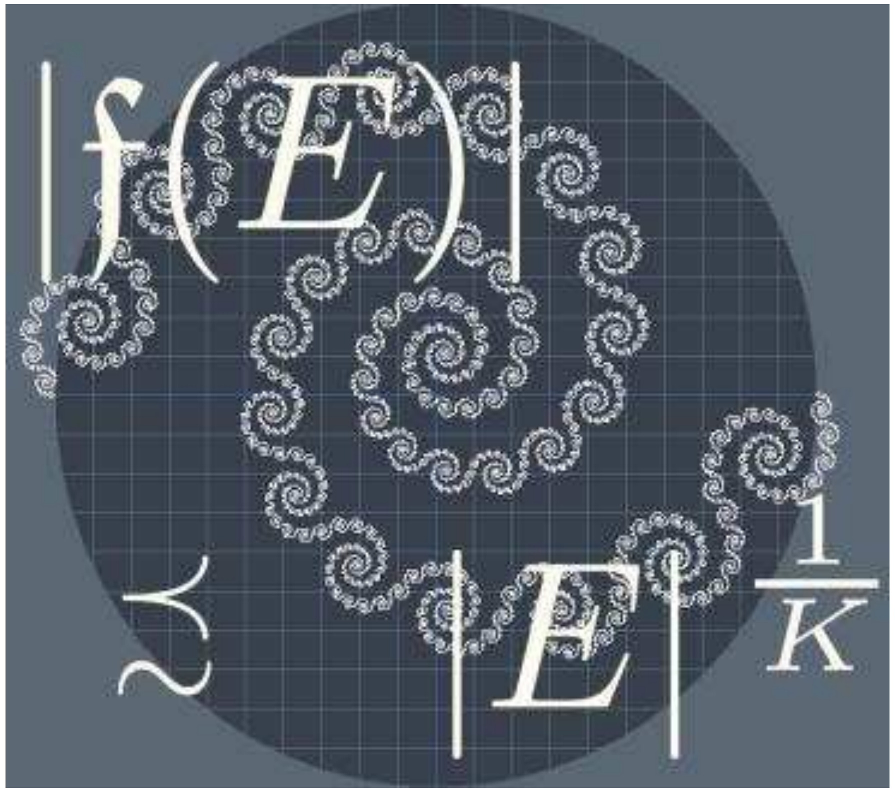
UNIVERSITY OF HELSINKI

Los Angeles – May 2013

INSTITUTE FOR PURE AND APPLIED



Los A

QUASICONFORMAL
AND ELLIPTIC PDEs

May 20 – 24, 2013

ORGANIZING COMMITTEE: John Garnett (UCLA), **Tadeusz Iwan**
Steffen Rohde (University of Washington), **Eero Saksman** (Univers
(University of Washington)

Scientific Overview

The theories of quasiconformal mappings and elliptic partial differential equa
dating back the work of Vekua, Bers, Bojarski, and others. During the last ten ye
revitalized through new methods and breakthroughs and surprising applications
methods. These include the solution of Calderón's problem of impedance tomo

HOLOMORPHIC MOTIONS

$$\Phi\colon \mathbb{D} \times E \to \mathbb{C}, \quad E \subset \mathbb{C}$$

- $z \mapsto \Phi(\lambda, z) = \Phi_\lambda(z)$ is injective for all $\lambda \in \mathbb{D}$,
- $\lambda \mapsto \Phi(\lambda, z)$ is holomorphic for all $z \in E$,
- $\Phi(0, z) \equiv z$.

Mañé-Sad-Sullivan, Slodkowski's λ-lemma:

"holomorphic motions = quasiconformal maps"

$\{\Phi_\lambda(z)\}$ (extends to) an analytic family of qc maps

BELTRAMI EQUATION

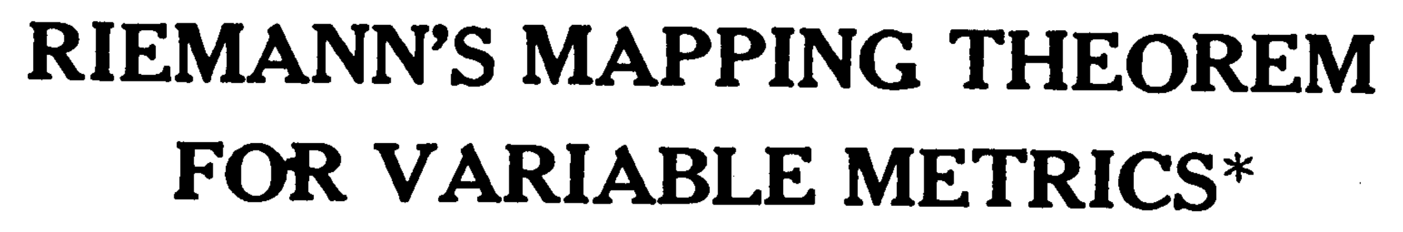
$$f_{\bar z} = \mu(z)\, f_z, \quad |\mu(z)| \leqslant k\, \chi_D(z), \quad 0 \leqslant k < 1$$

$f(z) = z + \mathcal{O}(1/z)$ principal *quasiconformal mapping*

$f \in W^{1,2}_{loc}(\mathbb{C})$ homeomorphism

$$K = \frac{1+k}{1-k}$$

$J(z,f) > 0$ a.e.



RIEMANN'S MAPPING THEOREM FOR VARIABLE METRICS*

existence and uniqueness

Morrey

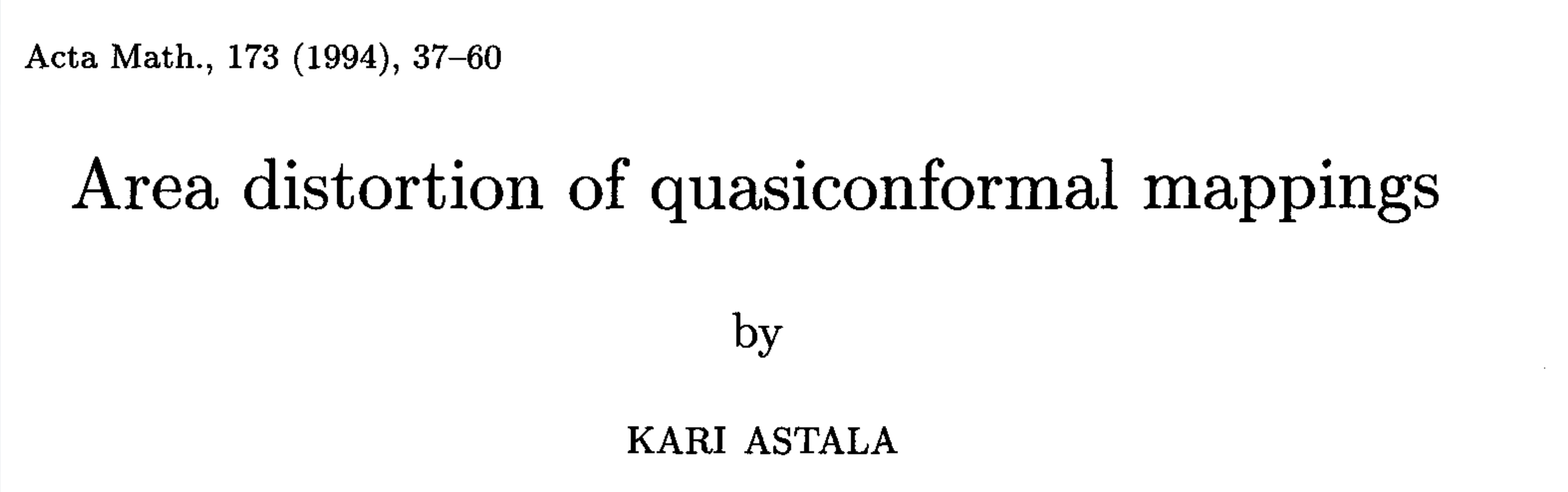
regularity

Bojarski, Astala

analytic dependence

Ahlfors-Bers

Beurling transform



Acta Math., 173 (1994), 37–60

Area distortion of quasiconformal mappings

by

KARI ASTALA

area distortion

$$|fE| \lesssim |E|^{\frac{1}{K}}$$

dim distortion

$$\dim E = 1$$

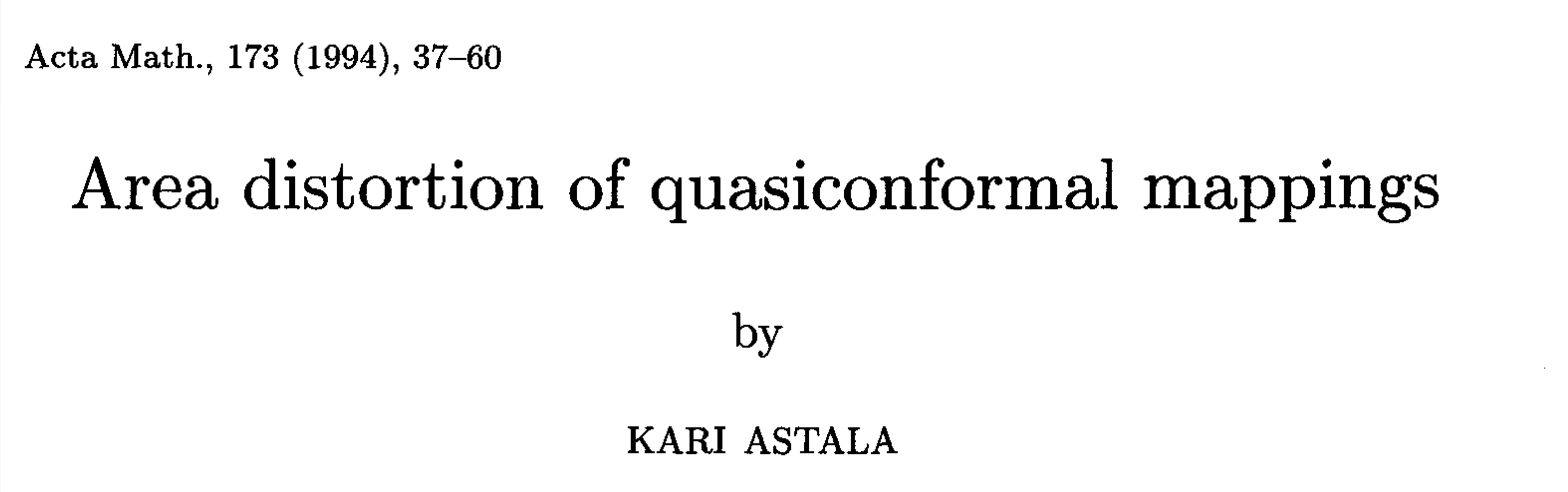
$$1 - k \leqslant \dim fE \leqslant 1 + k$$

higher integrability

$$f \in W^{1,p}_{loc}(\mathbb{C})$$

$$p < \frac{2K}{K-1}$$

Beurling transform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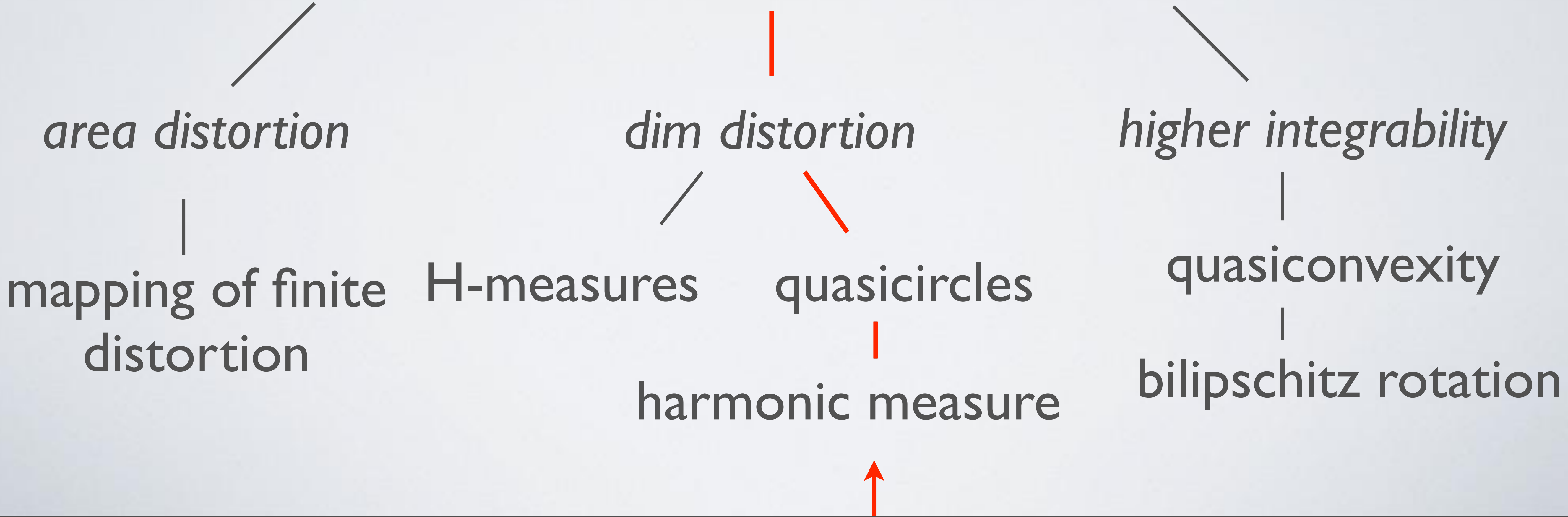

area distortion

mapping of finite distortion

dim distortion

H-measures

quasicircles

harmonic measure

higher integrability

quasiconvexity

bilipschitz rotation

Quasiconvexity result

Astala-Iwaniec-Prause-Saksman

$$B_p(z,w) = \big(|z| - (p-1)\,|w|\big)\cdot\big(|z| + |w|\big)^{p-1} \qquad p \geqslant 2$$

$$B_p(Df) = B_p(f_z, f_{\bar z})$$

Theorem: $f(z) \in z + C_0^\infty(\Omega), \quad B_p(Df) \geqslant 0, \; z \in \Omega$

$$\int_\Omega B_p\,(Df) \leqslant \int_\Omega B_p\,(Id) = |\Omega|$$

Burkholder’s martingale inequality

$$\mathbb{E}\,B_p(X_n, Y_n) \leqslant 0 \qquad \text{for} \qquad X_n \prec Y_n$$

Stretching vs Rotation

harmonic dependence *“conjugate harmonic”*

stretching	rotation
quasiconformal	**bilipschitz**
Grötzsch problem	John’s problem
Hölder exponent	rate of spiralling
$\log J(z,f) \in \mathrm{BMO}$	$\arg f_z \in \mathrm{BMO}$
higher integrability	exponential integrability
multifractal spectrum	

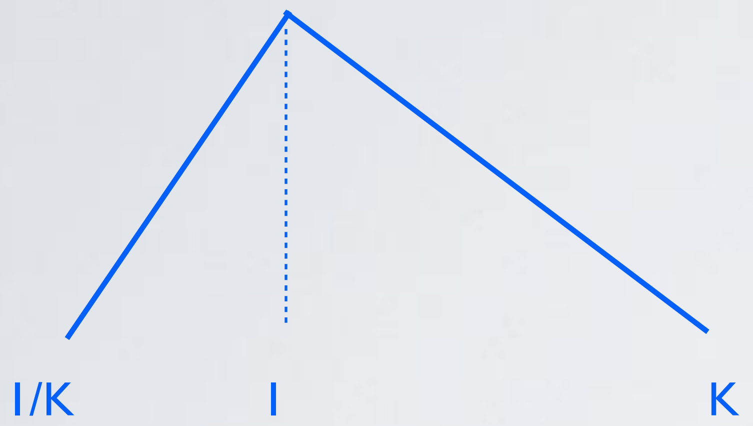
Multifractal spectra

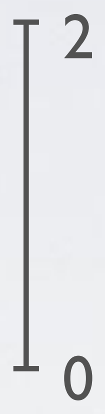
Astala-Iwaniec-Prause-Saksman

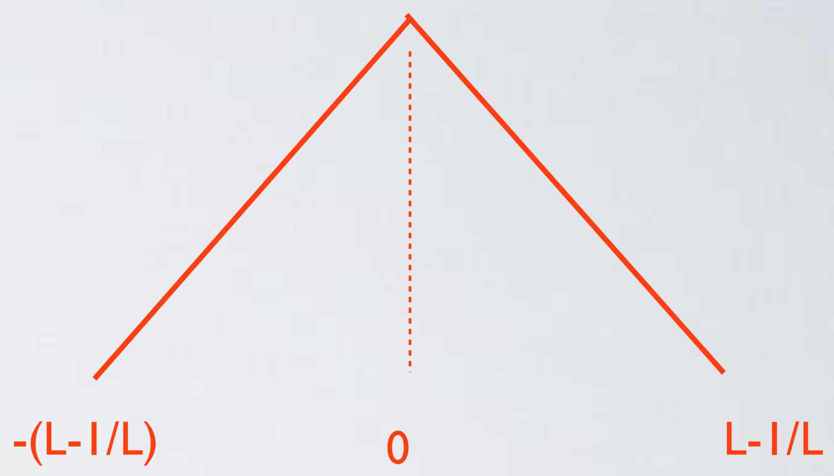
K-quasiconformal

$f\colon \mathbb{C} \to \mathbb{C}$

L-bilipschitz







$$\dim_H\{z \in \mathbb{C} : \alpha(z) = \alpha\} \leq 1 + \alpha - \frac{|1-\alpha|}{k}$$

$$\dim_H\{z : \gamma(z) = \gamma\} \leq 2 - \frac{2L}{L^2 - 1}|\gamma|$$

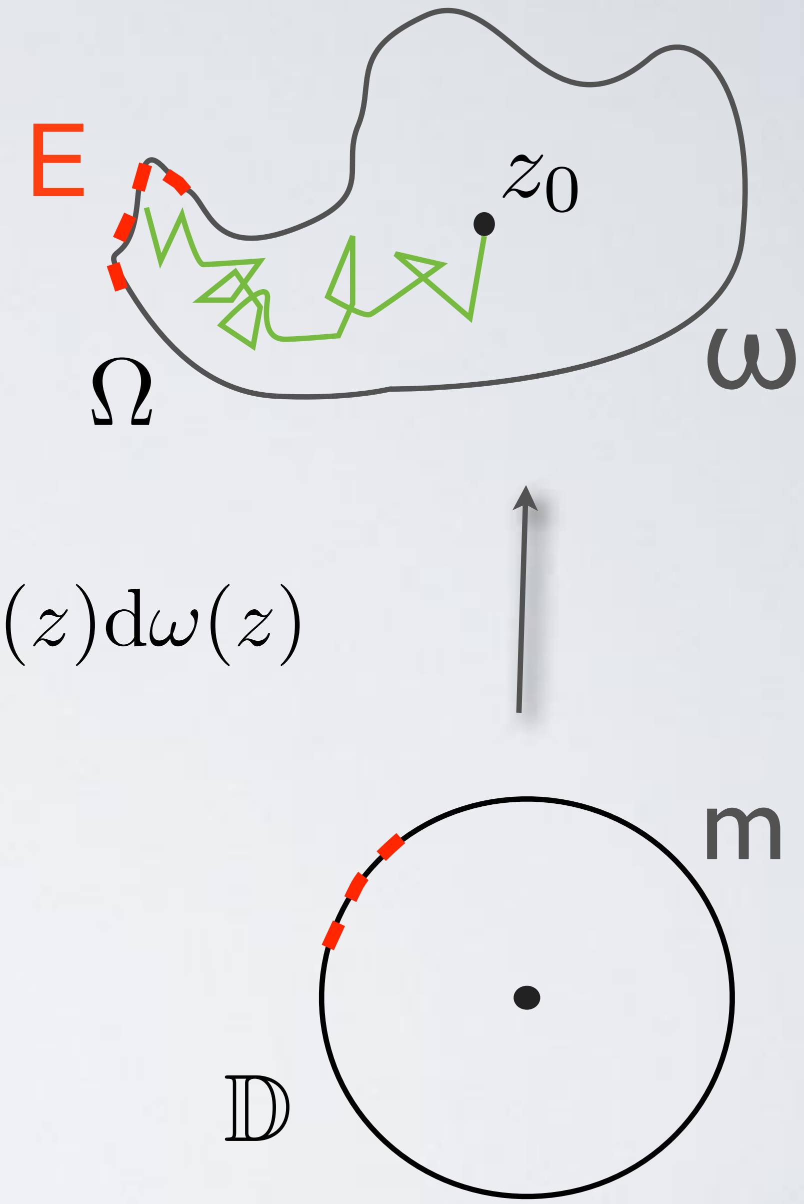
$$\alpha(z_0) = \lim_{|z-z_0|=r_n \to 0} \frac{\log|f(z) - f(z_0)|}{\log|z - z_0|}$$

$$\gamma(z_0) = \lim_{r_n \to 0} \frac{\arg(f(z_0 + r_n) - f(z_0))}{\log|f(z_0 + r_n) - f(z_0)|}$$

What about conformal maps?

HARMONIC MEASURE

$$\omega_{\Omega,z_0}(E)$$



- Brownian motion
  hitting probability
- potential theory
  equilibrium measure

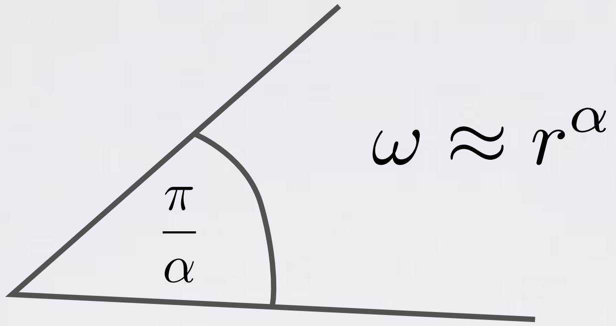
$$u(z_0) = \int_{\partial\Omega} u(z)\mathrm{d}\omega(z)$$

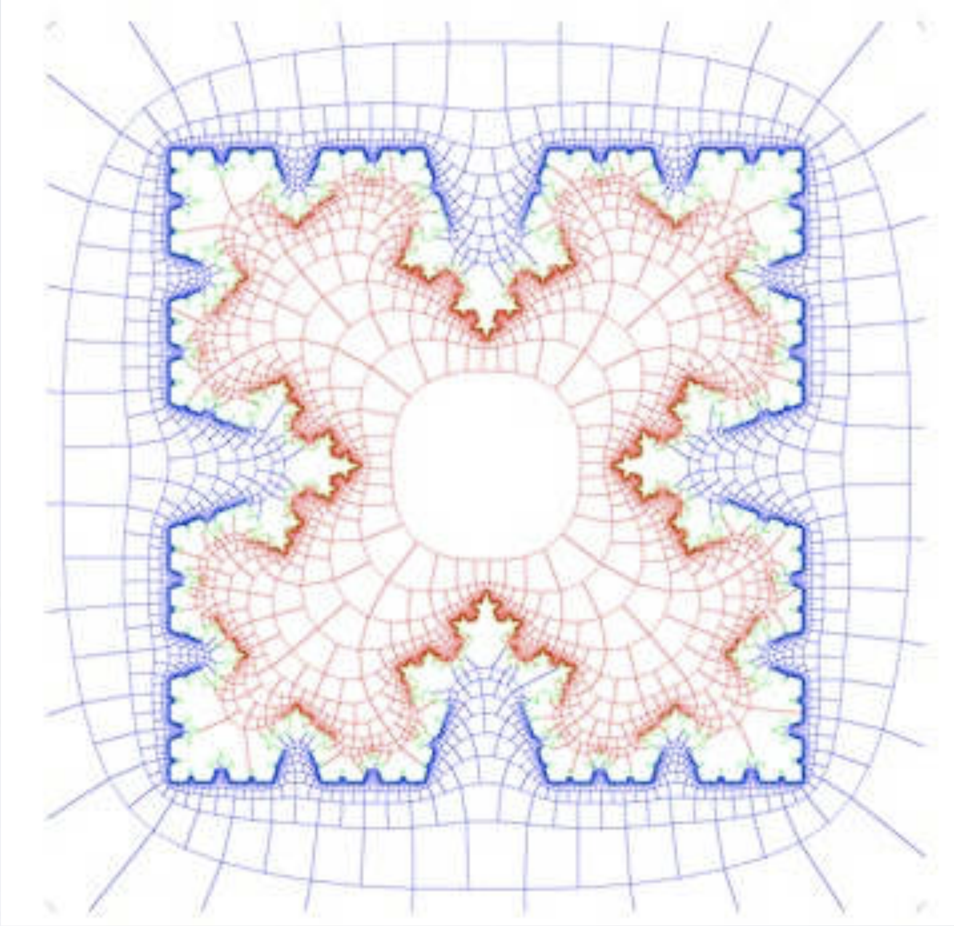
- conformal map
  image of arclength

MULTIFRACTALITY OF ω

'fjords and spikes'

$\mathcal{F}_\alpha$ scaling: $\omega B(z,r) \approx r^\alpha$





multifractal spectrum: $f_\Omega(\alpha) = \dim \mathcal{F}_\alpha$

Beurling's estimate: $\alpha \geq 1/2$

Makarov's theorem: $\dim \omega = 1$

$\alpha = 1$ ω-a.e.

Courtesy of D. Marshall

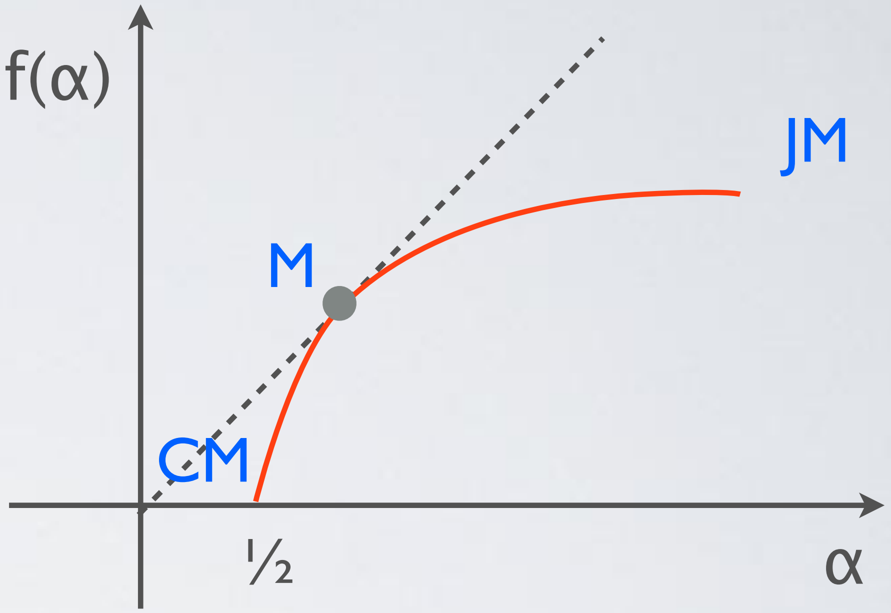
$$\dim \omega \overset{\text{def}}{=} \inf\{\dim E : \omega(E) = 1\}$$

UNIVERSAL SPECTRUM

related to many problems in complex analysis

$$f(\alpha) = \sup_{\Omega} f_{\Omega}(\alpha)$$

over all simply connected domains



Universal Spectrum Conjecture:

$$\boxed{f(\alpha) = 2 - \frac{1}{\alpha}}$$

Brennan-Carleson-Jones-Krätzer-...

INTEGRAL MEANS

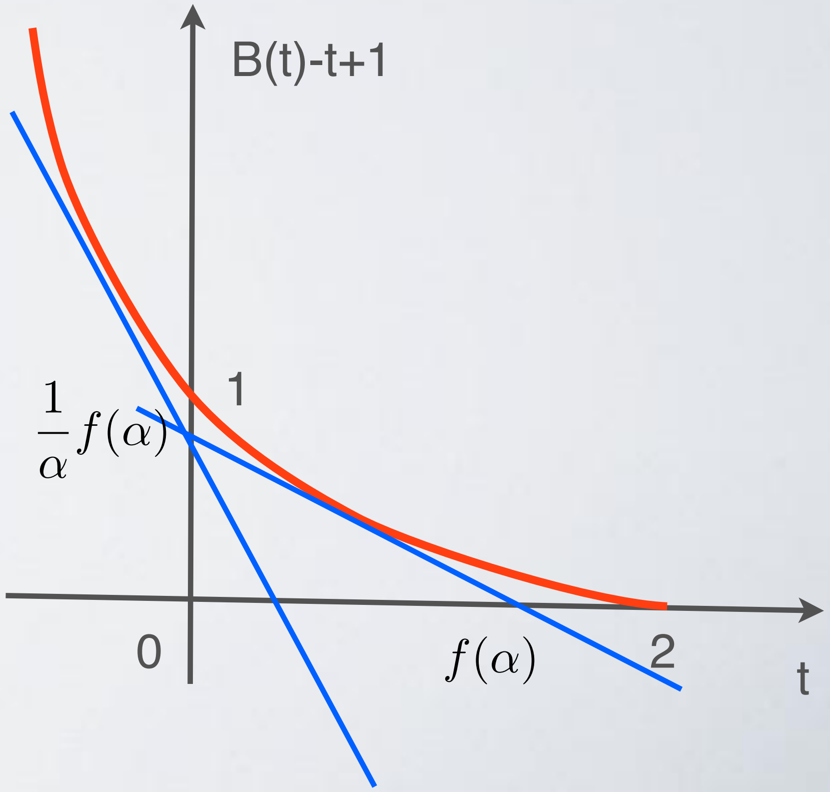
$\varphi\colon \mathbb{D} \to \Omega$ (bounded) conformal map

$$\beta_\varphi(t) = \inf\left\{\beta\colon \int |\varphi'(re^{i\theta})^t|\,d\theta = O((1-r)^{-\beta})\right\}, \quad t \in \mathbb{R}$$

$$B(t) = \sup_\Omega\{\beta_\varphi(t)\}, \quad \pi(t) = B(t) - t + 1$$

Thm (Makarov): Legendre transforms:

$$f(\alpha) = \inf_t\{\alpha\pi(t) + t\} \quad \longleftrightarrow \quad \pi(t) = \sup_\alpha\{(f(\alpha) - t)/\alpha\}$$



Universal Spectrum Conjecture

$$B(t) = \frac{|t|^2}{4}, \quad |t| \le 2$$

CLASSICAL PROBLEMS

- Brennan's conjecture: $B(-2) = 1$ $\quad t = -2 \leftrightarrow \alpha = 1/2$

  $$f(\alpha) \leq 4(\alpha - 1/2)$$

- Carleson-Jones' conjecture: $B(1) = 1/4$ $\quad t = 1 \leftrightarrow \alpha = 2$

  coeff decay for bdd univalent fns $|a_n| \leq n^{-3/4+\epsilon}$

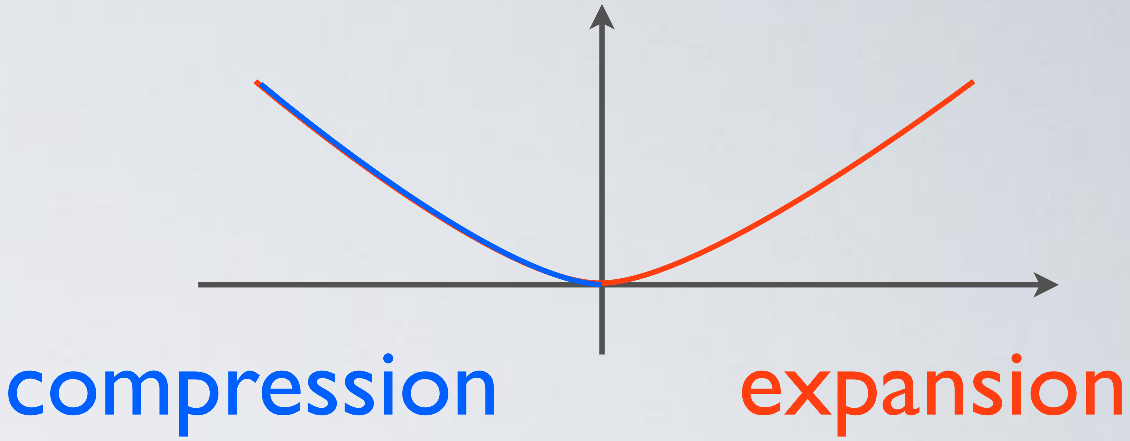
- Hölder domain conjecture: $\dim_H \partial\Omega \leq 2 - \delta,$

  $t \geqslant 0 \leftrightarrow \alpha \geqslant 1$ $\quad \Omega$ is δ-Hölder

- Makarov's thm: $\dim \omega = 1$ $\quad t = 0 \leftrightarrow \alpha = 1$

DIFFICULTIES

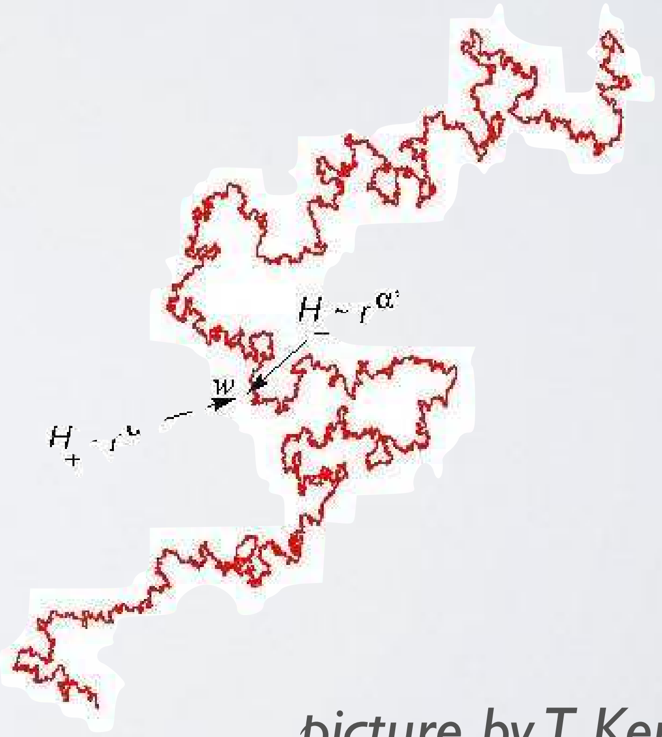
- one-sided, no symmetry



- no quasicircles

- no physical intuition

  not *SLE*



picture by T. Kennedy

DIM OF QUASICIRCLES

Smirnov’s theorem:

$$\dim \Gamma \leqslant 1 + k^2$$

for any K-quasicircle

Prause-Tolsa-Uriarte-Tuero:

$$H^{1+k^2}(\Gamma \cap B(z,r)) \leq C(K) r^{1+k^2}$$

for any $z \in \mathbb{C}, r > 0$

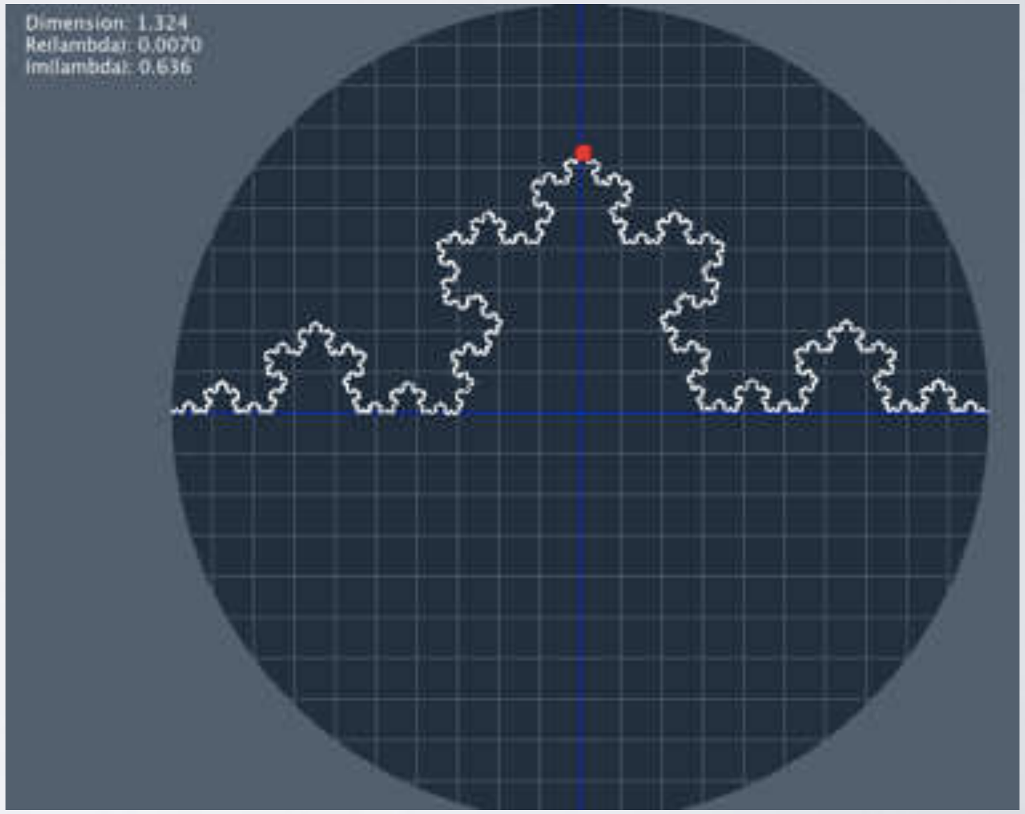
Sharpness?

the best c for

$$\dim \Gamma \geqslant 1 + c\, k^2$$

examples of

$$(k \to 0)$$



Dimension: 1.324
Re(lambda): 0.0070
Im(lambda): 0.636

snowflake: c≈0.69

SPECTRUM OF QUASIDISKS

$\varphi\colon \mathbb{D} \to \Omega$ conformal with *K*-quasiconformal extension

$B_K(t)$ - *integral means spectrum*

Conjecture:
(≈universal spectrum)

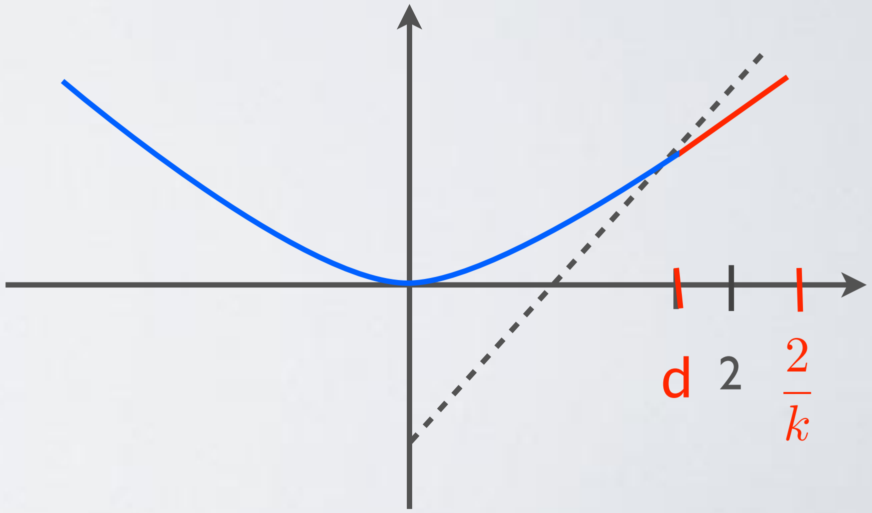
$$B_K(t) \leq \frac{k^2t^2}{4} \quad \text{for } |t| \leqslant \frac{2}{k}, \qquad k = \frac{K-1}{K+1}$$

Theorem:
(Prause-Smirnov)

$$B_K(t) \leq \frac{k^2t^2}{4} \quad \text{for } t \geqslant d, \qquad 1 \leq d \leq 2 \qquad \frac{k^2d^2}{4} = d-1.$$

$t = d$ $\quad 1 + k^2$-bound for quasicircles

$t = \frac{2}{k}$ $\quad$ higher integrability up to $p < \frac{2(K+1)}{K-1}$



inside/outside symmetry

HARMONIC MEASURE AND HOLOMORPHIC MOTIONS

joint project with
Kari Astala and Stas Smirnov

TWO-SIDED SPECTRUM

two-sided spectrum

symmetry

$$f(\alpha_-,\alpha_+) = \dim \mathcal{F}_{\alpha_-,\alpha_+}$$

rotation (Binder)

‘complexification’

$$f(\alpha_-,\alpha_+,\gamma) = \dim \mathcal{F}_{\alpha_-,\alpha_+,\gamma}$$

γ-spiralling

Beurling’s estimate

$$\frac{1}{\alpha_-} + \frac{1}{\alpha_+} \leq \frac{2}{1+\gamma^2}$$

BI-MOTION

Take Beltrami $\|\mu\|_\infty = 1$ in the upper halfplane $\mathbb{H}_+$

$$\mu_{\lambda,\eta}(z) = \begin{cases} \lambda\mu(z) & \text{in } \mathbb{H}_+ \\ \eta\overline{\mu(\bar{z})} & \text{in } \mathbb{H}_- \end{cases} \qquad (\lambda,\eta) \in \mathbb{D}^2$$

$$\bar{\partial}\varphi_{\lambda,\eta} = \mu_{\lambda,\eta}\,\partial\varphi_{\lambda,\eta}$$

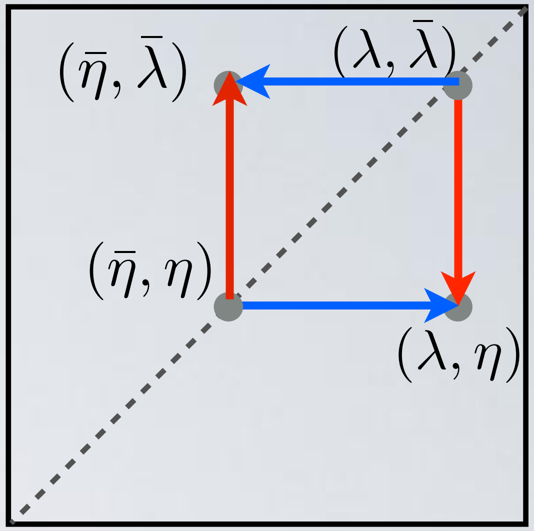
(normalized) homeomorphic solution

$$\big(z,(\lambda,\eta)\big) \mapsto \varphi_{\lambda,\eta}(z)$$

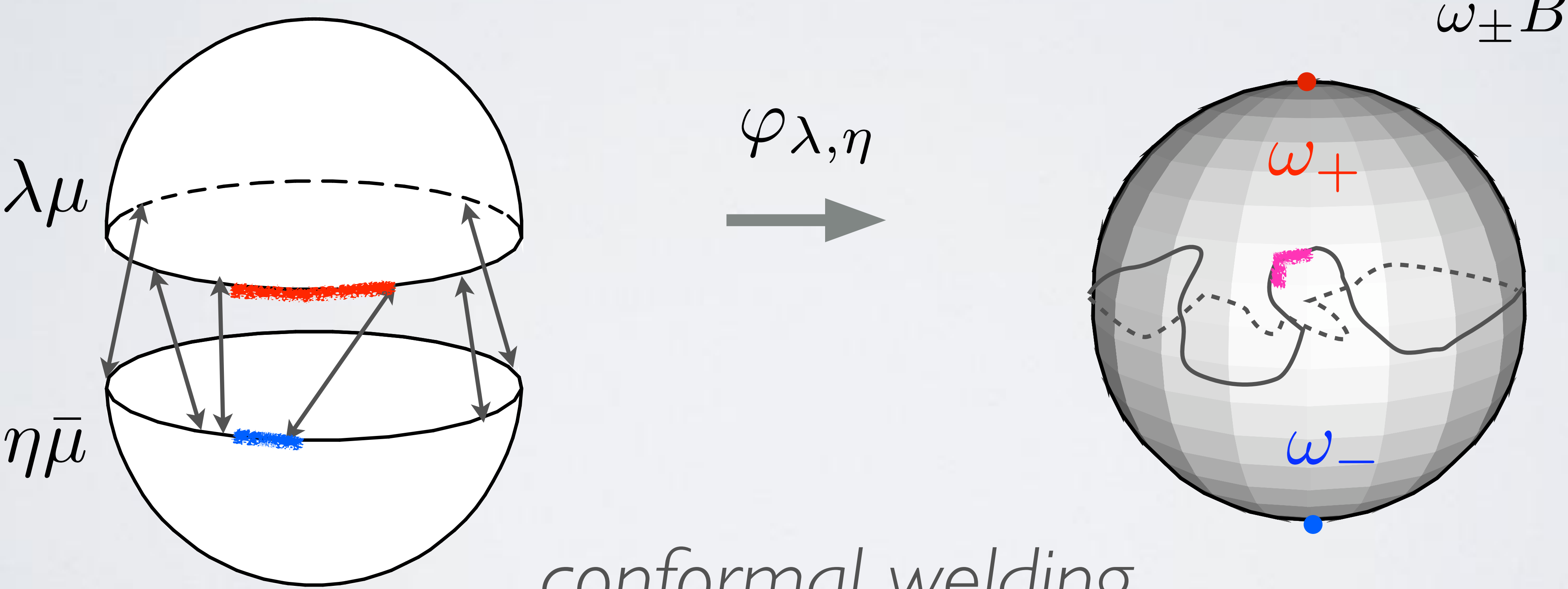
holomorphic motion of $\hat{\mathbb{C}}$
parametrized by the *bidisk*

BIDISK STRUCTURE

- reflection symmetry $\varphi_{\lambda,\eta}(z) = \overline{\varphi_{\bar\eta,\bar\lambda}(\bar z)}$
- diagonal (weldings) $(\lambda,\bar\lambda)$ $\varphi_{\lambda,\bar\lambda}(\mathbb{R}) = \mathbb{R}$
- projections $(\lambda,\eta)_+ = (\lambda,\bar\lambda),\ (\lambda,\eta)_- = (\bar\eta,\eta)$



$\omega_\pm B \approx r(B_\pm)$



conformal welding

K^2-quasisymmetric welding ↔ K-quasicircle

CANTOR SETS

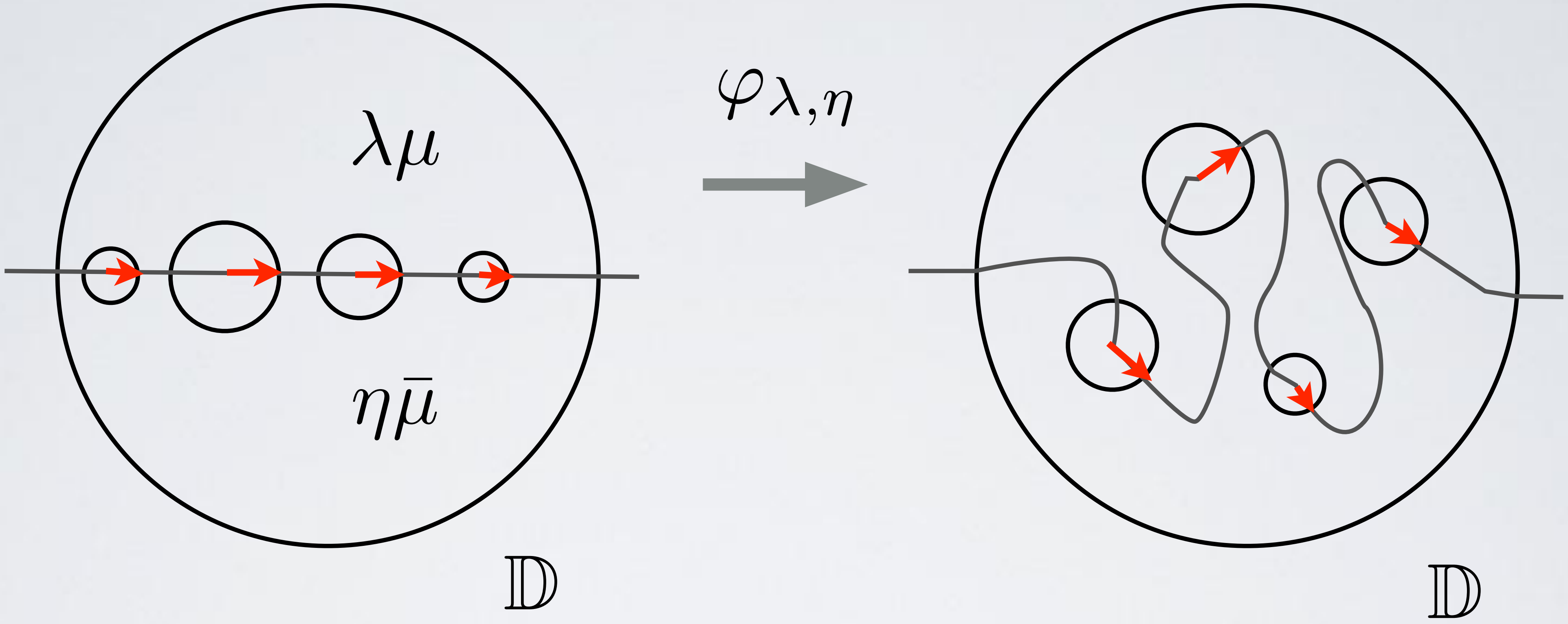
discretization

$E \approx C_0$

$\dim C_0 = t$

$\varphi(E) \approx C_{\lambda,\eta}$

$\dim C_{\lambda,\eta} = ?$



a packing of disks	$\{B_{\lambda,\eta}\}$
“complex radius”	$r_i(\lambda,\eta) := \varphi_{\lambda,\eta}(z_i + r_i) - \varphi_{\lambda,\eta}(z_i)$
Cantor sets	$C_{\lambda,\eta}$

THERMODYNAMICS

variational principle (Ruelle, Bowen)

$$P_{\lambda,\eta}(t) := \log\left(\sum |r_i(\lambda,\eta)|^t\right) = \sup_{p\in\mathrm{Prob}} \{I_p - t\,\mathrm{Re}\,\Lambda_p(\lambda,\eta)\}$$

$$\mathrm{I}_p = \sum p_i \log\frac{1}{p_i}$$

entropy

$$\Lambda_p(\lambda,\eta) = \sum p_i \log\frac{1}{r_i(\lambda,\eta)}$$

(complex) Lyapunov exponent

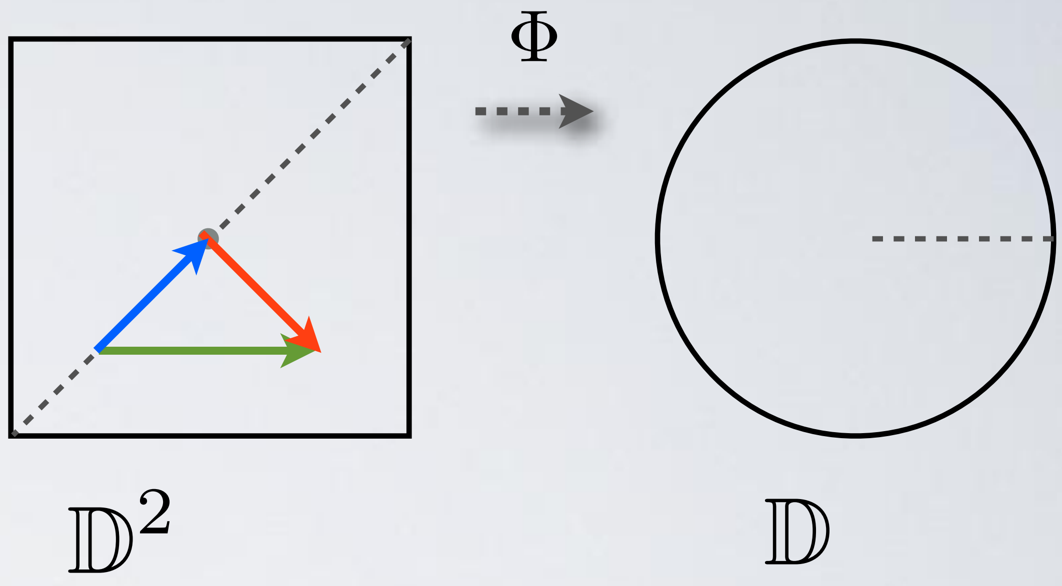
$$\dim C_{\lambda,\eta} = \text{root of } P_{\lambda,\eta} = \sup_p \frac{I_p}{\mathrm{Re}\,\Lambda_p(\lambda,\eta)}$$

$$\boxed{(\lambda,\eta) \mapsto \frac{I_p}{\Lambda_p(\lambda,\eta)} \quad \text{holomorphic!}}$$

APRIORI BOUNDS

$$\Phi(\lambda, \eta) = 1 - \frac{I_p}{\Lambda_p(\lambda, \eta)}$$

- $\Phi : \mathbb{D}^2 \to \mathbb{D} \quad \dim C_{\lambda,\eta} \leq 2$
- $\Phi(\lambda, \bar{\lambda}) \geq 0 \quad \dim C_{\lambda,\bar{\lambda}} \leq 1$
- $\Phi(\lambda, \eta) = \overline{\Phi(\bar{\eta}, \bar{\lambda})}$



quasicircles: *antisymmetric diagonal*

qs singularity: *symmetric diagonal*

quasidisk spectrum: *decomposition*

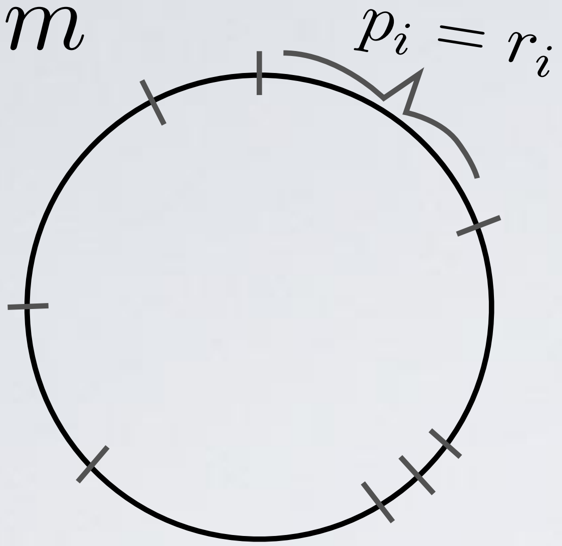
use *Schwarz-lemma/ Harnack/ 3-point Schwarz etc.*

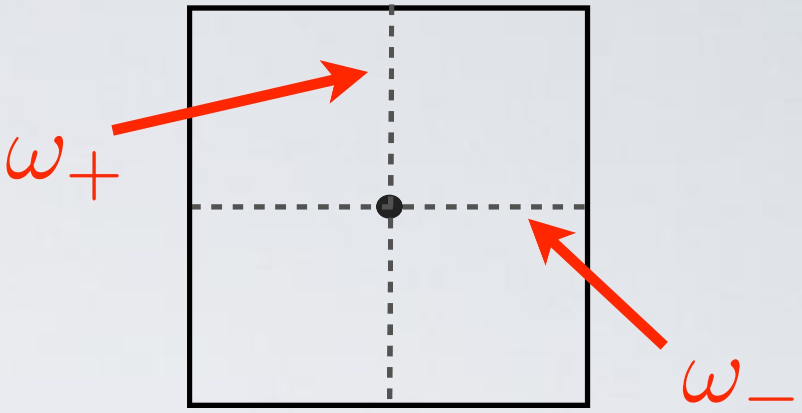
MAKAROV'S THEOREM

Take $p = m$

$$\mathrm{I}_p = \Lambda_p$$

$$\Phi = 1 - \frac{I_p}{\Lambda_p}$$





$\dim \omega = 1$ & $\gamma = 0$ ω-a.e.

Makarov Binder

$$\Phi(0,0) = 0 \longrightarrow \Phi(\cdot, 0) = \Phi(0, \cdot) = 0$$

can factor out zeroes on the diagonal

Schwarz lemma on $\mathbb{D}^2$: $|\Phi(\lambda, \eta)| \leq |\lambda\eta|$

$$1 - k^2 \leqslant \dim \varphi_{k,k}(m) \leqslant 1 + k^2$$

RIGIDITY

Astala's conjecture

existence of quasicircle with dim=$1+k^2$

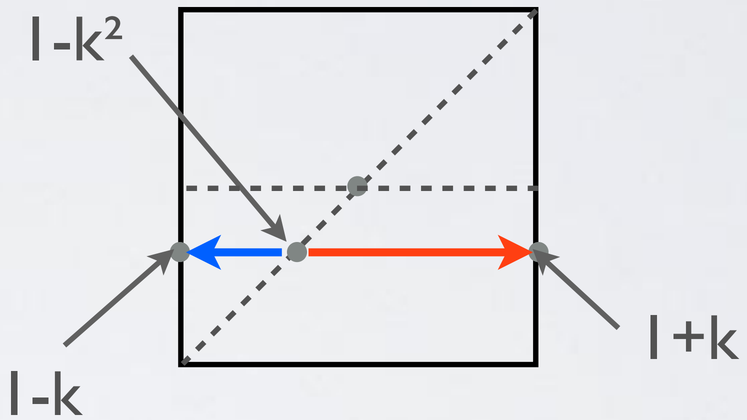
→

Universal Spectrum conjecture, *lower bounds*

$$B(t) \geqslant t^2/4$$

$$\Phi(\lambda, \eta) = \lambda\eta$$

extremal motion



compressing/

expanding

conformal map

dim=$1+ck^2$ →

$$\liminf_{t \to 0} \frac{B(t)}{t^2/4} \geqslant c$$

WELDING CONJECTURE

Conjecture: $\Phi(\lambda, \bar{\eta})$ is a kernel (i.e. positive definite).

$$\Phi(\lambda, \bar{\eta}) = \langle k(\lambda), k(\eta) \rangle_{\mathcal{H}}$$

Cauchy-Schwarz: $|\Phi|^2 \leq \Phi_- \Phi_+$

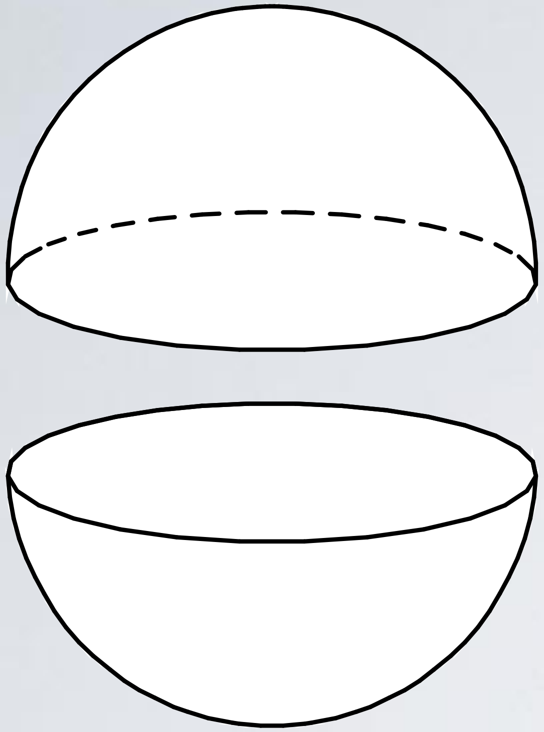
$$f(\alpha_-, \alpha_+, \gamma) \leq \frac{2 - (1+\gamma^2)\left(\frac{1}{\alpha_-} + \frac{1}{\alpha_+}\right)}{1 - \frac{1+\gamma^2}{\alpha_- \alpha_+}}$$

welding conjecture → universal spectrum conjecture

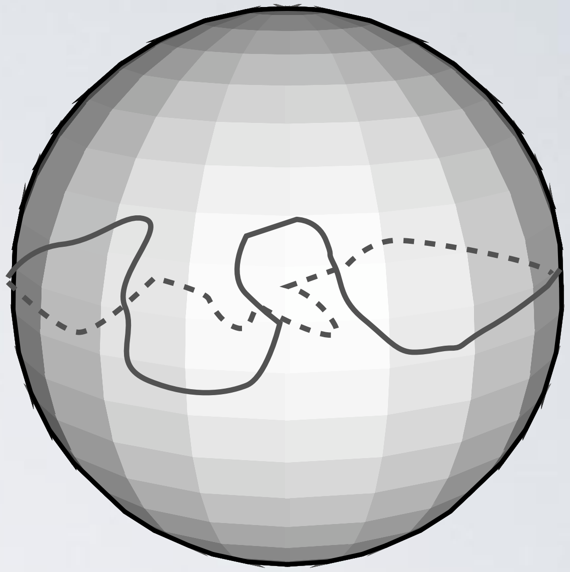
perturbative

BLASCHKE PRODUCTS

$$f_{\lambda,\bar\lambda}(z) = \frac{z^2+\lambda z}{1+\bar\lambda z} \qquad \lambda \in \mathbb{D} \qquad d=2$$



$$\frac{z^2+\lambda z}{1+\bar\eta z}$$



$$F\colon \mathbb{C}^2 \to \mathbb{C}^2$$ hom. polynomial

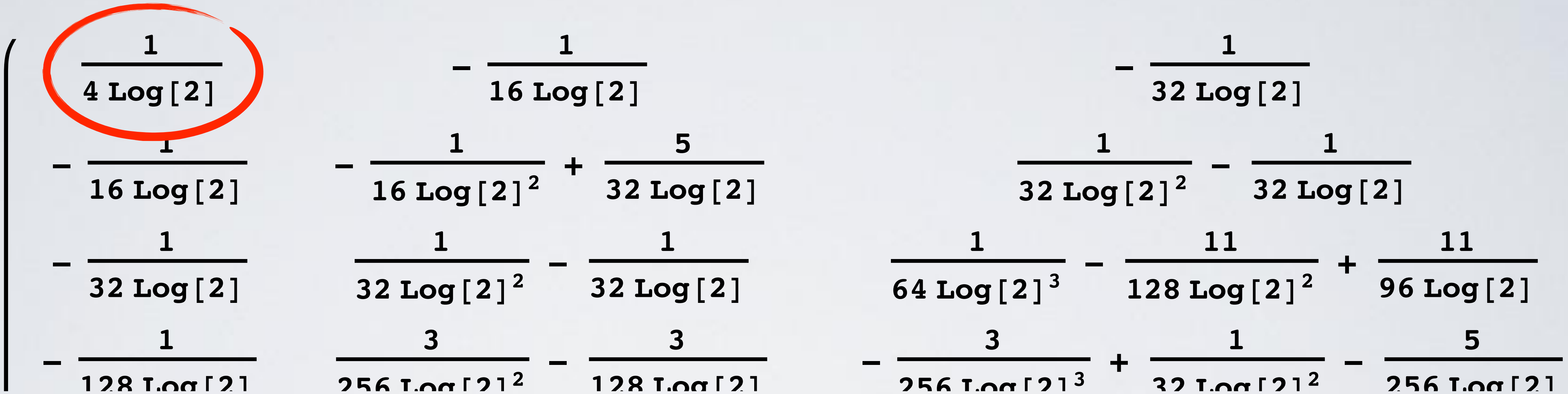
DeMarco

$$\Lambda(f) = \log d + \sum G_F(c_j) - \frac{2}{d}\log|\mathrm{Res}(F)|$$ pluriharmonic

$$G_F(z) = \lim_{n\to\infty} \frac{\log\|F^n(z)\|}{d^n}$$ escape rate

ASYMPTOTIC EXPANSION

$$\Phi(\lambda,\bar\lambda) = 1 - \dim(m_{\lambda,\bar\lambda}) = \sum_{k,l\geq 1} a_{k,l}\lambda^k\,\bar\lambda^l$$



$$\left(\begin{array}{ccc}
\boxed{\frac{1}{4\,\text{Log}[2]}} & -\frac{1}{16\,\text{Log}[2]} & -\frac{1}{32\,\text{Log}[2]} \\
-\frac{1}{16\,\text{Log}[2]} & -\frac{1}{16\,\text{Log}[2]^2}+\frac{5}{32\,\text{Log}[2]} & \frac{1}{32\,\text{Log}[2]^2}-\frac{1}{32\,\text{Log}[2]} \\
-\frac{1}{32\,\text{Log}[2]} & \frac{1}{32\,\text{Log}[2]^2}-\frac{1}{32\,\text{Log}[2]} & \frac{1}{64\,\text{Log}[2]^3}-\frac{11}{128\,\text{Log}[2]^2}+\frac{11}{96\,\text{Log}[2]} \\
-\frac{1}{128\,\text{Log}[2]} & \frac{3}{256\,\text{Log}[2]^2}-\frac{3}{128\,\text{Log}[2]} & -\frac{3}{256\,\text{Log}[2]^3}+\frac{1}{32\,\text{Log}[2]^2}-\frac{5}{256\,\text{Log}[2]}
\end{array}\right.$$

Oleg Ivrii

Question: positive semidefinite ?

OK up to 10x10.

ASYMPTOTIC VARIANCE

McMullen

$$-\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\dim(m_{t,t}) = 2\frac{d^2}{dt^2}\Big|_{t=0}\dim(J(f_t))$$

$$= \lim_{r\to 1}\frac{1}{2\pi|\log(1-r)|}\int_{|z|=r}|v'(z)|^2|dz| \qquad v = \frac{d\varphi_t}{dt}\Big|_{t=0}$$

$$|\mu| \leqslant \chi_{\mathbb{D}}$$

$$\sigma^2_{S\mu} = \limsup_{r\searrow 1}\frac{1}{2\pi|\log(1-r)|}\int_{|z|=r}|S\mu|^2 d\theta$$

Problem: $\sup_{\mu}\sigma^2_{S\mu} = 1$?

LACUNARY SERIES

Ruelle

$$f_\lambda = z^2 + \lambda z$$

$$v'(z) \approx -\frac{1}{2}\sum_{n=0}^{\infty} z^{-2^n} \longrightarrow \dim J(f_\lambda) = 1 + \frac{1}{4\log 2}\left(\frac{|\lambda|}{2}\right)^2 + \cdots$$

$$\sigma_{v'}^2 = \frac{1}{4\log 2} = 0.36\ldots$$

$$\mu(re^{i\theta}) = e^{-i\theta}, \quad r_0^d \leqslant r \leqslant r_0$$

$$\mu(z) = \mu(z^d)$$

Astala-Perälä-Prause: Example of σ^2=0.879...
(d=20)

Do perturb my quasicircles!

Hyvää syntymäpäivää, Kari!