Cheeger constants and L^2-Betti numbers

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The Cheeger constant

The Cheeger constant of a manifold X is defined by

$$h(X) = \inf\left\{ \frac{\operatorname{area}(\partial M)}{\operatorname{vol}(M)} : \ M \text{ compact } \subset X, \operatorname{vol}(M) \leq \operatorname{vol}(X)/2 \right\}.$$

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Main Problem: Determine $h(X|\mathcal{C})$ in some interesting cases.

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Motivation: the bass note of Schottky groups

A Schottky group is a subgroup $\Gamma < Isom(\mathbb{H}^n)$ freely generated by elements $g_1, \ldots, g_m$ such that there are conformally round pairwise disjoint balls $B_1, \ldots, B_m, B'_1, \ldots, B'_m \subset \partial \mathbb{H}^n$ such that $g_i B_i = \partial \mathbb{H}^n \setminus B'_i$ for each i.

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Theorem (Philips-Sarnak, Doyle)

For every $n \geq 3$ there is a $c_n > 0$ such that if $\Gamma < Isom(\mathbb{H}^n)$ is a finitely generated Schottky group then $\lambda_0(\mathbb{H}^n/\Gamma) > c_n$.

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$\Rightarrow h(\mathbb{H}^n | \text{Schottky}) > 0$ for $n \geq 3$.

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Is it possible to deform such a subgroup so that it is geometrically finite and has limit set with Hausdorff dimension close to 3?

If so then $h(\mathbb{H}^4|\text{Surface}) = 0$.
(PS theory $\Rightarrow \lambda_0(\mathbb{H}^n/\Gamma) = \dim(L\Gamma)(n - 1 - \dim(L\Gamma))$)

Main results

Definition

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Corollary

For any even number $n \geq 2$, $h(\mathbb{H}^n | \mathcal{G}_{n/2}) > 0$.

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For any even number $n \geq 2$, $h(\mathbb{H}^n|\mathcal{G}_{n/2}) > 0$.

Lemma

If Γ is isomorphic to a subgroup of a fundamental group of a complete finite volume hyperbolic 3-manifold then $\Gamma \in \mathcal{G}_d$ for all $d \geq 2$. So $h(\mathbb{H}^{2n}|\mathrm{Free}) > 0$ for all $n \geq 2$.

Benjamini-Schramm convergence

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We say $\{K_i\}$ is *Benjamini-Schramm convergent* if

$$\lim_{i \to \infty} \frac{\rho_i(L, v)}{|V(K_i)|} =: \bar{\rho}(L, v)$$

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Lemma (G. Elek, 2010)

Suppose $\{K_i\}$ is BS-convergent and suppose there is a uniform bound on the vertex degrees of $\{K_i\}$. Then for every $d > 0$,

$$\lim_{i \to \infty} \frac{b_d(K_i)}{|V(K_i)|}$$

exists.

Lück Approximation

Theorem (W. Lück, 1994)

Let Λ be a finitely generated group, $\{\Lambda_n\}_{n=1}^{\infty}$ a decreasing sequence of finite-index normal subgroups of Λ with $\bigcap_n \Lambda_n = \{e\}$. Then

$$\lim_{n \to \infty} \frac{b_d(\Lambda_n)}{[\Lambda : \Lambda_n]} = b_d^{(2)}(\Lambda).$$

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Given a finite connected simplicial complex K, define

$$\mu_K = \sum_{v \in V(K)} \delta_{[K,v]}.$$

Lemma

$\{K_i\}$ *is BS-convergent* $\Leftrightarrow \mu_{K_i}$ **converges in the weak* topology on** $\mathcal{M}_1(RSC(\Delta))$.

Metric measure spaces

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Let $\mathbb{M}$ denote the set of isomorphism classes of pointed mm-spaces.

A sequence $[M_i, p_i] \in \mathbb{M}$ converges to $[M_\infty, p_\infty]$ iff there exist a complete separable metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ $(1 \leq i \leq \infty)$ such that

- $(\varphi_i(M_i), \varphi_i(p_i))$ converges to $(\varphi_\infty(M_\infty), \varphi_\infty(p_\infty))$ in the pointed Hausdorff topology;
- $(\varphi_i)_* \mathrm{vol}_{M_i}$ converges to $(\varphi_\infty)_* \mathrm{vol}_{M_\infty}$ in the weak* topology.

A variation on Elek's Theorem

Let $\mathcal{M}_1(\mathbb{M}) =$ the space of Borel probability measures on $\mathbb{M}$.

Theorem

Let M_i be a sequence of path-connected finite-volume mm-spaces. Let μ_i be the push-forward of vol_{M_i} under the map $p \mapsto [M_i, p]$. Suppose μ_i is converges in $\mathcal{M}_1(\mathbb{M})$ (weak*). Suppose also that there exist constants $\epsilon, v_0, v_1, \Delta > 0$ such that

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Then

$$\lim_{i \to \infty} \frac{b_d(M_i)}{vol_{M_i}(M_i)} \text{ exists for every } d > 0.$$

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- *vol_{M_i} is purely non-atomic and fully supported;*
- *every sphere in M_i has measure zero;*
- *$0 < v_0 \leq vol_{M_i}(B_{M_i}(p, \epsilon/2)) \leq vol_{M_i}(B_{M_i}(p, 10\epsilon)) \leq v_1 \; \forall p \in M_i$;*
- *$B_{M_i}(p, r)$ is strongly convex for every $r \leq 10\epsilon$;*
- *the max. cardinality of an ϵ-separated subset of $B_{M_i}(p, 20\epsilon)$ is $\leq \Delta$.*

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Proof Sketch of the Main Theorem

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It suffices to show that if $\Gamma_i < \mathrm{Isom}(X)$ is a sequence of residually finite subgroups which act freely and properly discontinuous and $h(X/\Gamma_i) \to 0$ then there exist subgroups $\Gamma_i' < \Gamma_i$ s.t. $b_d^{(2)}(\Gamma_i') > 0$ for all but finitely many i.

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Let us also pretend that $\pi_1(M_i) \to \Gamma_i$ is injective. Then $b_d(M_i) = b_d(\Gamma_i')$ for some $\Gamma_i' < \Gamma_i$.

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By Lück Approximation, we may pass to a finite-index subgroup $\Gamma_i'' < \Gamma_i'$ such that

$$\frac{b_d(\Gamma_i'')}{[\Gamma_i' : \Gamma_i'']} \approx b_d^{(2)}(\Gamma_i').$$

It follows that $b_d^{(2)}(\Gamma_i') > 0$ for all but finitely many i.

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Buser's Hair-cutting Lemma $\implies$ we can choose M_i so that

$$\lim_{i \to \infty} \frac{\mathrm{vol}(N_r(\partial M_i))}{\mathrm{vol}(M_i)} = 0 \quad \forall r > 0.$$

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Lift M_i to a subset $M_i' \subset X/\Gamma_i'$. We still have

$$\lim_{i \to \infty} \frac{\text{vol}(N_r(\partial M_i'))}{\text{vol}(M_i')} = 0 \quad \forall r > 0.$$

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By Lück Approximation, we can also choose Γ_i'' so that

$$\liminf_{i\to\infty} \frac{b_d(\Gamma_i'')}{\mathrm{vol}(M_i'')} = \liminf_{i\to\infty} \frac{b_d^{(2)}(\Gamma_i')}{\mathrm{vol}(M_i')}.$$

Benjamini-Schramm convergence implies:

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The Mayer-Vietoris sequence

$$\cdots \to H_d(\partial M_i'') \to H_d(M_i'') \oplus H_d(X/\Gamma_i'' - M_i'') \to H_d(X/\Gamma_i'') \to \cdots$$

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Benjamini-Schramm convergence implies:

$$\liminf_{i \to \infty} \frac{b_d^{(2)}(\Gamma_i')}{\operatorname{vol}(M_i')} = \liminf_{i \to \infty} \frac{b_d(\Gamma_i'')}{\operatorname{vol}(M_i'')} \geq \lim_{i \to \infty} \frac{b_d(M_i'')}{\operatorname{vol}(M_i'')} = \frac{b_d^{(2)}(\Lambda)}{\operatorname{vol}(X/\Lambda)} > 0.$$

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But $b_d(\partial M_i'') \leq C\operatorname{vol}(N_r(\partial M_i''))$ for some $r, C > 0$. So

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- What is $h(\mathbb{H}^4|\mathcal{G}_2)$?