# Ricci Curvature and Bochner Formula on Alexandrov Spaces


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(work with Prof. Xi-Ping Zhu)

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- Alexandrov Spaces
- Generalized Ricci Curvature
- Geometric and Analytic consequences
- Dirichlet Form, Laplacian and Harmonic Functions
- Bochner Formula and its Applications

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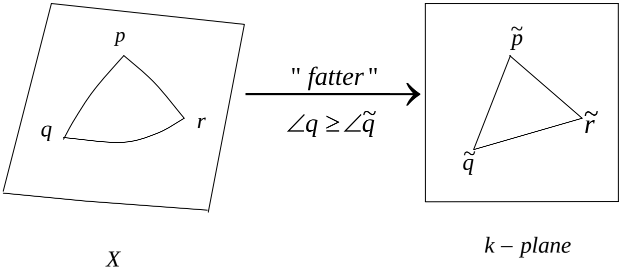
# 1. Alexandrov Spaces

Alexandrov spaces are locally compact geodesic spaces with the concept of (sectional) curvature bounded below.

- A geodesic space $X$ is called to have curvature $\geqslant k$ if



Basic examples of Alexandrov spaces:

- Orbifolds
- Polyhedrons
- The limit spaces (Gromov-Hausdroff topology) of a family of smooth manifolds with controlled curvature, diameter and dimension.

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Burago–Gromov–Perelman:

The following basic concepts and facts are well established on Alexandrov spaces

- The dimension is always an integer
- Tangent cones.
  Regular points is dense (but irregular points might be dense too)
- Volume comparison (Bishop–Gromov type)
- Parallel transportation
- Compactness theorem.

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Some important results on Riemanian manifolds with lower bounds of sectional curvature have been extended to Alexandrov spaces.

- (Petrunin) Extended Synge theorem
- (Milka) Extended Toponogov splitting theorem
- (Perelman, Shioya-Yamaguchi, Rong-Xu, etc) Extended soul theorem

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In PDE and Riemannian geometry, the differential calculus is the most fundamental tools. Now we recall the calculus on Alexandrov spces.

- Ostu–Shioya:

  Established a $C^1$-structure and a corresponding $C^0$-Riemannian structure on the set of regular points

- Perelman:

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Under Ostu-shioya-Perelman's Riemannian structure.

- let point $p$ be regular, Reimannian structure $g_{ij}$ around $p$, then

$$|g_{ij}(x) - \delta_{ij}| = o(1), \qquad \text{as } |px| \to 0.$$

- let point $p$ be "smooth" in the sense of Perelman, then

$$|g_{ij}(x) - \delta_{ij}| = o(|px|), \qquad \text{as } |px| \to 0.$$

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Many fundamental results in Riemannian geometry assume only the lower bounds on Ricci curvature. For example:

- Bonnet–Myers diameter theorem
- Bishop–Gromov volume comparison and Levy–Gromov isoperimetric inequality
- Cheeger–Gromoll splitting theorem and Cheng maximal diameter theorem
- Lichnerowicz's eigenvalue estimate and Obata theorem
- Yau's gradient estimates and Li-Yau's estimates

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An interesting problem is to give a generalized notion of lower bounds of Ricci curvature on singular spaces, and study its geometric and analytic comsequences.

Nowadays, several such generalizations have appeared. To recall them, let us recall the equivalent conditions for lower Ricci bounds on Riemannian manifolds.

# 2. Ricci curvature

In a Riemannian manifold, the condition $Ric \geqslant K$ is equivalent to each one of the following properties:

- Bochner formula
- Displacement $K-$convexity
- Volume comparison
- (average) Second variation formula of arc-length.

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At last, clearly

- $$sec \geqslant K \Longrightarrow Ric \geqslant (n-1)K.$$

(1) Bochner formula:
For each $C^3$ function $f$,

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta f\rangle + Ric(\nabla f, \nabla f)$$
$$\geqslant \frac{(\Delta f)^2}{n} + \langle \nabla f, \nabla \Delta f\rangle + K|\nabla f|^2$$

- Lin–Yau: define Ricci curvature on locally finite graphs
- Ambrosio–Gigli–Savaré: define Bakry-Emery curvature-dimension condition on Dirichlet form

(2) Displacement $K-$convexity:
$M^n$— Riemannian manifold.
$P_2(M^n, d_W, vol)$— $L^2-$Wasserstein space

$$Ent(\mu) = \int_{M^n} \frac{d\mu}{dx} \cdot \log \frac{d\mu}{dx} dvol(x)$$

is $K-$convex in $P_2(M^n, d_W, vol)$.

- Sturm, Lott–Villani: define $CD(K, n)$ on metric measure spaces
- Ambrosio–Gigli–Savaré: improve $CD(K, n)$ to $RCD(K, n)$.

(3) Volume comparison:
Denote $A_p(r,\xi)$ the density of the Riemannian measure on $\partial B_p(r)$. Then

$$\frac{A_p(r,\xi)}{\big(s_K(r/\sqrt{n-1})\big)^{n-1}}$$

is non-increasing on $r$ (for each $\xi \in \Sigma_p$), where $\big(s_K(r/\sqrt{n-1})\big)^{n-1}$ is the corresponding density of the space form.

- Sturm, Ohta: define $MCP(K,n)$ on metric measure spaces.
- Kuwae–Shioya: define $BG(K)$ on Alexandrov spaces.

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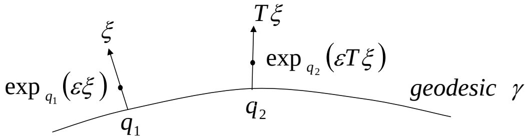
(4)Second variation formula of arc-length:




$$\sum_{i=1}^{n-1} d\big(\exp_{q_1}(\varepsilon\xi_i), \exp_{q_2}(\varepsilon T\xi_i)\big)$$

$$\leqslant (n-1)d(q_1,q_2) - K\frac{d(q_1,q_2)}{2!}\varepsilon^2 + o(\varepsilon^2)$$

where $\xi_1, \cdots, \xi_{n-1}$ is an $(n-1)-$orthonormal frame at $q_1$ and orthogonal to $\gamma$.

- Zhang–Zhu: $Ric \geqslant K$ on Alexandrov spaces.

- On an $n$-dim Alexandrov space, the above notions "$CD(K,n)$", "$RCD(K,n)$", "$MCP(K,n)$" and "$Ric \geqslant K$" make sense.

  Relation:

$$Ric \geqslant K \Longrightarrow CD(K,n)$$

$$\text{Petrunin } (K=0), \ \text{Zhang} - \text{Zhu } (K \neq 0)$$

$$\Longleftrightarrow RCD(K,n) \qquad \text{Gigli} - \text{Kuwada} - \text{Ohta}$$

$$\Longrightarrow MCP(K,n) \qquad \text{Sturm}$$

$$\Longleftrightarrow BG(K) \qquad \text{Kuwae} - \text{Shioya}$$

- Question:

$$Ric \geqslant K \Longleftrightarrow CD(K,n)?$$

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Now let us recall the geometric consequences of above generalized Ricci conditions.
From now on, we always consider $M$ to be an $n$-dim Alexandrov space.

- $BG(K) \Longrightarrow$ Bishop–Gromov volume comparison
  Sturm, Lott–Vinalli, Kuwae–Shioya, Ohta
- $BG(n-1) \Longrightarrow$ Bonnet–Myers theorem: diameter $\leq \pi$
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**Theorem**

- Topological splitting theorem (Kuwae–Shioya):
  $MCP(0, n)$, $\exists$ *a line* $\Longrightarrow M \overset{hemeo}{\cong} \mathbb{R} \times N$.
- Topological maximal diameter theorem (Ohta):
  $MCP(n-1, n)$, $diam = \pi$
  $$\Longrightarrow M \overset{hemeo}{\cong} \text{a spherical suspension } [0, \pi] \times_{\sin} N.$$

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Under our definition of Ricci curvature, we gave a new geometric argument to prove:

**Theorem (Zhang–Zhu)**

- Splitting theorem:

  *Alexandov space* $M$, $Ric \geqslant 0$, $\exists$ *a line* $\Longrightarrow M \overset{isom}{\cong} \mathbb{R} \times N$.

- Maximal diameter theorem:

  $n-\dim$ *Alexandov space* $M$, $Ric \geqslant n-1$, $diam = \pi$,

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Since our defined Ricci condition is local, it can be lifted to the covering spaces. Some topological consequences are given

## Corrollaries

- *Alexandov space $M$, $Ric \geqslant 0$*
  $\Longrightarrow$
  - *$M$ compact, $\pi_1(M)$ has a finite index Bieberbach subgroup;*
  - *any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leqslant n$.*
- *Alexandov space $M$, $Ric \geqslant (n-1)K$, $diam \leqslant D$*
  $\Longrightarrow$

$$b_1(M) \leqslant C(n, KD^2).$$

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# 3. Geometric Results

Question:

Levy–Gromov isoperimetric inequality for Alexandrov spaces?

i.e.,

Alexandrov space $M$ with $Ric \geqslant n-1$. Set a surface $\sigma_\alpha$ dividing the volume of $M$ in ratio $\alpha$. Let $s_\alpha$ be a geodesic sphere in $\mathbb{S}^n$ dividing the volume of $\mathbb{S}^n$ in the same ratio $\alpha$. Can one prove

$$\frac{vol(\sigma_\alpha)}{vol(M)} \geqslant \frac{vol(s_\alpha)}{vol(\mathbb{S}^n)} \ ?$$

Remark: Petrunin sketched a proof of the inequality under assumption sectional curvature $\geqslant 1$.

# 4. Dirichlet Form, Laplacian and Harmonic functions

Now let us consider the analysis on Alexandrov spaces. We will begin from the canonical Dirichlet form and the definition of Laplace operator.

Recall the following basic facts

- A Lipschitz function has derivative almost everywhere (Cheeger)
- Sobolev spaces $W^{1,p}(M)$ are well-defined (Cheeger, Shanmugalingam) etc.
- Canonical Dirichlet form

$$\mathscr{E}(u,v)=\int_M \langle \nabla u, \nabla v\rangle \, dvol, \quad \forall u,v\in W^{1,2}(M).$$

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# 4. Dirichlet Form, Laplacian and Harmonic Functions

We will understand the Laplician of a Sobolev function as a Radon measure.

- If $u \in W^{1,2}(M)$ such that

$$\mathscr{L}_u(\phi) := -\int_M \langle \nabla u, \nabla \phi \rangle \, dvol \underset{(\geqslant)}{\leqslant} \int_M f\phi dvol,$$

for all $\phi \in Lip_0(M), \phi \geqslant 0$,
$\Longrightarrow$ $\mathscr{L}_u$ is a signed Radon measure, say

$$\Delta u \underset{(\geqslant)}{\leqslant} f \cdot vol.$$

- Poisson equation

$$\Delta u = f \cdot vol.$$

Under Lebesgue decomposition, it is

$$\Delta^{reg} u = f \qquad \text{and} \qquad \Delta^{sing} u = 0.$$

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To determine the regularity of harmonic functions on an Alexandrov space $M$, one has the followings facts (from volume comparison):

- doubling property
- Poincaré inequality and Sobolev inequality
- De Giorgi, Nash–Moser iteration argument works
- Every harmonic function is Hölder continuous

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The same regularity problem for harmonic maps have considered:

- Jost, Lin: harmonic maps between Alexandrov spaces are Hölder continuous

- Korevaar–Schoen: harmonic maps from smooth manifolds to Alexandrov spaces are Lipschitz continuous

- A generalized Liouville theorem have proved:
  Hua: $CD(0, n) \implies$ the space of harmonic function with polynomial growth of degree $\leqslant d$ is finite dimensional for any $d \in \mathbb{R}^+$.

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*Question*: Is any harmonic function on an Alexandrov space Lipschitz continuous?

By using Ostu–Shioya, Perelman's coordinate system, there exists a $BV^0_{loc}-$Riemannian metric $(g_{ij})$. A harmonic function $u$ is a solution of

$$\sum_{i,j=1}^{n} \partial_i(\sqrt{g}g^{ij}\partial_j u) = 0$$

in the sense of distribution.
The difficulty is the coefficient $\sqrt{g}g^{ij}$ might be not continuous on a dense subset.

- Petrunin (1996, announced):
  Let $u$ be a harmonic function on an Alexandrov space with (sectional) curvature $\geqslant 0$

$$\Longrightarrow u \text{ is Lipschitz continuous.}$$

- Gigli–Kuwada–Ohta, Zhang–Zhu:
  Let $u$ be a harmonic function on an Alexandrov space,

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Open Question (Lin's conjecture):

Is any harmonic maps between Alexandrov spaces Lipschitz continuous?

# 5. Bochner Formula and Its Applications

Bochner formula is one of most important tools in Riemannian geometry.
On a Riemannian manifold with $Ric \geqslant K$, there holds that for any $C^3$ function $u$,

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u\rangle + Ric(\nabla u, \nabla u)$$
$$\geqslant \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla \Delta u\rangle + K|\nabla u|^2$$

- In Alexandrov spaces, $\Delta u$ is understood as a Radon measure, but there is no sense of Hessian $\nabla^2 u$
- It is not know if an Alexandrov space can be approximated (in GH distance) by smooth manifolds.

# 5. Bochner Formula and Its Applications

**Theorem (Bochner formula, Zhang–Zhu)**

*Let $M$ be an Alexandrov space with $Ric \geqslant K$. Suppose $f \in Lip(M)$ and $u$ satisfies*

$$\Delta u = f \cdot vol.$$

*Then we have $|\nabla u|^2 \in W^{1,2}_{loc}(M)$ and*

$$-\frac{1}{2}\int_M \langle \nabla |\nabla u|^2, \nabla \phi \rangle \, dvol \geqslant \int_M \phi \Big( \frac{f^2}{n} + \langle \nabla u, \nabla f \rangle + K|\nabla u|^2 \Big) dvol$$

*for every $0 \leqslant \phi \in W^{1,2}_0(M) \cap L^\infty(M)$.*

# Sketch of the proof

Step 1. Consider Hamilton-Jacobi semigroup

$$u_t(x) = \min_y \{u(y) + \frac{|xy|^2}{2t}\}, \qquad x \in M, \; t > 0.$$

- For any $t > 0$ and almost any $x \in M$,

$$\lim_{s \to 0^+} \frac{u_{t+s}(x) - u_t(x)}{s} = -\frac{|\nabla_x u_t|^2}{2}$$

- For any $t > 0$ and $x$ regular,

$$\Rightarrow \exists \text{ unique} \qquad y = \exp_x(-t\nabla_x u_t)$$

satisfying

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Step 2.

- For each $t > 0$, define a map $F_t : M \to M$ by $F_t(x)$ to be one of point such that

$$u_t(x) = u\big(F_t(x)\big) + \frac{|xF_t(x)|^2}{2t}.$$

- For any $t > 0$ sufficiently small,

$$a^2 \Delta u_t \leqslant \Big(f \circ F_t + \frac{n(a-1)^2}{t} - \frac{Kt}{3}(a^2+a+1)|\nabla u_t|^2\Big) vol \quad (*)$$

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## Sketch of the proof

To conclude the inequality (*), two essential facts of Alexandrov spaces are used:

- Comparison Property:
  Let $u$ be semi-concave, $\Delta u \geqslant 0$ on $\Omega$ if and only if for any small ball $B \Subset \Omega$, we have $u \geqslant u_B$, where $u_B$ is harmonic on $B$ with the same boundary data of $u$.
- For Perelman's "smooth" points $x$ and $y$, the assumption $Ric \geqslant 0$ implies

$$\int_{B_o(\delta_j)} \left( |\exp_x(a\eta)\exp_y(T\eta)|^2 - |xy|^2 \right) dH^n(\eta)$$

$$\leqslant (1-a)^2 \frac{\omega_{n-1}}{n+2} \cdot \delta_j^{n+2} + o(\delta_j^{n+2})$$

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# Sketch of the proof

A contradiction argument implies the inequality (*). Write the RHS of (*) as $w(x)$.

(i) If inequality (*) is wrong at point $x$, the continuity of $f$ implies that $\Delta u_t > w$ near $x$, say $\Omega$.

(ii) solve a function $v$ such that $\Delta v = -w$ in $\Omega$, and with the same boundary data as $u_t$. So, from the above comparison property, $v + u_t$ have *strict* minimum in $\Omega$. Then the function

$$H(x, y) := u(y) + v(x) + |xy|^2/2t$$

has *strict* minimum in $\Omega \times \Omega$.

(iii) By Petrunin's perturbation argument, we can assume that the minimum of $H(x, y)$, say $(x_0, y_0)$, is Perelman's "smooth". Now an contradiction come from the combination of the facts: $H(x, y)$ has mimimum at $(x_0, y_0)$ and mean inequalities for $u$ and $v$.

# Sketch of the proof

Step 3.

- by suitable choosing $a$ in the inequality (*), we get

$$\frac{\Delta u_t(x) - f(x)}{t} \leqslant \frac{f(F_t(x)) - f(x)}{t} - \frac{1}{n} f(x) f(F_t(x)) - K|\nabla u_t(x)|^2 + C|f(F_t(x)) - f(x)| + Ct$$

  for some positive constant $C$.

- By using the Lipschitz continuity of $f$ and taking $t \to 0^+$, we get

$$\frac{1}{2}\Delta|\nabla u|^2 \geqslant \left(\frac{f^2}{n} + \langle \nabla u, \nabla f \rangle + K|\nabla u|^2\right) vol.$$

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# Application 1: Yau's gradient estimate

**Theorem (Yau's gradient estimate, Zhang–Zhu)**

*Let $M$ be an Alexandrov space with Ric $\geqslant -(n-1)k$, $(k \geqslant 0)$ and $u$ be a positive harmonic function. Then*

- $$\max_{x \in B_p(\frac{R}{2})} |\nabla \log u| \leqslant C(n, \sqrt{k}R)(\sqrt{k} + \frac{1}{R}),$$
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# Application 2: Eigenvalue estimate, Obata type theorem

**Theorem (Qian–Zhang–Zhu)**

*Let $M$ be a compact n-dim Alexandrov space, and denote by $\lambda_1(M)$ the first non-zero eigenvalue. Then*

$$Ric(M) \geqslant (n-1)K \Longrightarrow \lambda_1(M) \geqslant \lambda_1(K, n, d),$$

*where $d$ is the diameter of $M$, and $\lambda_1(K, n, d)$ is the first non-zero Neumann eigenvalue of following 1-dim model :*

$$v''(x) - (n-1)T(x)v'(x) = -\lambda v(x), \qquad x \in (-\tfrac{d}{2}, \tfrac{d}{2}),$$
$$v'(-\tfrac{d}{2}) = v'(\tfrac{d}{2}) = 0$$

*and* $T(x) = \begin{cases} \sqrt{K}\tan(\sqrt{K}x), & K \geqslant 0; \\ -\sqrt{-K}\tanh(\sqrt{-K}x), & K < 0. \end{cases}$

Remark: For smooth Riemannian manifolds, the theorem proved by Bakry–Qian (analytic method), Chen–Wang (coupling method) and Andrews–Clutterbuck (parabolic method).

## Corrollaries (Qian–Zhang–Zhu)

- $$\lambda_1(M) \geqslant 4s(1-s)\frac{\pi^2}{d^2} + sK$$

  *for all* $s \in [0,1]$*;*

- *if in addition* $K > 0$*, then*

  $$\lambda_1(M) \geqslant \frac{n}{n-1}K$$

  *and "=" holds if and only if* $M$ *is a spherical suspension.*

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Remark:

- Lichnerouicz estimate, $\lambda_1(M) \geqslant \frac{n}{n-1}K$, was earlier obtained by Petrunin and Lott–Villani.

# Application 3: Li–Yau's estimate

**Theorem (Qian–Zhang–Zhu)**

*Let $M$ be a compact Alexandrov space with $Ric \geqslant 0$. Assume $u(x,t)$ is a positive solution of heat equation. Then*

- $$|\nabla \log u|^2 - \frac{\partial}{\partial t} \log u \leqslant \frac{n}{2t},$$

  *for any $t > 0$.*

- *(Sharp Harnack estimate)*

  $$u(x_1, t_1) \leqslant u(x_2, t_2) \Big(\frac{t_2}{t_1}\Big)^{\frac{n}{2}} \exp\Big(\frac{|x_1 x_2|^2}{4(t_2 - t_1)}\Big)$$

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*Thank You !*