Branching geodesics in metric spaces with Ricci curvature lower bounds

Tapio Rajala
University of Jyväskylä
tapio.m.rajala@jyu.fi

Interactions Between Analysis and Geometry: Analysis on Metric Spaces

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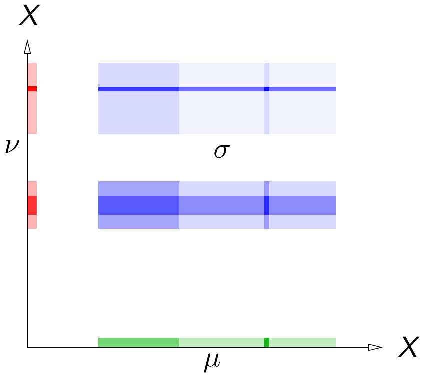
The space $(\mathcal{P}(X), W_2)$

For $\mu, \nu \in \mathcal{P}(X)$ define

$$W_2(\mu,\nu) = \left(\inf\left\{\int_{X\times X} \mathrm{d}(x,y)^2 \,\mathrm{d}\sigma(x,y) \;\middle|\; \begin{array}{l}(\mathrm{p}_1)_\#\sigma = \mu \\ (\mathrm{p}_2)_\#\sigma = \nu\end{array}\right\}\right)^{1/2}.$$



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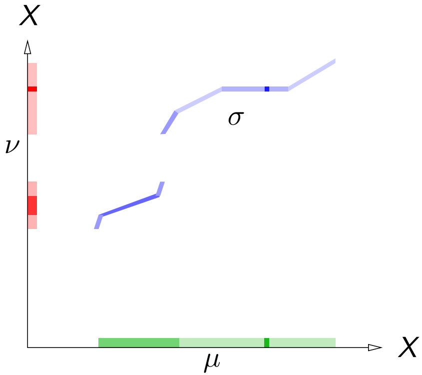

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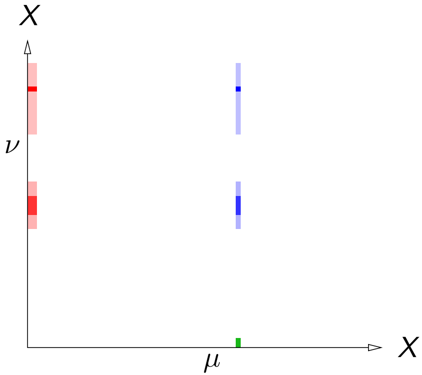


The space $(\mathcal{P}(X), W_2)$

Question (Existence of optimal maps)

When is the/an optimal plan induced by a map? ($\sigma = (id, T)_\# \mu$ for some map $T\colon X \to X$)

The usual steps in the proof are (given two measures $\mu, \nu \in \mathcal{P}(X)$):

1. Prove that any optimal plan from μ to ν is induced by a map.
2. Linear combinations of optimal plans are optimal.
3. The combination of two different optimal maps is not a map. Hence there is only one optimal map.

Existence of optimal maps

The existence of optimal maps for the quadratic cost has been obtained for example by

- Brenier (1991) in $\mathbb{R}^n$ with $\mu \ll \mathcal{L}^n$. (Earlier steps by Brenier 1987 and Knott & Smith 1984)
- McCann (1995) in $\mathbb{R}^n$ with $\mu(E) = 0$ for $n-1$-rectifiable E.
- Gangbo & McCann (1996) in $\mathbb{R}^n$ with $\mu(E) = 0$ for all $c-c$-hypersurfaces E.
- McCann (2001) for Riemannian manifolds M with $\mu \ll \mathrm{vol}$.
- Gigli (2011) for Riemannian manifolds M there exists an optimal map from μ to every μ if and only if $\mu(E) = 0$ for all $c-c$-hypersurfaces E.

Existence of optimal maps

and ...

- Bertrand (2008) for finite dimensional Alexandrov spaces with $\mu \ll \mathcal{H}^d$.
- Gigli (2012) for non-braching $CD(K,N)$-spaces, $N < \infty$, with $\mu \ll \mathfrak{m}$. For non-branching $CD(K,\infty)$-spaces with $\mu, \nu \in D(\mathrm{Ent}_{\mathfrak{m}})$.
- Rajala & Sturm (2012) for $RCD(K,\infty)$-spaces with $\mu, \nu \ll \mathfrak{m}$.
- Cavalletti & Huesmann (2013) for non-branching $MCP(K,N)$-spaces with $\mu \ll \mathfrak{m}$.

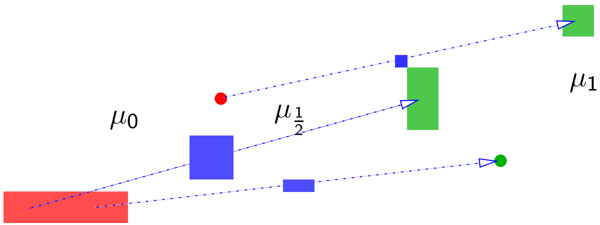
The space $(\mathcal{P}(X), W_2)$ for (X, d) geodesic

(All the geodesics in this talk are constant speed geodesics parametrized by $[0,1]$.)

- If (X, d) is geodesic, then $(\mathcal{P}(X), W_2)$ is geodesic.
- A geodesic $(\mu_t) \subset \mathrm{Geo}(\mathcal{P}(X))$ can be realized as a probability measure $\boldsymbol{\pi} \in \mathcal{P}(\mathrm{Geo}(X))$ in the sense that for all $t \in [0,1]$ we have $(e_t)_\# \boldsymbol{\pi} = (\mu_t)$.

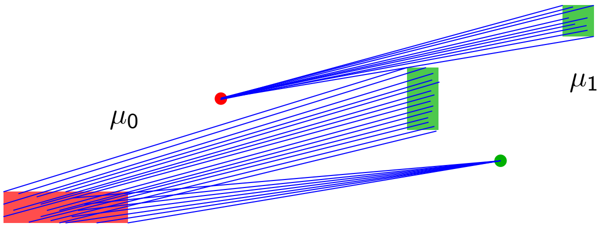
From $\mathrm{Geo}(\mathcal{P}(X))$ to $\mathcal{P}(\mathrm{Geo}(X))$

Optimal plan between μ_0 and μ_1 as a geodesic in $\mathcal{P}(X)$.



From $\mathrm{Geo}(\mathcal{P}(X))$ to $\mathcal{P}(\mathrm{Geo}(X))$

Optimal plan between μ_0 and μ_1 as a measure in $\mathcal{P}(\mathrm{Geo}(X))$.



Optimal plans σ vs. $\mathrm{Geo}(\mathcal{P}(X))$ vs. $\mathcal{P}(\mathrm{Geo}(X))$

Denote by $\mathrm{OptGeo}(\mu_0, \mu_1) \subset \mathcal{P}(\mathrm{Geo}(X))$ the set of all $\boldsymbol{\pi}$ for which $(e_t)_\# \boldsymbol{\pi}$ is a geodesic connecting μ_0 and μ_1. The space $\mathrm{OptGeo}(\mu_0, \mu_1)$ has the most information on transports between μ_0 and μ_1: Usually neither of the maps

$$\mathrm{OptGeo}(\mu_0, \mu_1) \to \mathrm{Geo}(\mathcal{P}(X)) \colon \boldsymbol{\pi} \mapsto (t \mapsto (e_t)_\# \boldsymbol{\pi}),$$

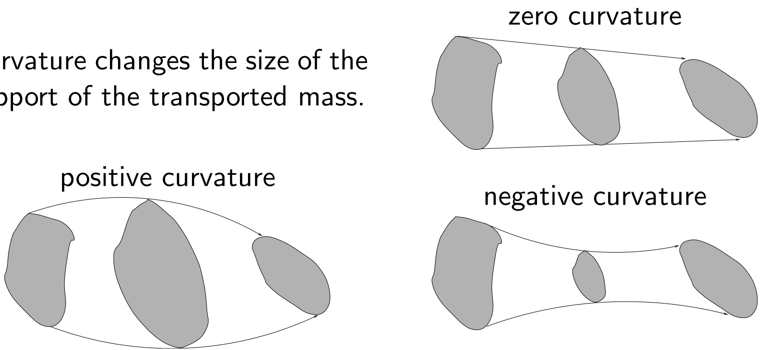
$$\mathrm{OptGeo}(\mu_0, \mu_1) \to \mathcal{P}(X \times X)) \colon \boldsymbol{\pi} \mapsto (e_0, e_1)_\# \boldsymbol{\pi}$$

is injective.

Moreover, a geodesic $(\mu_t)_{t=0}^1$ does not (in general) define an optimal plan σ, nor does σ define $(\mu_t)_{t=0}^1$.

Optimal transport on Riemannian manifolds

Curvature changes the size of the support of the transported mass.



Theorem (Otto & Villani 2000, Cordero-Erausquin, McCann & Schmuckenschläger 2001, von Renesse & Sturm 2005)

For any smooth connected Riemannian manifold M and any $K \in \mathbb{R}$ the following properties are equivalent:

1. $\mathrm{Ric}(M) \geq K$ *in the sense that* $\mathrm{Ric}_x(v,v) \geq K|v|^2$ *for all* $x \in M$ *and* $v \in T_xM$.
2. $\mathrm{Ent}_{\mathrm{vol}}$ *is* K*-convex on* $\mathcal{P}_2(M)$.

Here $\mathrm{Ent}_{\mathrm{vol}}(\rho\mathrm{vol}) = \int_M \rho \log \rho \,\mathrm{dvol}$ and K-convexity of $\mathrm{Ent}_{\mathrm{vol}}$ on $\mathcal{P}_2(M)$ means that along every geodesic $(\mu_t)_{t=0}^1 \subset \mathcal{P}_2(M)$ we have

$$\mathrm{Ent}_{\mathrm{vol}}(\mu_s) \leq (1-s)\mathrm{Ent}_{\mathrm{vol}}(\mu_0) + s\mathrm{Ent}_{\mathrm{vol}}(\mu_1) - \frac{K}{2}s(1-s)W_2^2(\mu_0,\mu_1)$$

for all $s \in [0,1]$.

Ricci-curvature bounds in metric spaces

Definition (Sturm (2006))

$(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K,\infty)$-space ($K \in \mathbb{R}$) if between any two absolutely continuous measures there exists a geodesic $(\mu_t) \in \mathrm{Geo}(\mathcal{P}(X))$ along which the entropy $\mathrm{Ent}_{\mathfrak{m}}(\rho\mathfrak{m}) = \int \rho \log \rho \, d\mathfrak{m}$ is K-convex:

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_s) \leq (1-s)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + s\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}s(1-s)W_2^2(\mu_0,\mu_1)$$

for all $s \in [0,1]$.

There is also a slightly stronger definition by Lott and Villani.

Ricci limit spaces

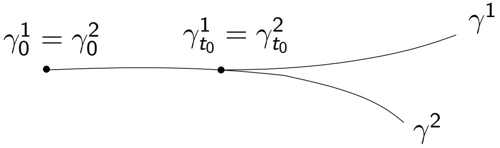
Riemannian manifolds with $\mathrm{Ric} \geq K$

$CD(K, \infty)$ of Lott-Villani

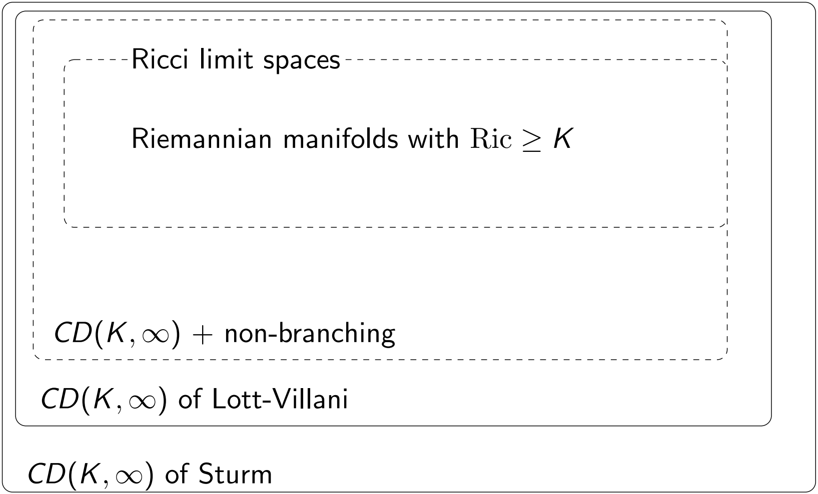
$CD(K, \infty)$ of Sturm

Branching geodesics

One problematic feature of the definition is that $CD(K,\infty)$-spaces include also spaces with branching geodesics; like $(\mathbb{R}^2, ||\cdot||_\infty, \mathcal{L}^2)$. We say that two geodesics $\gamma^1 \neq \gamma^2$, branch if there exists $t_0 \in (0,1)$ so that $\gamma^1_t = \gamma^2_t$ for all $t \in [0, t_0]$.



A space where there are no branching geodesics is called non-branching.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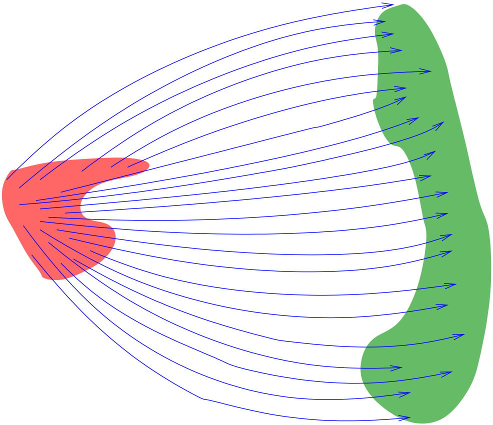
Localization without branching



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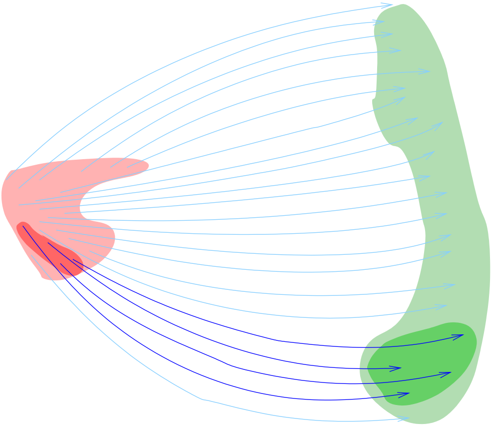

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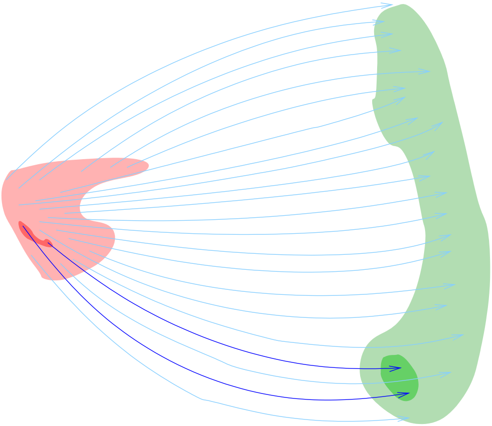

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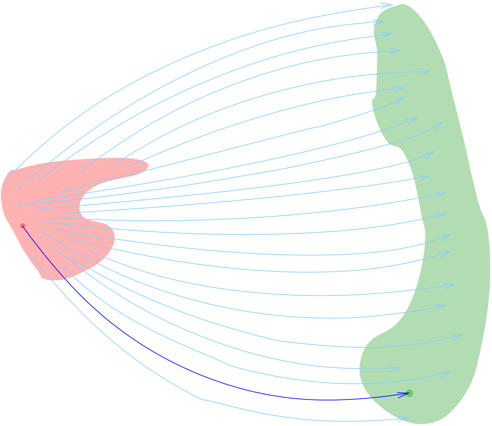

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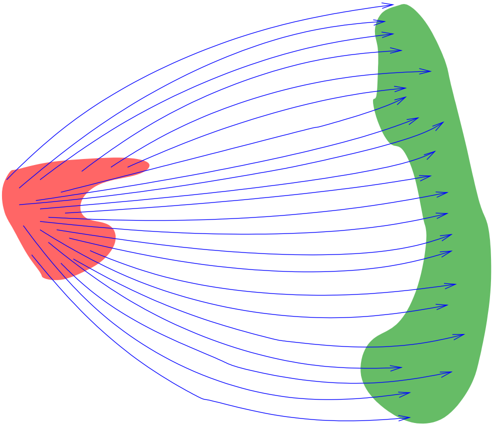

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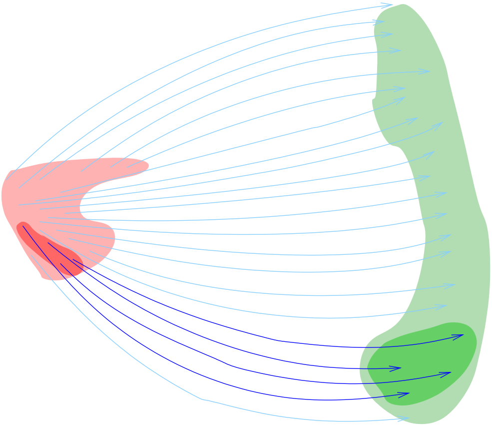

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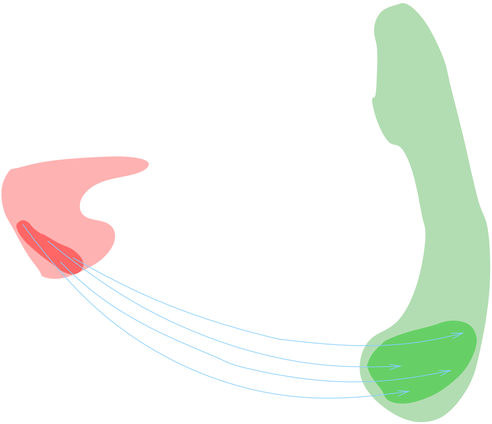

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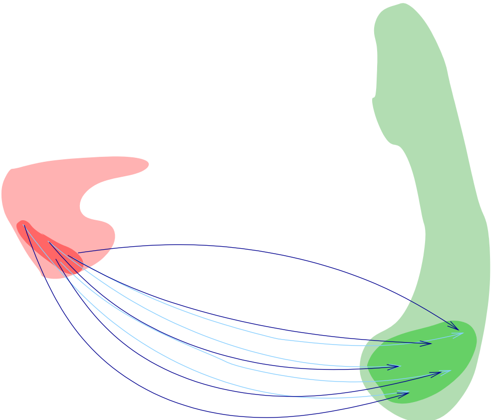
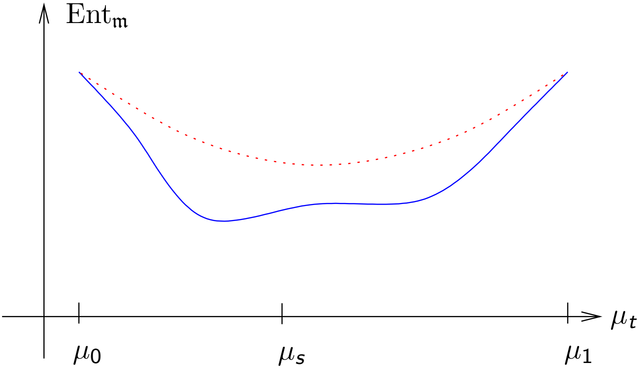


Localization in time with branching



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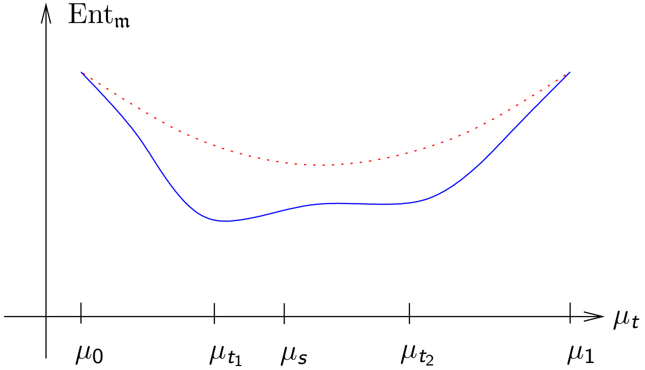

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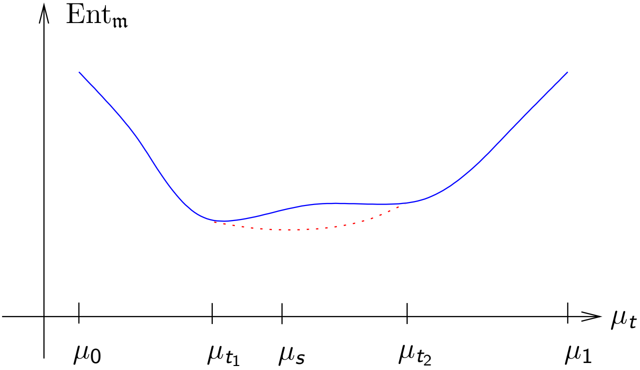

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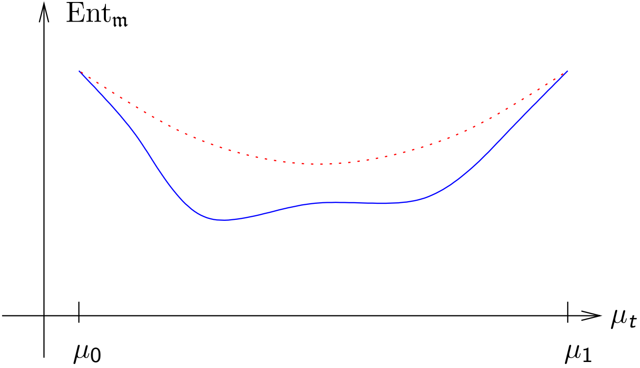

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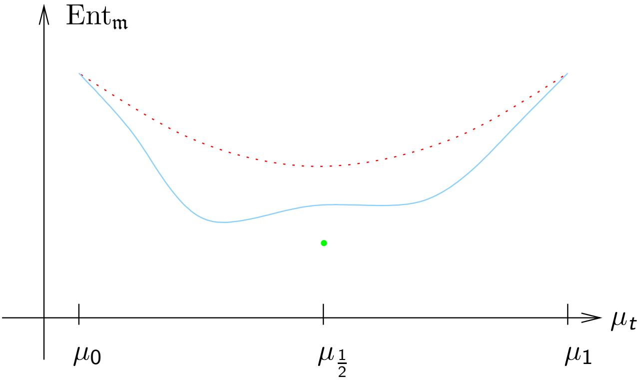

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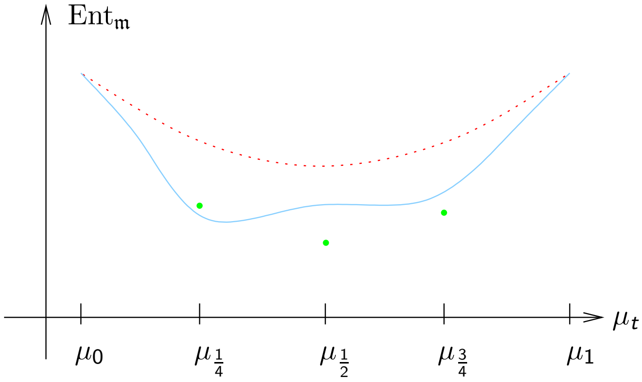

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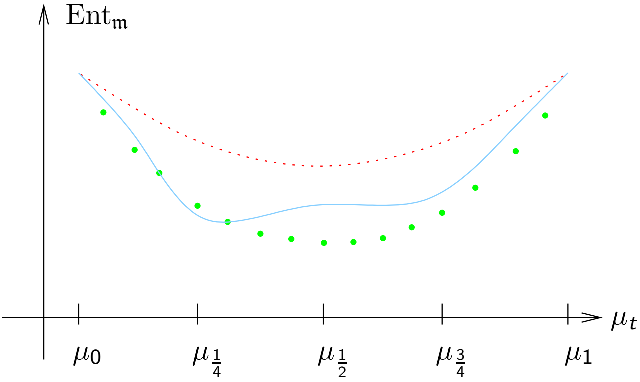

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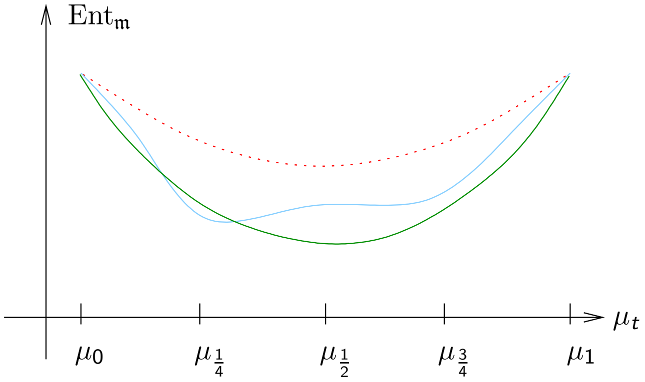
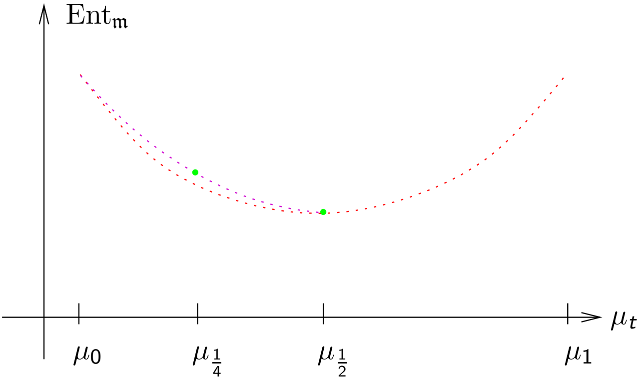


Localization in time with branching

Not only does the new geodesic satisfy the K-convexity inequality between any three times $0 < t_1 < s < t_2 < 1$, but also the measures along the geodesic have bounded densities (under some assumptions on the initial and final measures). This is true for instance if we start with two measures μ_0 and μ_1 having bounded densities and bounded supports.

Iterating minimization fails for $CD(K,N)$

In $CD(K,N)$-spaces, minimizing the (Rényi) entropy $\int_X \rho^{1-1/N}\,\mathrm{d}\mathfrak{m}$ does not always produce a geodesic along which the inequalities required by $CD(K,N)$-condition hold.



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However, if one of the measures is singular with respect to $\mathfrak{m}$, we get the correct $CD(K,N)$-bounds when taking intermediate points towards this measure.

Combining this observation with the density bounds gives

Theorem

$CD(K,N) \Rightarrow MCP(K,N)$ (by Ohta).

Definition

A space (X, d, m) is said to satisfy the measure contraction property $MCP(K, N)$ (in the sense of Ohta) if for every $x \in X$ and $A \subset X$ (and $A \subset B(x, \pi\sqrt{(N-1)/K})$ if $K > 0$) with $0 < m(A) < \infty$ there exists

$$\pi \in \mathrm{GeoOpt}\left(\delta_x, \frac{1}{m(A)} m|_A\right)$$

so that

$$dm \geq (e_t)_\# \left(t^N \beta_t(\gamma_0, \gamma_1) m(A) d\pi(\gamma)\right).$$

local Poincaré inequality in non-branching $MCP(K,N)$

Theorem (Lott & Villani, von Renesse, Sturm, Hinde & Petersen, Cheeger & Colding)

Suppose that $(X, \mathrm{d}, \mathfrak{m})$ is a nonbranching $MCP(K,N)$-space with $K \in \mathbb{R}$. Then the weak local Poincaré inequality

$$\fint_{B(x,r)} |u - \langle u \rangle_{B(x,r)}| \,\mathrm{d}\mathfrak{m} \leq C(N,K,r) r \fint_{B(x,2r)} g \,\mathrm{d}\mathfrak{m}$$

holds for any measurable function u defined on X, any upper gradient g of u and for each point $x \in X$ and radius $r > 0$.

local Poincaré inequality in $CD(K,\infty)$

Theorem

Suppose that $(X,\mathrm{d},\mathfrak{m})$ is a $CD(K,\infty)$-space (in the sense of Sturm) with $K \in \mathbb{R}$. Then the weak local Poincaré inequality

$$\int_{B(x,r)} |u - \langle u \rangle_{B(x,r)}| \, \mathrm{d}\mathfrak{m} \leq 4re^{K^- r^2} \int_{B(x,2r)} g \, \mathrm{d}\mathfrak{m}$$

holds for any measurable function u defined on X, any upper gradient g of u and for each point $x \in X$ and radius $r > 0$.

- Observe that we do not have average integrals in this theorem.

Proof of the local Poincaré inequality

Let us prove the $CD(0,\infty)$ case for simplicity. We have to show that

$$\int_{B(x,r)} |u - \langle u \rangle_{B(x,r)}| \,\mathrm{d}\mathfrak{m} \le 4r \int_{B(x,2r)} g \,\mathrm{d}\mathfrak{m}.$$

Abbreviate $B = B(x,r)$ and denote

$$M = \inf\left\{ a \in \mathbb{R} : \mathfrak{m}(\{u > a\}) \le \frac{\mathfrak{m}(B)}{2} \right\}.$$

Split the ball B into two Borel sets B^+ and B^- so that $B = B^+ \cup B^-$, $B^+ \cap B^- = \emptyset$, $\mathfrak{m}(B^+) = \mathfrak{m}(B^-)$ and

$$u(x) \le M \le u(y) \qquad \text{for all } x \in B^-, y \in B^+.$$

Proof of the local Poincaré inequality

Let $(\mu_t)_{t=0}^1$ be a geodesic between $\frac{1}{\mathfrak{m}(B^+)}\mathfrak{m}|_{B^+}$ and $\frac{1}{\mathfrak{m}(B^-)}\mathfrak{m}|_{B^-}$ along which we have the density bound (writing $\mu_t = \rho_t\mathfrak{m}$)

$$\rho_t(y) \le \frac{2}{\mathfrak{m}(B)}$$

for all $t \in [0,1]$ at $\mathfrak{m}$-almost every $y \in X$. Let $\boldsymbol{\pi}$ be a corresponding measure on the set of geodesics.

Proof of the local Poincaré inequality

From $u(z) \leq M \leq u(y)$ for all $(z,y) \in B^- \times B^+$ we get

$$|u(\gamma(0)) - u(\gamma(1))| = |u(\gamma(0)) - M| + |M - u(\gamma(1))|$$

for $\boldsymbol{\pi}$-almost every $\gamma \in \mathrm{Geo}(X)$. Therefore

$$\begin{aligned}
&\int_{\mathrm{Geo}(X)} |u(\gamma(0)) - u(\gamma(1))| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= \int_{\mathrm{Geo}(X)} |u(\gamma(0)) - M| \,\mathrm{d}\boldsymbol{\pi}(\gamma) + \int_{\mathrm{Geo}(X)} |M - u(\gamma(1))| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= \frac{2}{\mathfrak{m}(B)} \int_{B^+} |u(z) - M| \,\mathrm{d}\mathfrak{m}(z) + \frac{2}{\mathfrak{m}(B)} \int_{B^-} |M - u(z)| \,\mathrm{d}\mathfrak{m}(z) \\
&= \frac{2}{\mathfrak{m}(B)} \int_{B} |u(z) - M| \,\mathrm{d}\mathfrak{m}(z).
\end{aligned}$$

Proof of the local Poincaré inequality

$$\int_{B(x,r)} |u - \langle u\rangle_{B(x,r)}| \,\mathrm{d}\mathfrak{m} \le \frac{1}{\mathfrak{m}(B)} \iint_{B\times B} |u(z) - u(y)| \,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(y)$$

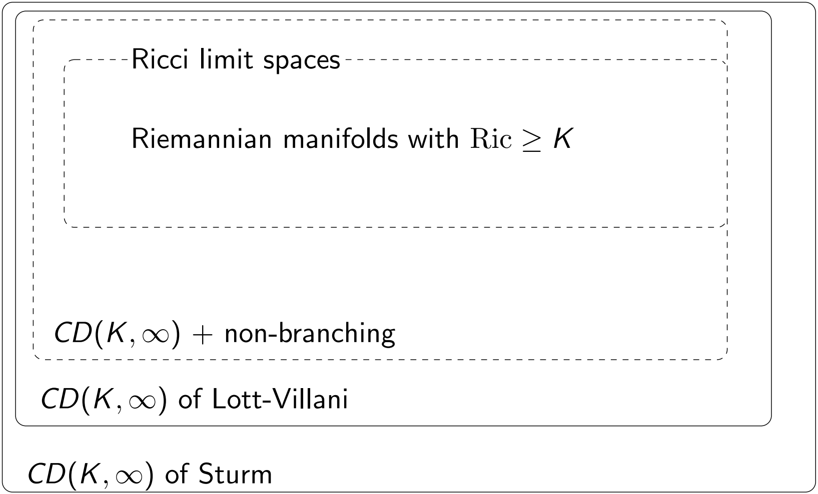
$$\le \frac{1}{\mathfrak{m}(B)} \iint_{B\times B} (|u(z) - M| + |M - u(y)|) \,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(y)$$

$$= 2\int_B |u(z) - M| \,\mathrm{d}\mathfrak{m}(z) = \mathfrak{m}(B) \int_{\mathrm{Geo}(X)} |u(\gamma(0)) - u(\gamma(1))| \,\mathrm{d}\boldsymbol{\pi}(\gamma)$$

$$\le 2r\mathfrak{m}(B) \int_{\mathrm{Geo}(X)} \int_0^1 g(\gamma(t))\, dt \,\mathrm{d}\boldsymbol{\pi}(\gamma)$$

$$= 2r\mathfrak{m}(B) \int_0^1 \int_X g(z)\rho_t(z) \,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}t$$

$$\le 4r \int_0^1 \int_{B(x,2r)} g(z) \,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}t = 4r \int_{B(x,2r)} g \,\mathrm{d}\mathfrak{m}.$$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$CD(K, \infty)$ of Sturm

Question

Does there exist a stable definition of Ricci curvature lower bounds that excludes branching?

$RCD(K,\infty)$

Definition (Ambrosio, Gigli & Savaré, 2011 (preprint))

A metric measure space $(X, \mathrm{d}, \mathfrak{m})$ has *Riemannian Ricci curvature bounded below* by $K \in \mathbb{R}$, or $RCD(K,\infty)$ for short, if one of the following equivalent conditions hold:

1. $(X, \mathrm{d}, \mathfrak{m})$ is a $CD(K,\infty)$ space and the Cheeger-energy $Ch(f) = \frac{1}{2}\int |Df|_w^2$ is a quadratic form on $L^2(X, \mathfrak{m})$.
2. $(X, \mathrm{d}, \mathfrak{m})$ is a $CD(K,\infty)$ space and the W_2 gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ is additive on $\mathcal{P}_2(X)$.
3. Any $\mu \in \mathscr{P}_2(X)$ is the starting point of an EVI_K gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$.

Theorem (Daneri & Savaré, 2008)

EVI_K implies strong displacement K-convexity.

$RCD(K, \infty) \Rightarrow$ essential non-branching

Theorem (R. & Sturm, 2012 (preprint))

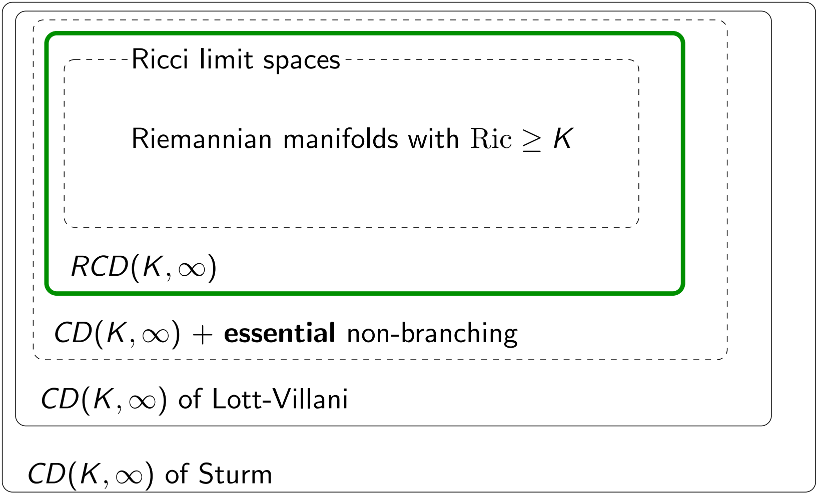
Strong $CD(K, \infty)$-spaces are essentially non-branching. In particular $RCD(K, \infty)$-spaces are essentially non-branching.

Corollary (of essential non-branching and Gigli's result)

There exist optimal transport maps in strong $CD(K, \infty)$-spaces between $\mu_0, \mu_1 \ll \mathfrak{m}$.

Definition

A space $(X, \mathrm{d}, \mathfrak{m})$ is called *essentially non-branching* if for every $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ that are absolutely continuous with respect to $\mathfrak{m}$ we have that any $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics. (Meaning that $\boldsymbol{\pi}(\Gamma) = 1$ for some $\Gamma \subset \mathrm{Geo}(X)$ so that there are no two branching geodesics in Γ.)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$CD(K,\infty)$ of Sturm

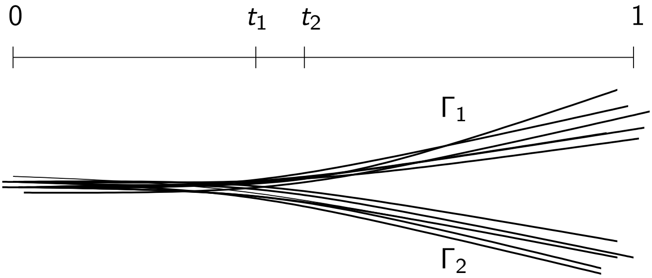
"Proof" Suppose that the claim is not true so that there exists a measure $\boldsymbol{\pi}$ that is not concentrated on non-branching geodesics. By restricting the measure $\boldsymbol{\pi}$ we may assume that there are $0 < t_1 < t_2 < 1$ with $|t_1 - t_2|$ small and two sets of geodesics Γ_1, Γ_2 so that

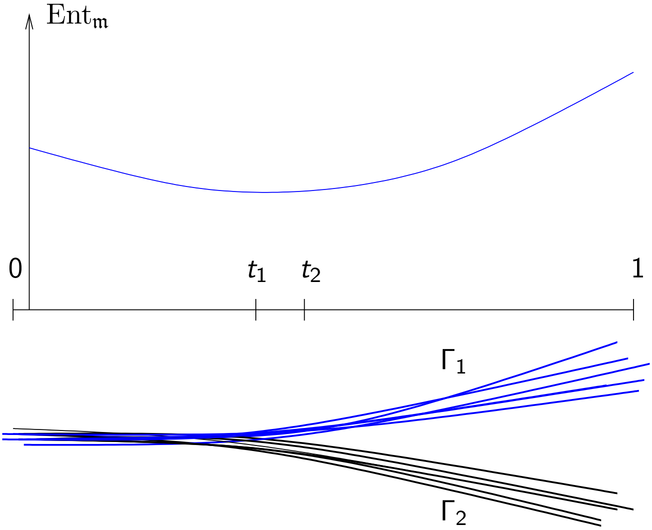
$$(e_t)_\#\boldsymbol{\pi}|_{\Gamma_1} = (e_t)_\#\boldsymbol{\pi}|_{\Gamma_2} \qquad \text{for all } t \in [0, t_1]$$

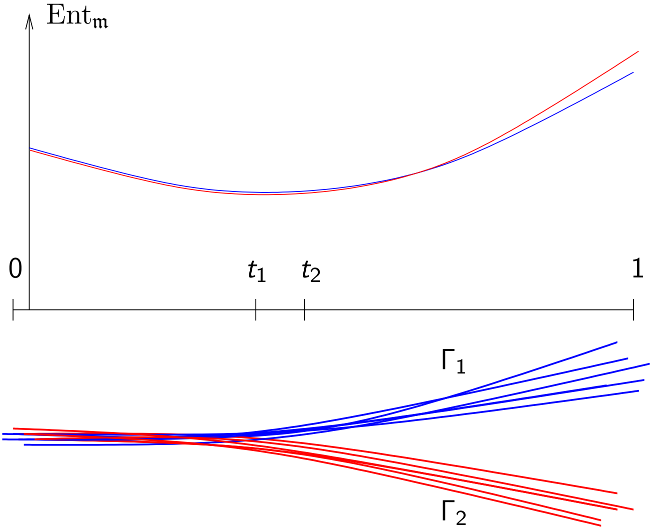
and

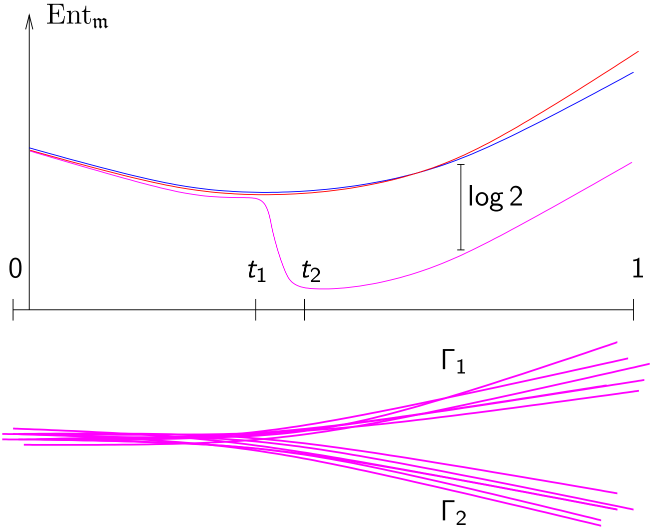
$$(e_t)_\#\boldsymbol{\pi}|_{\Gamma_1} \perp (e_t)_\#\boldsymbol{\pi}|_{\Gamma_2} \qquad \text{for all } t \in [t_2, 1]$$

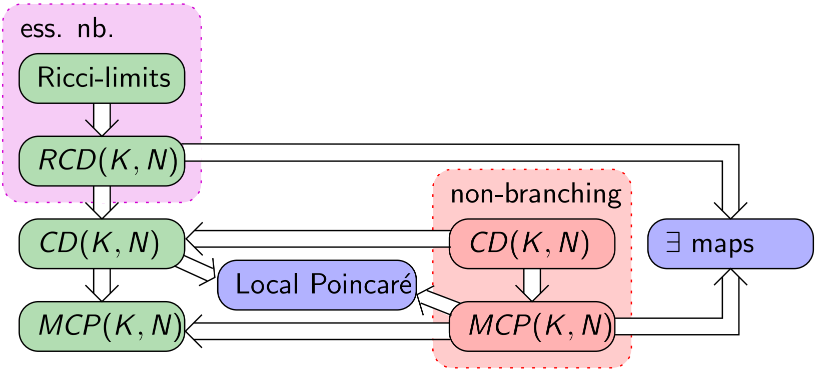
for all $t \in [t_2, 1]$.


0
t_1
t_2
1
Γ_1
Γ_2

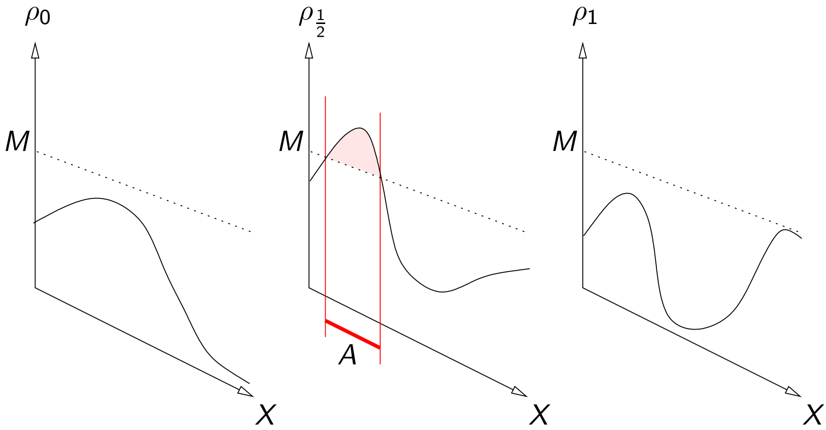

$\mathrm{Ent}_{\mathfrak{m}}$
0
t_1
t_2
1
Γ_1
Γ_2


$\mathrm{Ent}_{\mathfrak{m}}$
0
t_1
t_2
1
Γ_1
Γ_2


$\mathrm{Ent}_{\mathfrak{m}}$
log 2
0
t_1
t_2
1
Γ_1
Γ_2


ess. nb.
Ricci-limits
$RCD(K, N)$
$CD(K, N)$
$MCP(K, N)$
Local Poincaré
non-branching
$CD(K, N)$
$MCP(K, N)$
∃ maps

Why is $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\} =: M$?.



Consider the curve $\Gamma \in \mathcal{P}(\mathrm{Geo}(X))$ between the marginals corresponding to the part of the measure which we want to redistribute along which $\mathrm{Ent}_{\mathfrak{m}}$ is displacement convex. We have

$$\mathrm{Ent}_{\mathfrak{m}}(\Gamma_{\frac{1}{2}}) \leq \frac{1}{2}\mathrm{Ent}_{\mathfrak{m}}(\Gamma_0) + \frac{1}{2}\mathrm{Ent}_{\mathfrak{m}}(\Gamma_1) \leq \log M.$$

On the other hand, by Jensen's inequality we always have

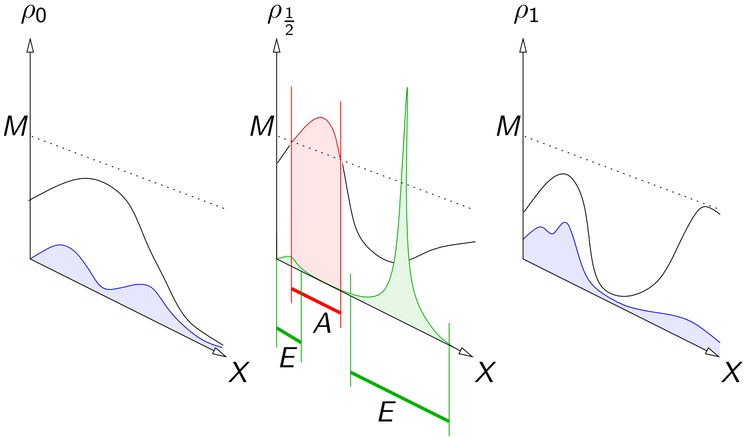
$$\begin{aligned}\mathrm{Ent}_{\mathfrak{m}}(\Gamma_{\frac{1}{2}}) &= \int_E \rho_{\frac{1}{2}} \log \rho_{\frac{1}{2}} \, \mathrm{d}\mathfrak{m} \\ &\geq m(E) \left(⨍_E \rho_{\frac{1}{2}} \, \mathrm{d}\mathfrak{m} \right) \log \left(⨍_E \rho_{\frac{1}{2}} \, \mathrm{d}\mathfrak{m} \right) \geq \log \frac{1}{\mathfrak{m}(E)},\end{aligned}$$

where $E = \{x \in X \ : \ \rho_{\frac{1}{2}}(x) > 0\}$ and $\Gamma_t = \rho_t \mathfrak{m}$. Thus

$$\mathfrak{m}(E) \geq \frac{1}{M}.$$

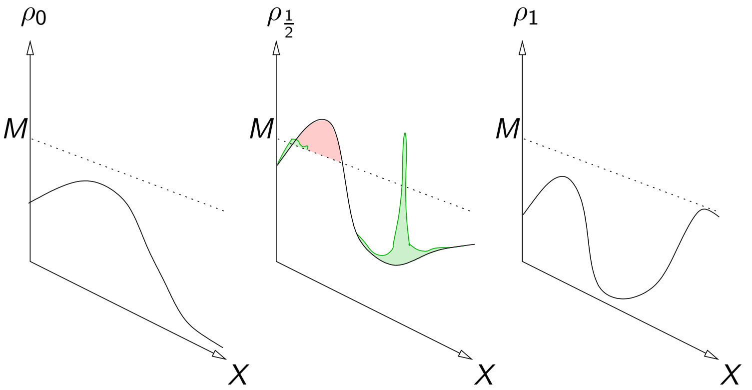
This is why $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\}$.

The $CD(K,\infty)$ condition gives a new well spread midpoint for the high-density part of the old midpoint.



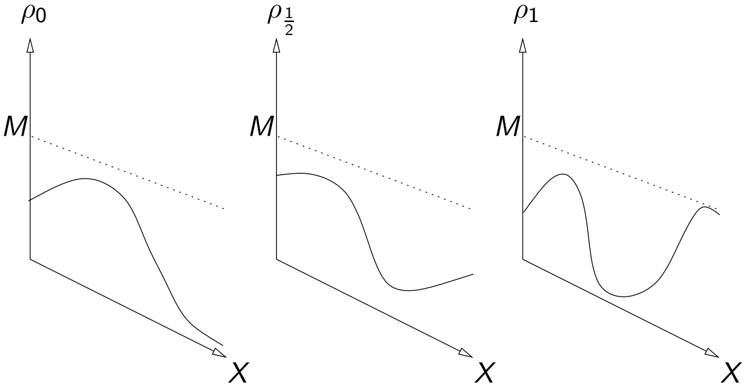
This is why $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\}$.

Taking a weighted combination of this new midpoint measure and the old one lowers the entropy.



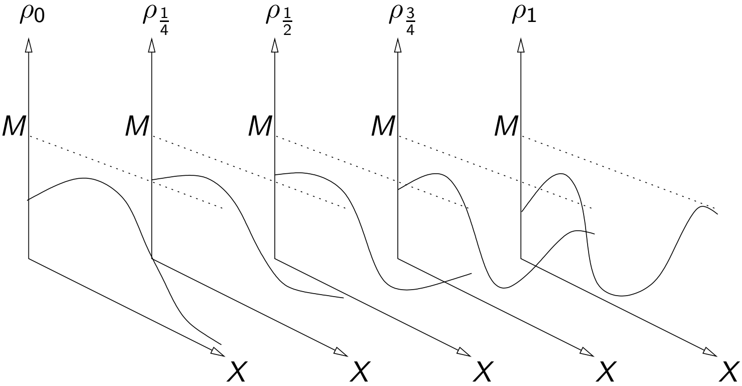
This is why $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\}$.

Therefore at the minimum of the entropy among the midpoints we have the density bound.



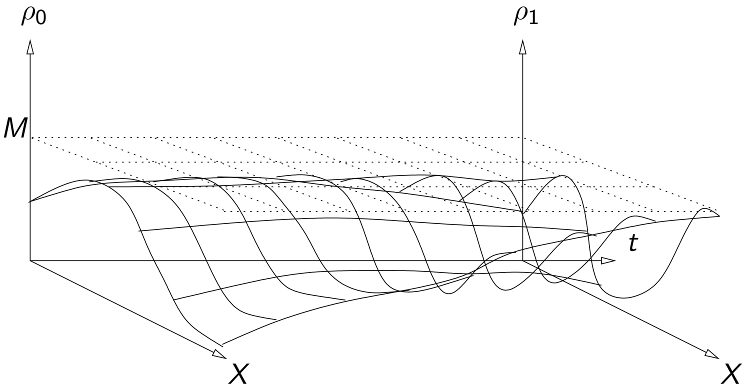
The rest of the geodesic.

When we continue taking minimizers in the next level midpoints the bound is preserved.



The rest of the geodesic.

Finally we end up with a complete geodesic with the density bound.



Question

$MCP(K,N) \Rightarrow$ Local Poincaré?

Question

Does there exist optimal maps in $CD(K,N)$-spaces from every $\mu \ll \mathfrak{m}$? (not all plans are given by maps)

Question

Local-to-global for $CD(K,\infty)$?

Question

Are $RCD(K,\infty)$-spaces non-branching?

Question

Are $RCD(K,\infty)$-spaces Ricci-limits?

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Thank you!