The Geometry of Last Passage Percolation

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Joint with Chris Janjigian and Timo Seppäläinen

Inviscid stochastic Burgers equation: $\partial_t u + u\,\partial_x u = \partial_x \mathcal{F}$

random forcing → $\mathcal{F}$

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Viscosity solution with initial condition $\mathcal{U}(t_0,x)=\mathcal{U}_0(x)$:

Lax-Oleinik: $\mathcal{U}(t,x) = \inf\limits_{\substack{\gamma:[t_0,t]\to\mathbb{R} \\ \gamma(t)=x}}$

abs. cont. ↗

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abs. cont. ↗ (γ)

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(minimizes the action between $(t_0,\gamma(t_0))$ & (t,x) $\forall t_0<t$)

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Yes, in compact & essentially compact space settings: Sinai et al
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Also for certain F's (kick forcing): Bakhtin, Cator, Khanin '14
Bakhtin '16

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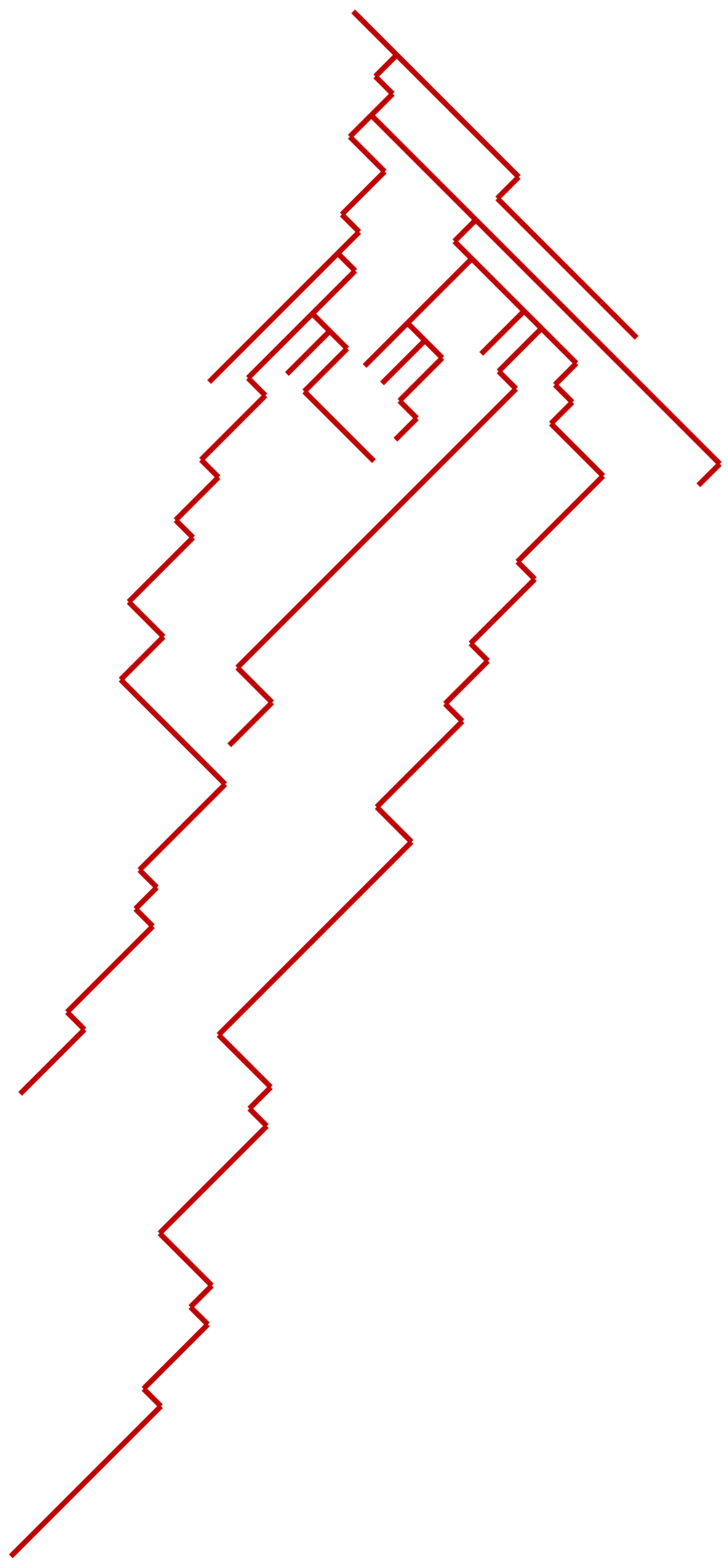
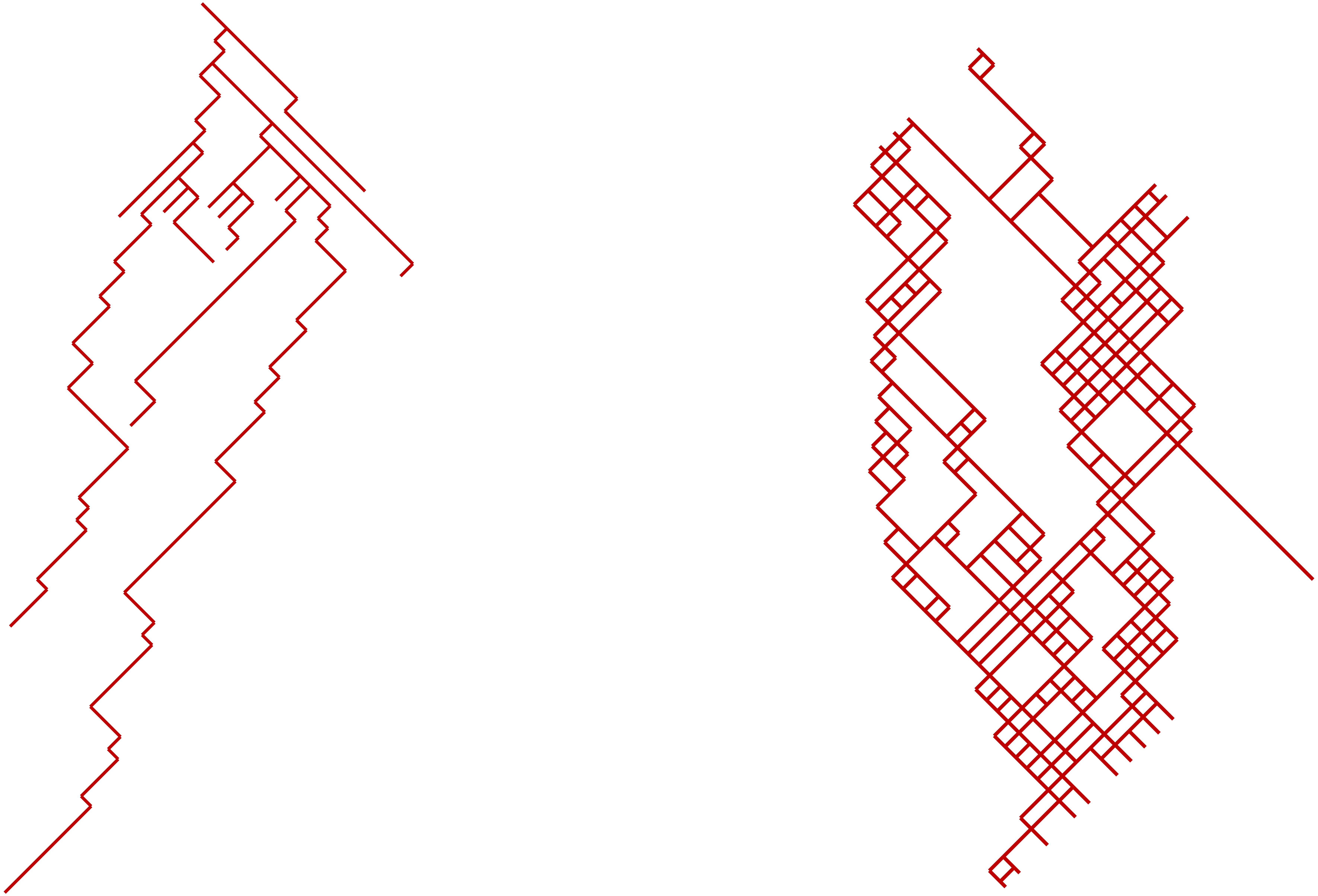
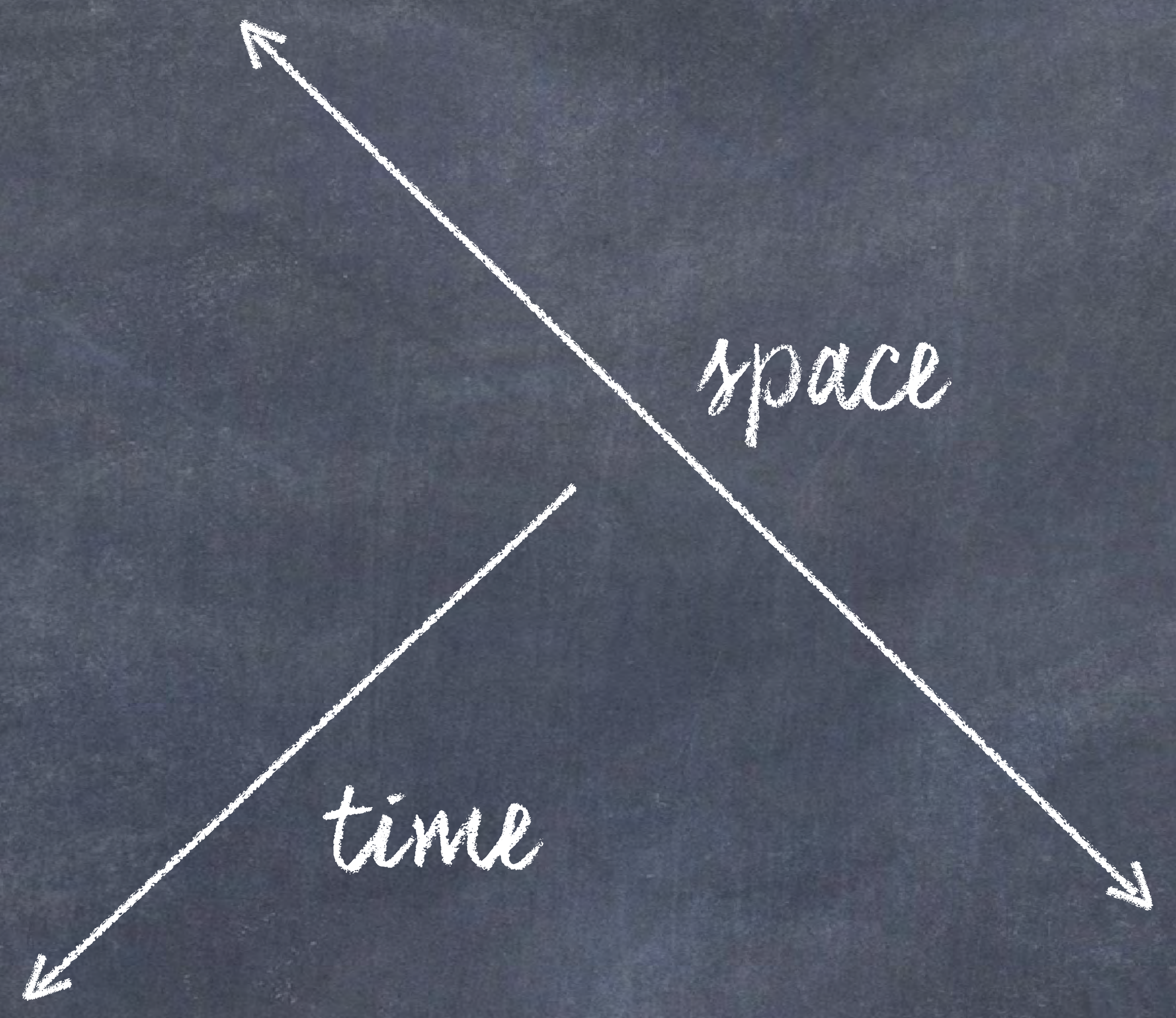
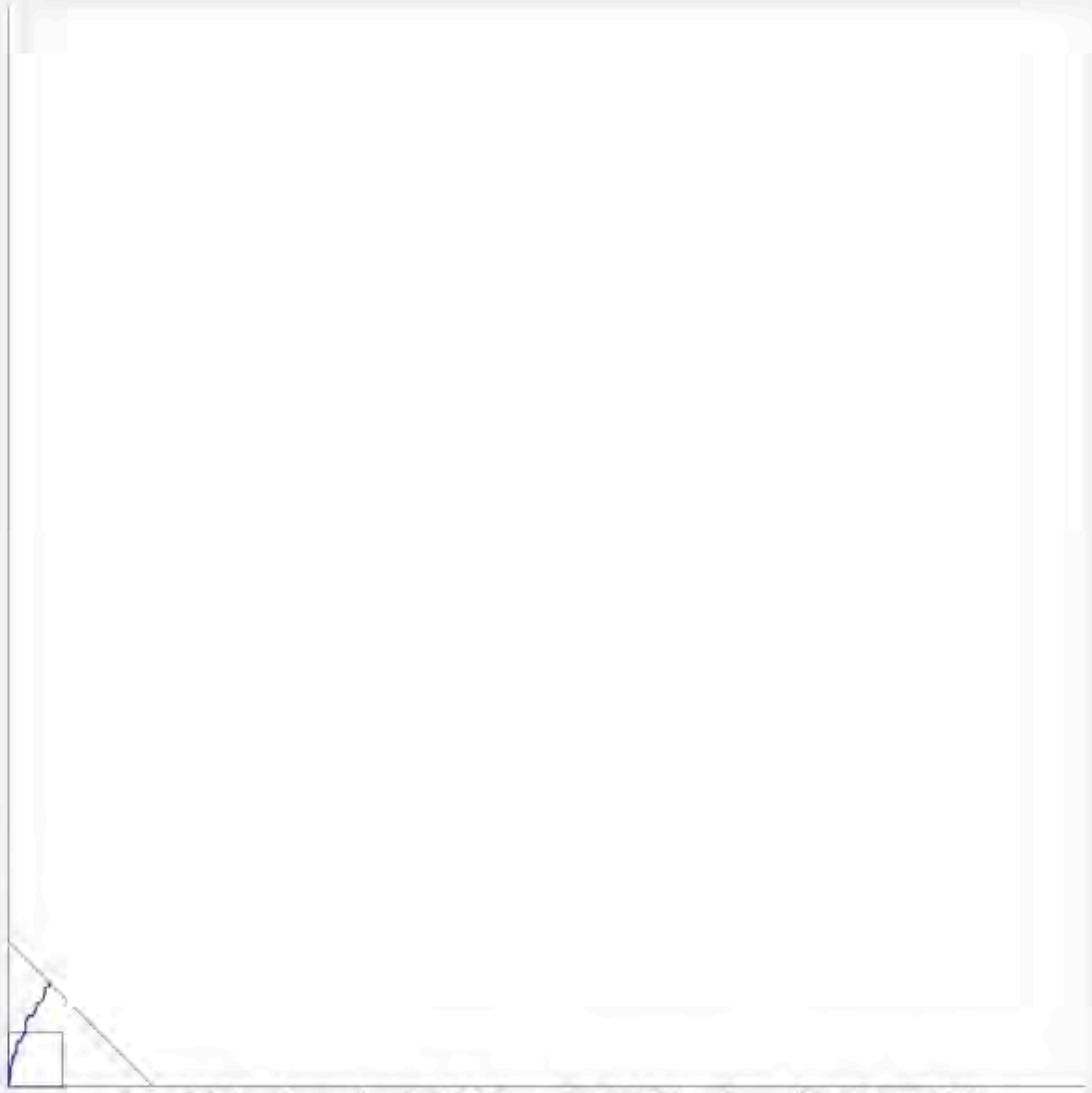
We find new instability structures for certain random v

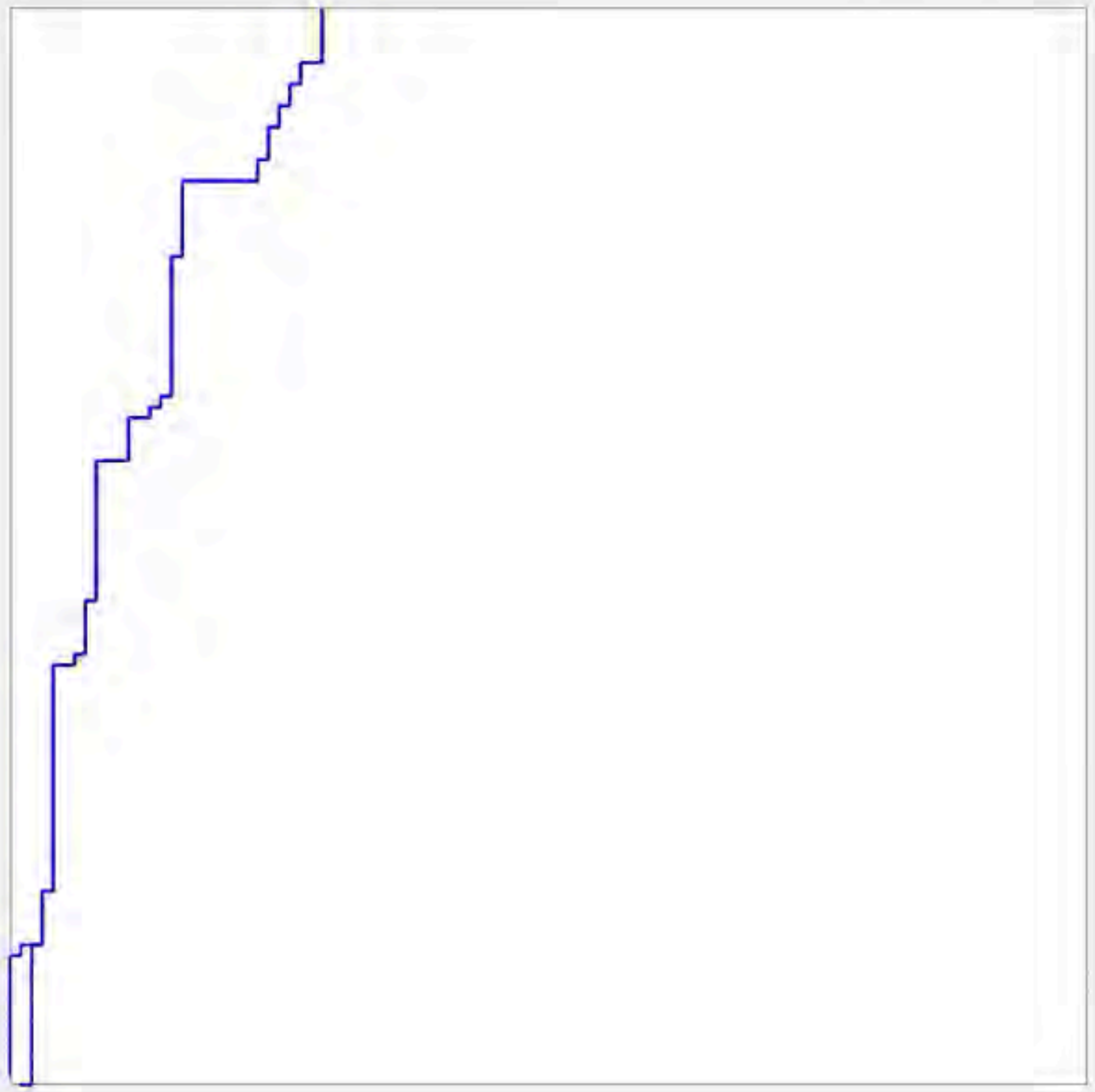


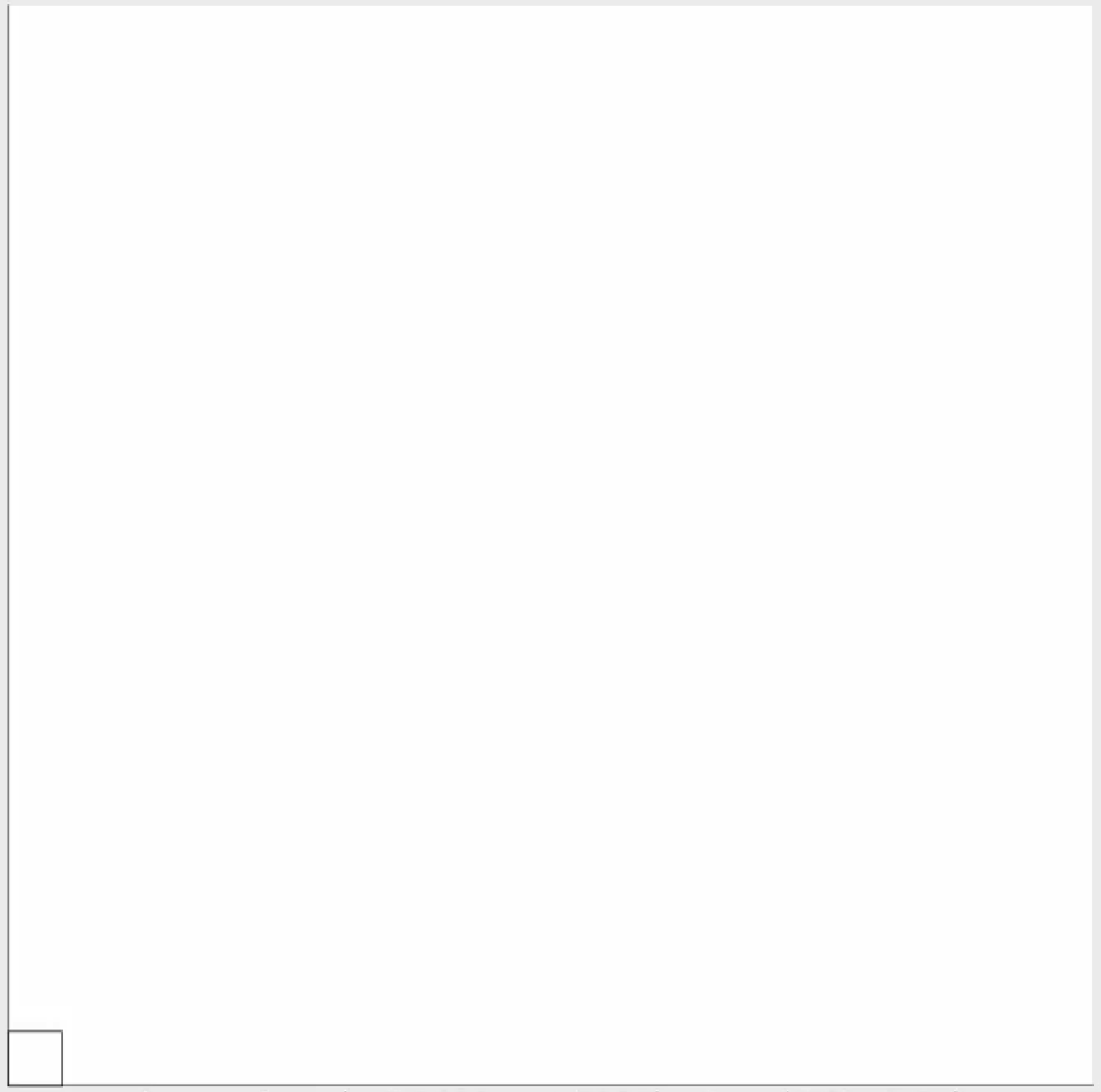



space
time

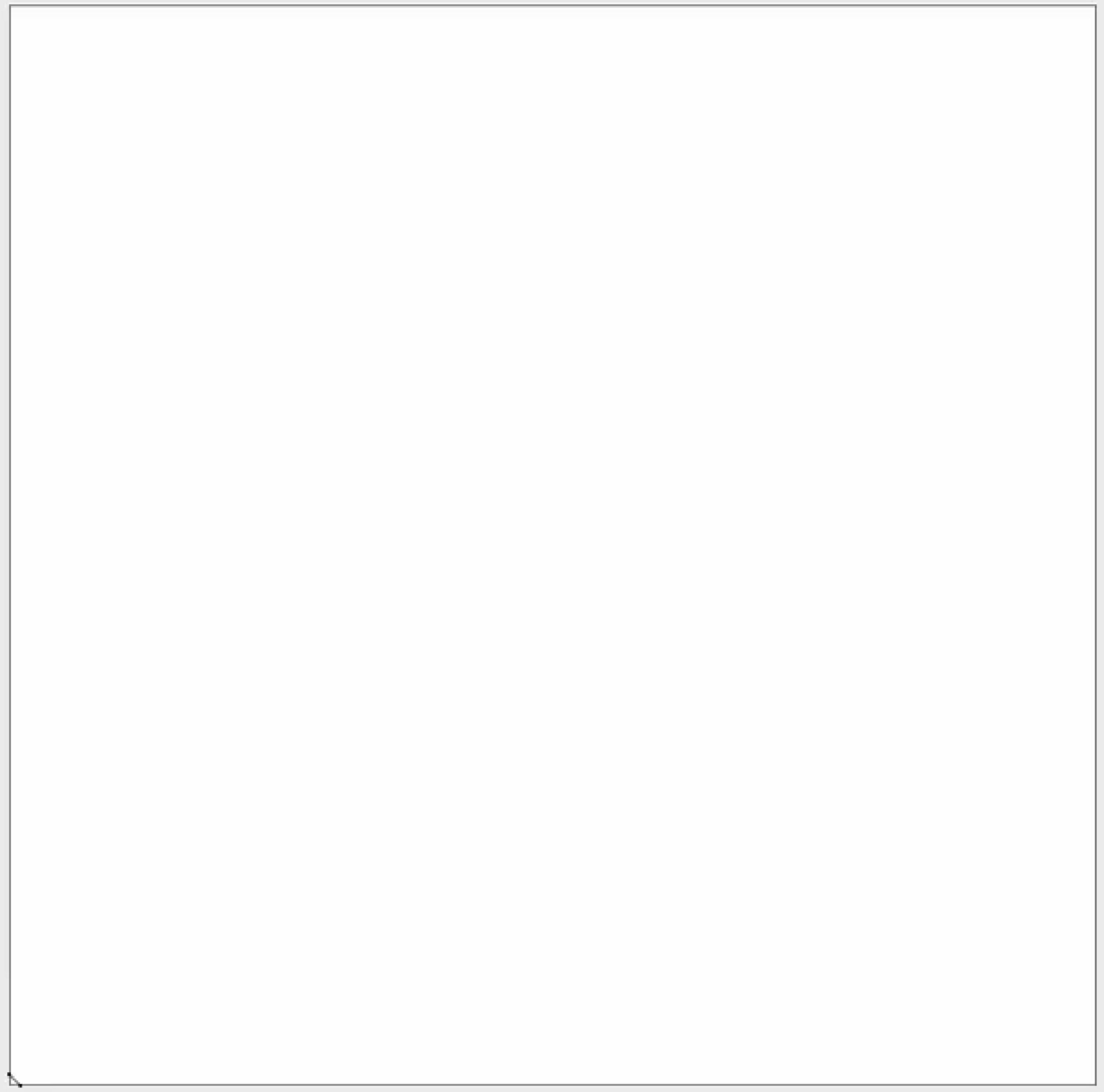


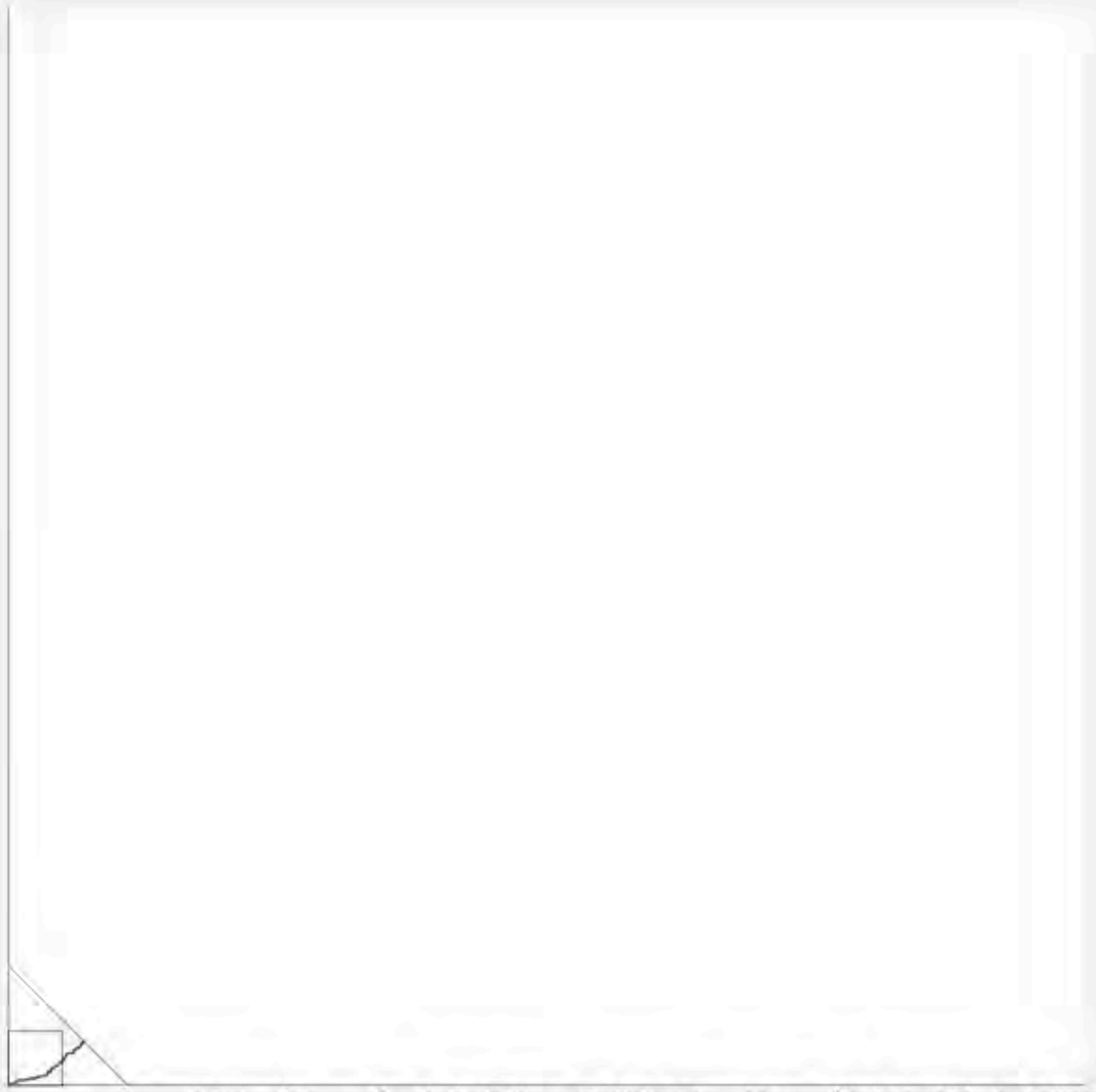
$\ell = 269,\ h = (-2.3628, -1.7338),\ \xi = (0.35, 0.65)$



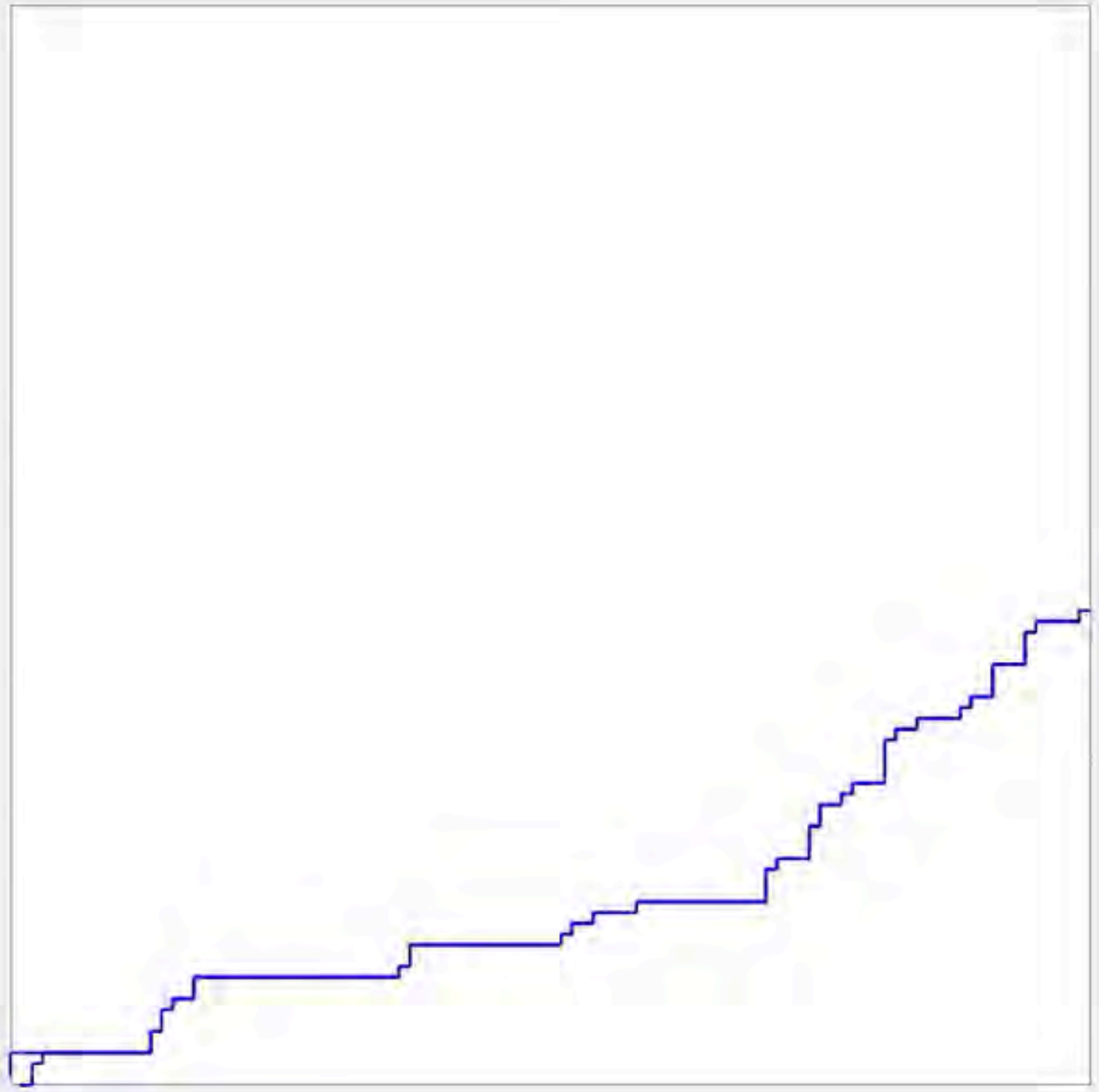


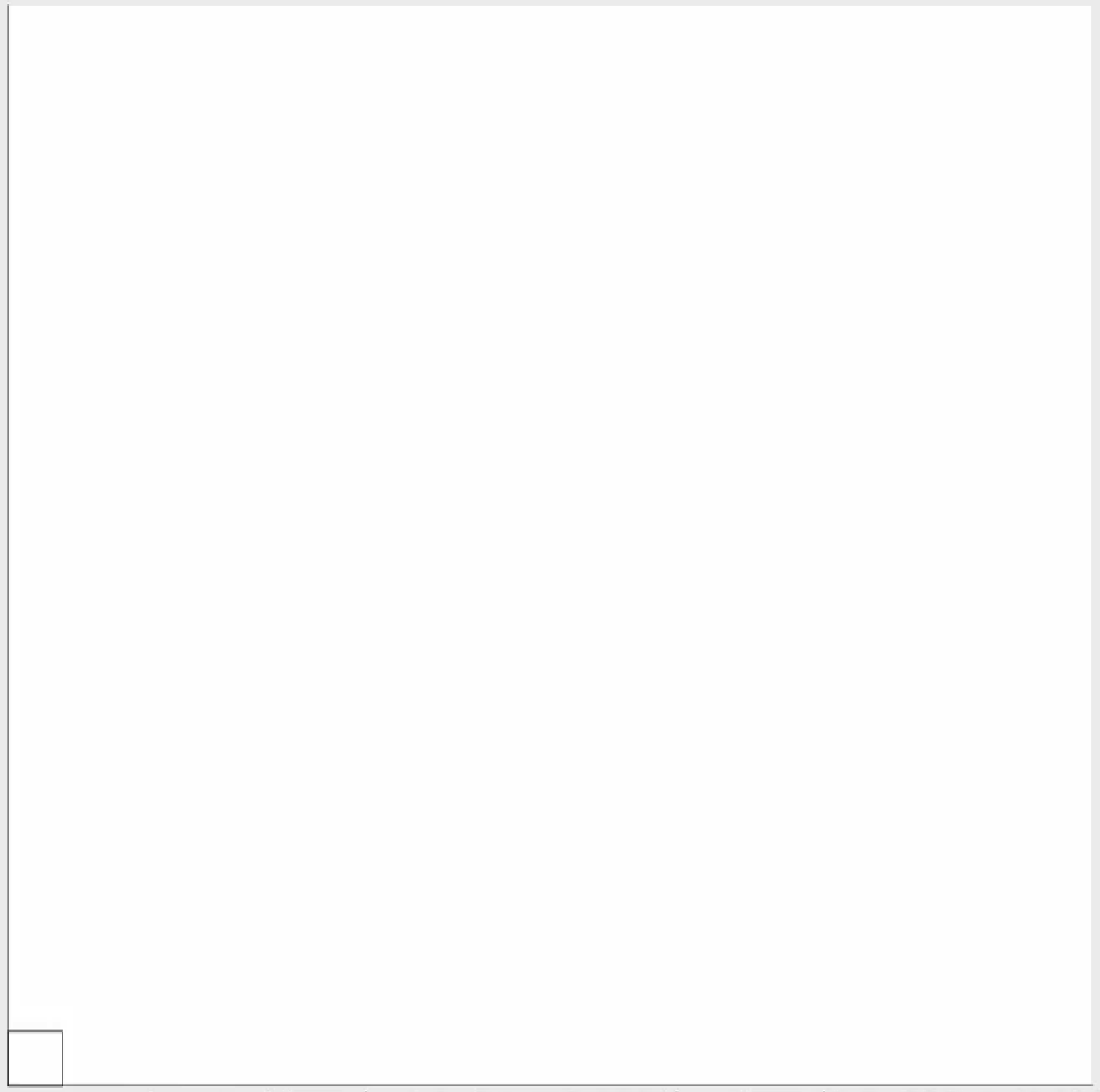
$\ell = 1, \ h = (-2.3628, -1.7338), \ \xi = (0.35, 0.65)$





$\ell = 223, \ h^* = (-1.9174, -2.0901), \ \xi^* = (0.54302, 0.45698)$





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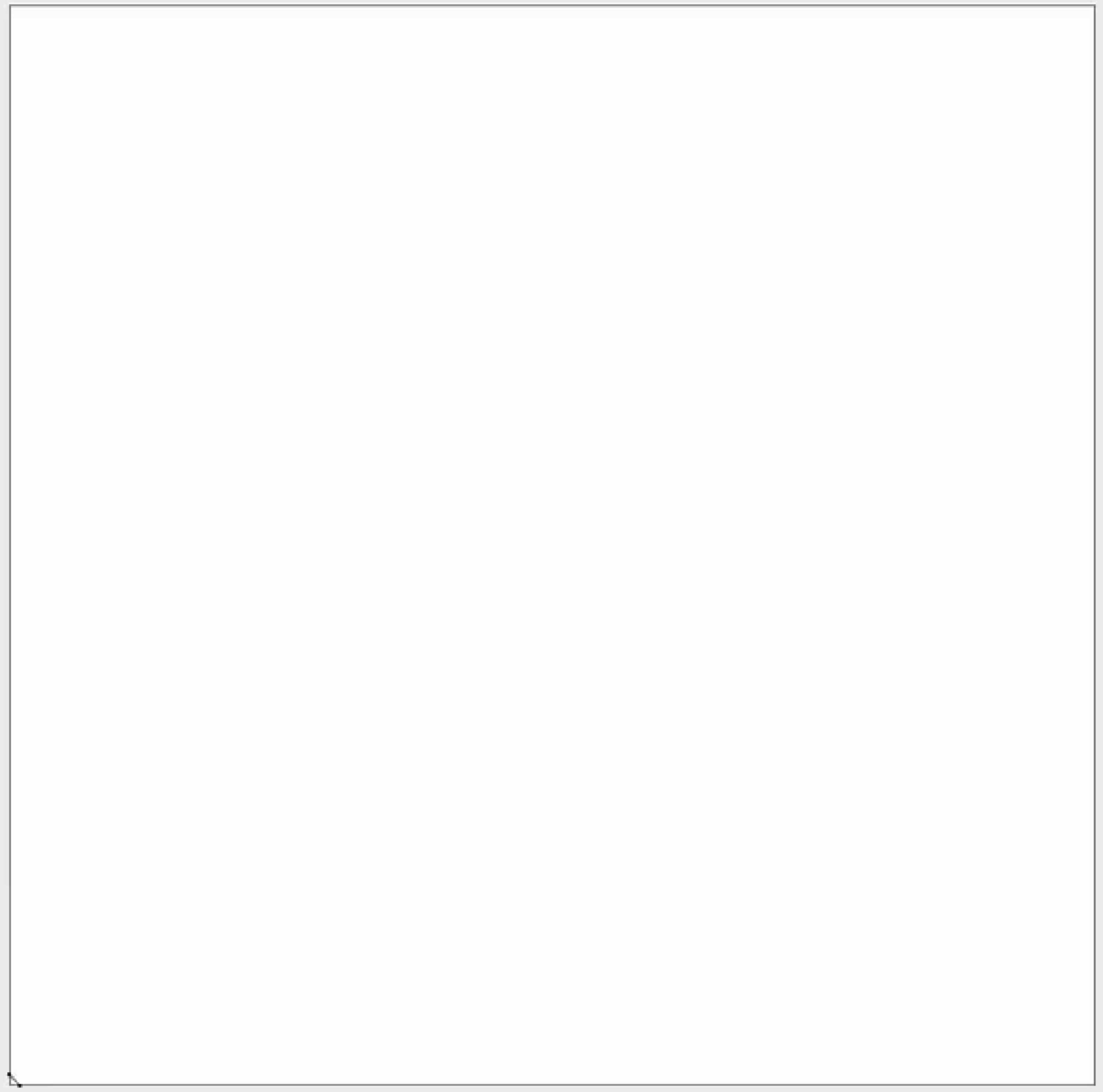
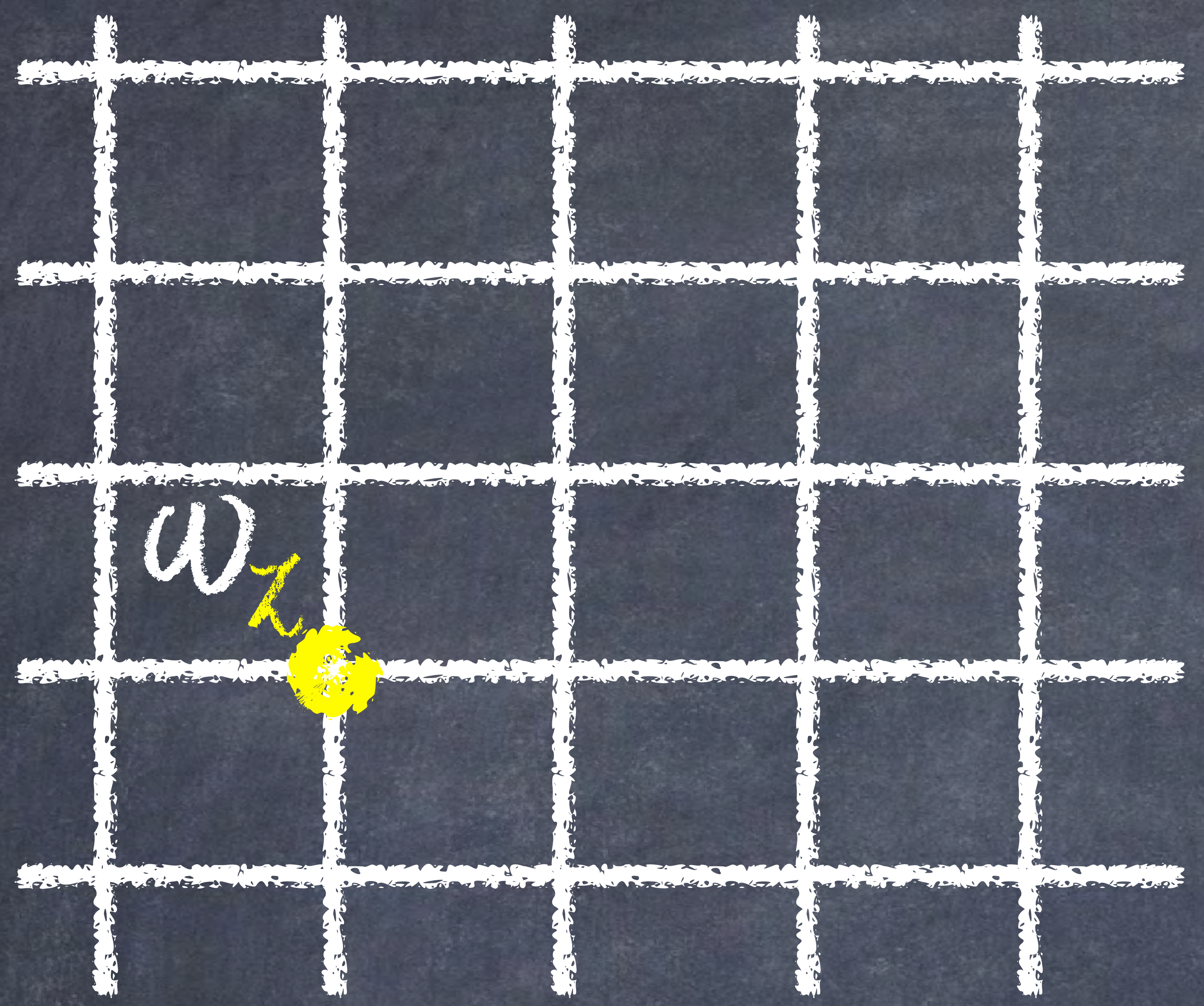
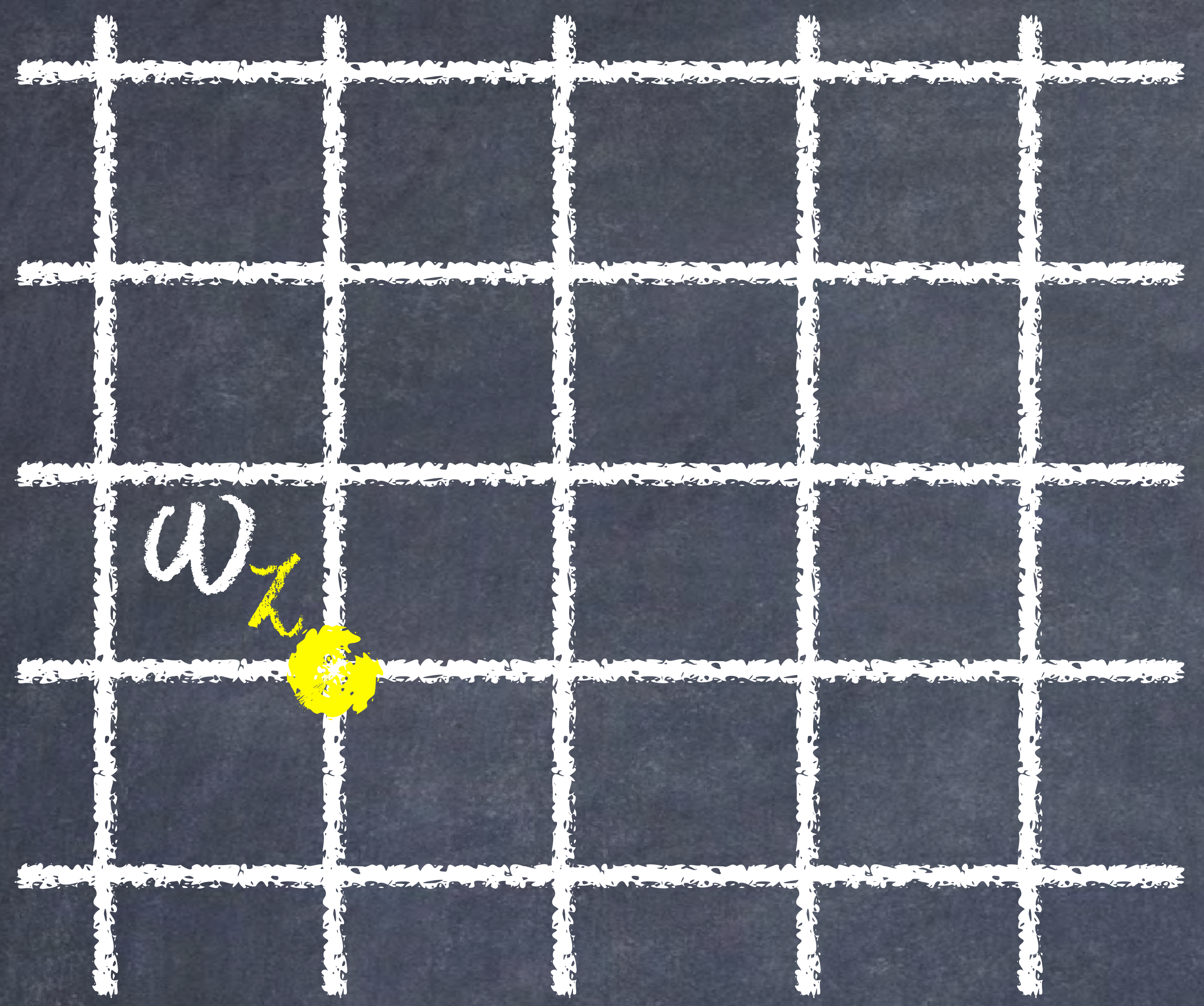
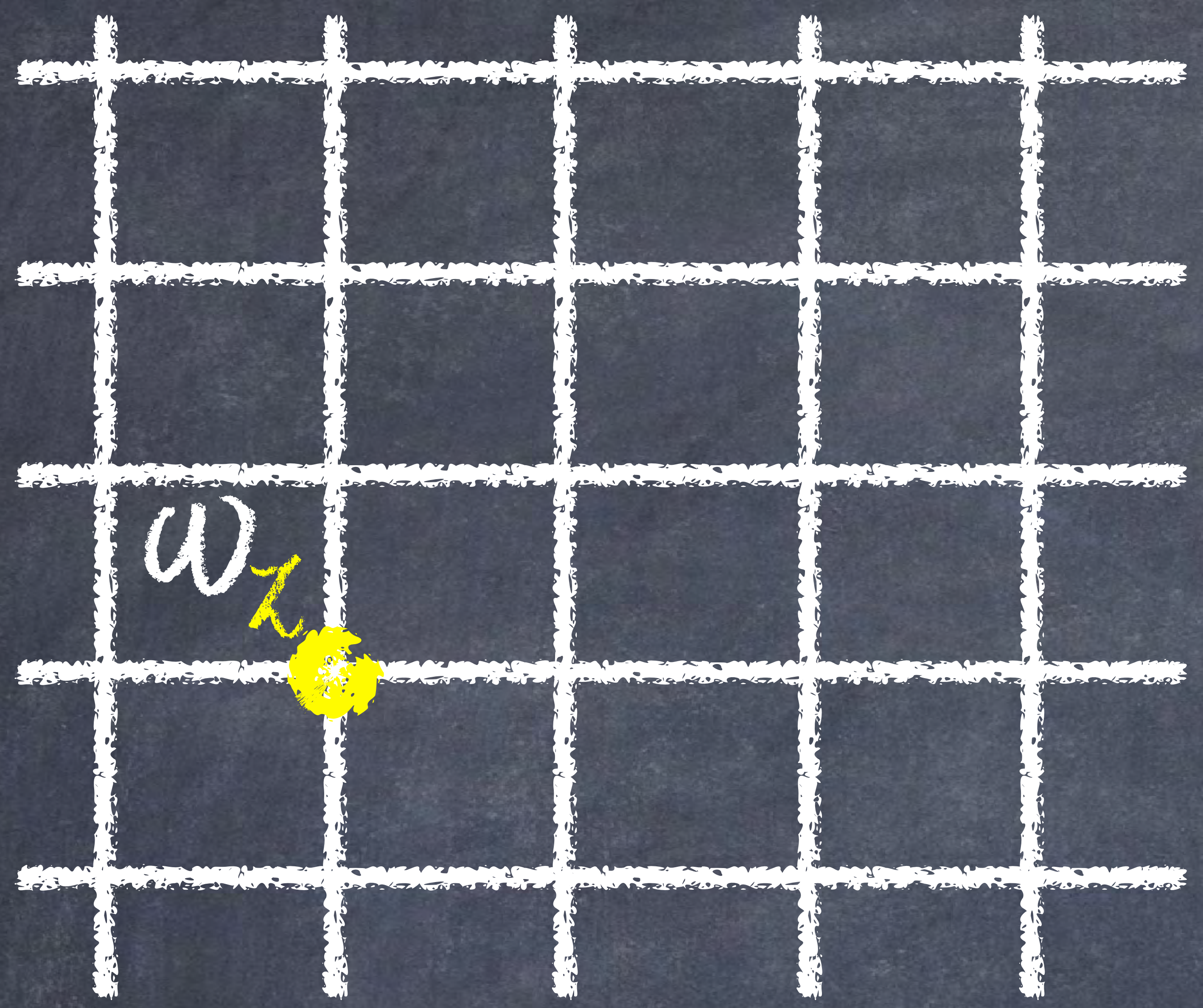




ω_λ



$d=2$, ω_x i.i.d., >2 moments



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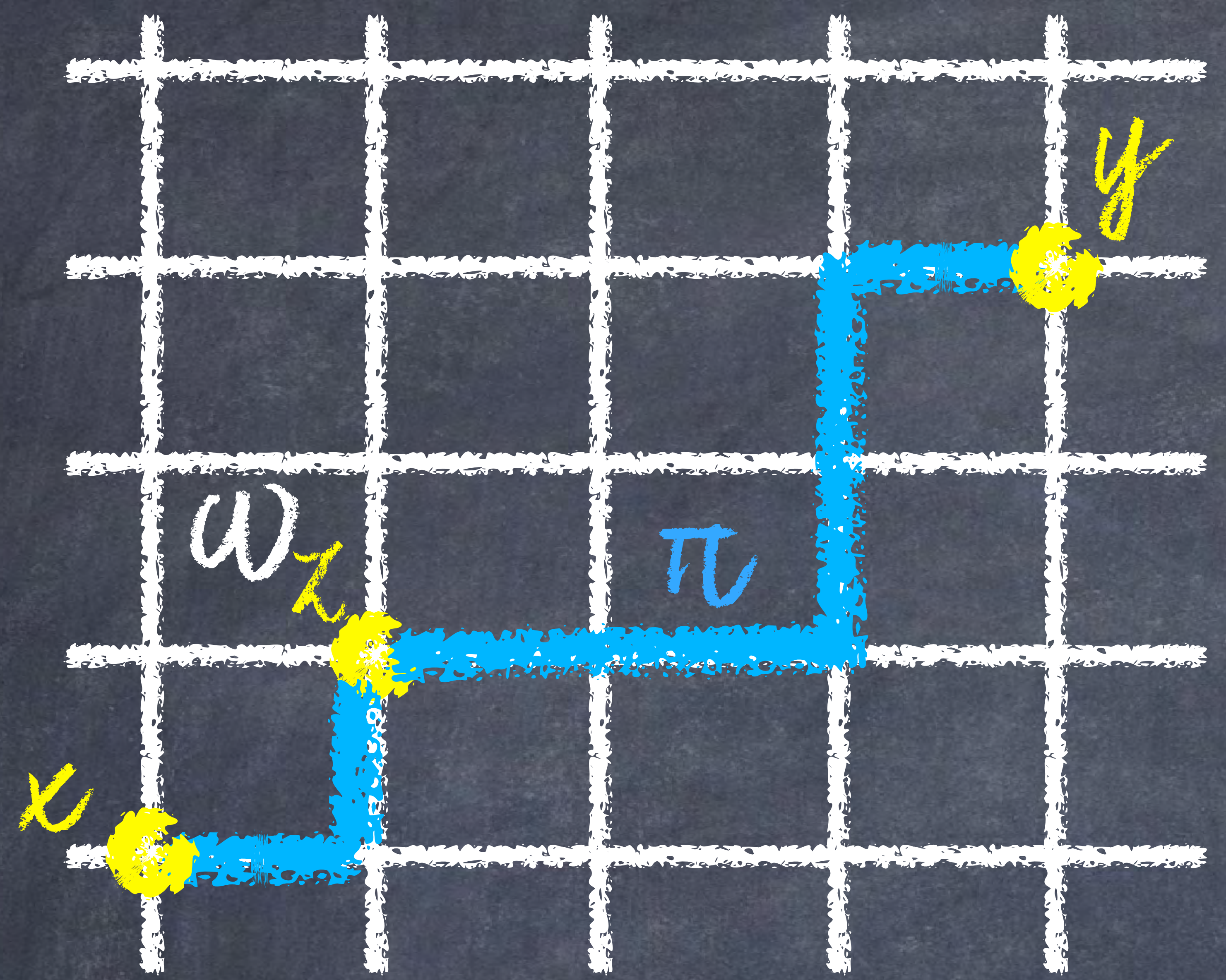
ω_x has a continuous CDF



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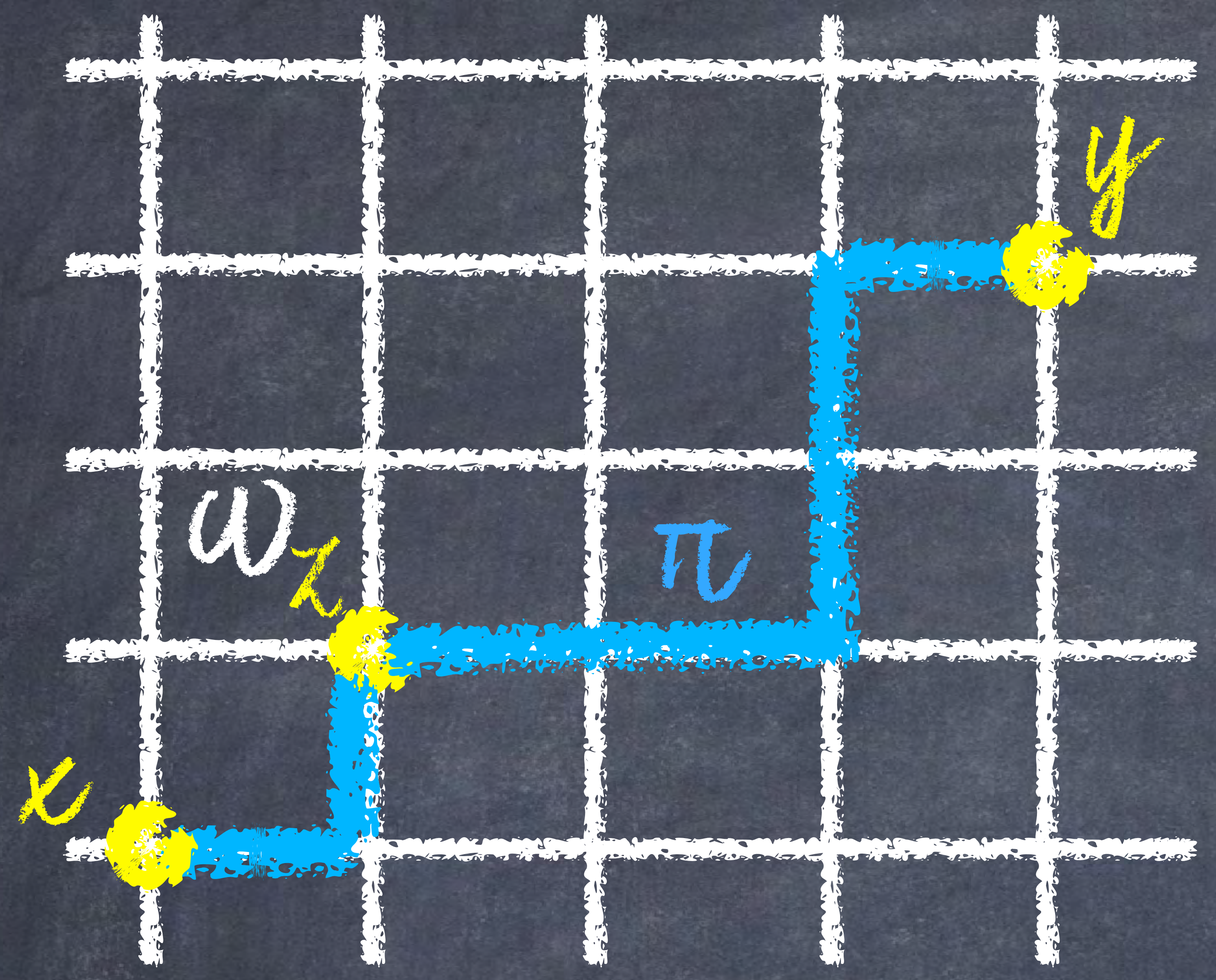
Passage Time:



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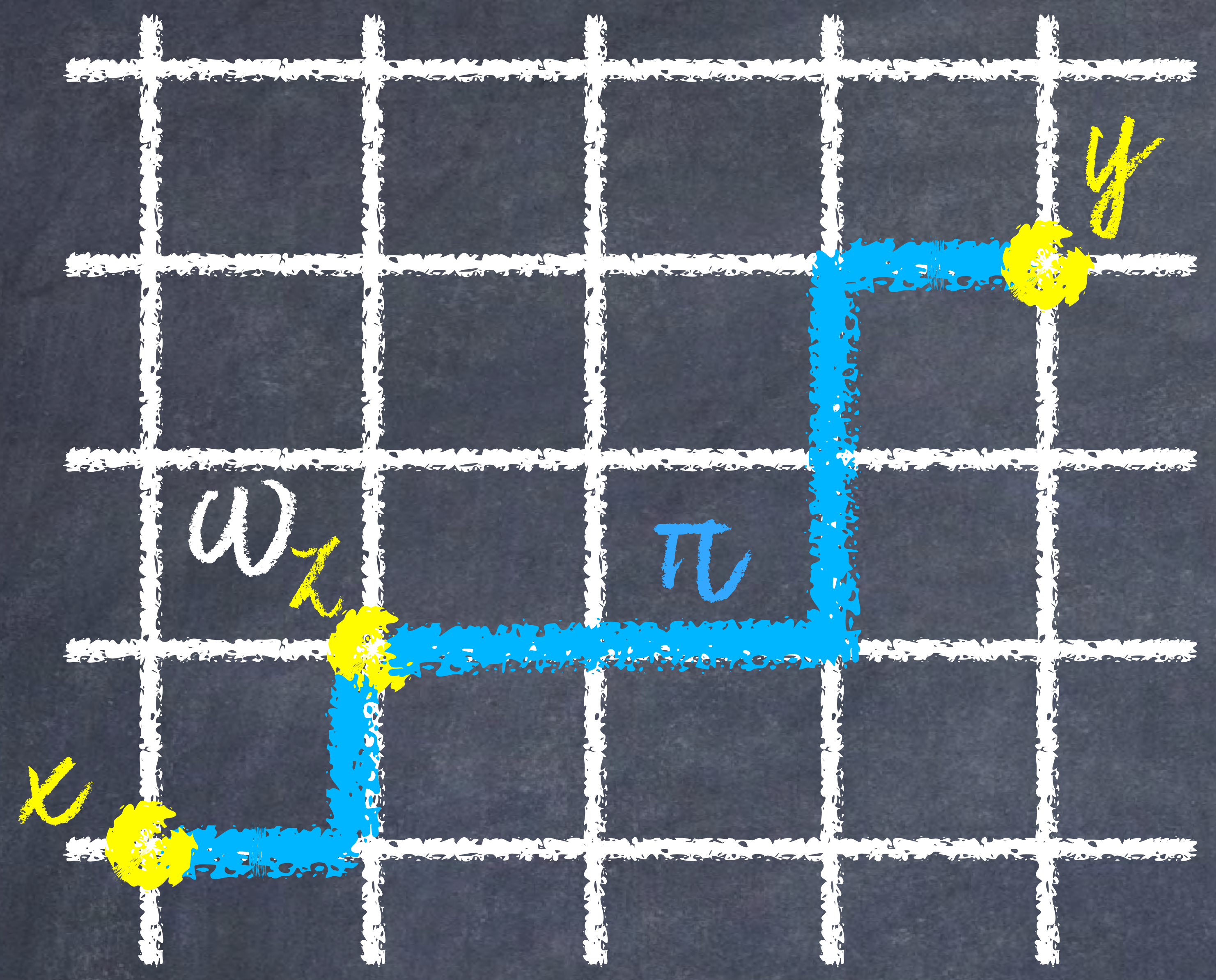
Passage Time:

$$\sum_{z\in\pi} \omega_z$$



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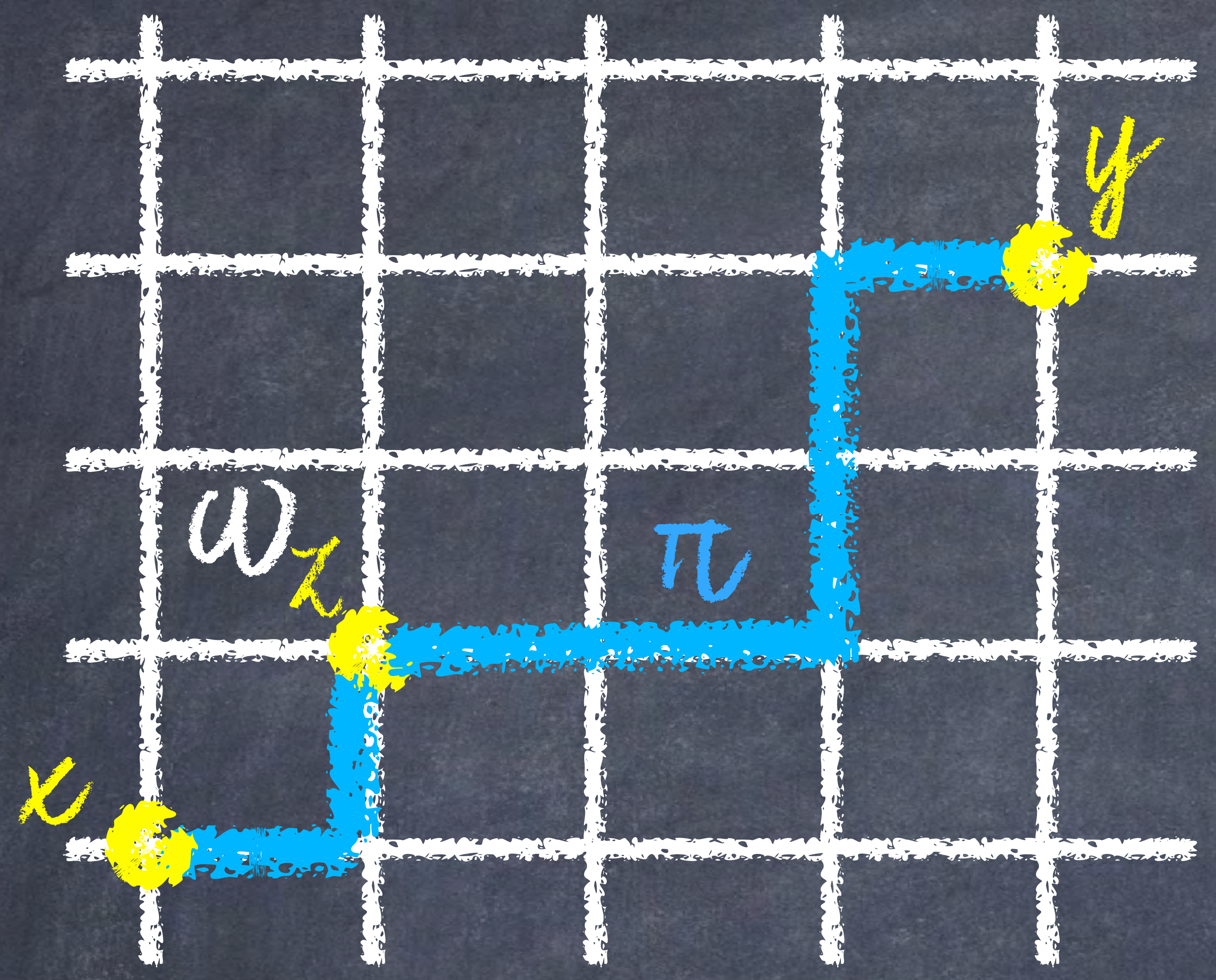
Last Passage Time:

$$G_{xy} = \max_{\pi} \sum_{z \in \pi} \omega_z$$



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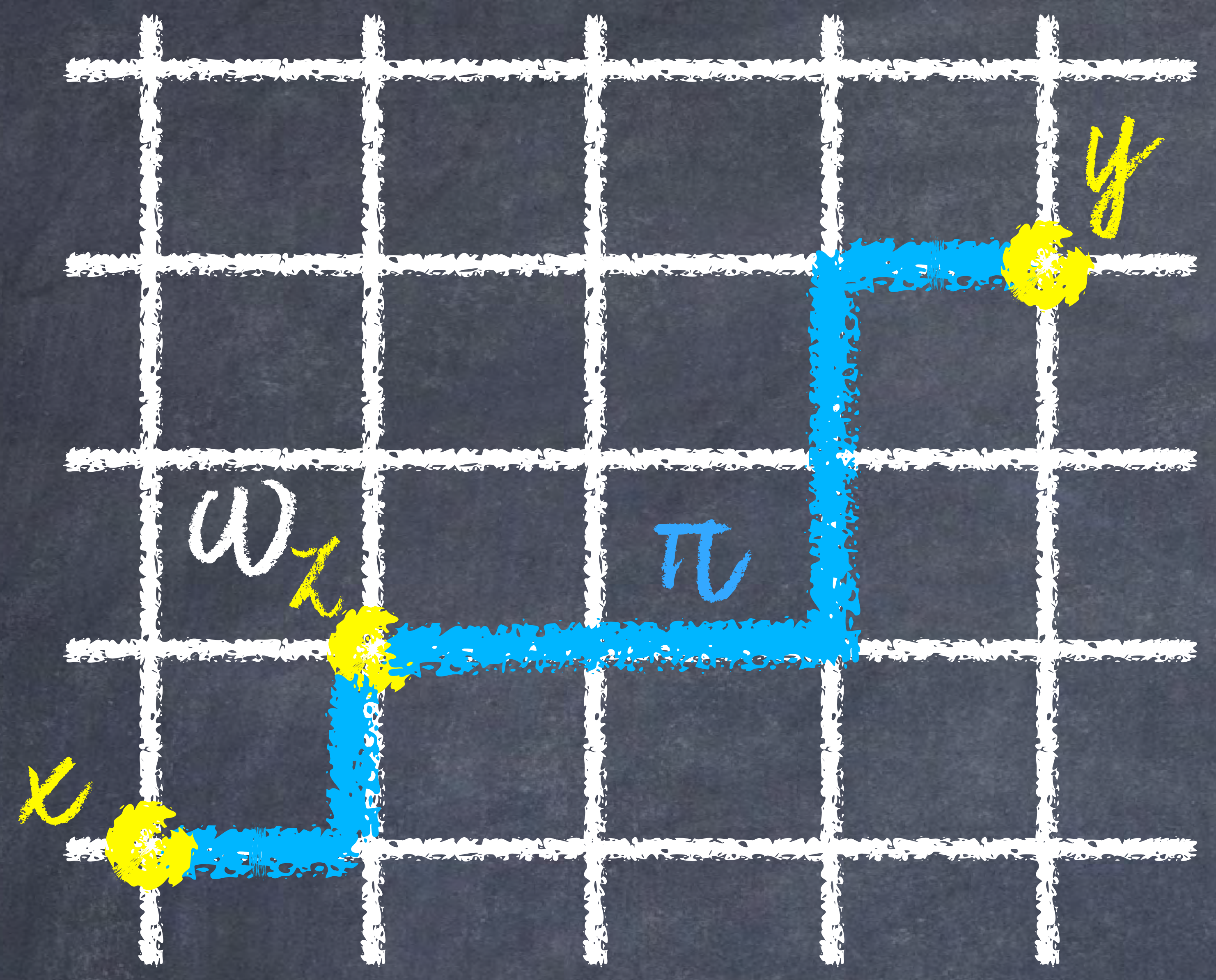
Point-to-point: $G_{xy} = \max_{\pi} \sum_{z \in \pi} \omega_z$



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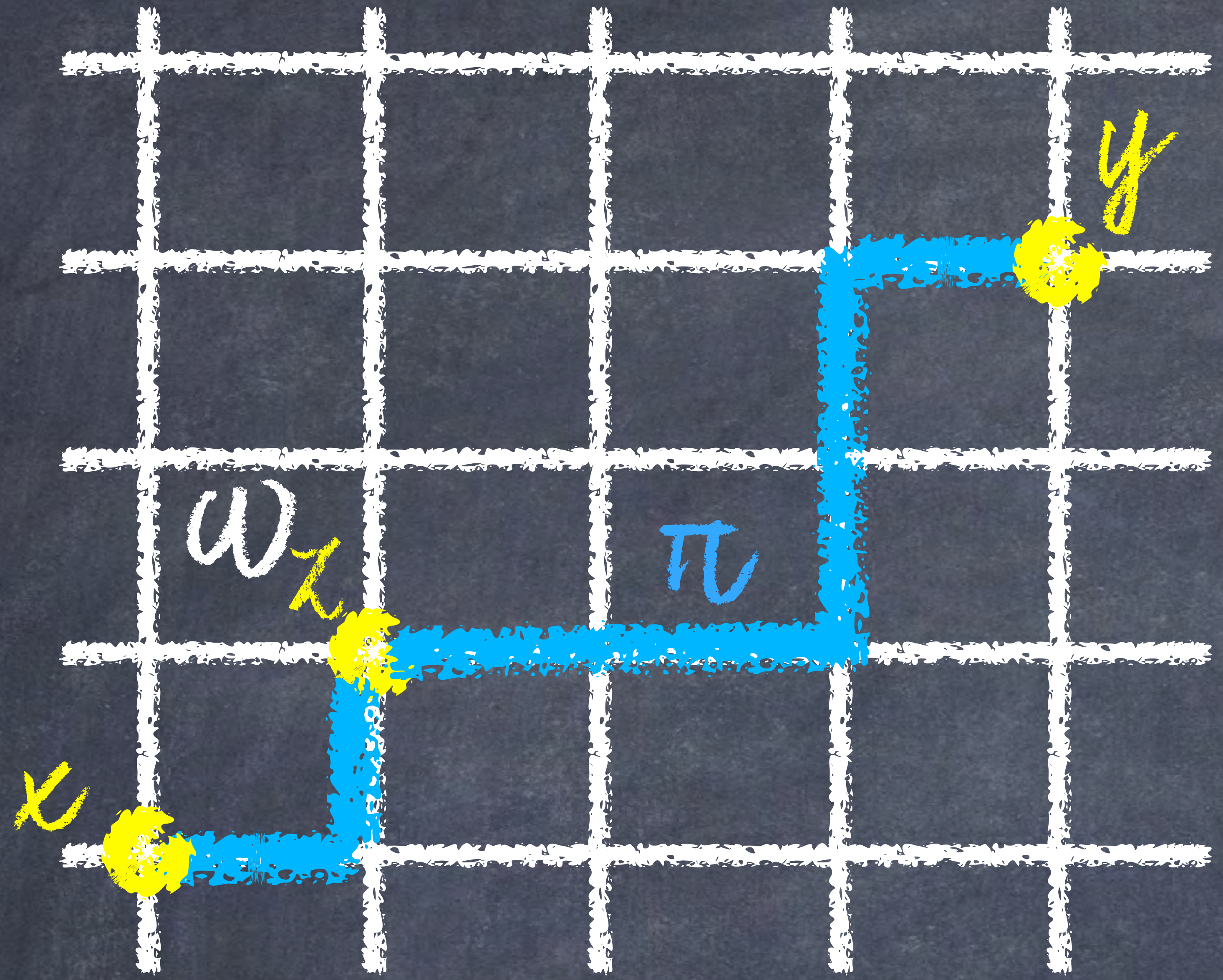
<u>Geodesic:</u> Maximizing path (unique)



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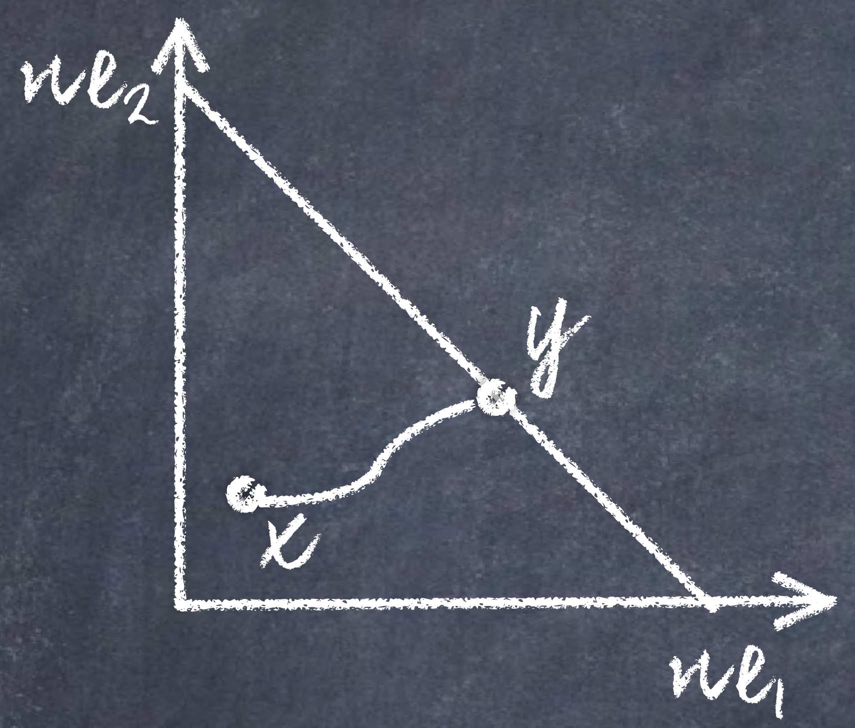
Geodesic: Maximizing path (unique)

Semi-infinite geodesic: each finite piece is a geodesic

Point-to-line: $G_{x,(n)}(h) = \max_{\pi, y} \left\{ h \cdot (y-x) + \sum_{z \in \pi} \omega_z \right\}$



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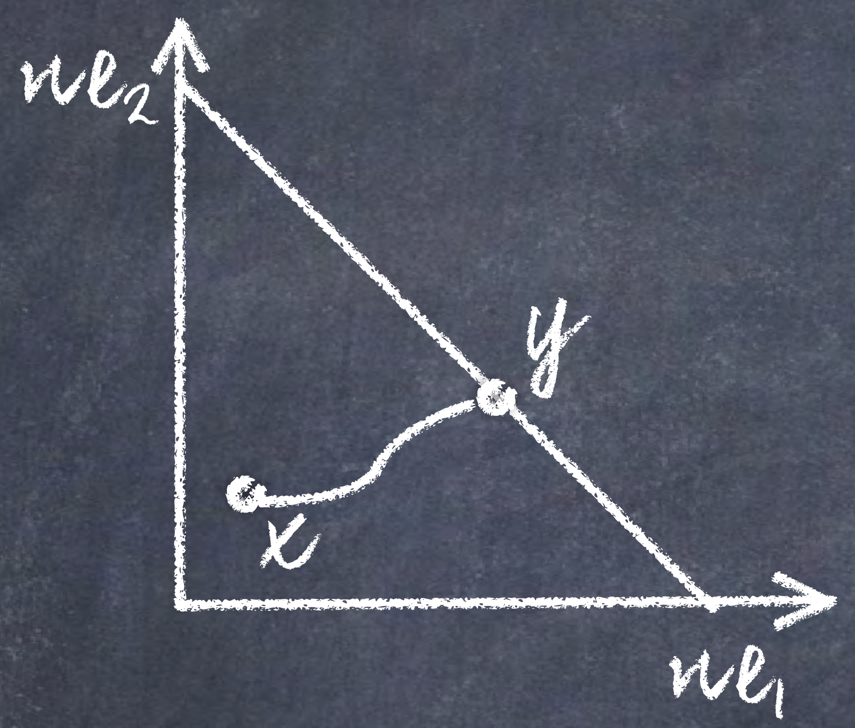
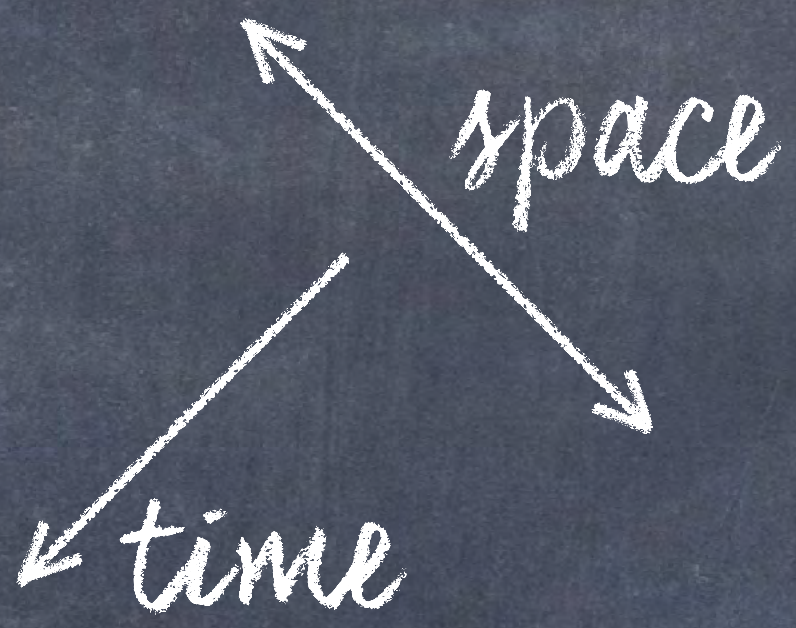
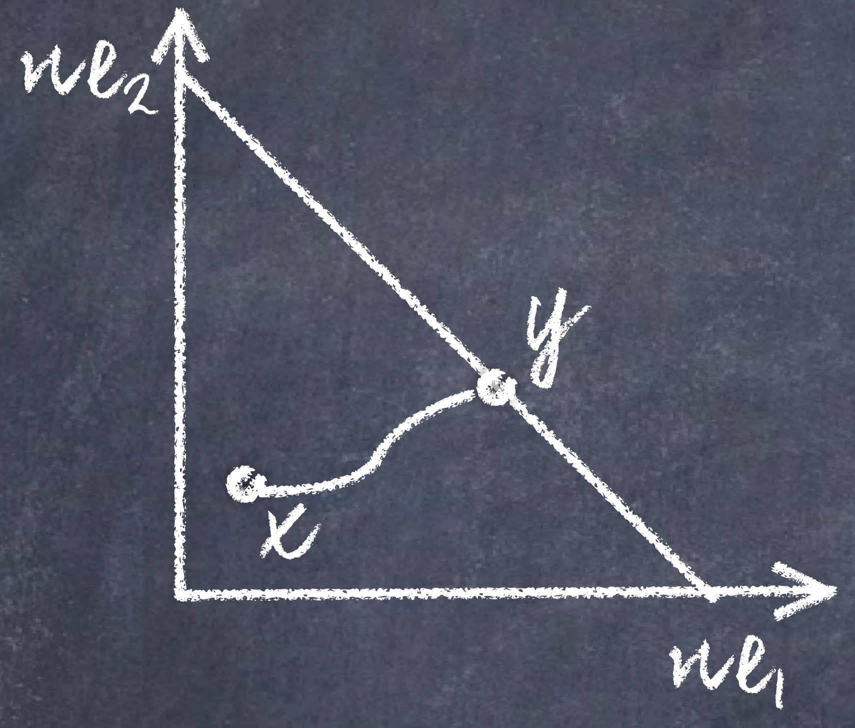


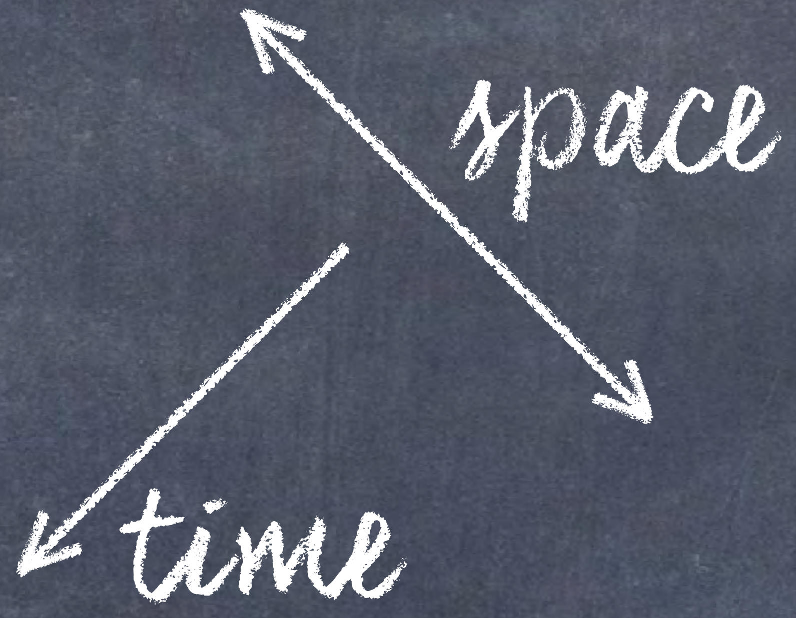


linear initial condition

Point-to-line: $G_{x,(n)}(h) = \max_{\pi,y} \{ \underbrace{h\cdot(y-x)} + \underbrace{\sum_{z\in\pi} \omega_z} \}$

Action





Lax-Oleinik

linear initial condition

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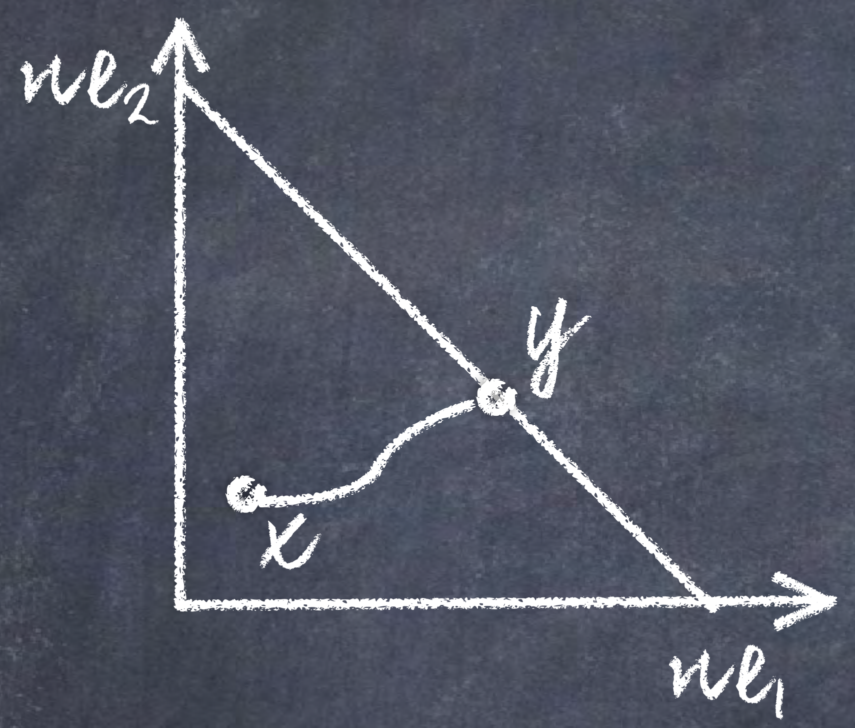

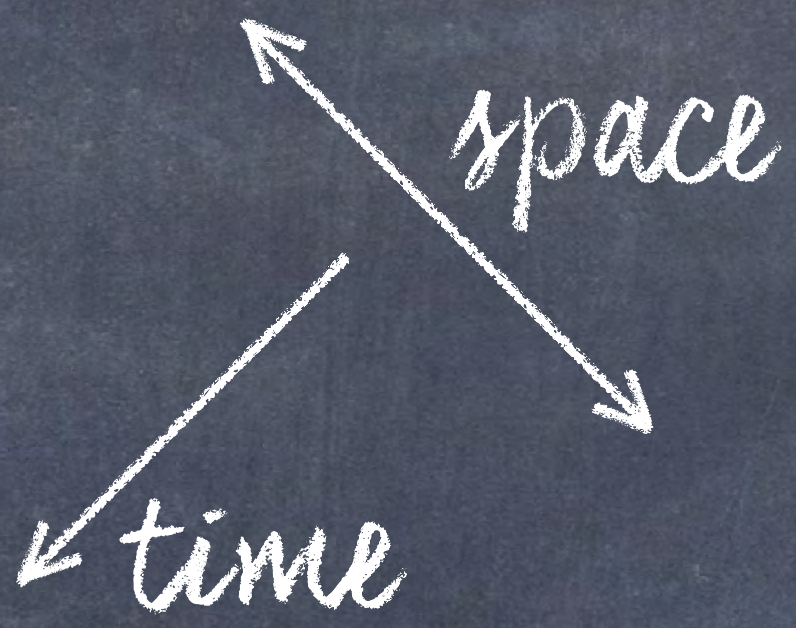

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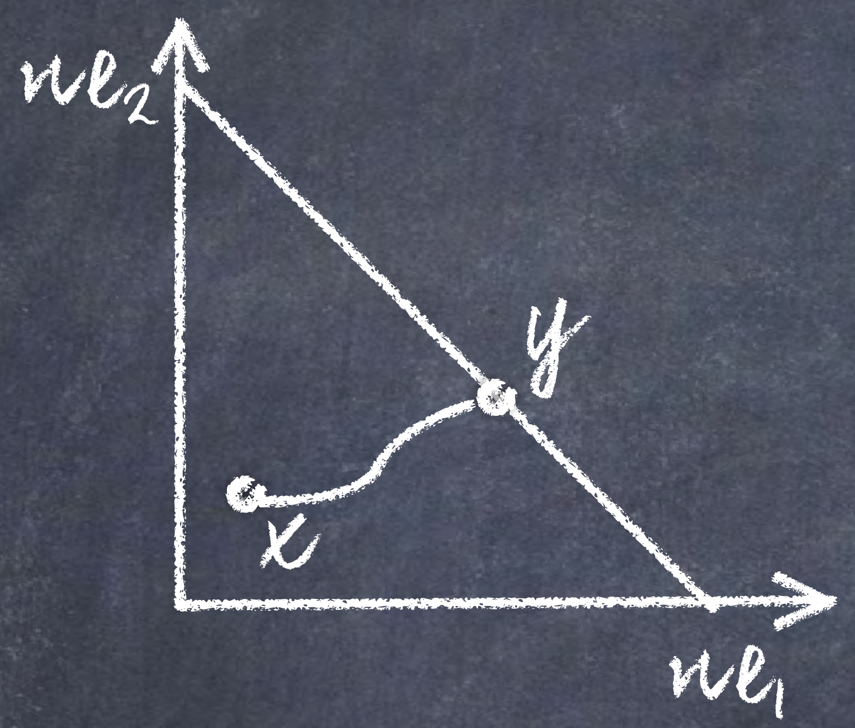
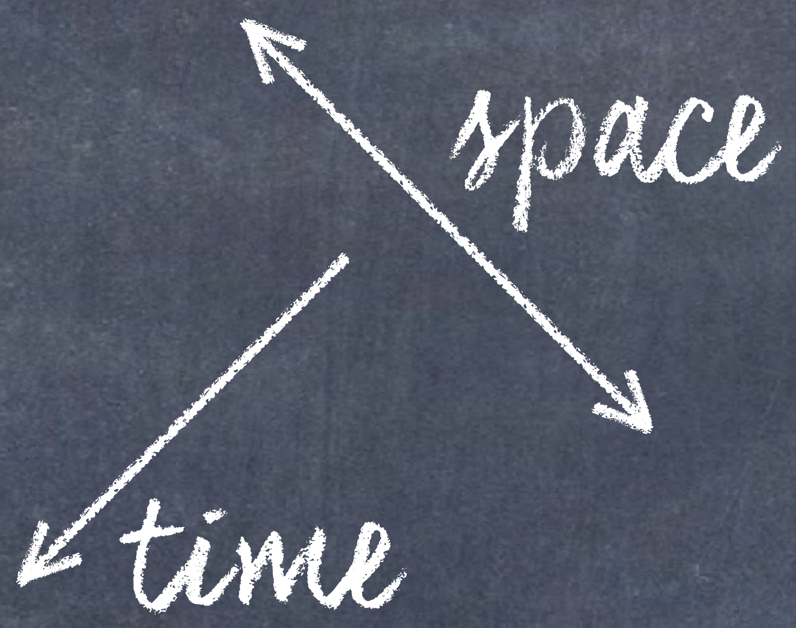
Action

conserved quantity





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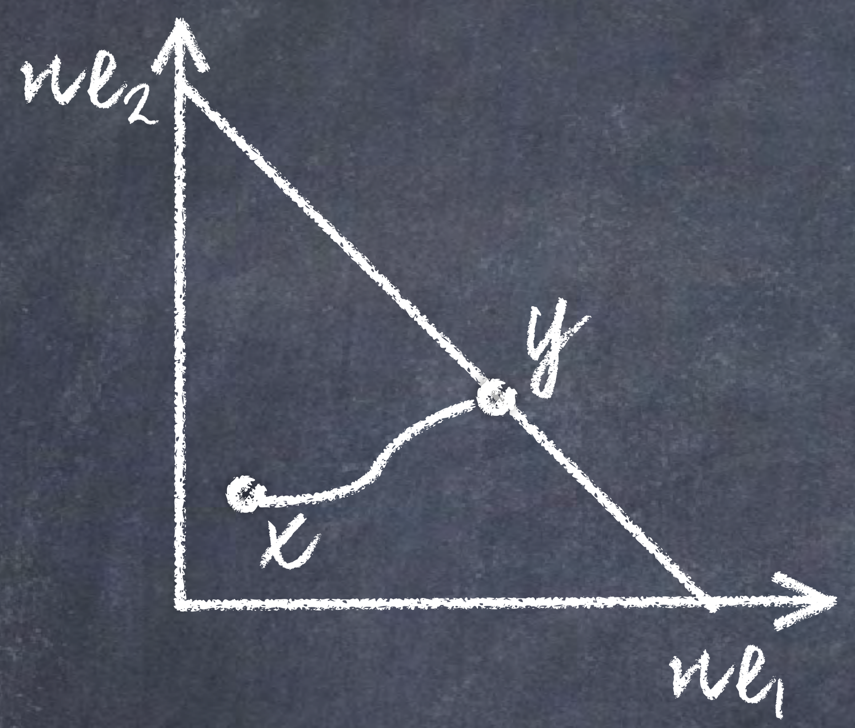

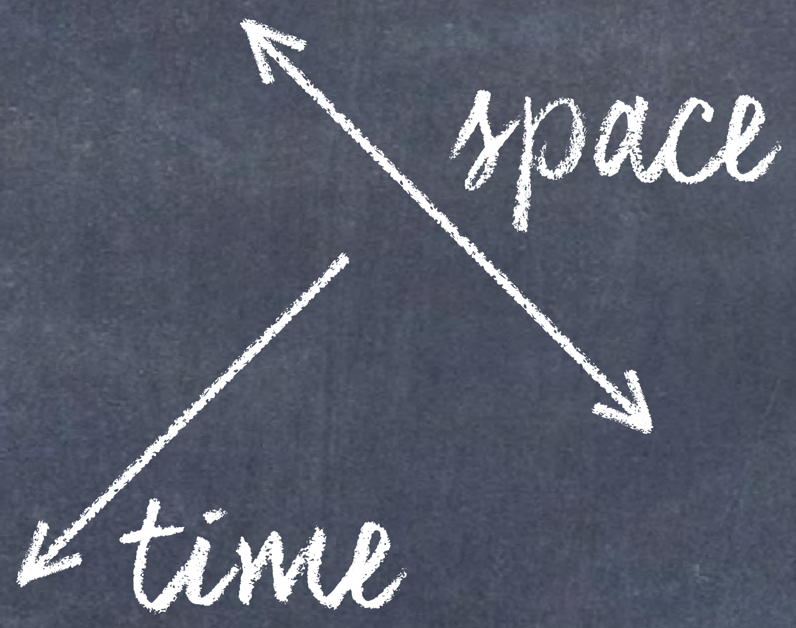

Both $G(x)=G_{xy}$ and $G_{x,(n)}(h)$ satisfy

$$G(x)=\omega_x+G(x+e_1)\vee G(x+e_2)$$

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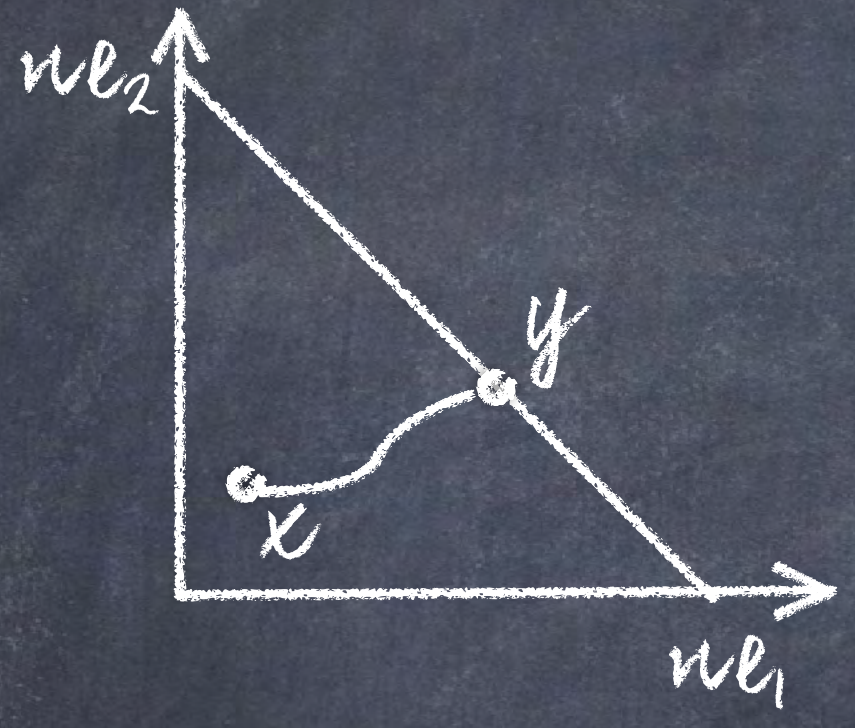

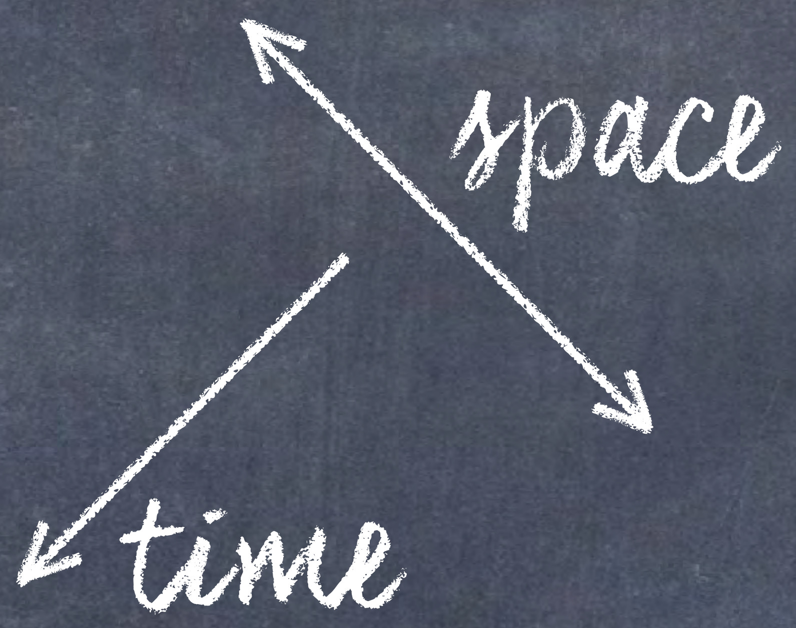

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Probabilistic discretization of HJB

Shape Theorem:

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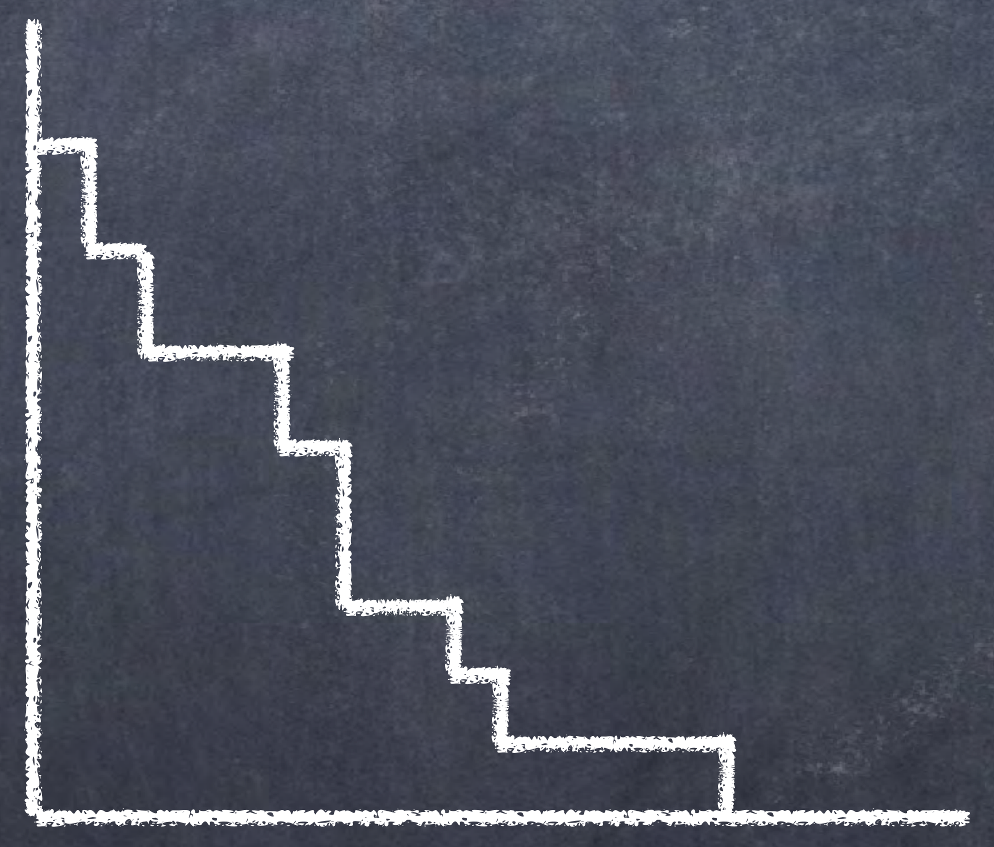

$\{x : G_{0x} \leq t\}$

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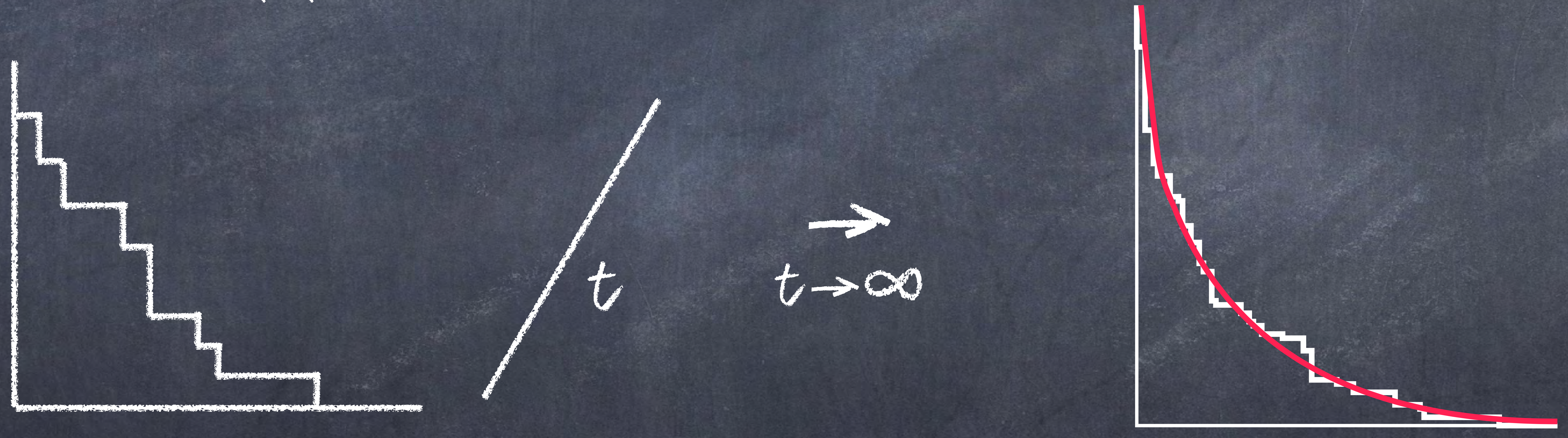

$\{x : G_{0x} \le t\}$

$\{\xi : g(\xi) \le 1\}$

Busemann functions:

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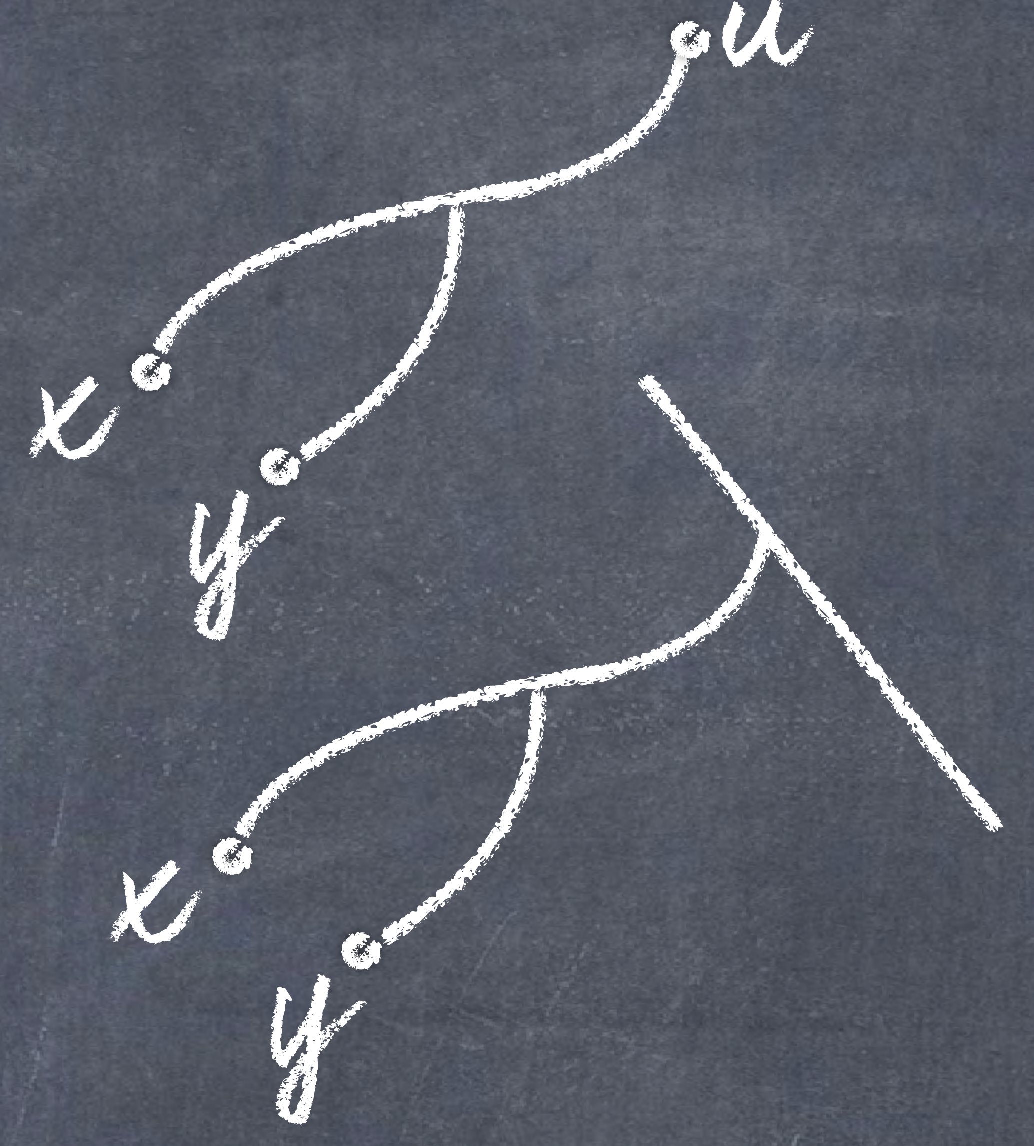
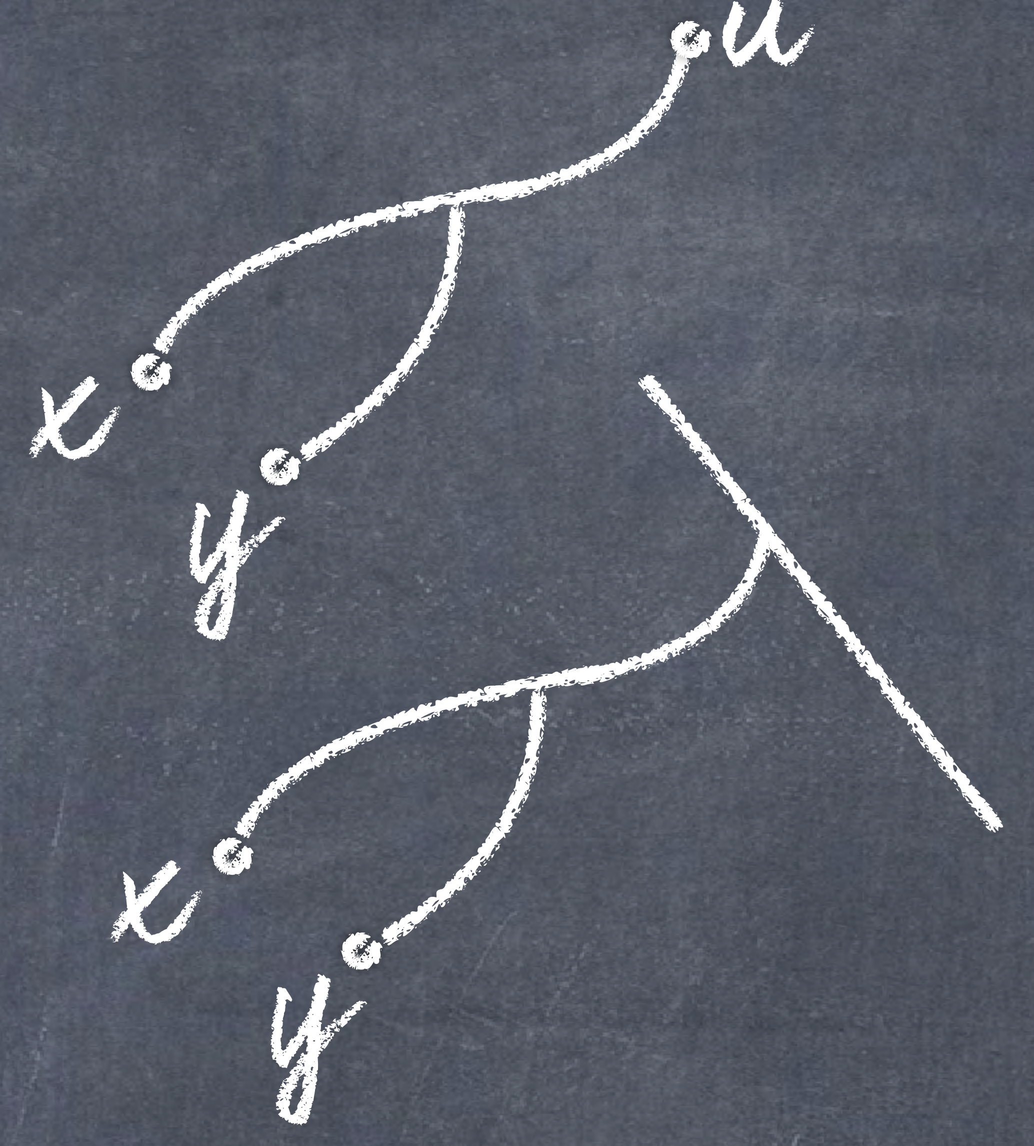
or $$B^h_n(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$$



Busemann functions:

$B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B_n^h(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



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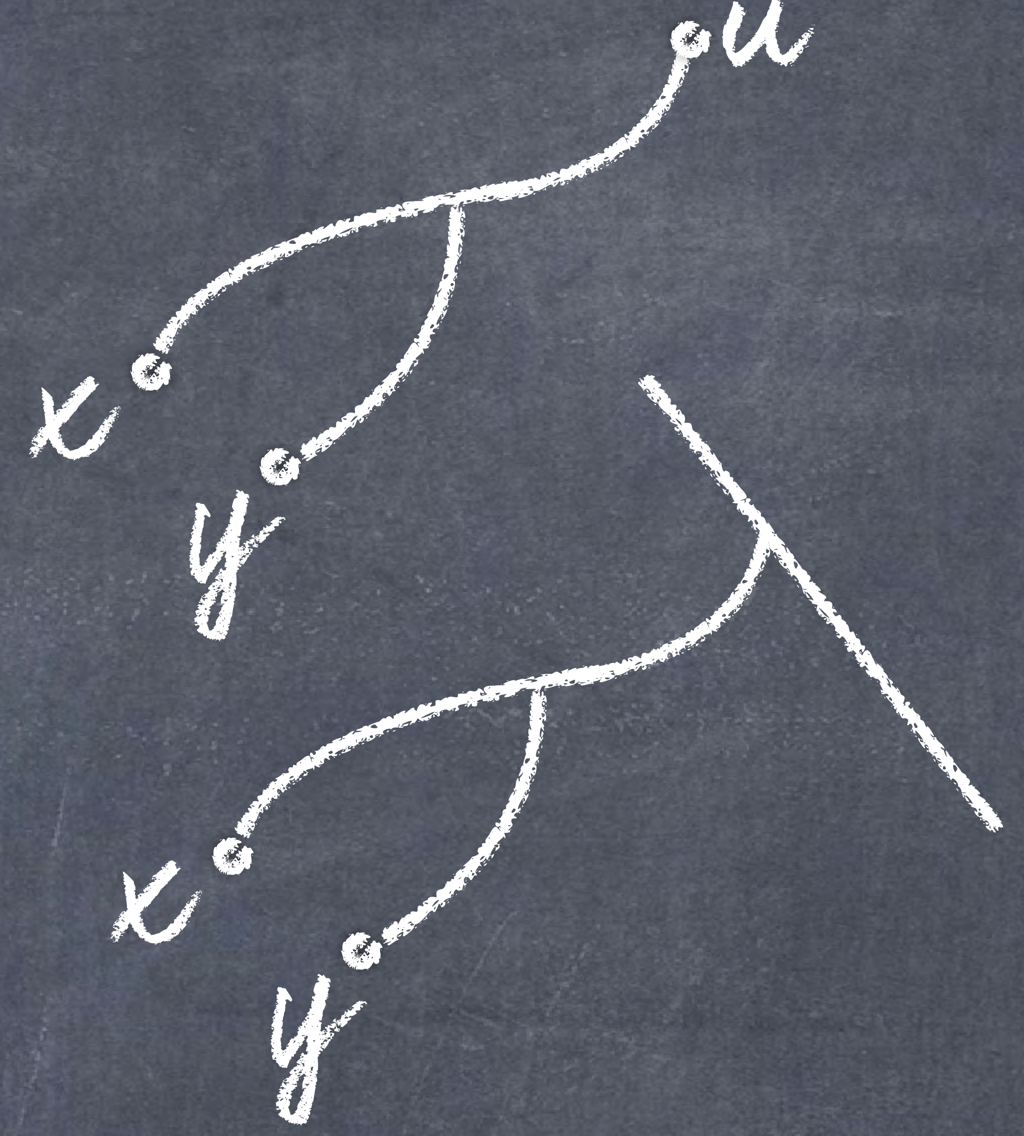

Cocycle: $B(x,y) + B(y,z) = B(x,z)$

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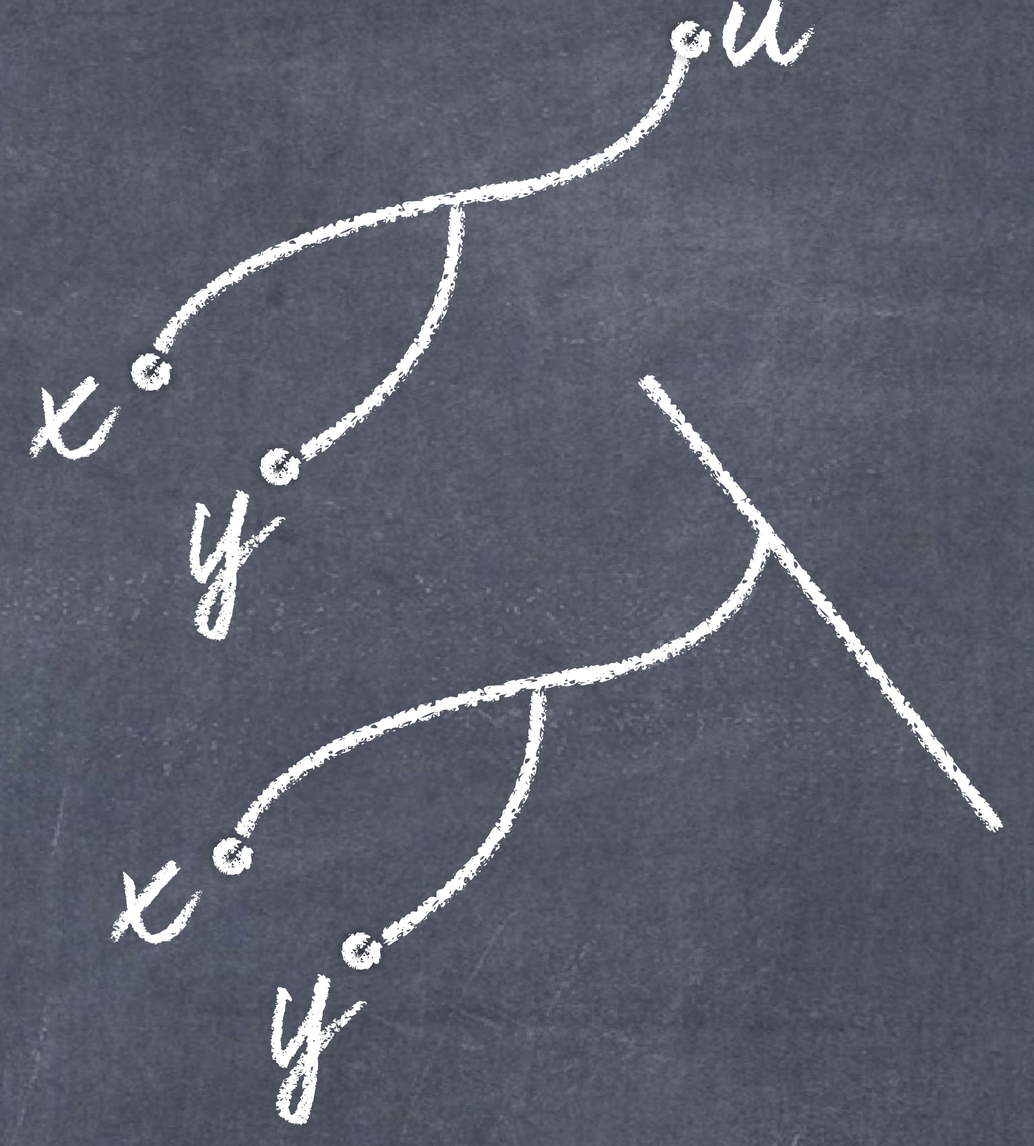

Cocycle: $B(x,y) + B(y,z) = B(x,z)$

Recovery: $B(x,x+e_1) \wedge B(x,x+e_2) = \omega_x$

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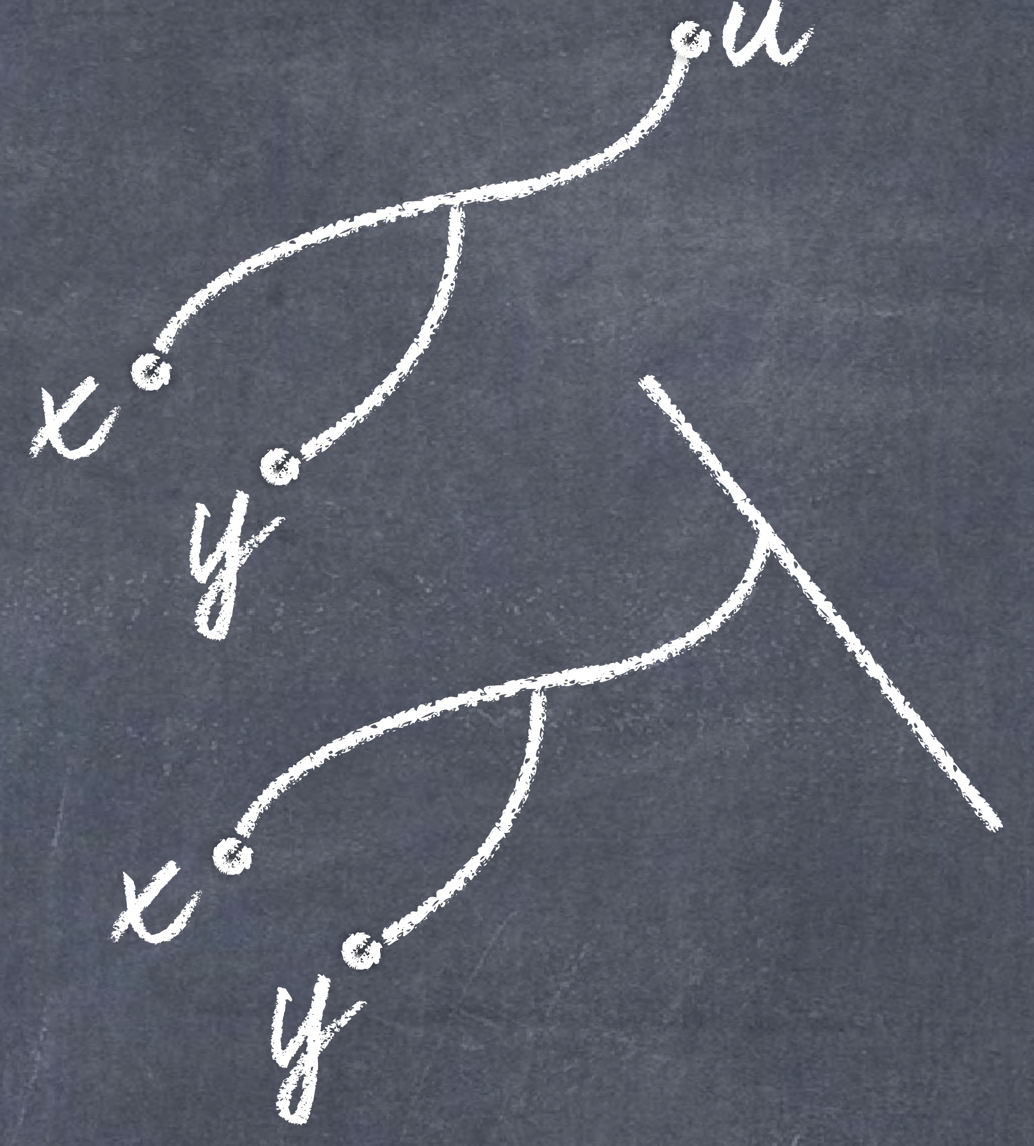

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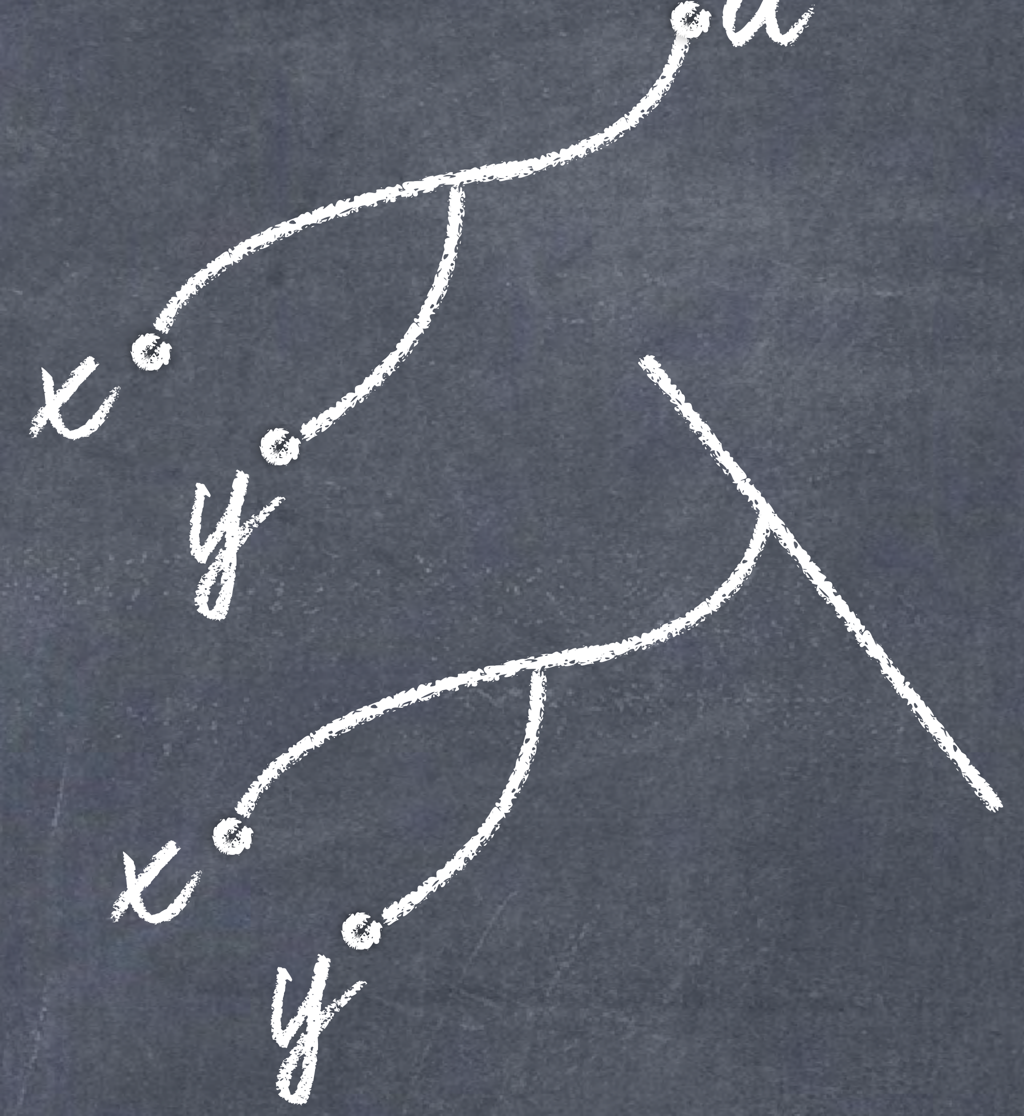


Cocycle: $B(x,y)+B(y,z)=B(x,z)$

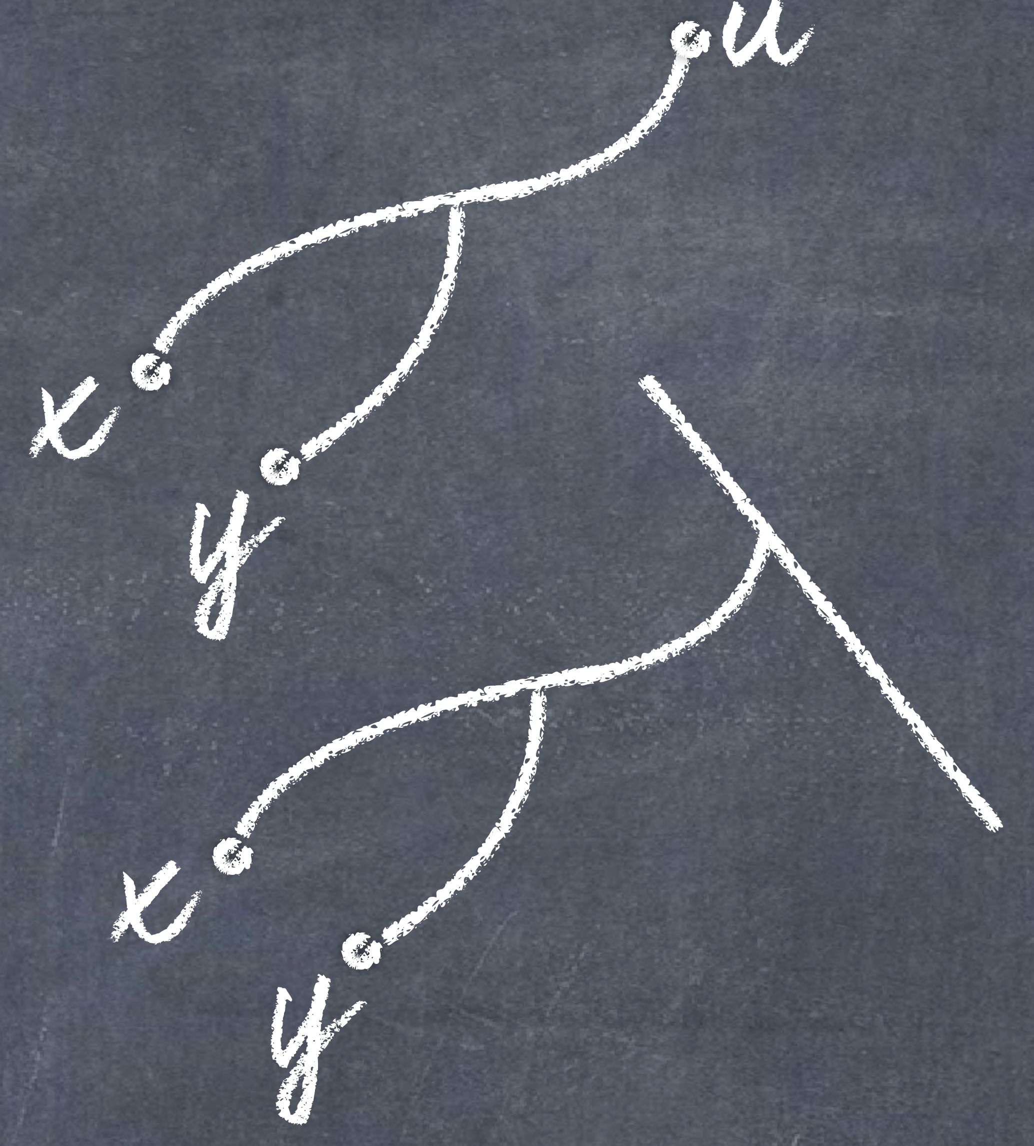
Recovery: $B(x,x+e_1)\wedge B(x,x+e_2)=\omega_x$

geodesic follows smallest increment

$$G(x)=\omega_x+G(x+e_1)\vee G(x+e_2)$$

Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B^h_n(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



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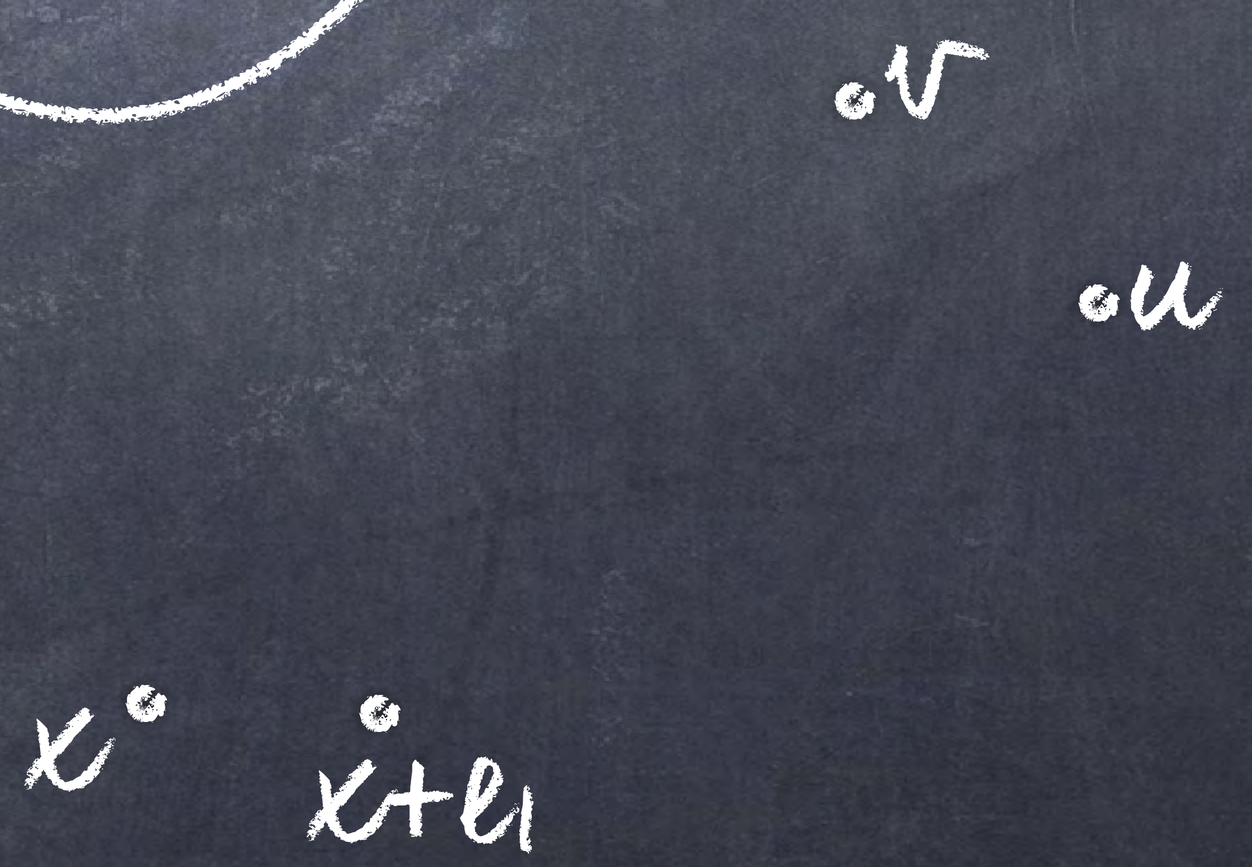

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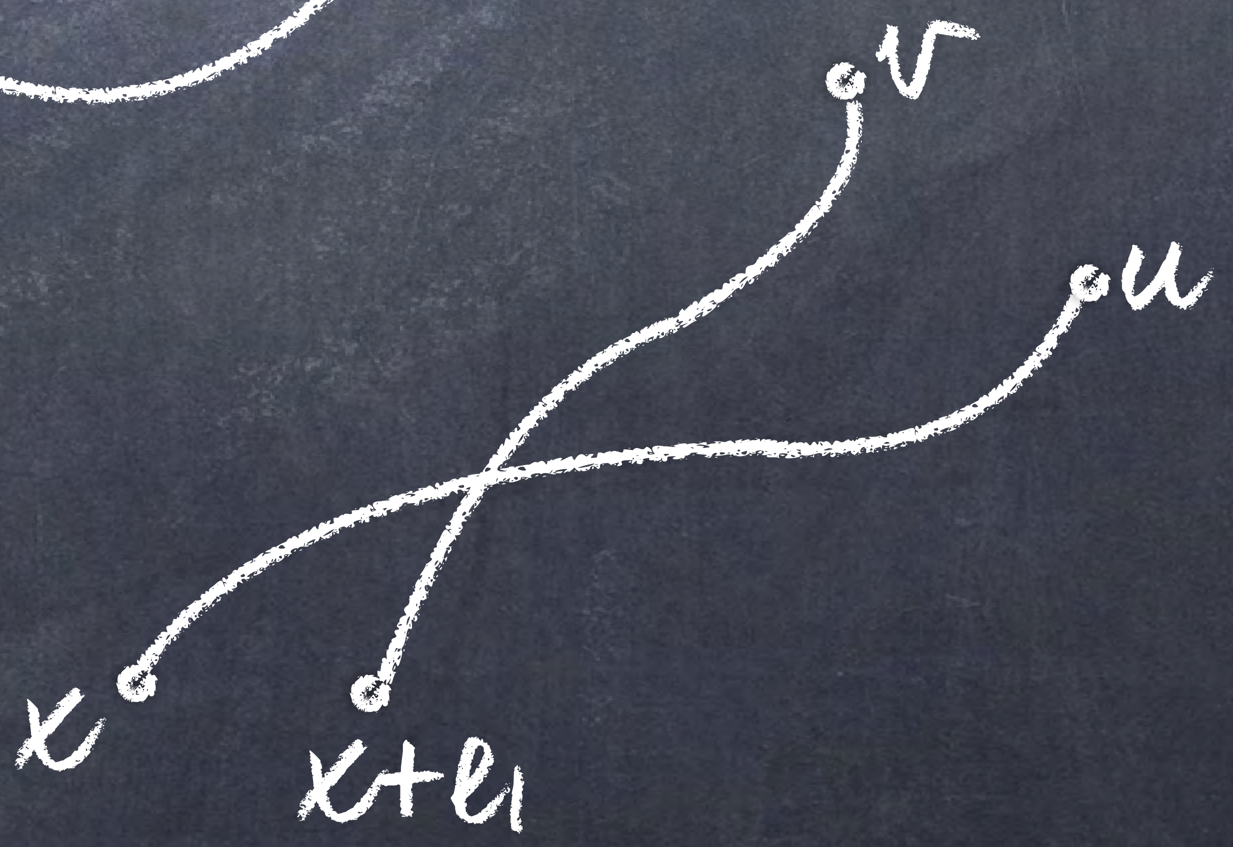

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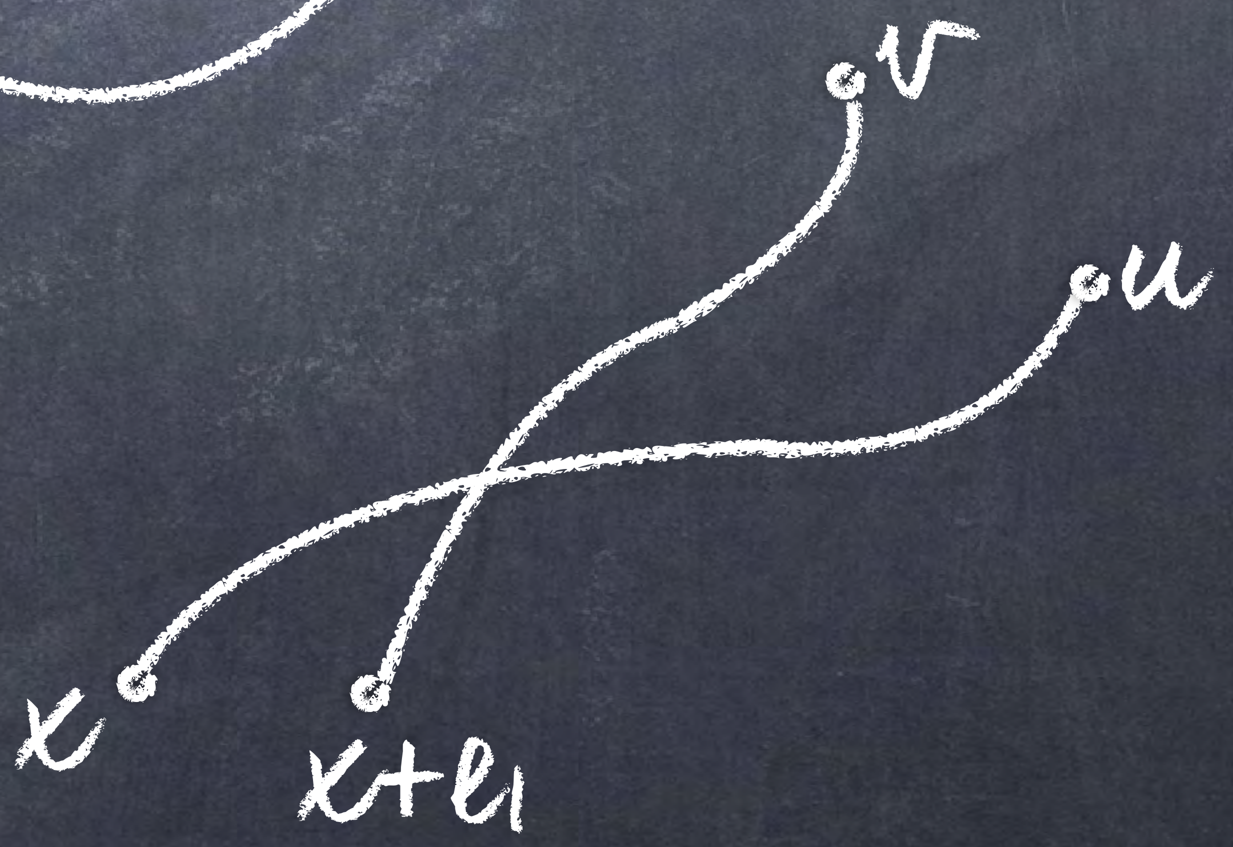

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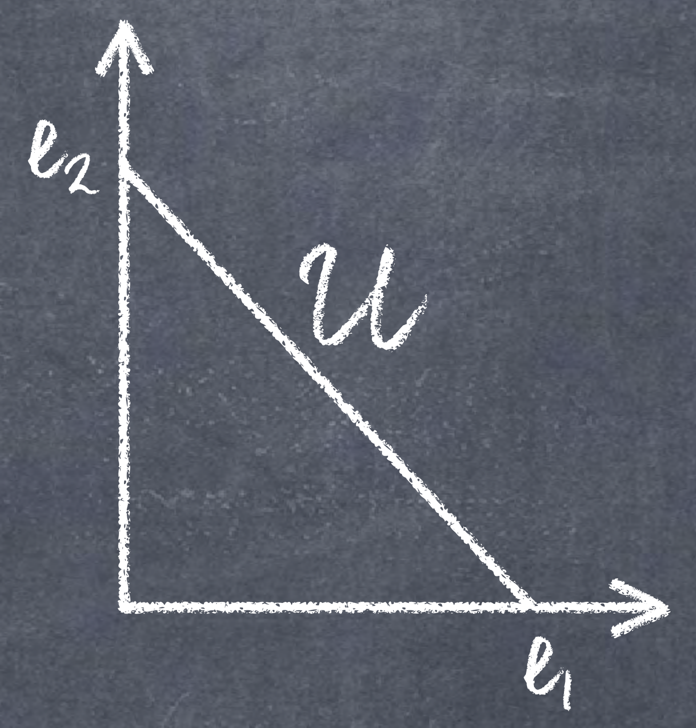

e2
U
e1

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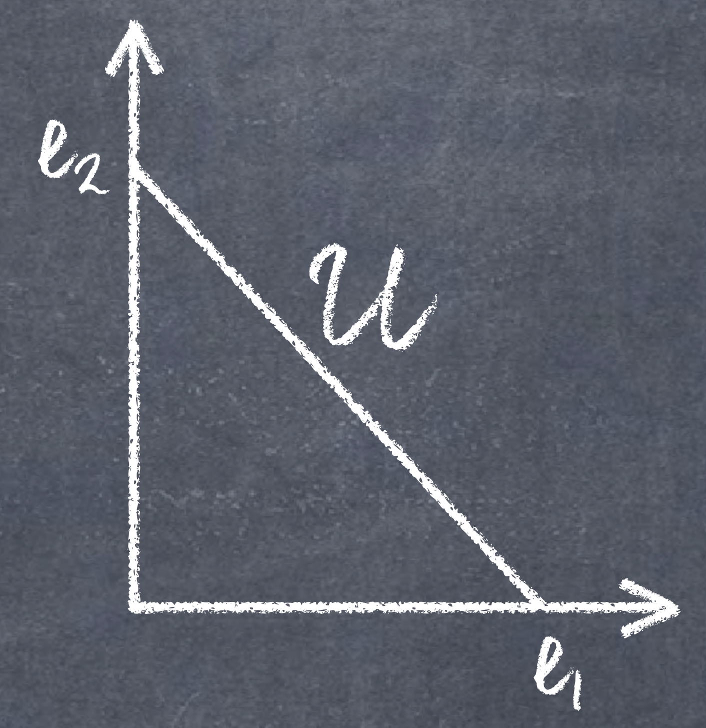

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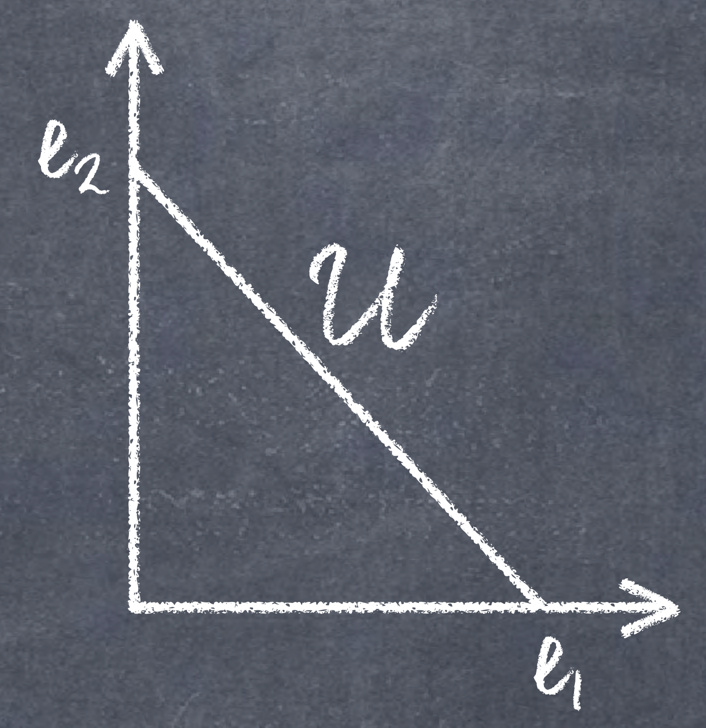

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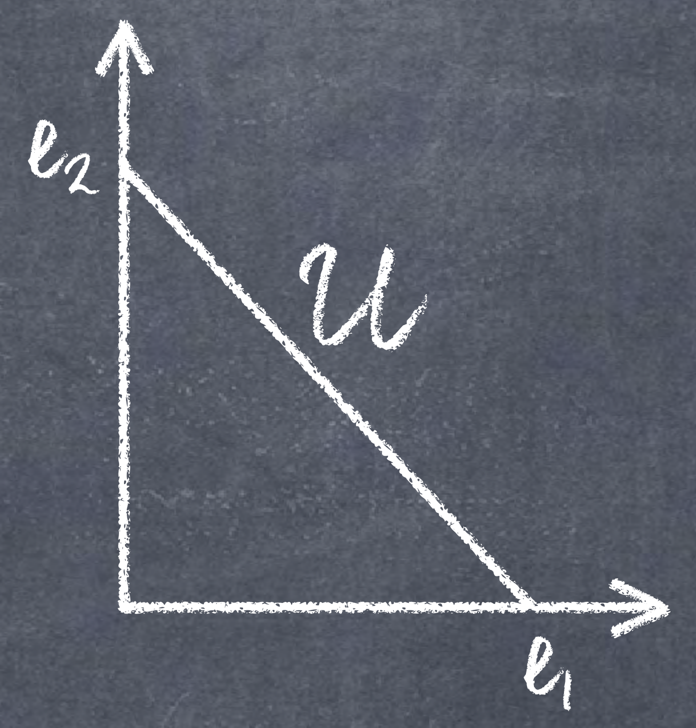

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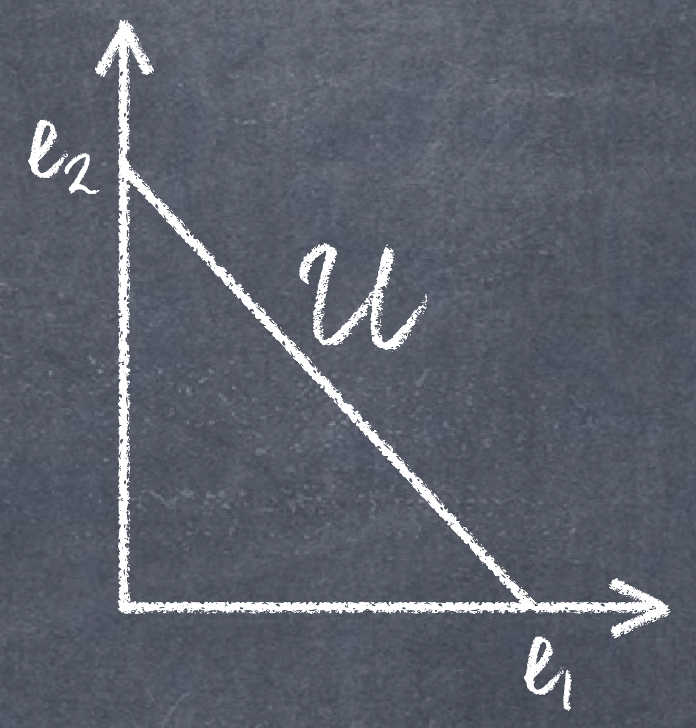

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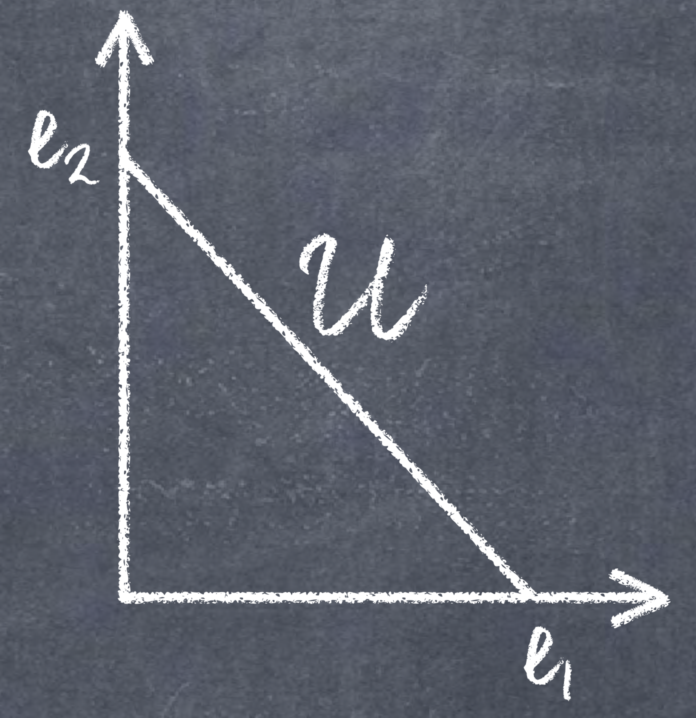

Licea-Newman '96: for standard FPP

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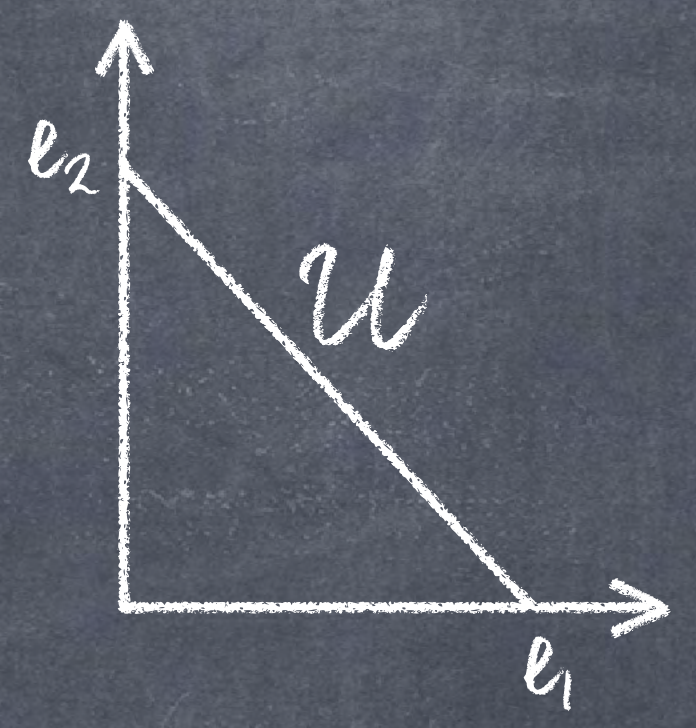

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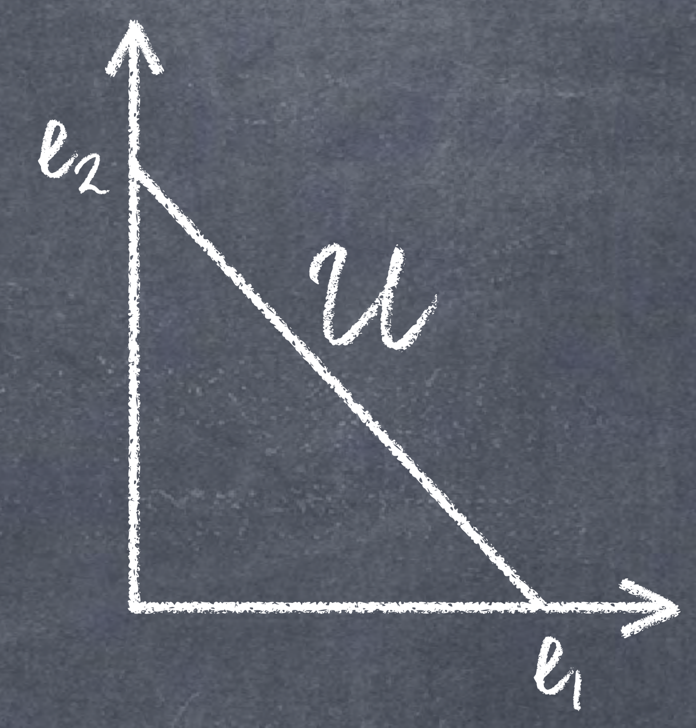

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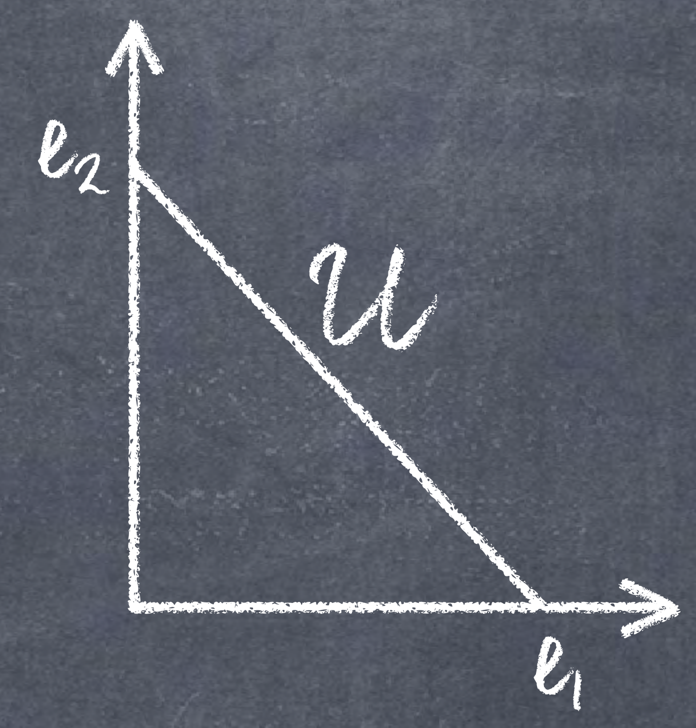

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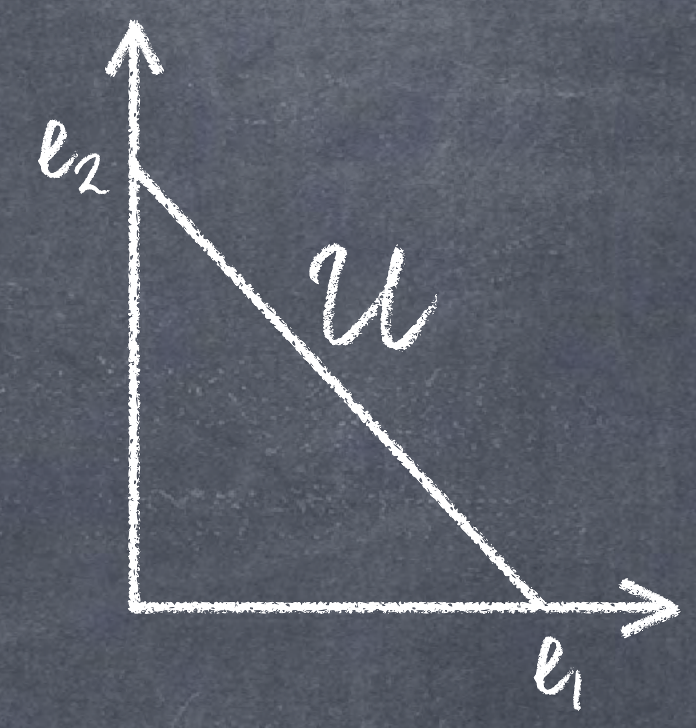

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Coalescence implies $B_{x_n}(x,y) \to B^{\xi}(x,y)$

Conjecture: g differentiable, strictly concave, and satisfies the curvature assumption.

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Burgers equation with kick forcing: Bakhtin-Cator-Khanin '14
Bakhtin '16

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Cocycle, recovery, monotonicity hold

Monotonicity implies $B^{\zeta\pm}(x,x+e_i)=\lim\limits_{\mathcal{D}_0\ni\eta\to\zeta} B^{\eta}(x,x+e_i)$

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Cocycle property extends this to $B^{\xi\pm}(x,y)$

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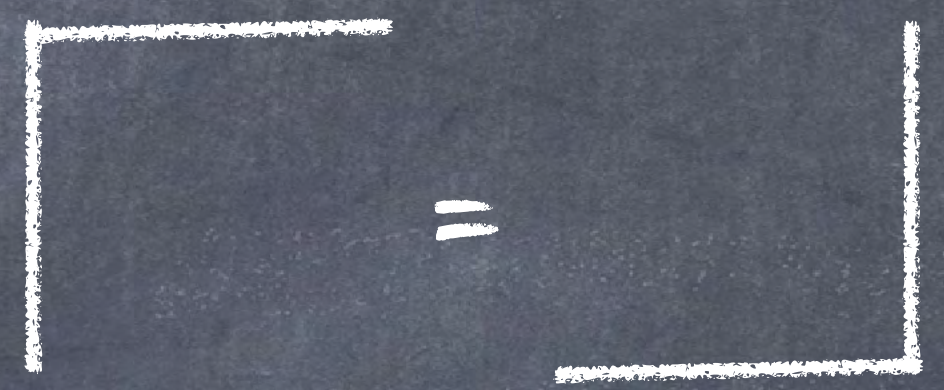

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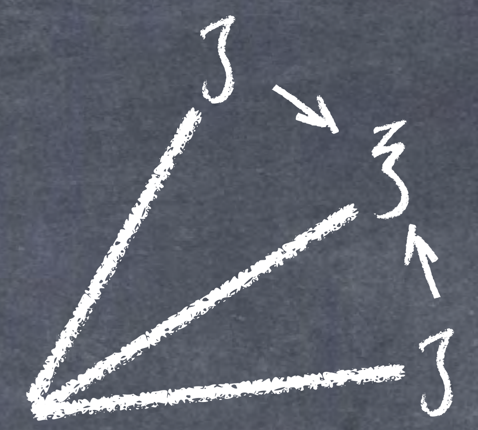

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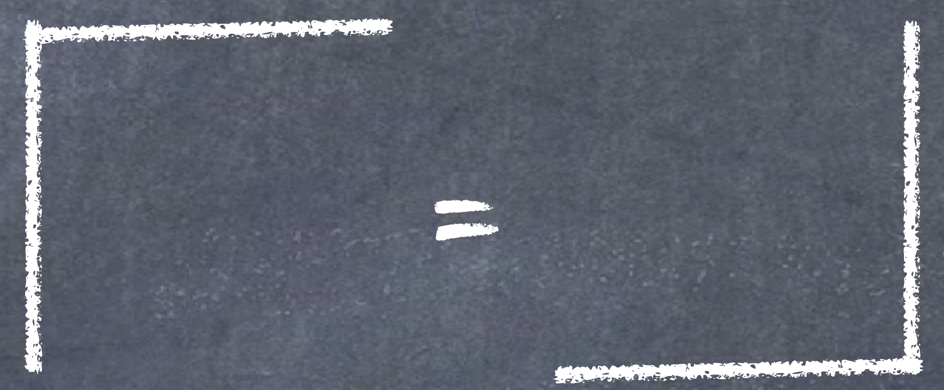

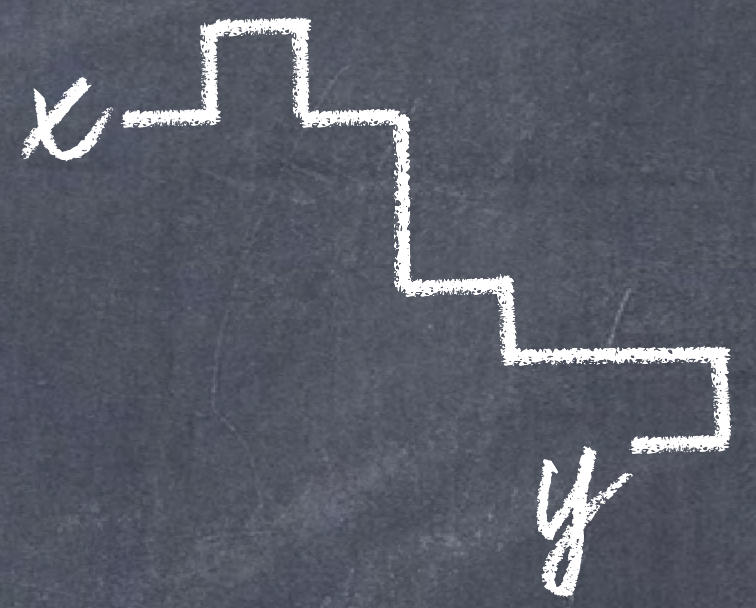

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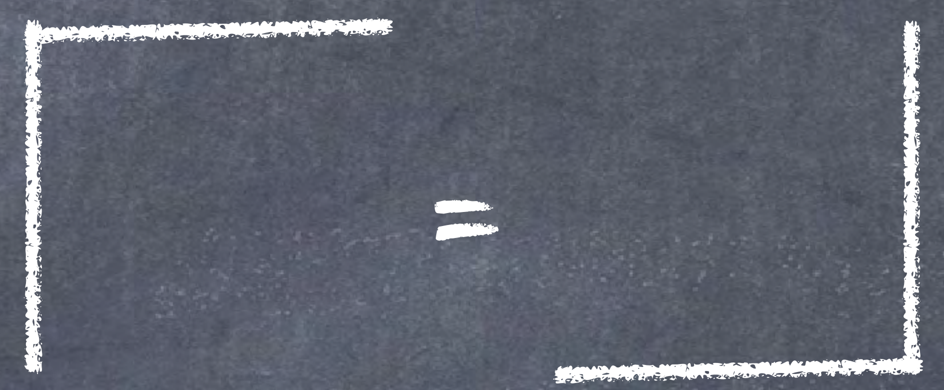

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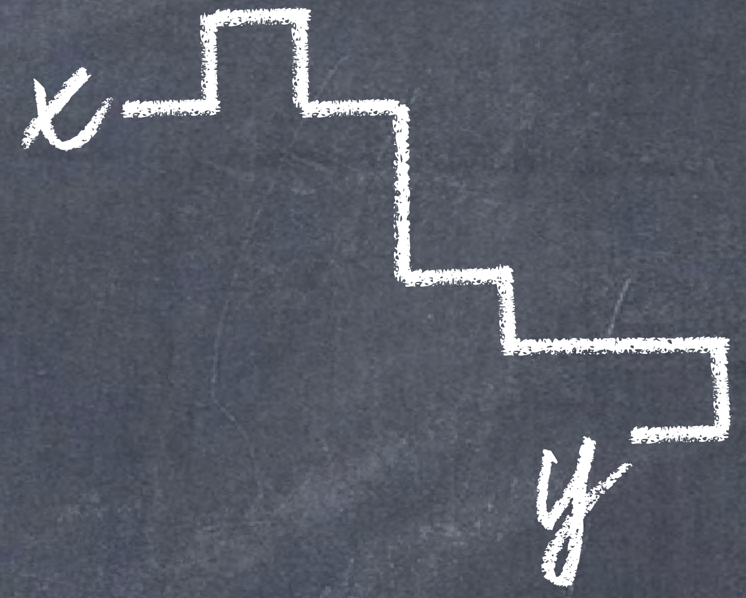

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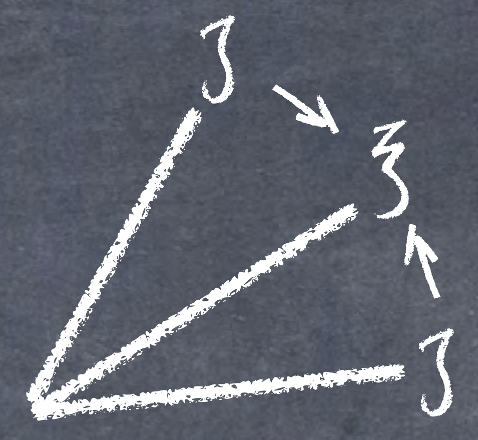

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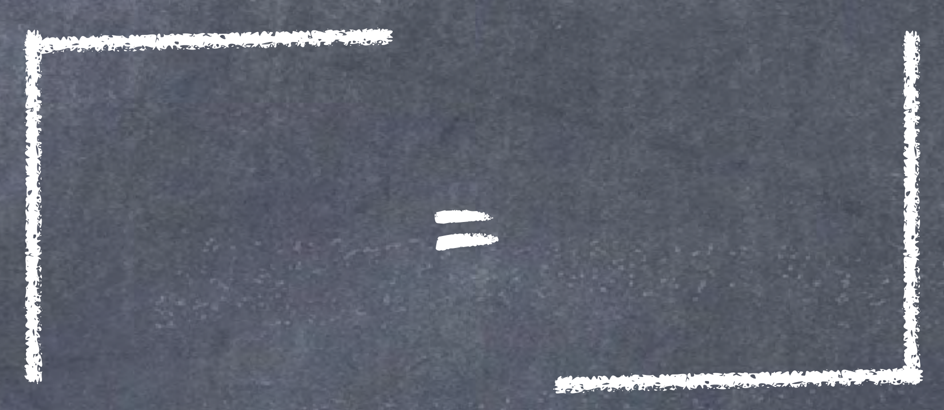

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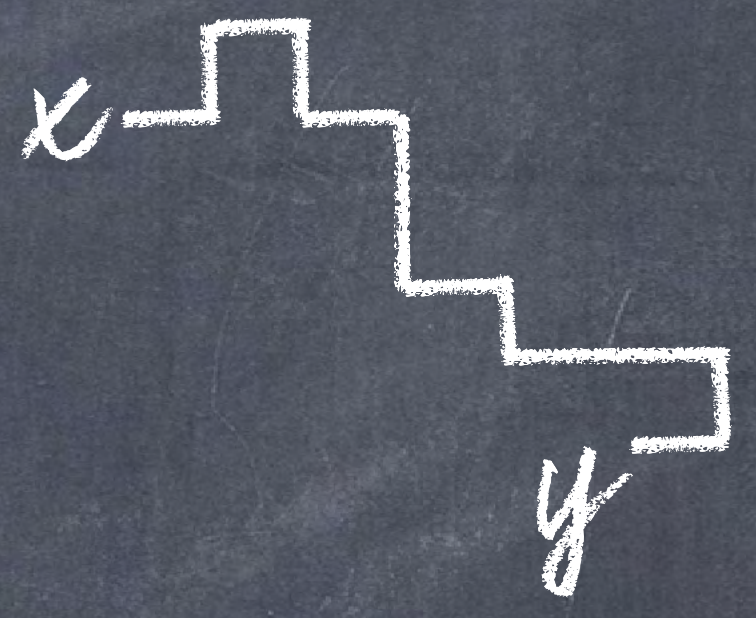

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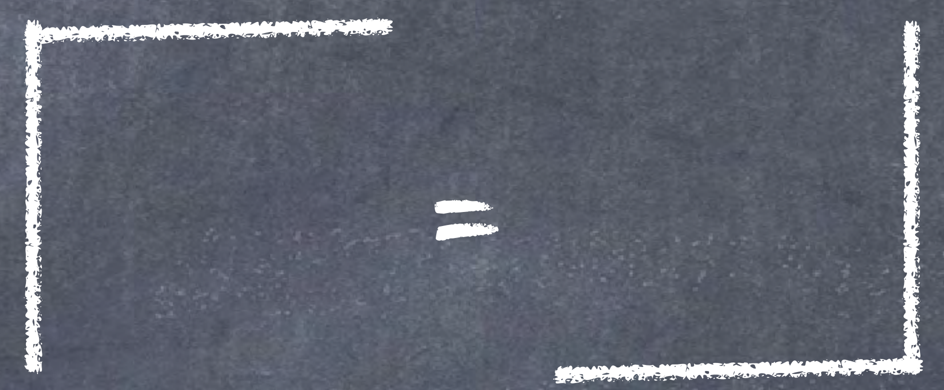

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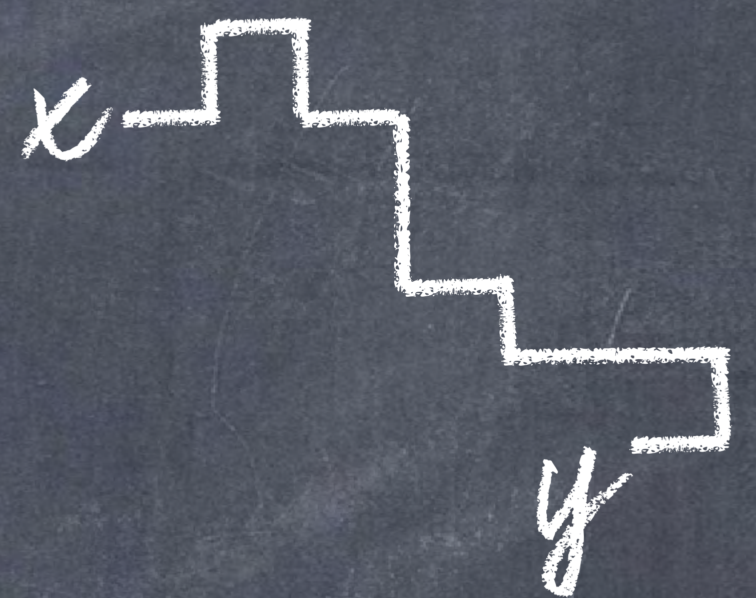

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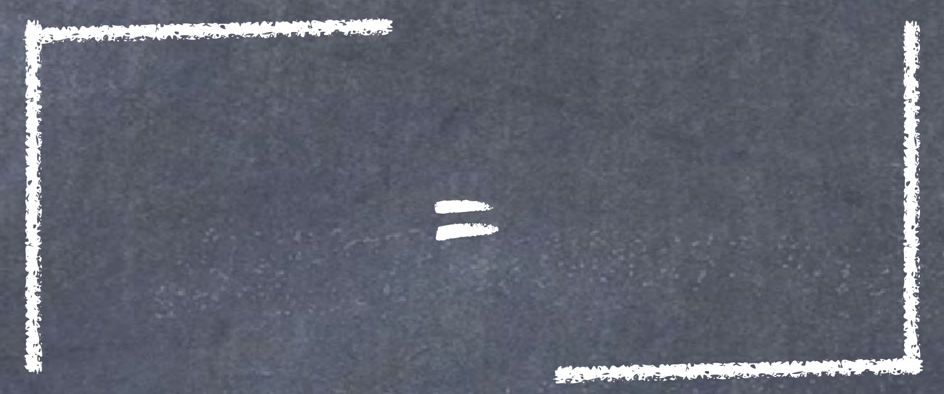

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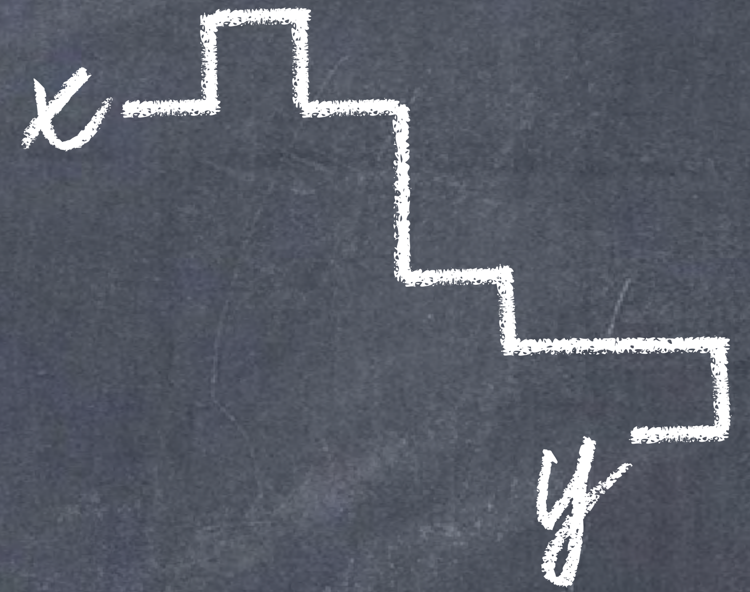

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(for example) holds for all ξ if g is everywhere differentiable

$\gamma^{x,\xi\pm}$: start at x and follow minimal $B^{\xi\pm}$-increment

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cocycle+recovery imply that $\gamma^{x,\xi\pm}$ are geodesics

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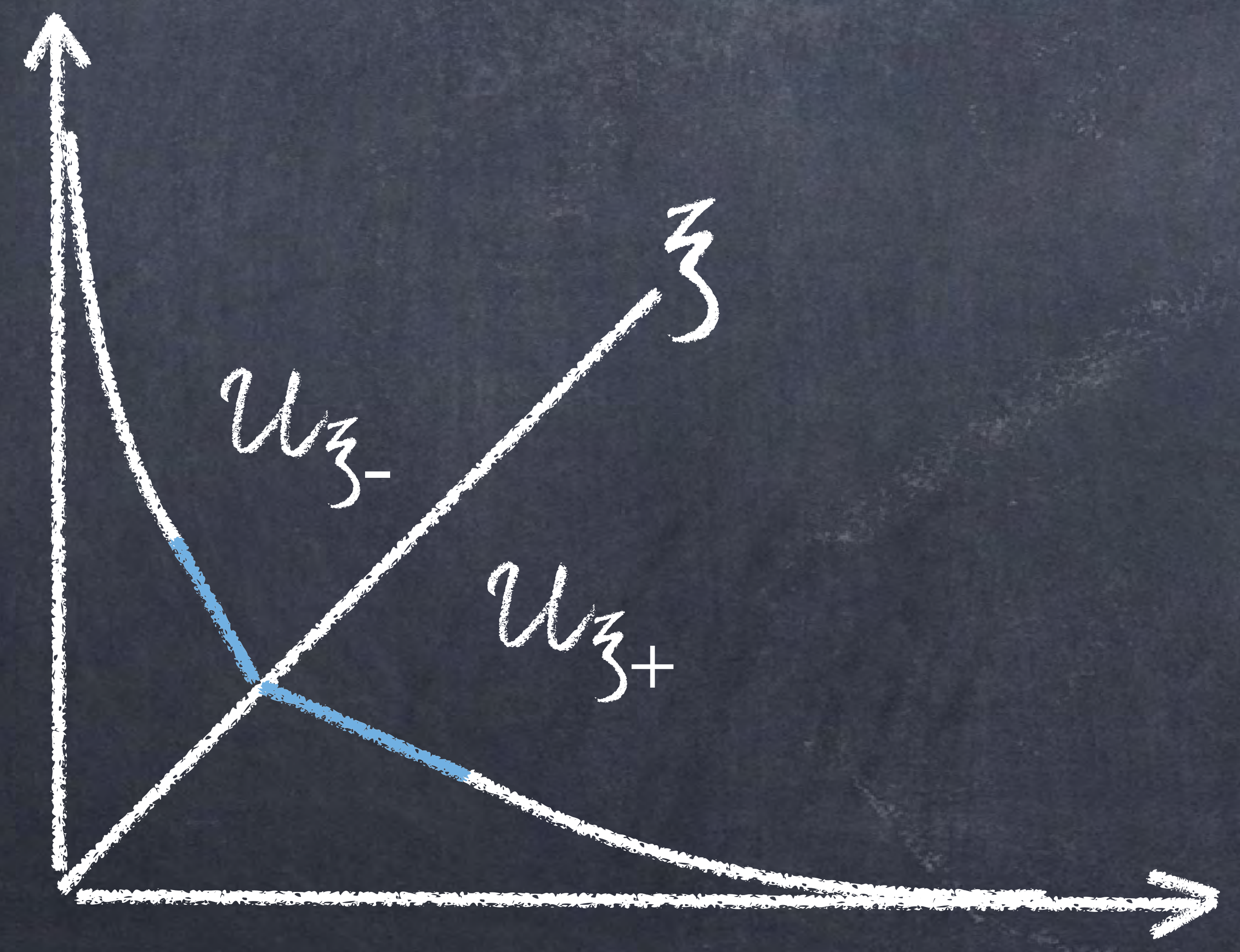
<u>Theorem:</u> a.s. $\forall x, \xi, \square \in \{-,+\}$, $\gamma^{x,\xi\square}$ is directed into $\underbrace{\{\zeta : \nabla g(\zeta) = \nabla g(\xi\square)\}}_{\mathcal{U}_{\xi\square}}$



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Then we can trap any geodesic between $\gamma^{x,\xi\pm}$ for some ξ

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<u>Corollary</u>: a.s. all geodesics are directed into $\mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+}$ for some ξ

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<u>Corollary:</u> a.s. all geodesics are directed into $\mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+}$ for some ξ

<u>Corollary:</u> If g is strictly concave then a.s. all geo. are ξ-directed for some ξ

__Theorem:__ $\forall \xi, x, y, \square \in \{-,+\}$: $\gamma^{x,\xi\square}$ and $\gamma^{y,\xi\square}$ a.s. coalesce

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Corollary: g diff. implies $\forall \xi, x$ a.s. $\exists!$ $\mathcal{U}_\xi$-directed geo. out of x

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Uniqueness and coalescence for all ξ?

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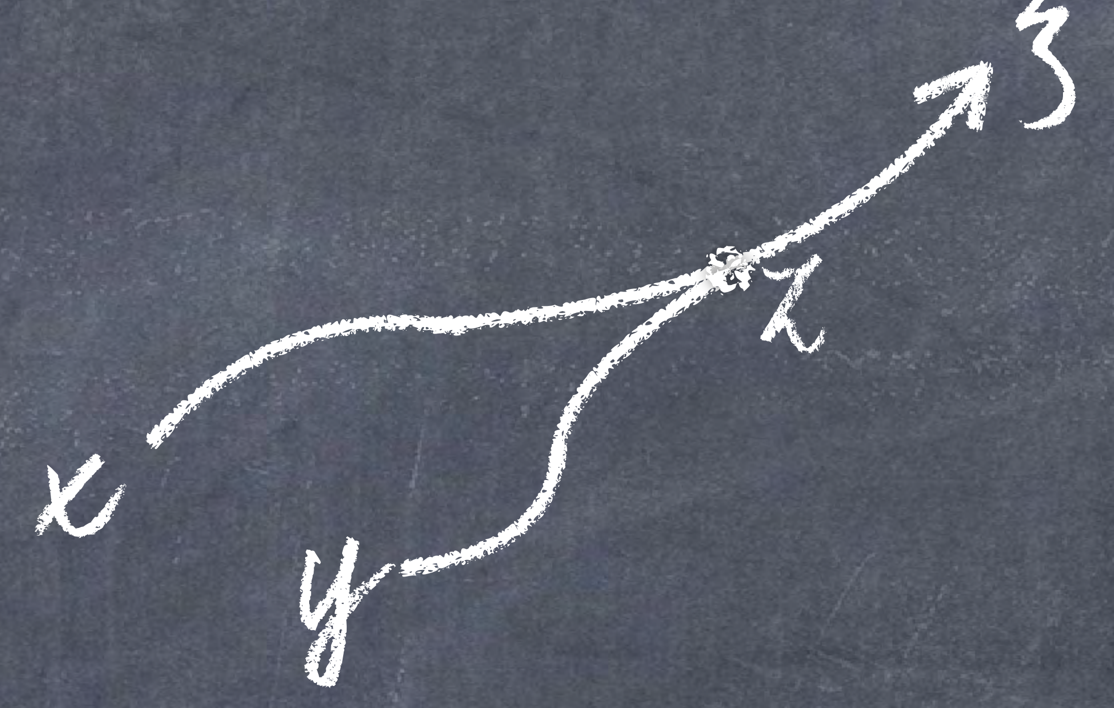
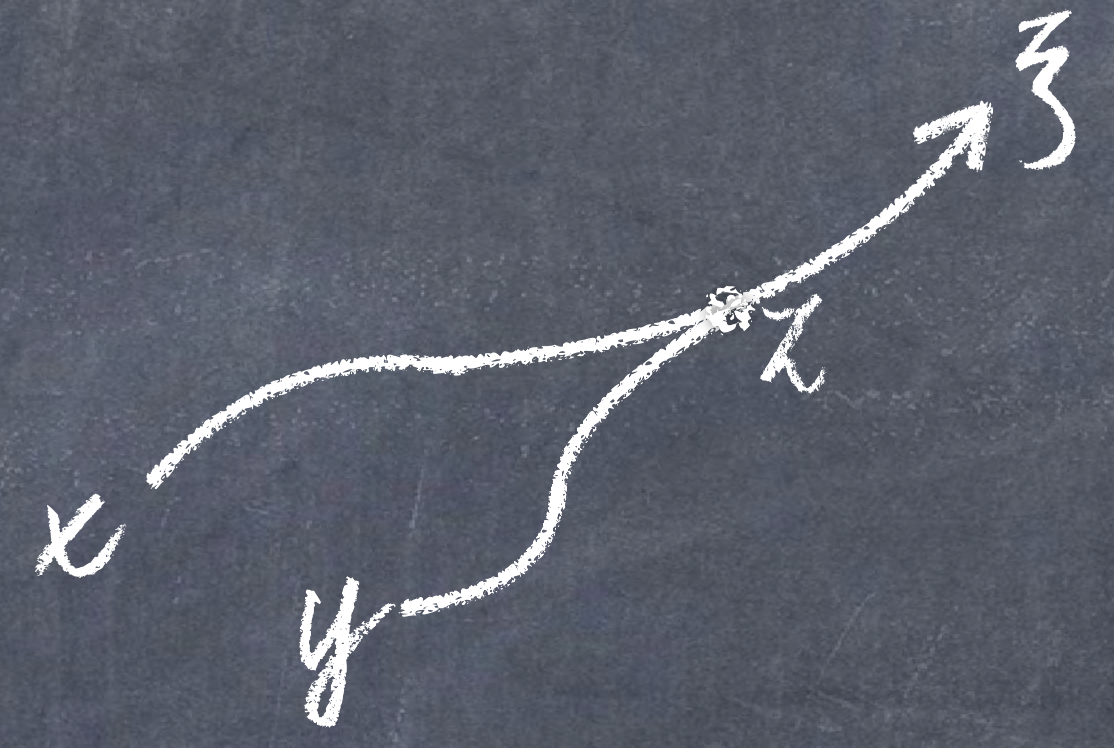
I.e. want to switch "$\forall \xi$ for a.e. ω" to "for a.e. ω $\forall \xi$"



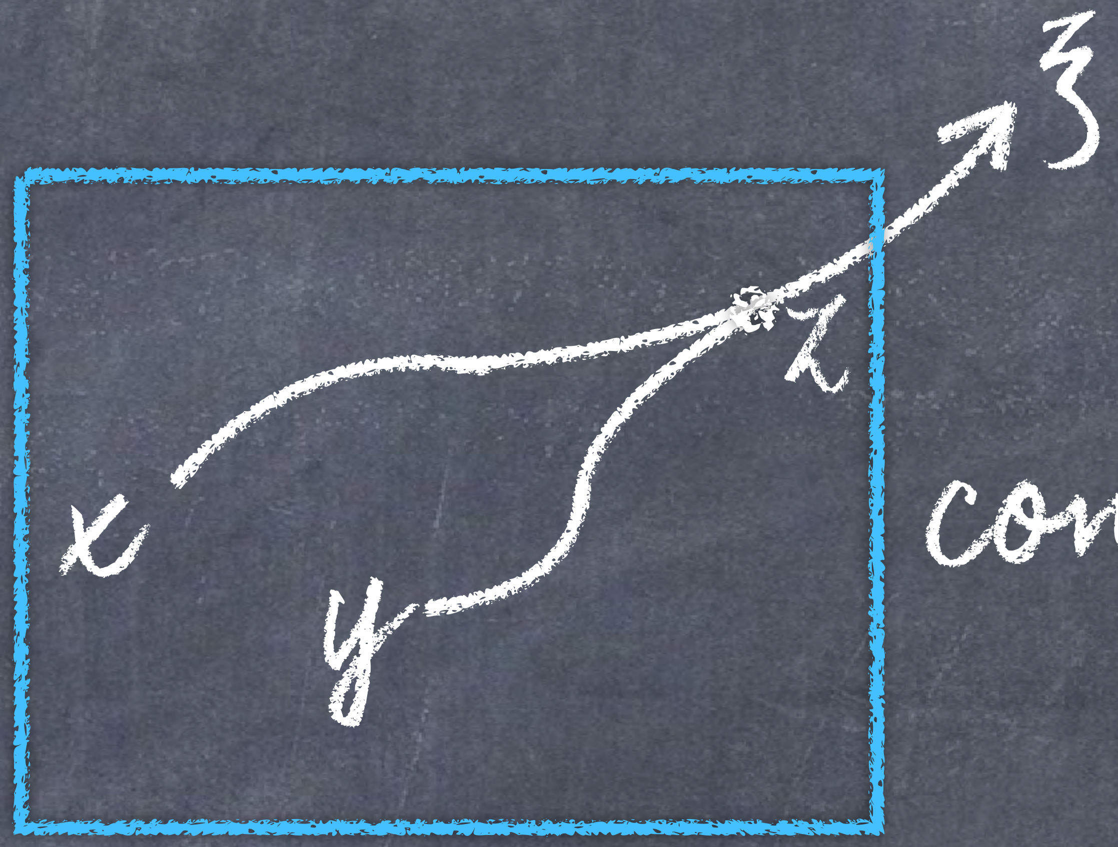
x
y
z
ξ

$$B^{\xi}(x,y) = G_{x,z} - G_{y,z}$$



$B^{\xi}(x,y) = G_{x,z} - G_{y,z}$

$B^{\xi}(x,y)$ constant over an interval of ξ's implies



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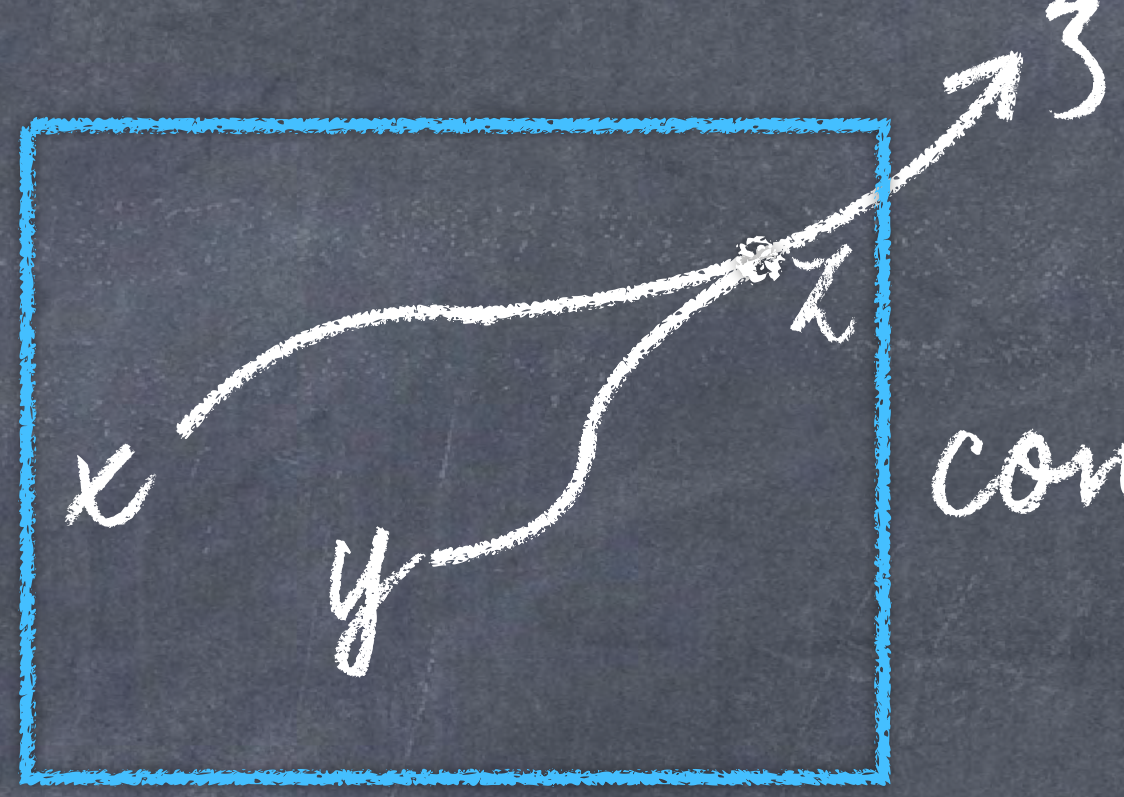

constant

$\mu_{x,y}$: Lebesgue-Stieltjes signed measure of $\xi \to B^{\xi}(x,y)$

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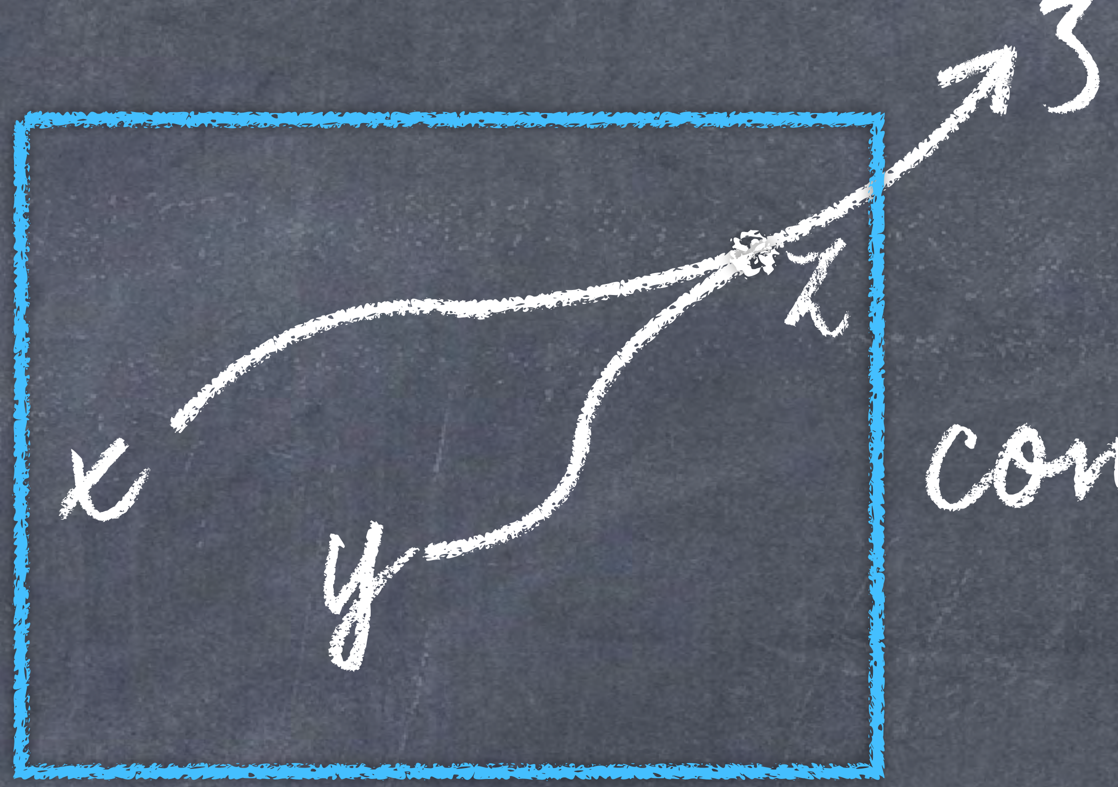

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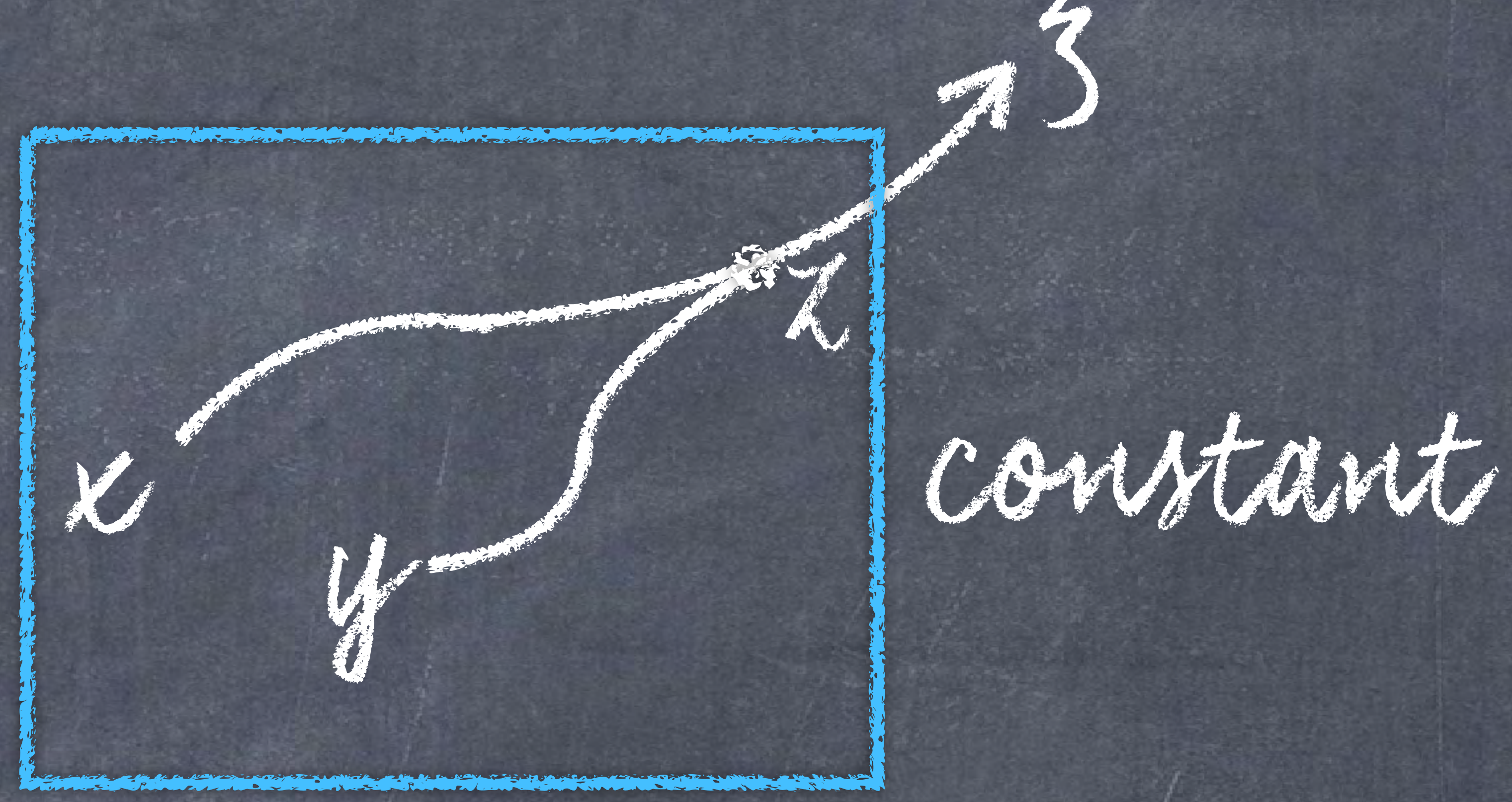
Set of exceptional directions $\mathcal{V}^{\omega} = \bigcup_{x,y} \operatorname{supp} \mu_{x,y}$

$B^{\xi}(x,y) = G_{x,z} - G_{y,z}$

$B^{\xi}(x,y)$ constant over an interval of ξ's implies



$\mu_{x,y}$: Lebesgue-Stieltjes signed measure of $\xi \to B^{\xi}(x,y)$

Set of exceptional directions $\mathcal{V}^{\omega} = \bigcup_{x,y} \operatorname{supp} \mu_{x,y}$

<u>Theorem:</u> $\xi \notin \mathcal{V}^{\omega} \iff \gamma^{x,\xi+} = \gamma^{x,\xi-} = \gamma^{x,\xi}$, $\gamma^{x,\xi}$ & $\gamma^{y,\xi}$ coalesce

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Complement is a countable intersection of dense open sets

Familiar members:

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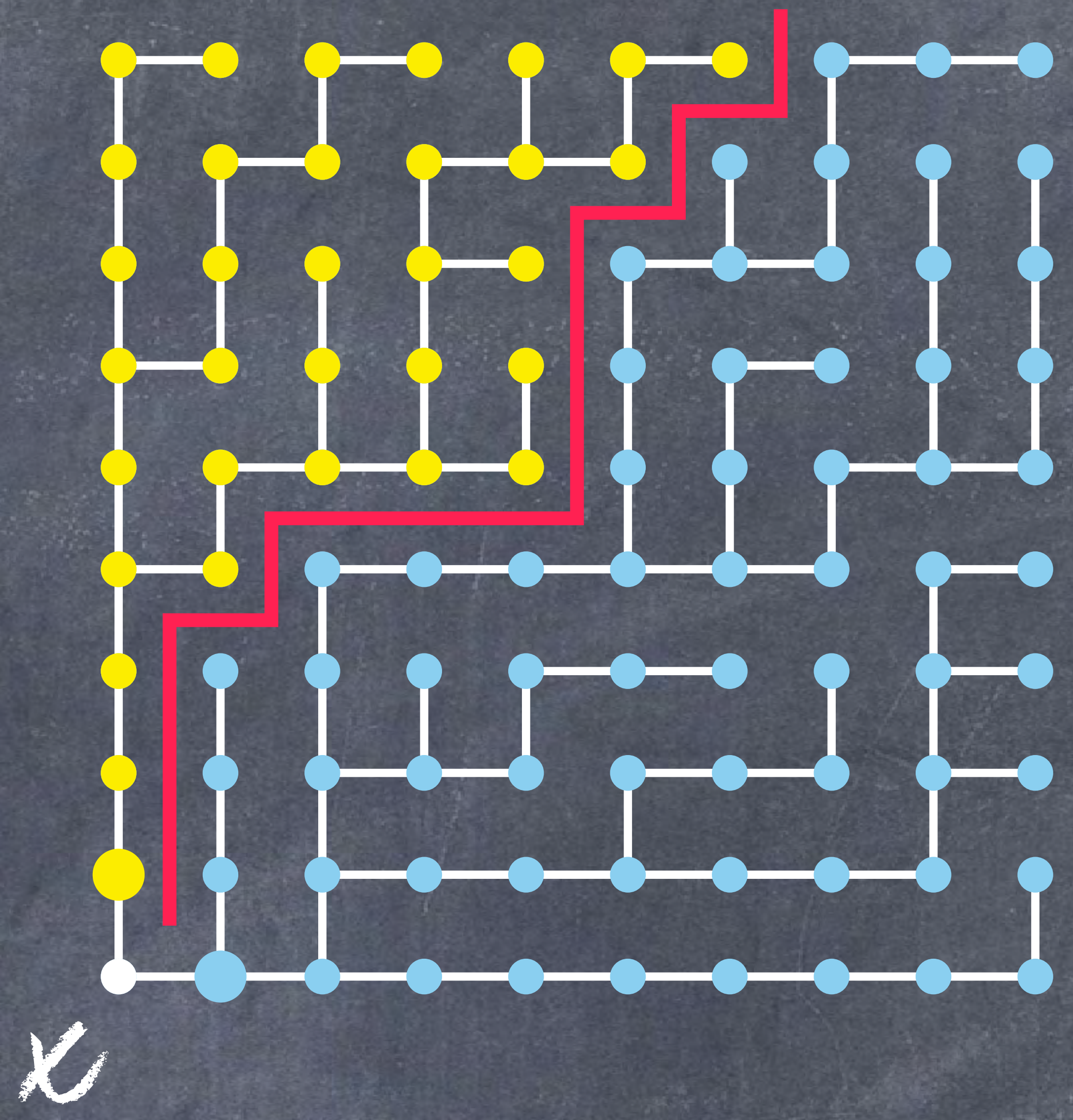
Tree of geodesics from x:



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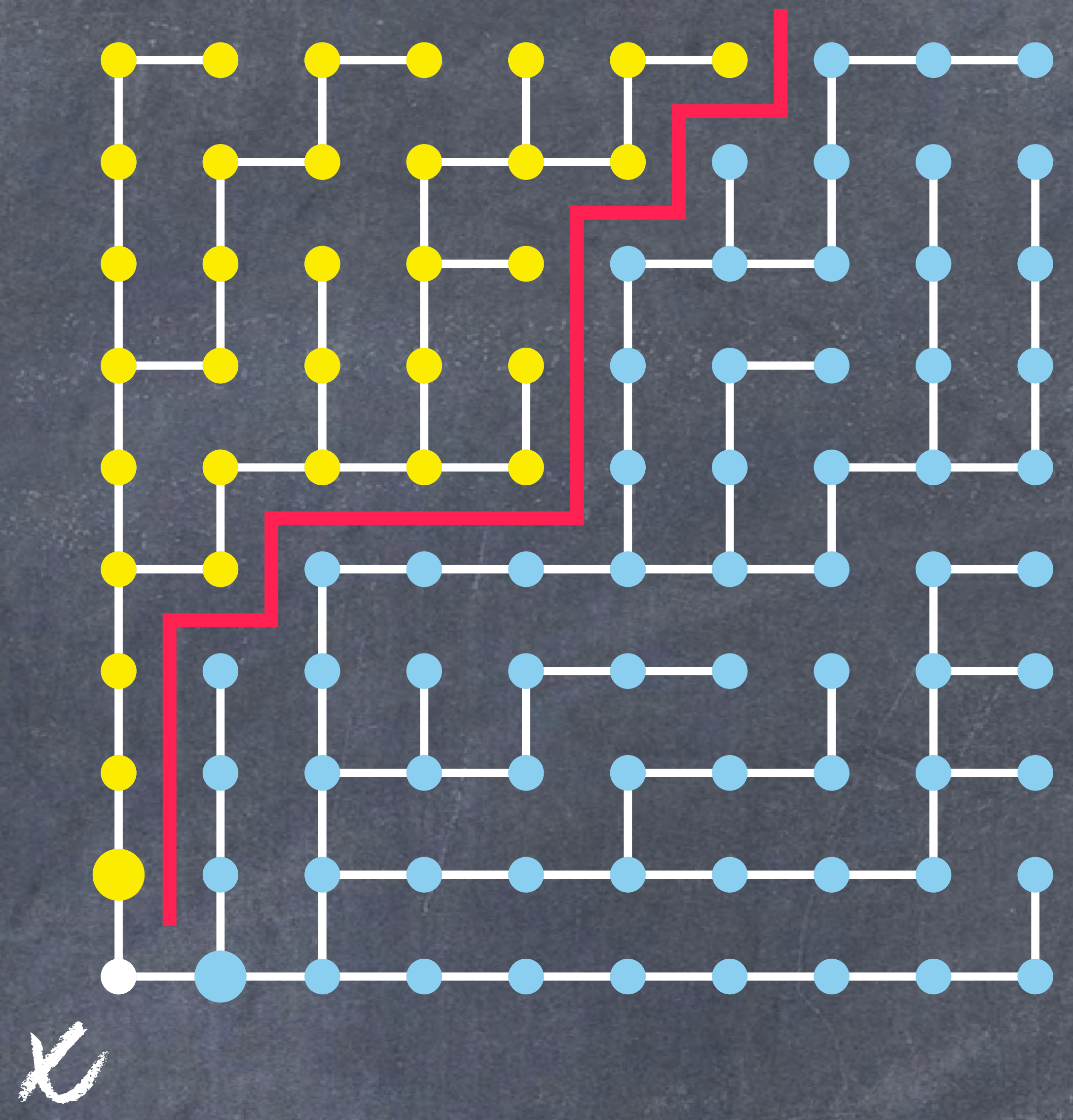
Under $(*)$: the competition interface

between subtrees rooted at $x+e_1$ and $x+e_2$

has asymptotic direction $\xi^*(x)$



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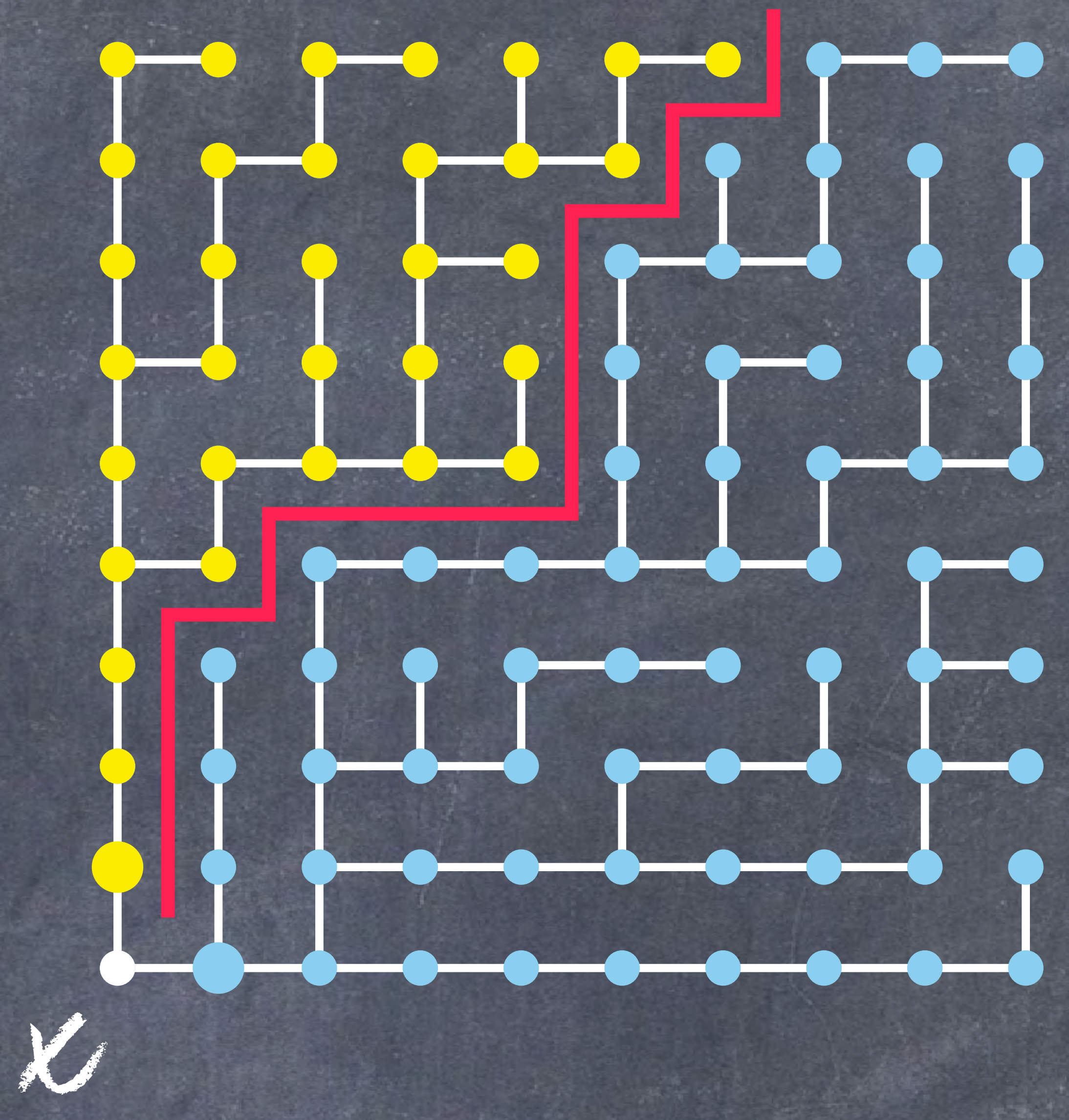

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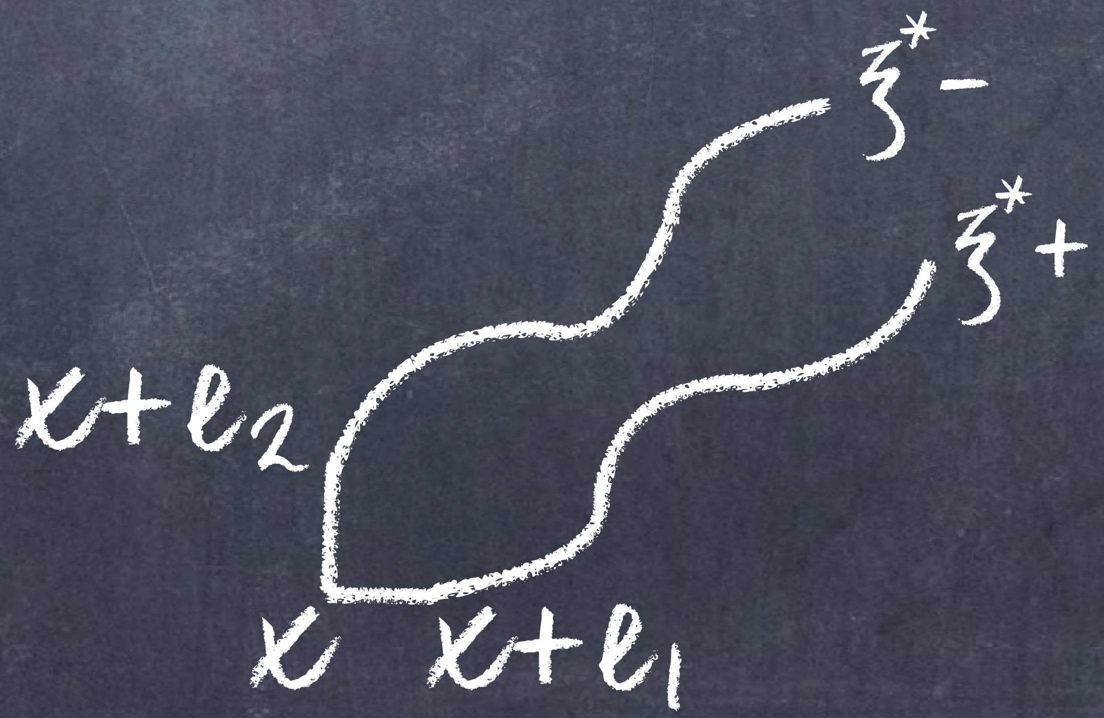

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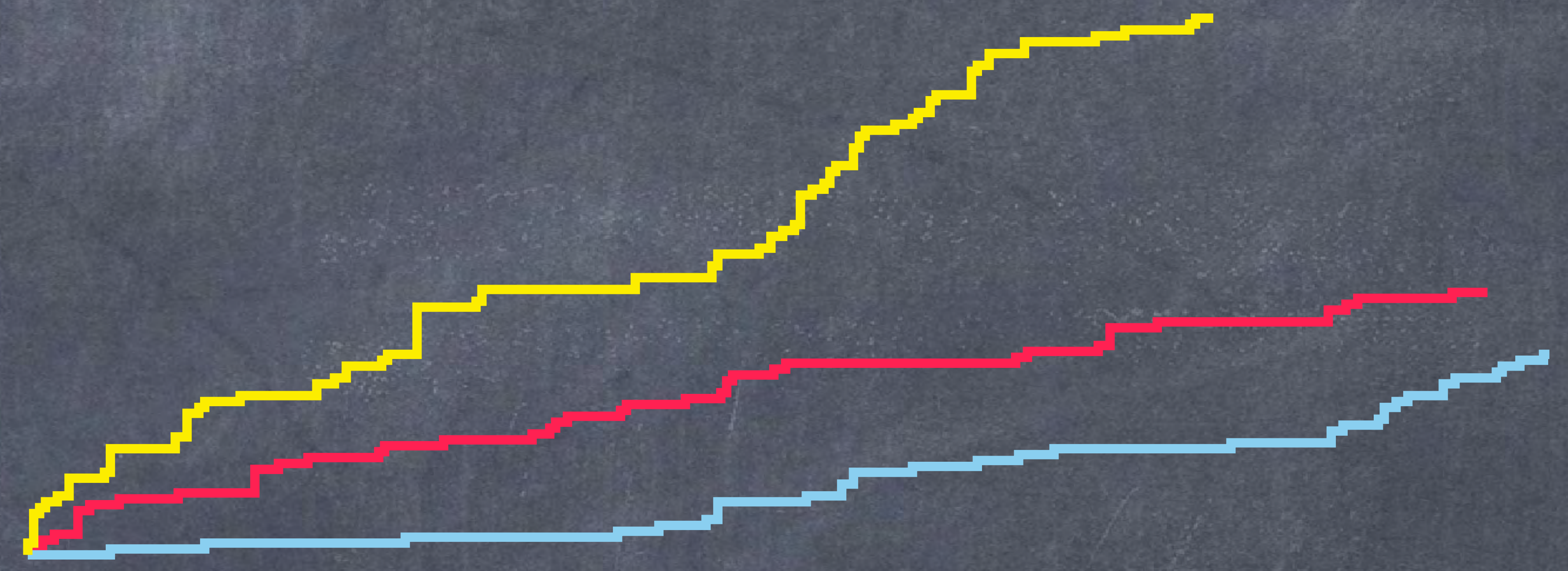

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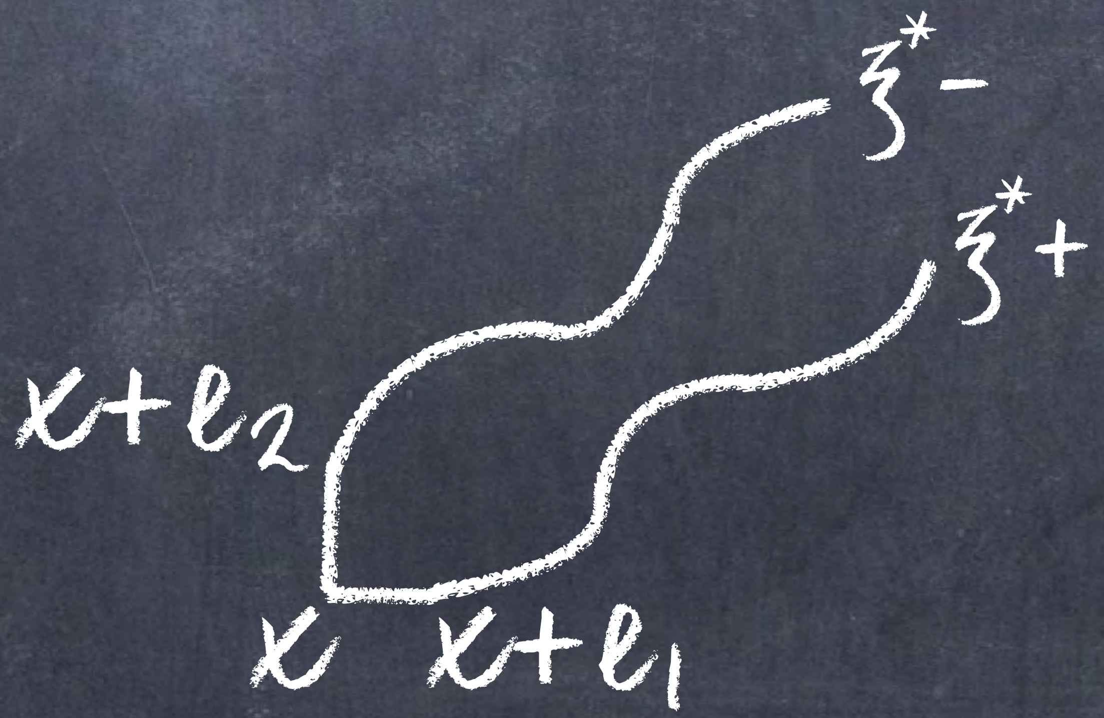

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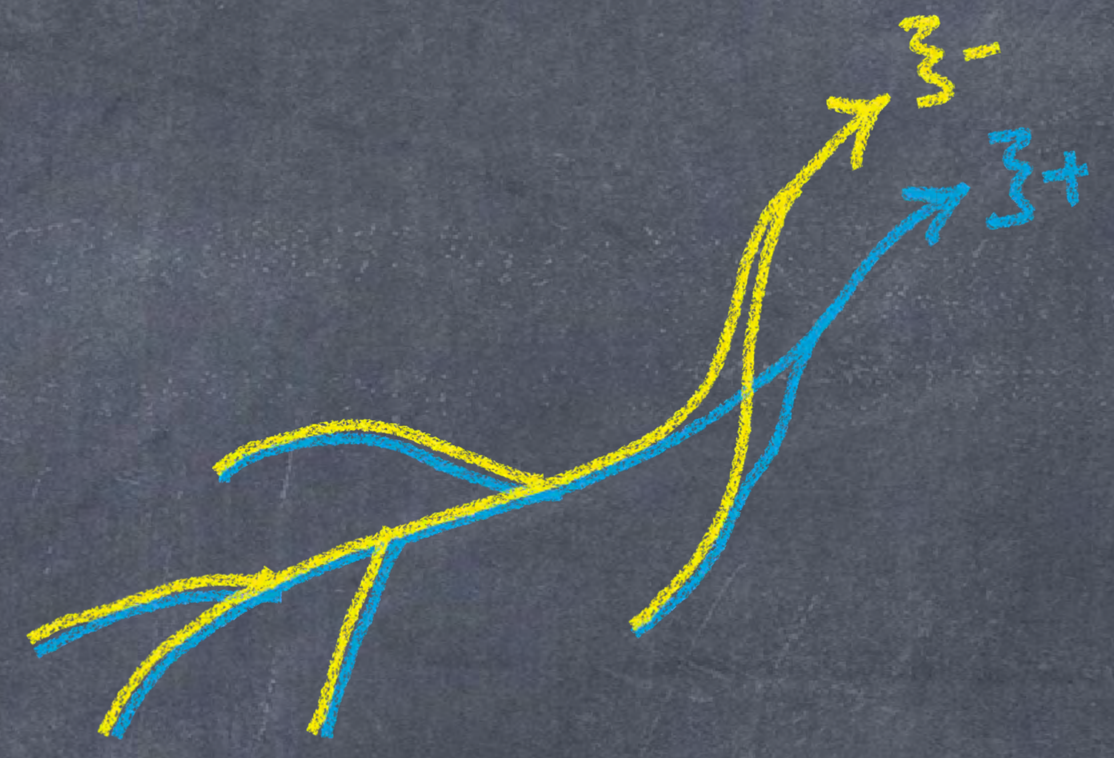

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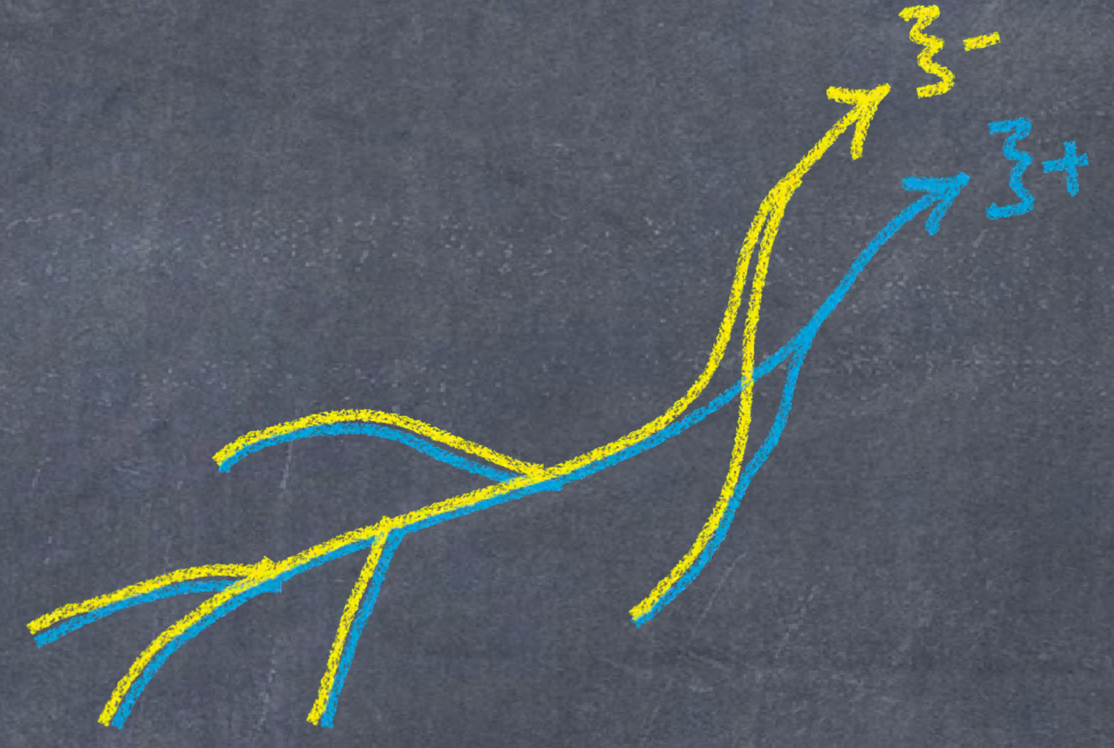

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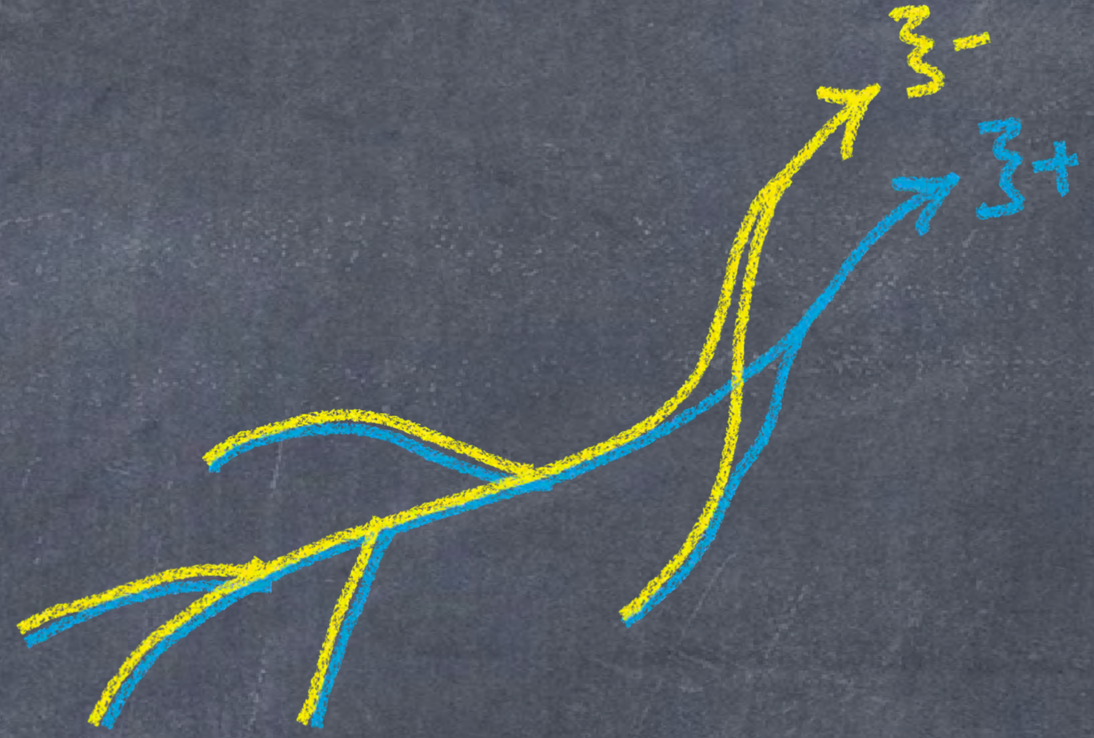

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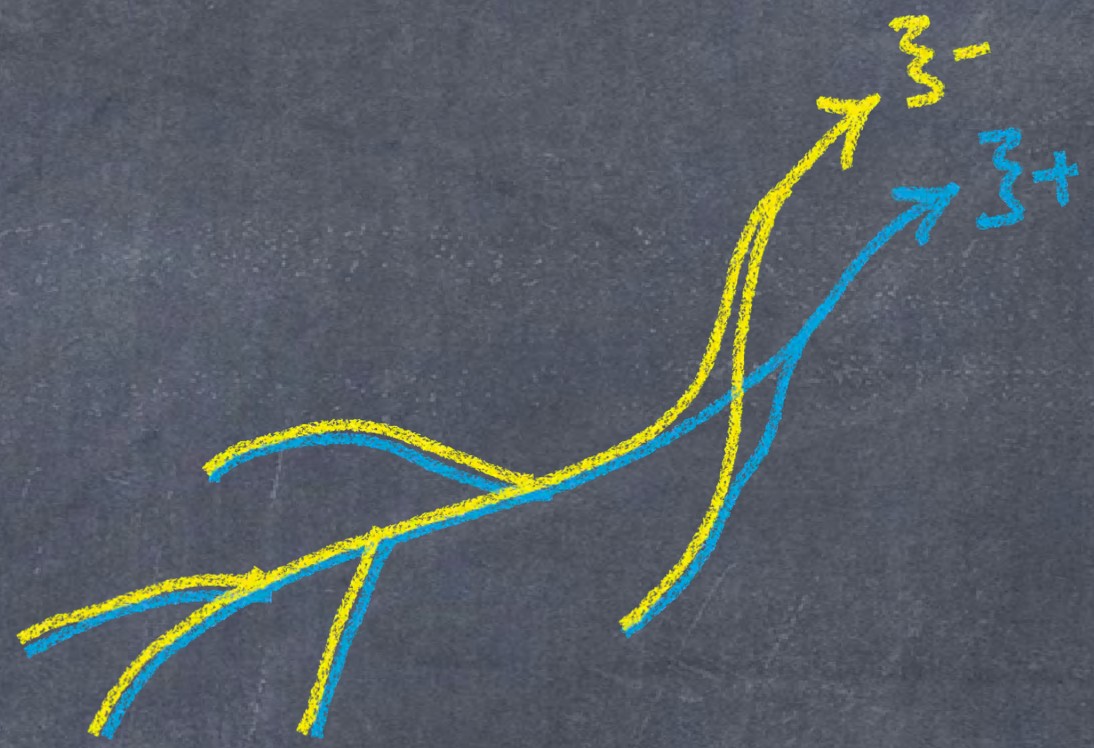

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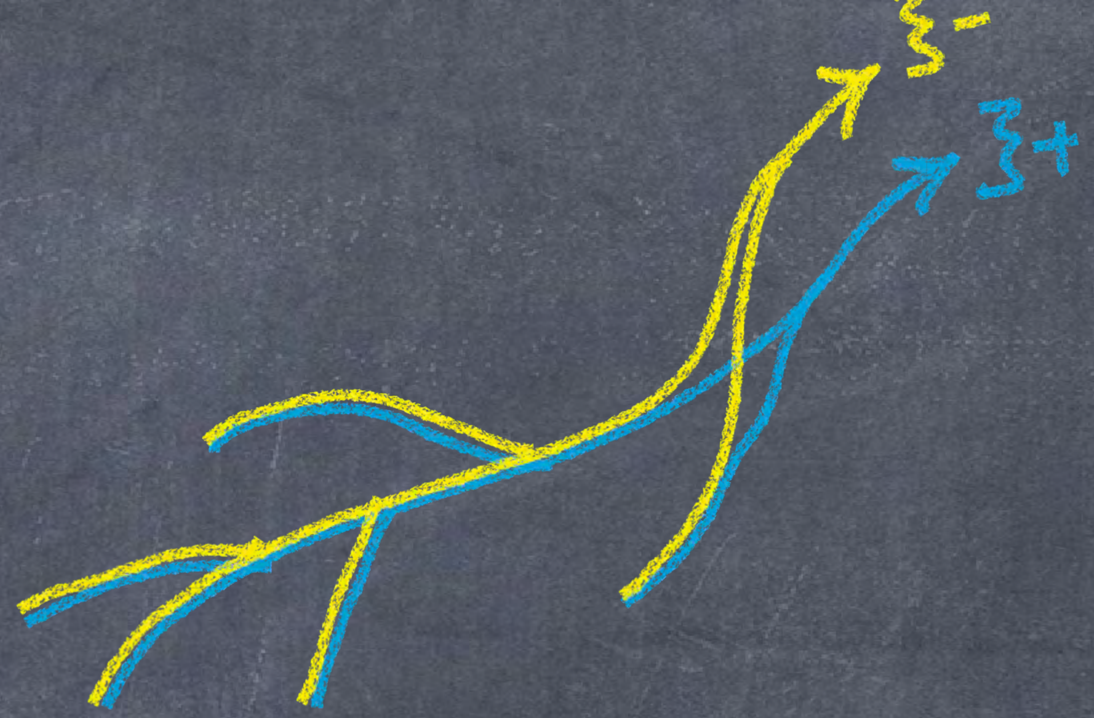

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- That's it! No other semi-infinite geodesics

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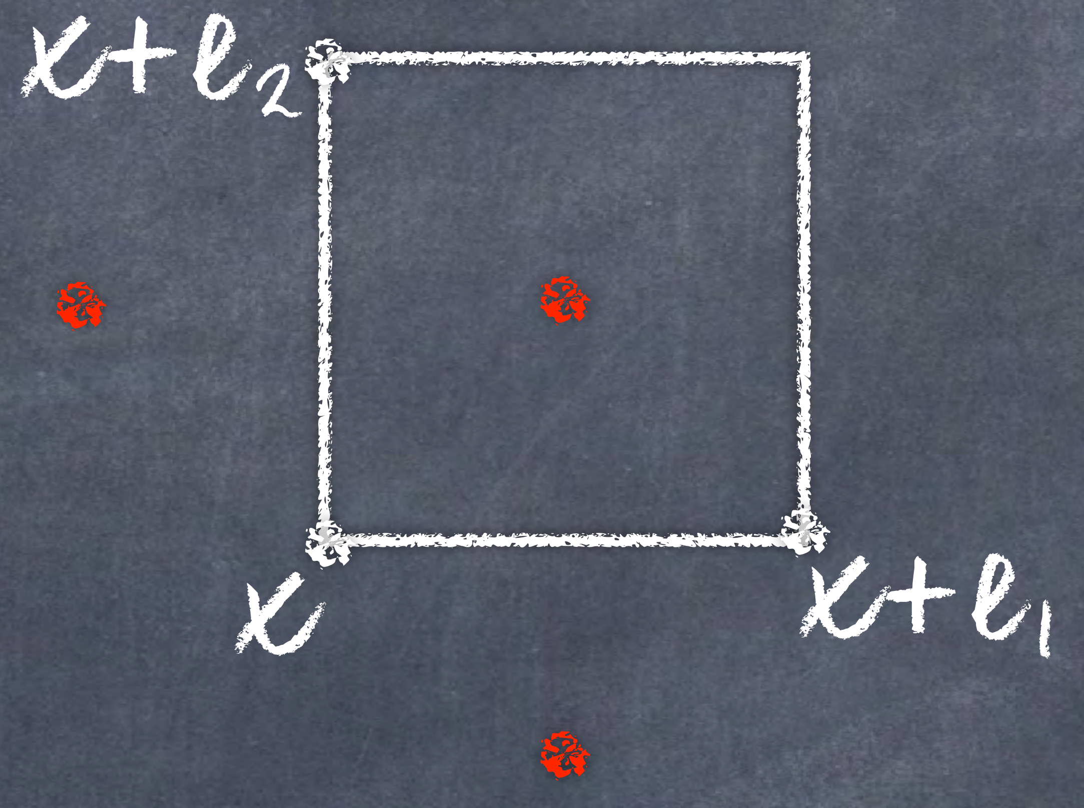

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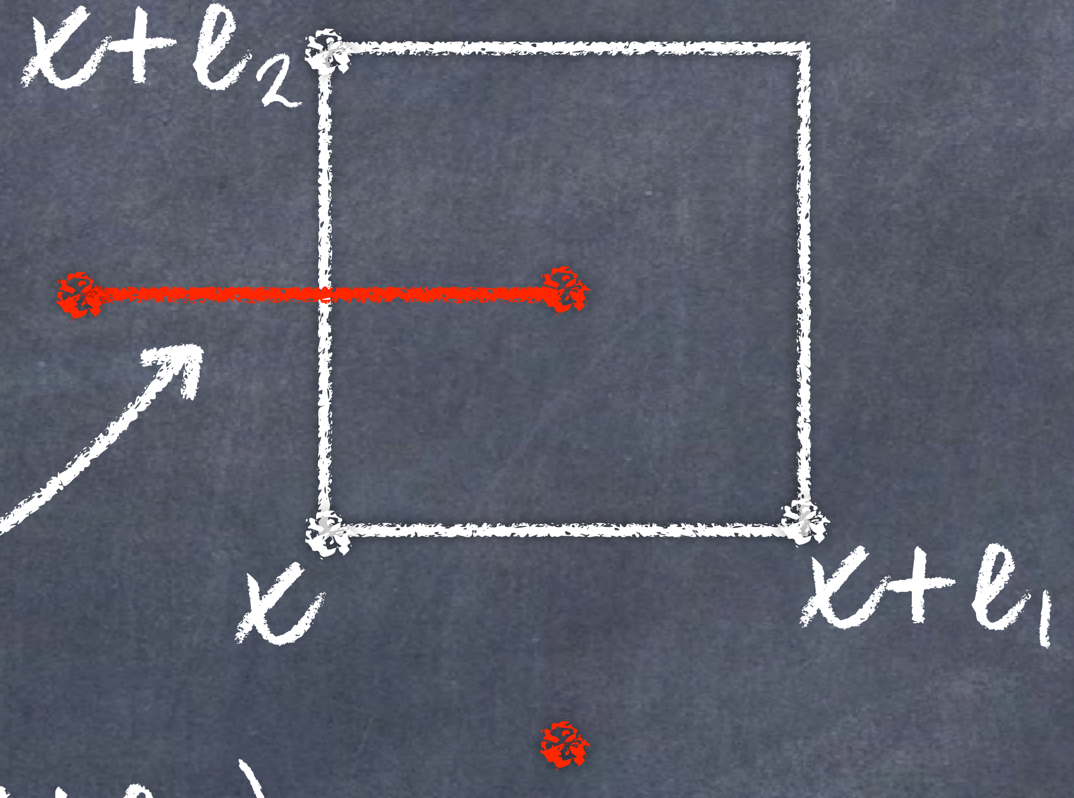

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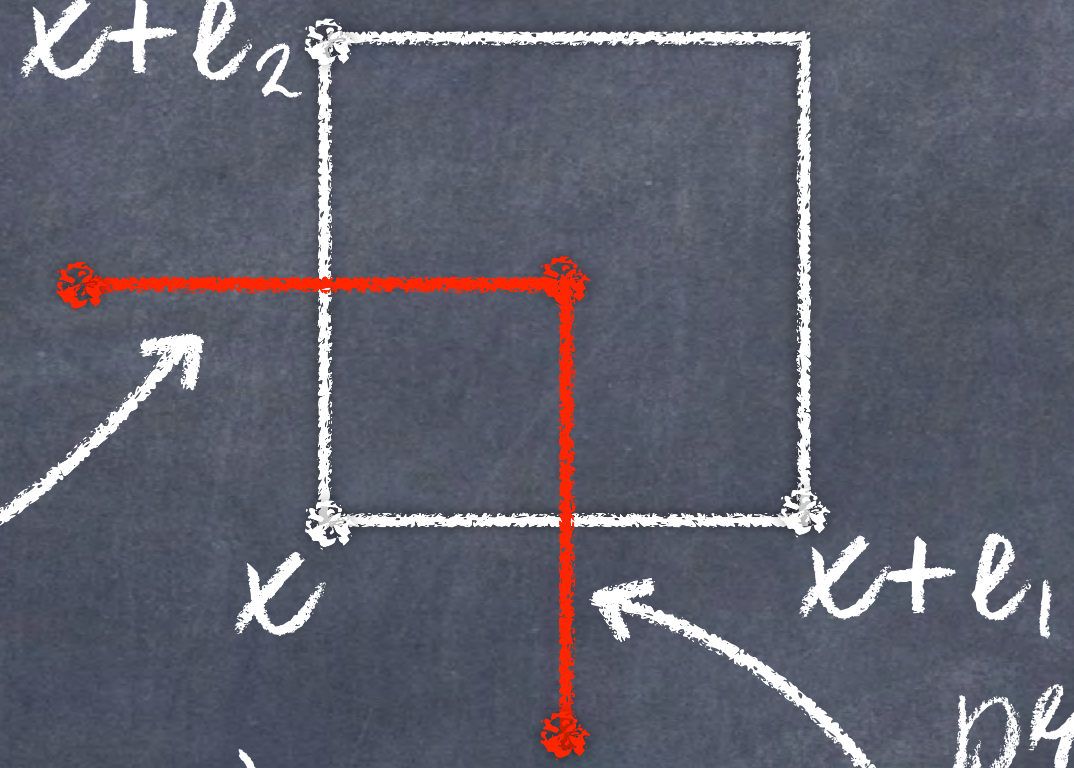

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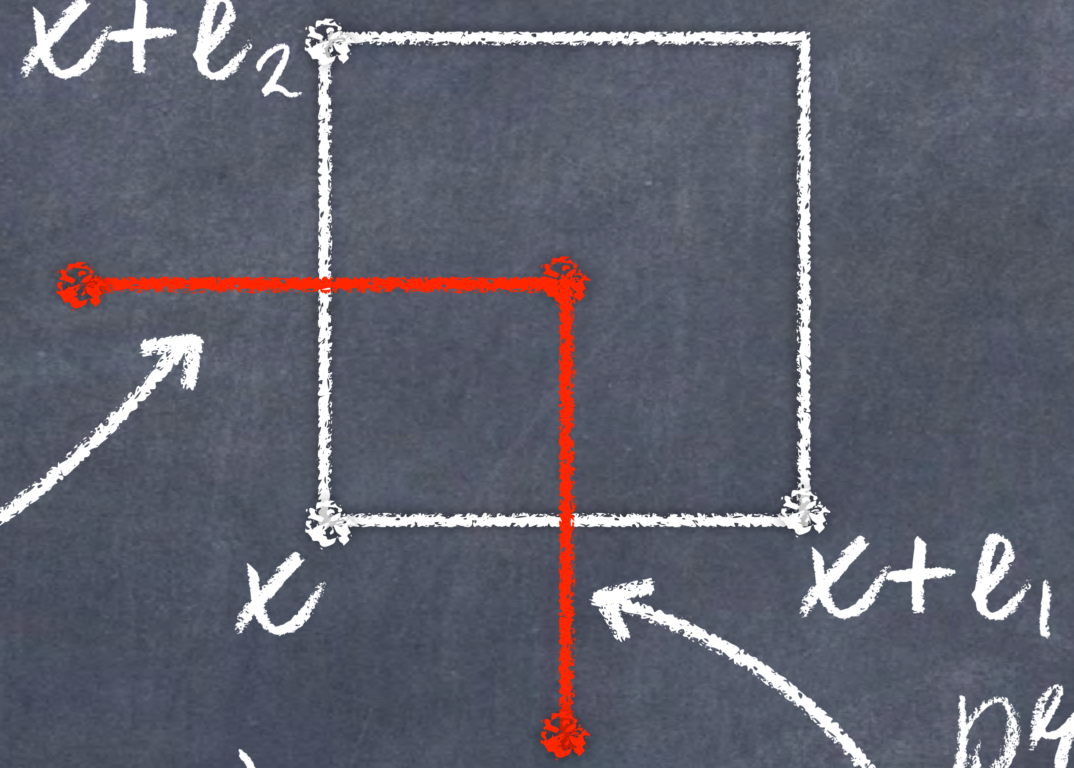

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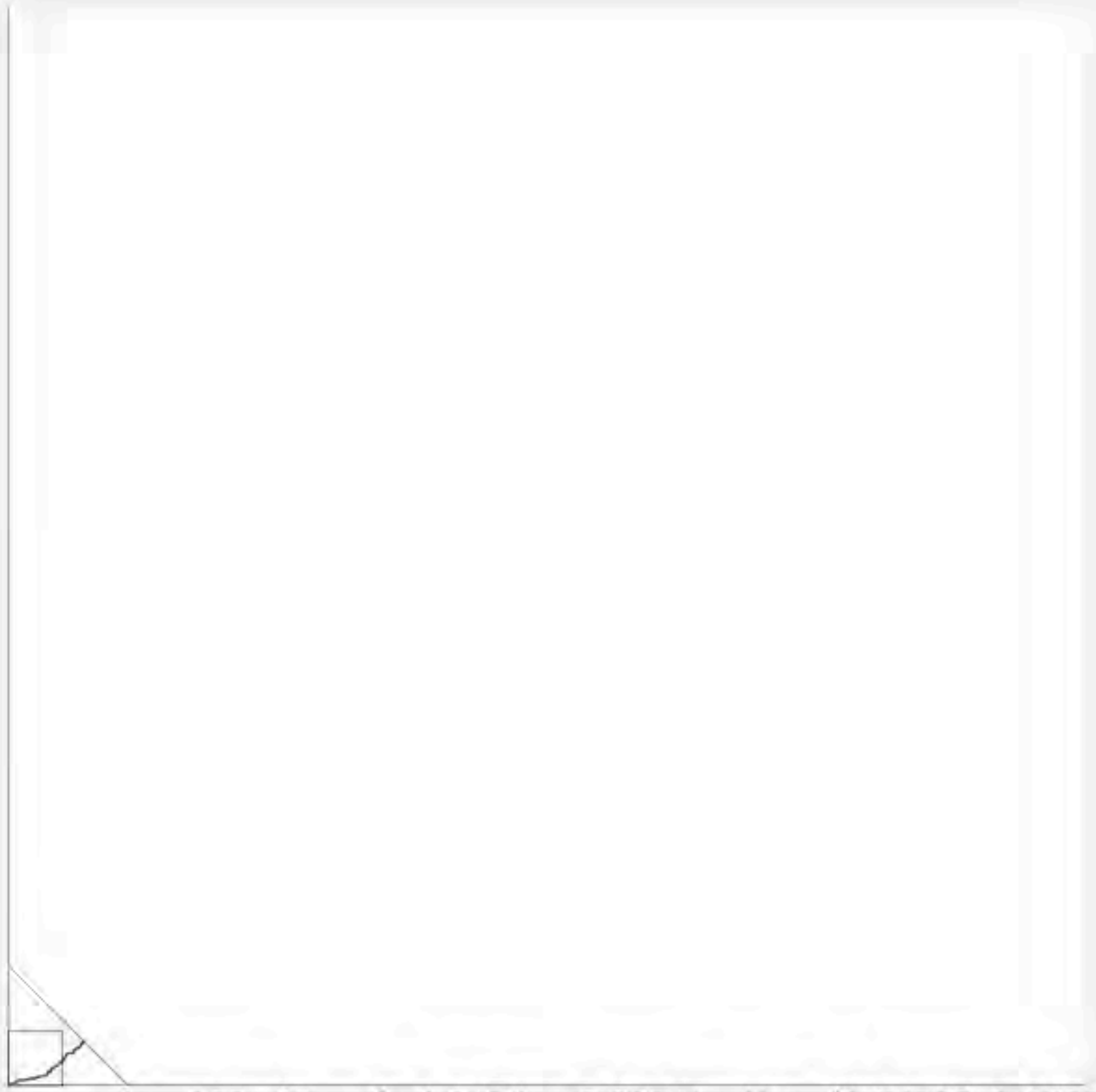
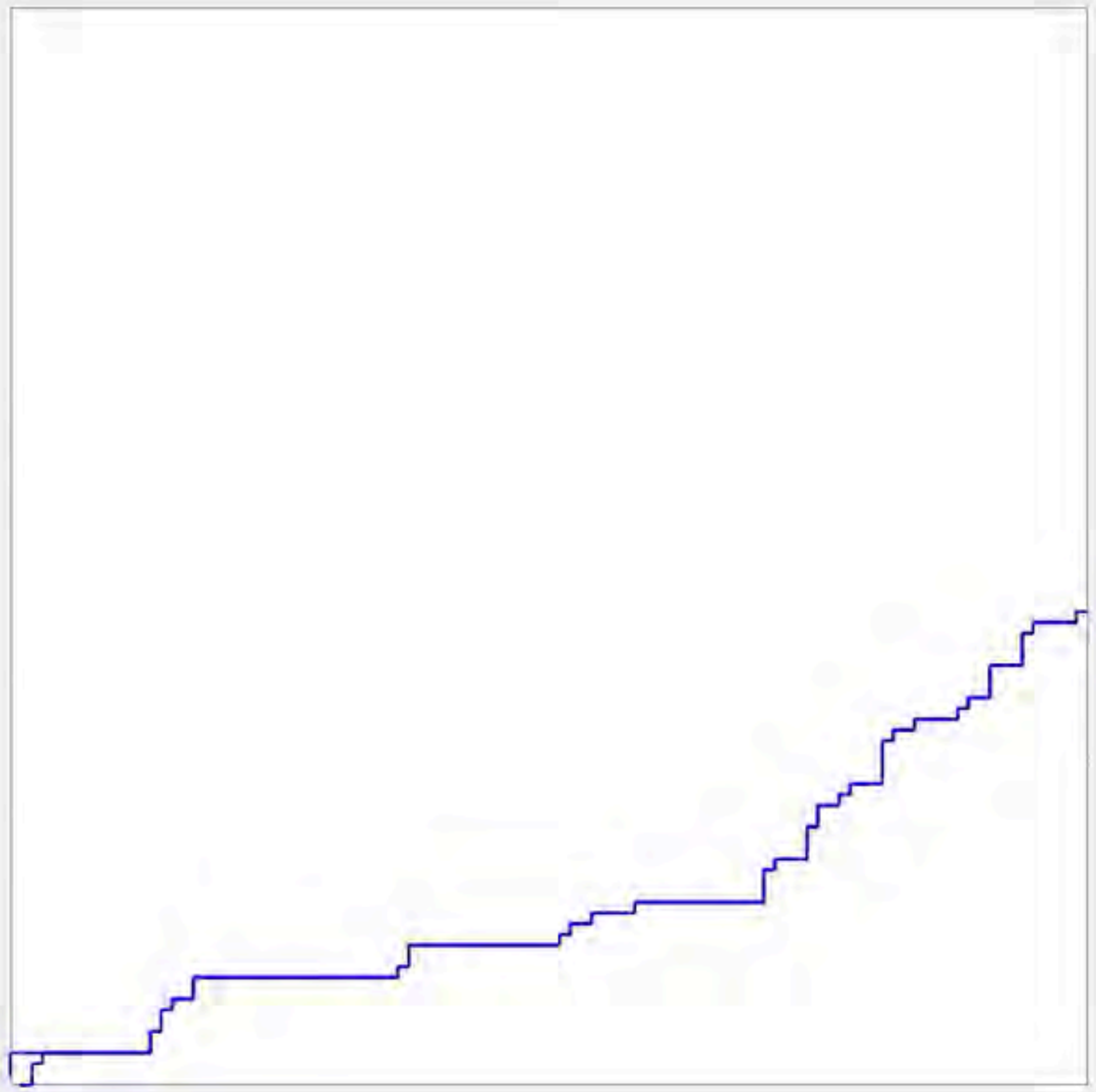
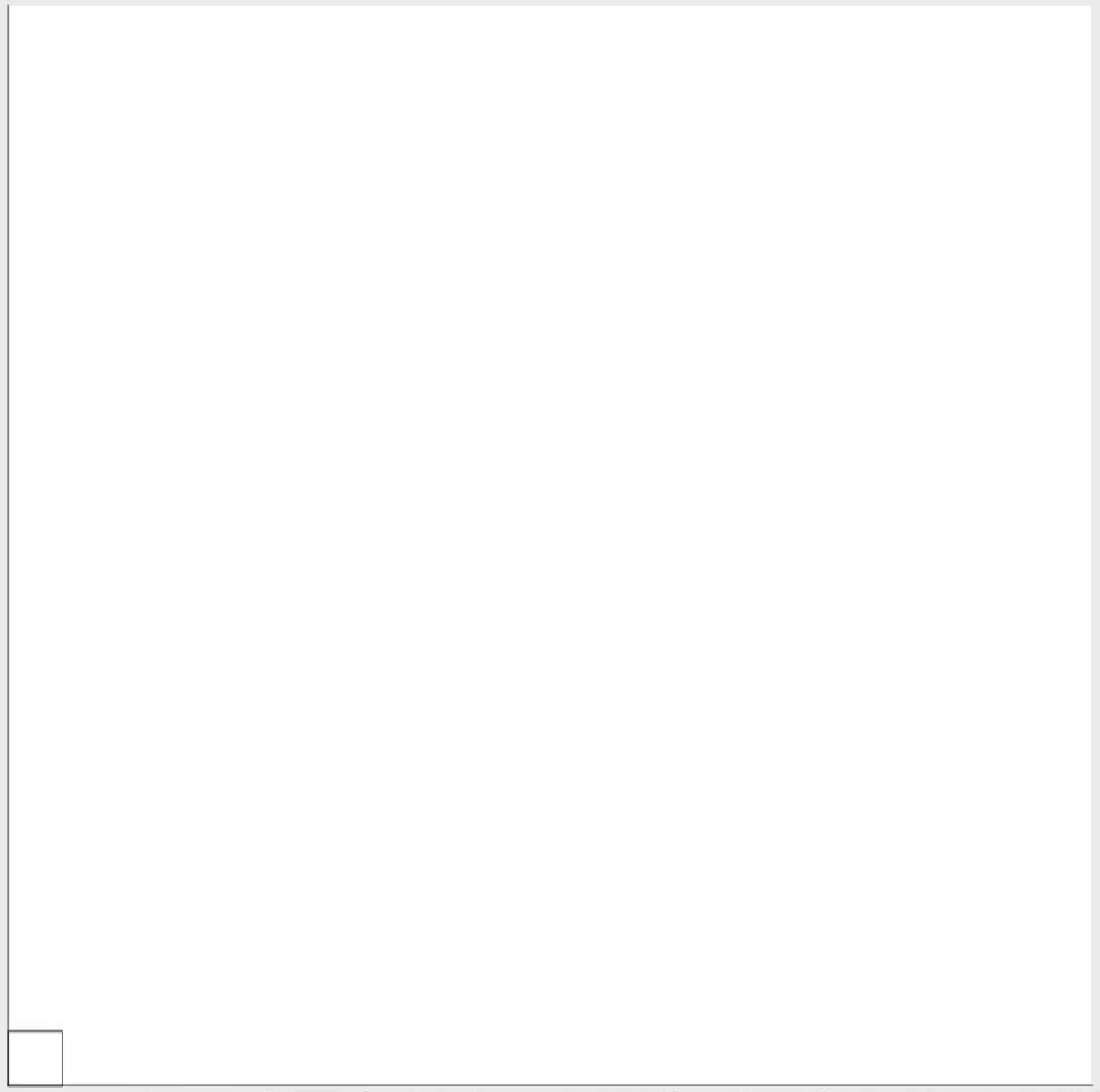


$\ell = 223, \ h^* = (-1.9174, -2.0901), \ \xi^* = (0.54302, 0.45698)$





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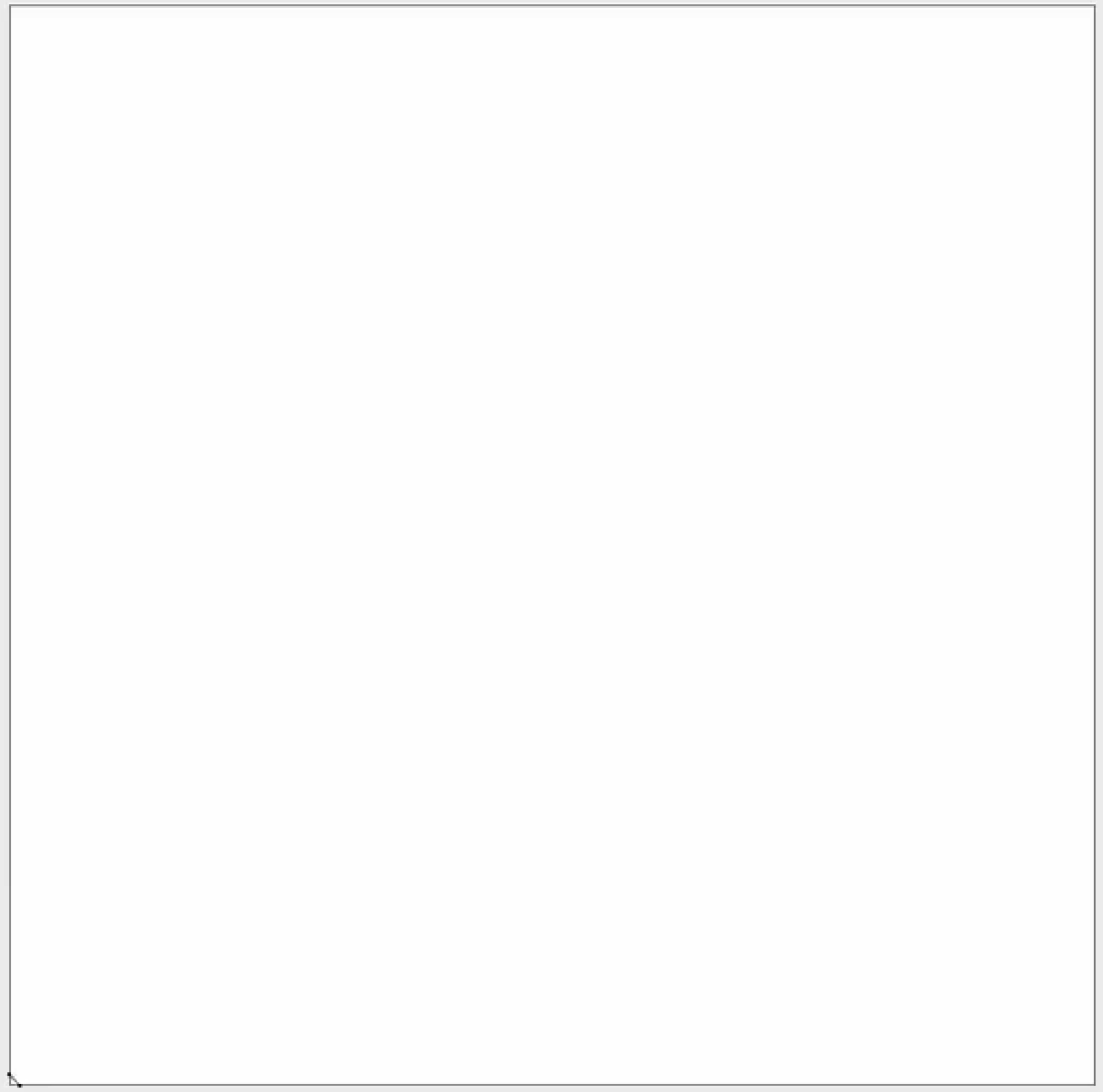

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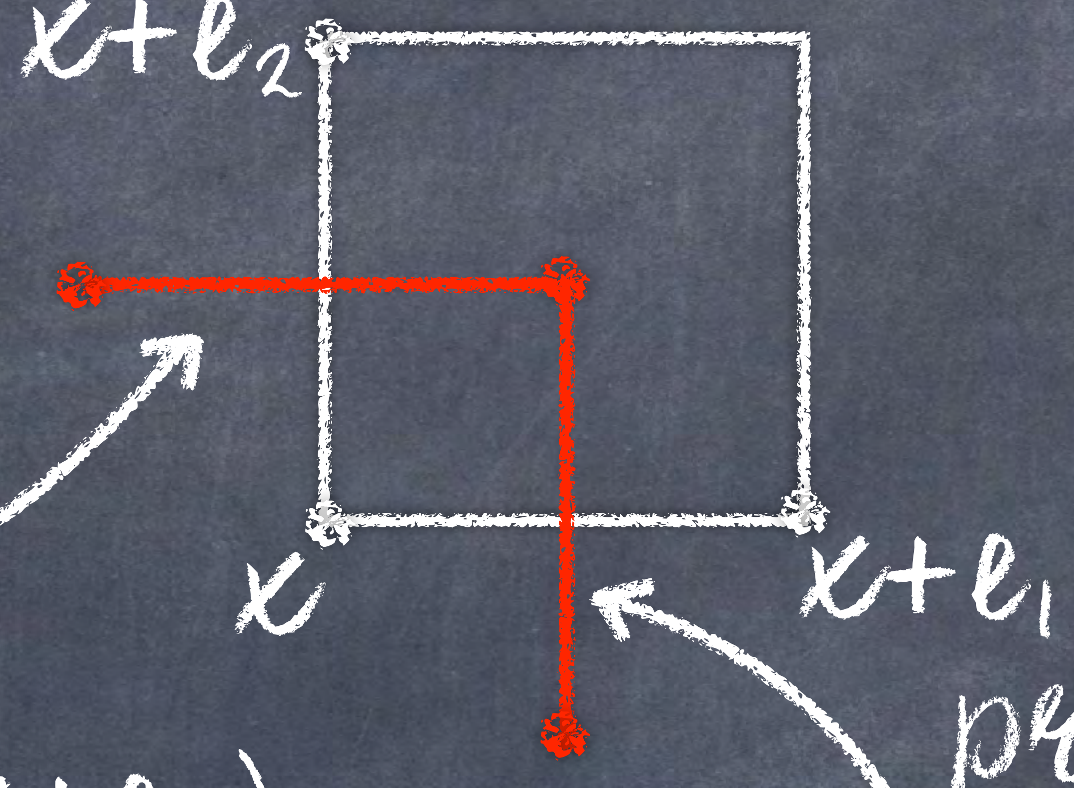

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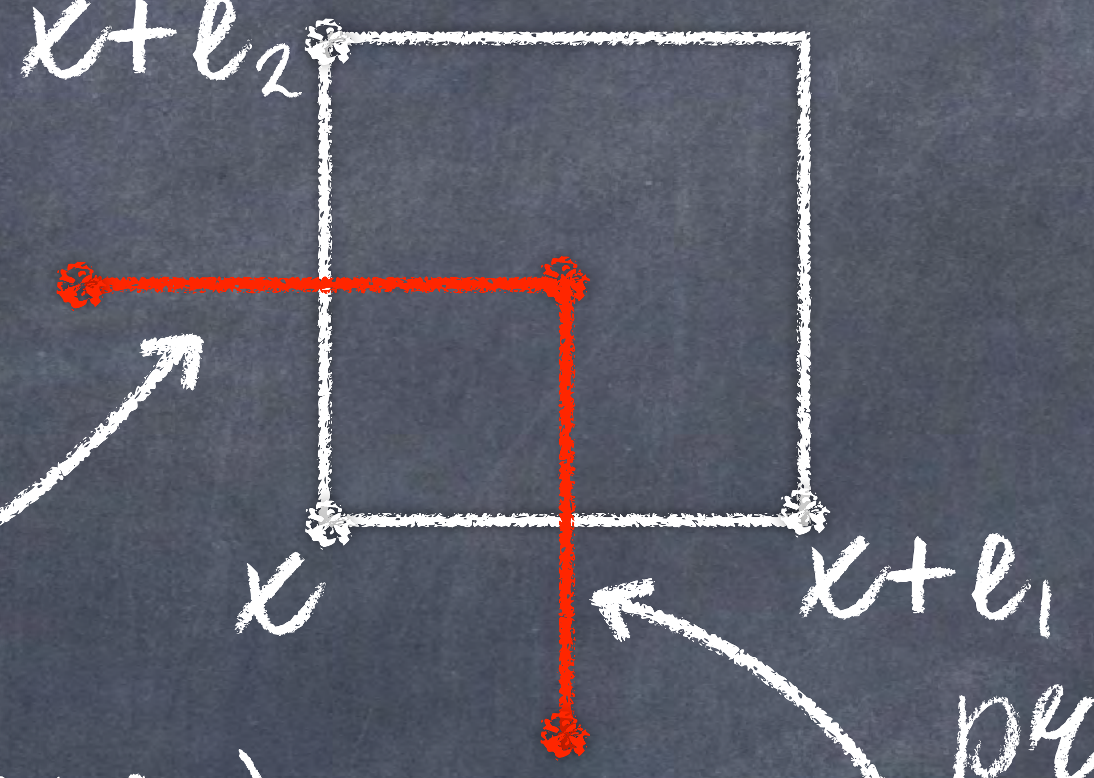

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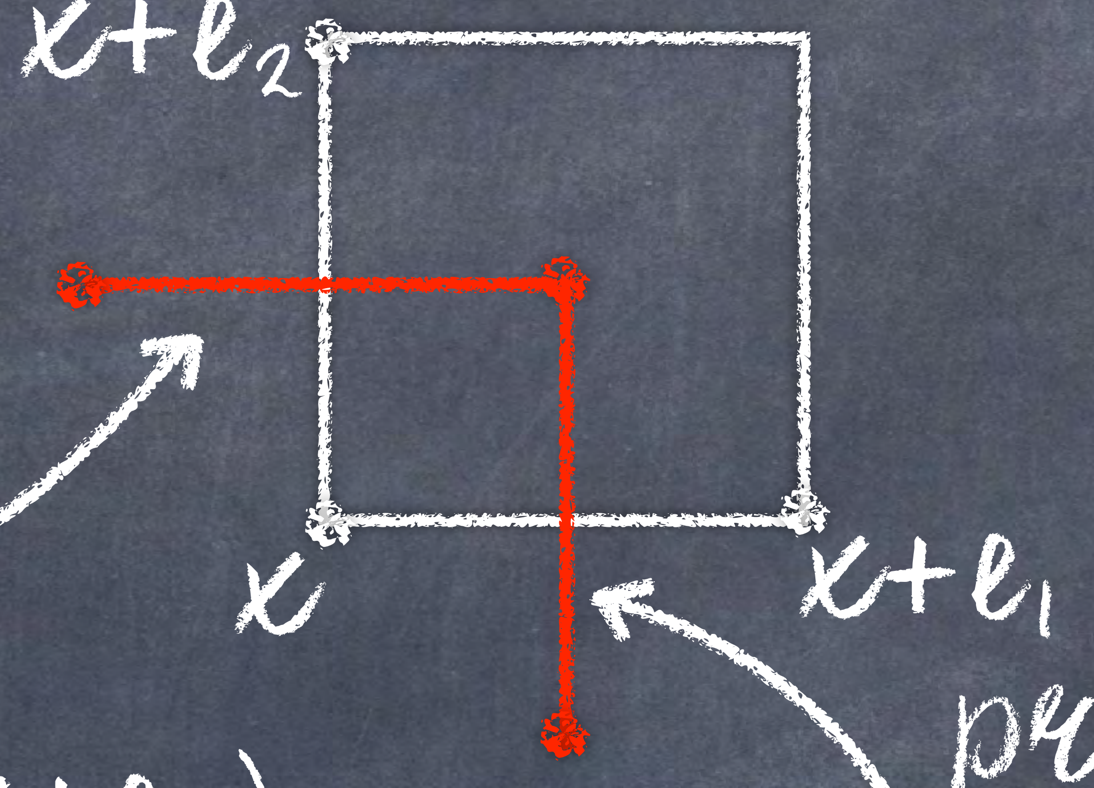

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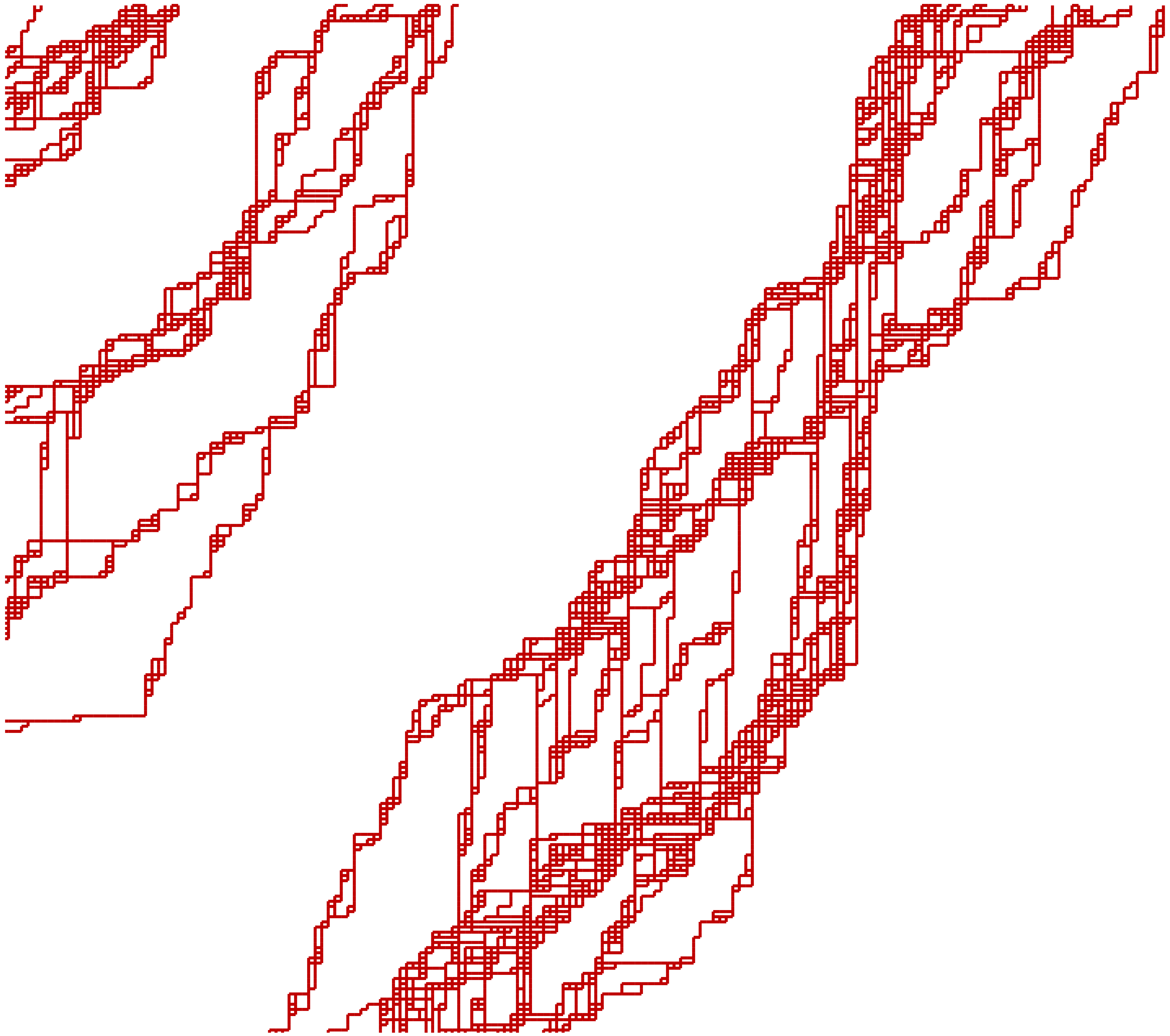
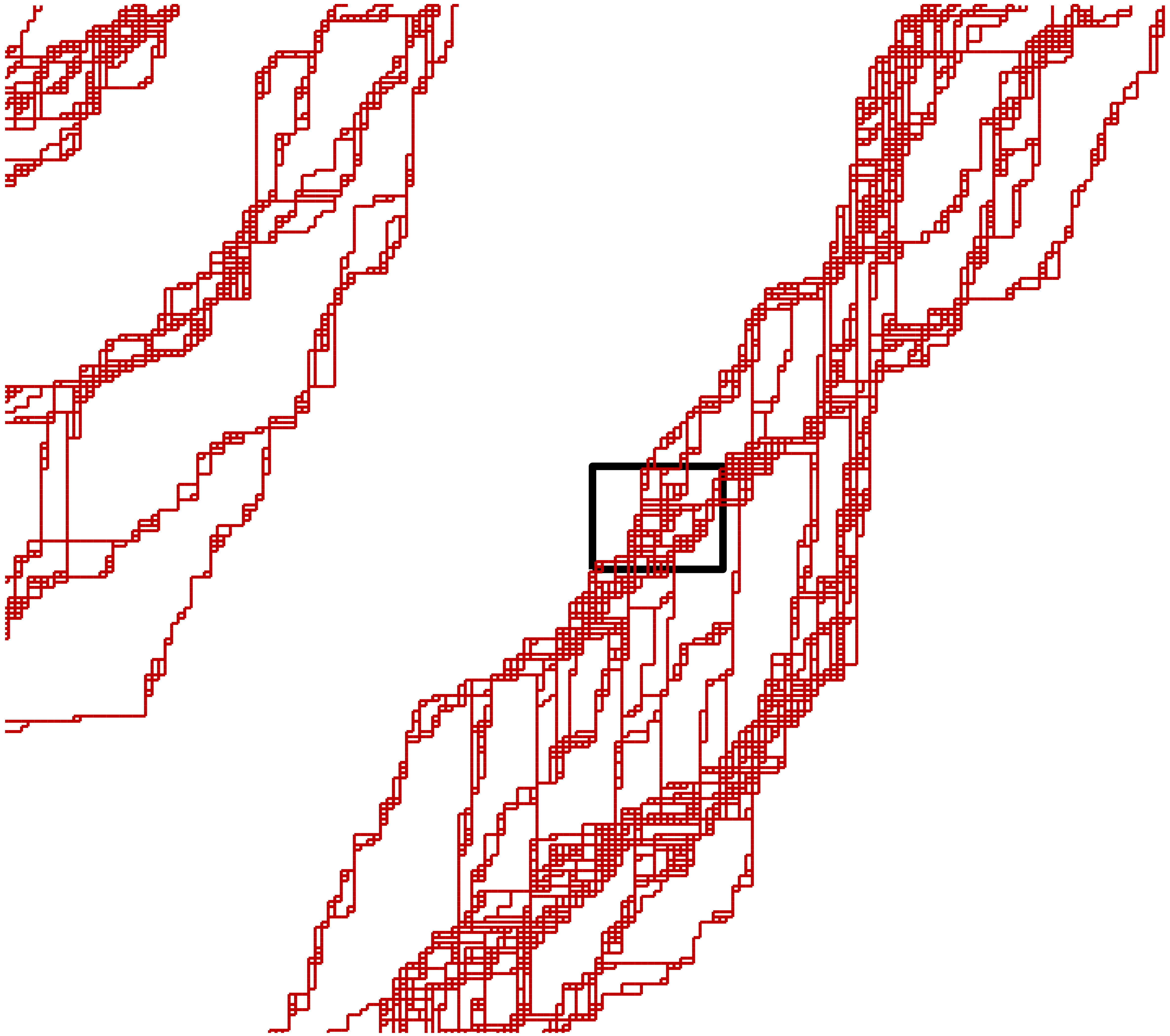
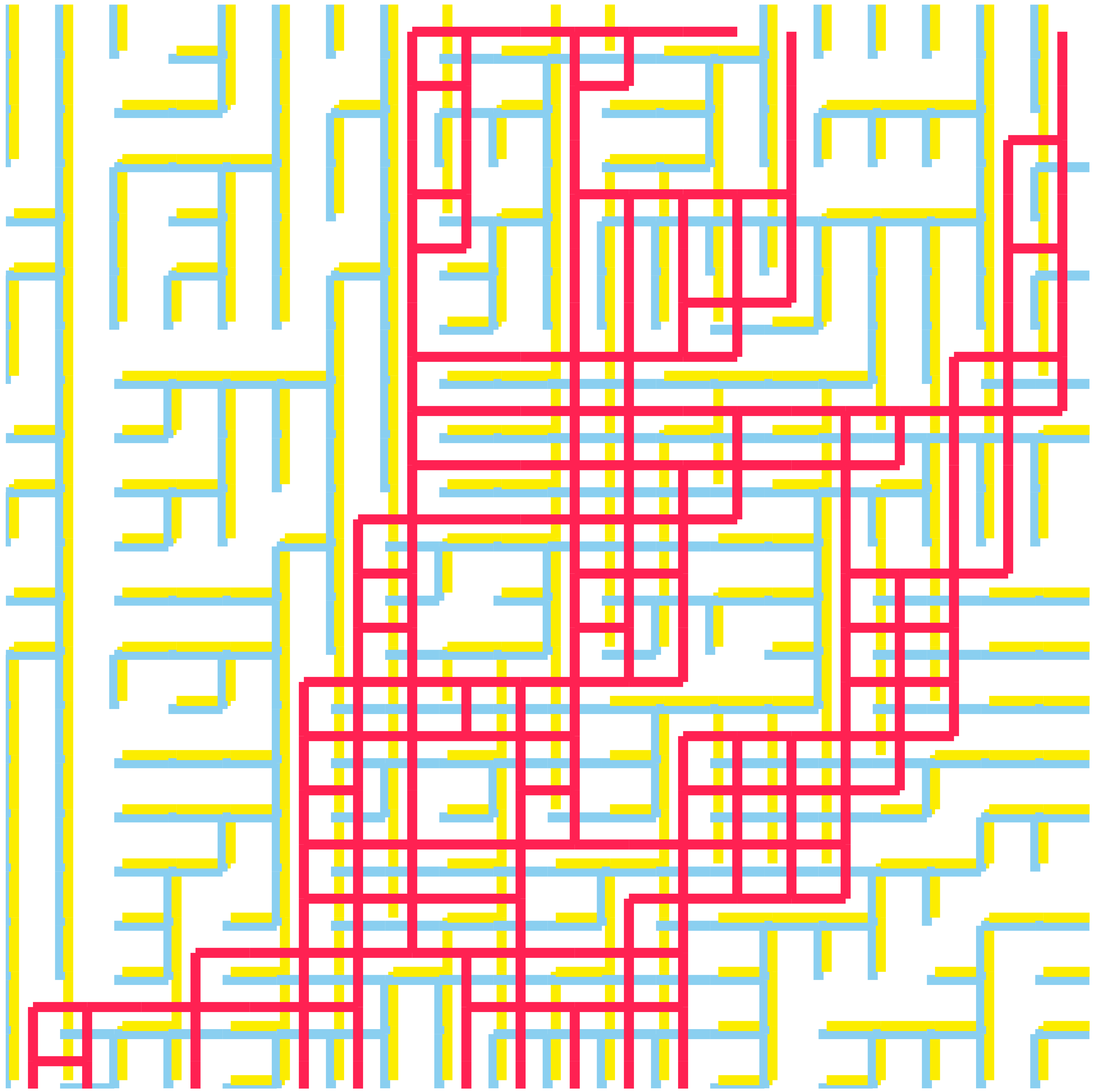
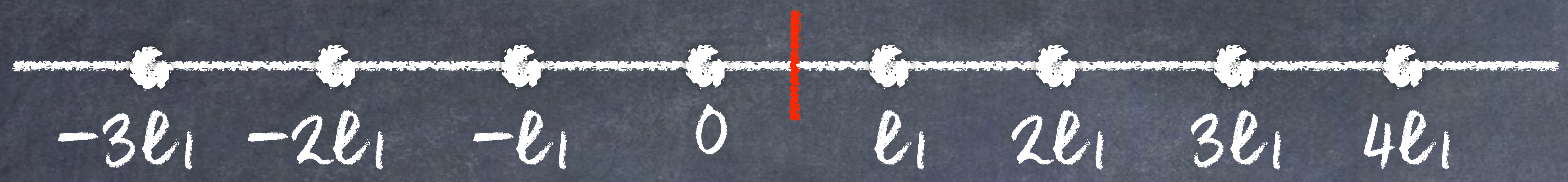
bi-infinite paths, coalescing both ↙ & ↗



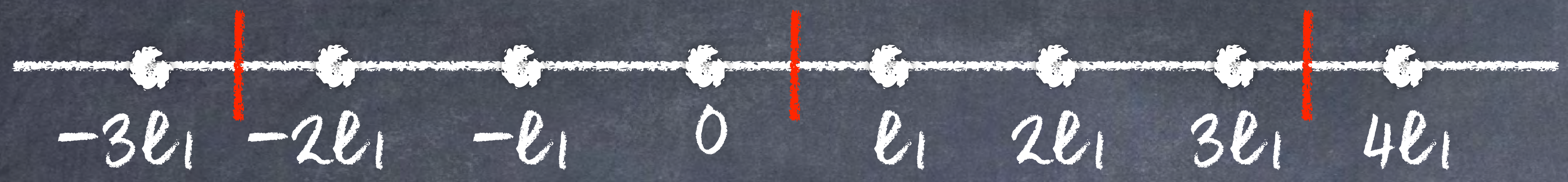




−3ℓ₁ −2ℓ₁ −ℓ₁ 0 ℓ₁ 2ℓ₁ 3ℓ₁ 4ℓ₁



Conditional (Palm sense) on $B^{\xi+}(0,\ell_1)\neq B^{\xi-}(0,\ell_1)$



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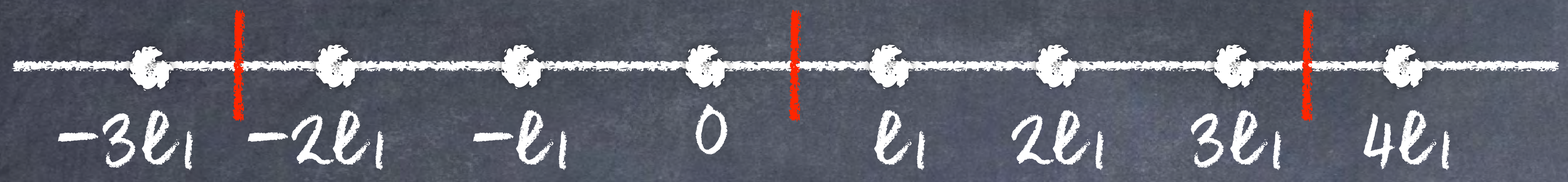

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the locations of the other jumps/crossings are $\{k: S_{2k}=0\}$ where S_n is a SSRW !