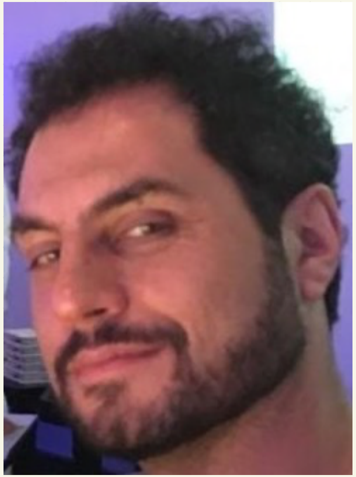
Stochastic homogenization of a class of nonconvex viscous Hamilton-Jacobi equations in one space dimension

Elena Kosygina
Baruch College and the CUNY Graduate Center
joint work with
Andrea Davini →
Sapienza University of Rome



ipam Workshop IV
May 18-22, 2020

We consider a viscous HJ equation

$$\partial_t u^\varepsilon = \varepsilon a\left(\frac{x}{\varepsilon},\omega\right)\partial^2_{xx} u^\varepsilon + G(\partial_x u^\varepsilon) + \beta V\left(\frac{x}{\varepsilon},\omega\right) \text{ in } (0,\infty)\times\mathbb{R};$$

$$u^\varepsilon\big|_{t=0} = g(x), \quad x\in\mathbb{R}$$

Random medium:

- $(\Omega, \mathcal{F}, P)$ probability space, Ω - Polish space, $\mathcal{F}$ - Borel σ-algebra on Ω, P invariant under the shifts by x and ergodic with respect to the shifts.
- $a, V : \mathbb{R}\times\Omega \longrightarrow [0,1]$ are stationary with respect to the shifts by $x\in\mathbb{R}$ and Lipschitz continuous in x.
- $\operatorname*{ess\,inf}_{x\in\mathbb{R}} V(x,\omega) = 0$ and $\operatorname*{ess\,sup}_{x\in\mathbb{R}} V(x,\omega) = 1$ a.s.

(an additional condition on V later)

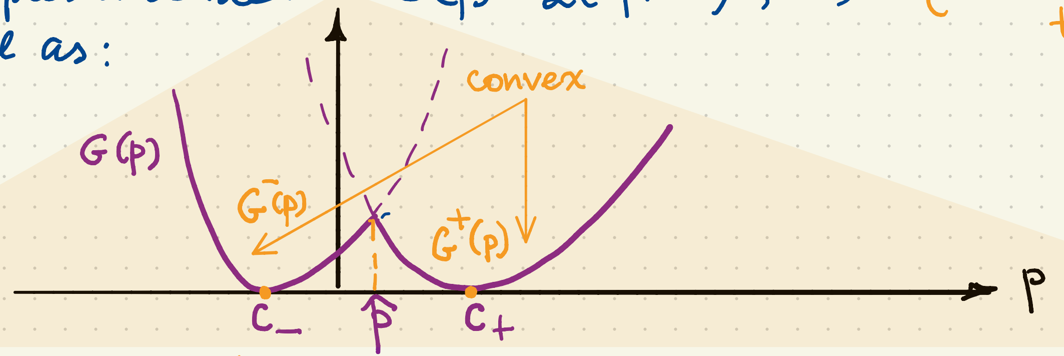
(1) $\partial_t u^\varepsilon = \varepsilon a(\frac{x}{\varepsilon},\omega)\partial^2_{xx}u^\varepsilon + G(\partial_x u^\varepsilon) + \beta V(\frac{x}{\varepsilon},\omega)$ in $(0,\infty)\times\mathbb{R}$

$u^\varepsilon|_{t=0} = g(x), \quad x\in\mathbb{R}.$

- $\beta \geq 0$ is a constant;
- G continuous, superlinear (precise conditions later)

Examples include: $G(p)=\frac{1}{2}(|p|-c)^2$, $c\geq 0$ (as in A. Yilmaz's talk)

as well as:



- "(1) homogenizes" if $\exists\,\overline{H}$ a continuous coercive function such that w.p. 1 $\forall g\in UC(\mathbb{R})$ $u^\varepsilon(t,x,\omega) \underset{loc}{\rightrightarrows} \bar{u}(t,x)$ as $\varepsilon\downarrow 0$

where $\bar{u}_t = \overline{H}(\partial_x\bar{u})$ in $(0,\infty)\times\mathbb{R}$, $\bar{u}(0,x)=g(x)$ in $\mathbb{R}$.

Remarks:

- It is known that (under conditions which we check) it is sufficient to show that for each $\theta \in \mathbb{R}$

$$u_\theta^\varepsilon(1,0,\omega) \xrightarrow[\varepsilon \to 0]{} \overline{H}(\theta) \quad \text{a.s.} \qquad (2)$$

where u_θ^ε is a solution of (1) with $g(x) = \theta x$.
Recall that $t\overline{H}(\theta) + \theta x$ solves $\partial_t \bar{u} = \overline{H}(\partial_x \bar{u})$ with $\bar{u}(0,x) = \theta x$.

- $u_\theta^\varepsilon(t,x,\omega) = \varepsilon u_\theta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega\right)$ where u solves

$$\begin{cases} \partial_t u_\theta = a(x,\omega)\,\partial_{xx} u_\theta + G(\partial_x u_\theta) + \beta V(x,\omega) \quad \text{in } (0,\infty)\times\mathbb{R} \\ u_\theta|_{t=0} = \theta x \end{cases}$$

$$\overline{H}(\theta) = \lim_{\varepsilon \to 0} \varepsilon u_\theta\left(\frac{1}{\varepsilon}, 0, \omega\right) = \lim_{t\to\infty} \frac{1}{t} u_\theta(t,0,\omega) \quad \text{if (2) holds}$$

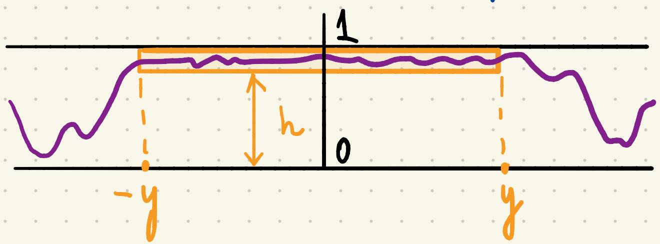
Notation: when the limit in (2) exists it defines a map: $\beta, G(\cdot) \longmapsto \overline{H}(\cdot)$; $\overline{H} = \mathcal{H}_\beta(G)$

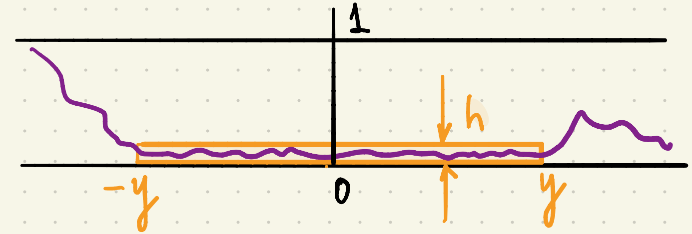
Conditions on V and $G(\cdot)$:

- Valley & hill : $\forall h \in (0,1)$ $\forall y > 0$ (V&H)

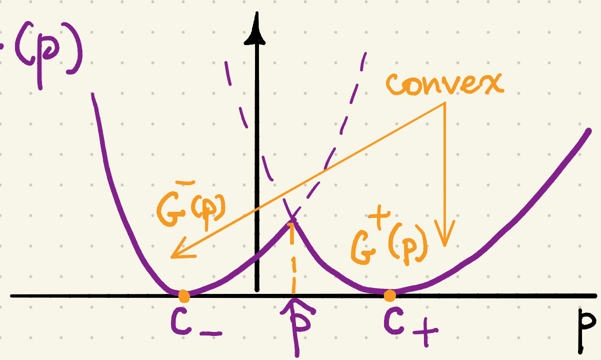
$$\mathbb{P}(\{[-y,y] \text{ is a valley (resp. hill)}\}) > 0$$

where an interval I is said to be an h-valley (resp. h-hill) if $V(x,\omega) \le h$ (resp. $V(x,\omega) \ge h$) $\forall x \in I$.





- $G(p)$



and $G^\pm$ satisfy for some $\alpha_0, \alpha_1 > 0$ and $\gamma > 1$

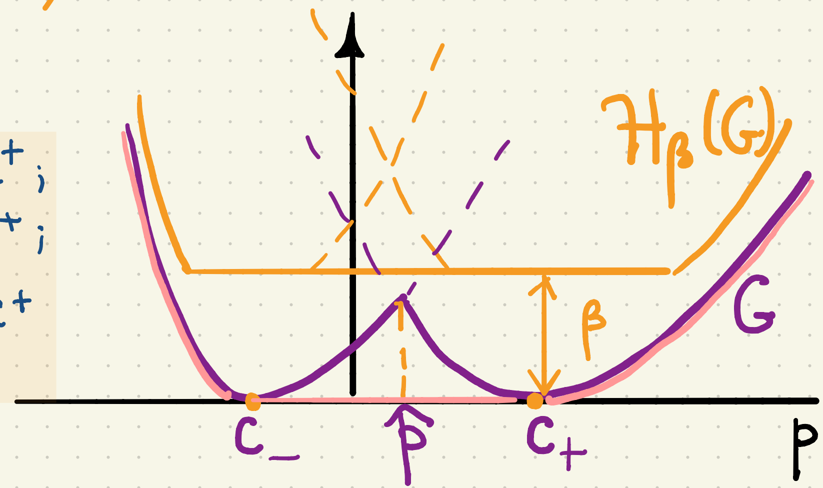
- $\alpha_0 |p|^\gamma - \frac{1}{\alpha_0} \le G^\pm(p) \le \alpha_1 (|p|^\gamma + 1)$
- $|G^\pm(p) - G^\pm(q)| \le \alpha_1 (|p| + |q| + 1)^{\gamma - 1} |p - q|$ $\forall p, q \in \mathbb{R}$

Theorem 1 (Davini, K, 2020+)

Under the above conditions on a, V, and G (1) homogenizes and the effective Hamiltonian $\mathcal{H}_\beta(G)$ is characterized as follows:

- if $\beta \geq G(\hat{p})$ (strong potential) then

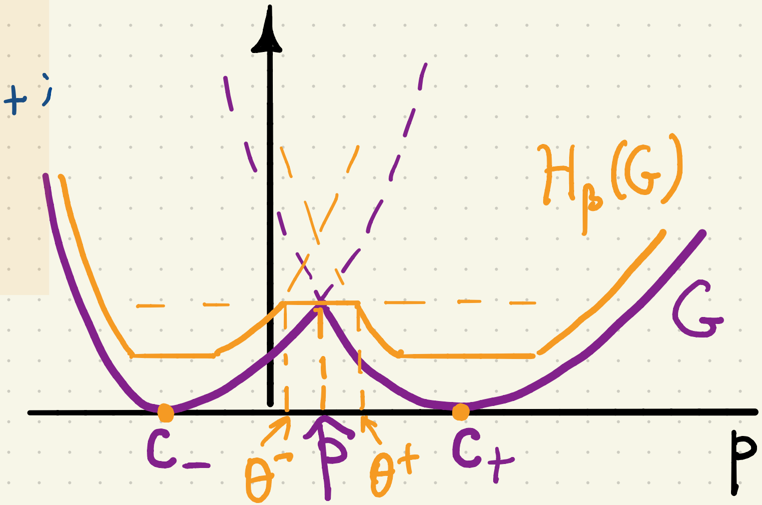
$$\mathcal{H}_\beta(G)(\theta) = \begin{cases} \mathcal{H}_\beta(G^+)(\theta) & \text{if } \theta > c^+; \\ \beta & \text{if } c^- \leq \theta \leq c^+; \\ \mathcal{H}_\beta(G^-)(\theta) & \text{if } \theta < c^+ \end{cases}$$



$$= \operatorname{conv}\left(\mathcal{H}_\beta(G^+) \wedge \mathcal{H}_\beta(G^-)\right) = \mathcal{H}_\beta\left(\operatorname{conv}(G^+ \wedge G^-)\right)$$

- if $\beta < G(\hat{p})$ (weak potential) then

$$\mathcal{H}_\beta(G)(\theta) = \begin{cases} \mathcal{H}_\beta(G^+) & \text{if } \theta > \theta_+ ; \\ G(\hat{p}) & \text{if } \theta_- \leq \theta \leq \theta_+ ; \\ \mathcal{H}_\beta(G^-) & \text{if } \theta < \theta_- , \end{cases}$$

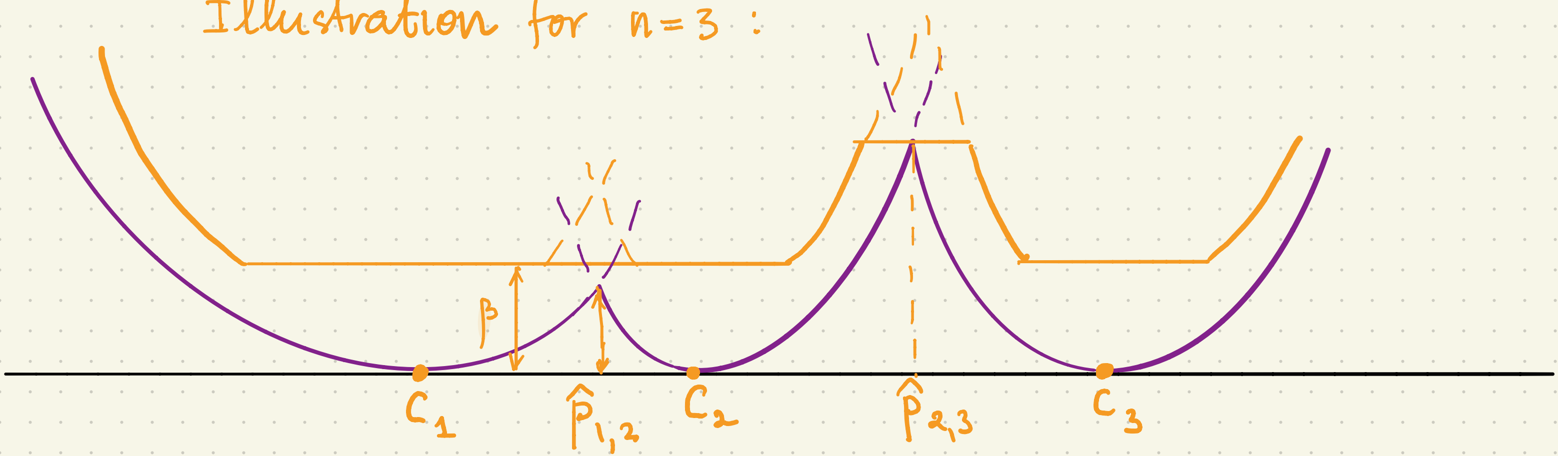


where θ_+ (resp. θ_-) is the unique solution in $[\hat{p}, c_+]$ (resp. $[c_-, \hat{p}]$) of the equation $H_\beta(G^+) = G(\hat{p})$ (resp $H_\beta(G^-) = G(\hat{p})$)

Theorem 2 (Davini, K, 2020+)

The last theorem generalizes to the case when G is a minimum of $n > 2$ convex functions with the same absolute minimum.

Illustration for $n=3$:



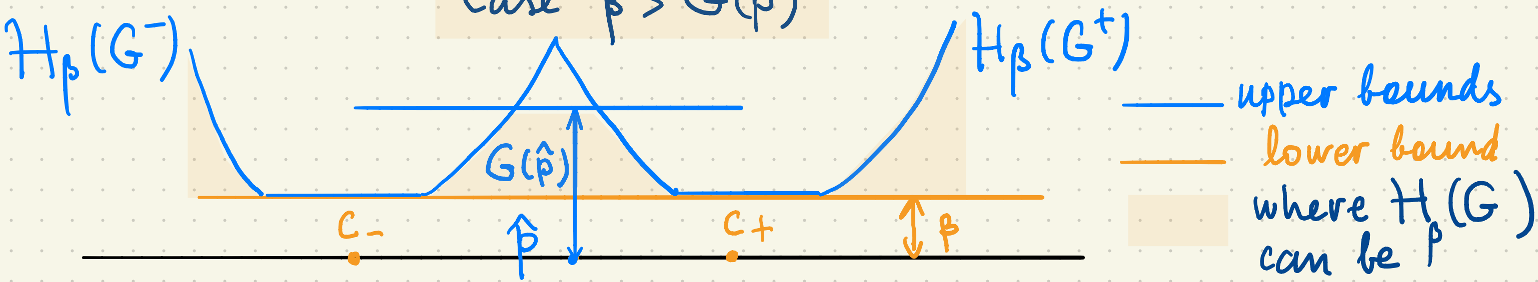
About proofs: for $\beta \geq 0$, $G(\cdot)$ let

$$H^U_\beta(G)(\theta) = \limsup_{\varepsilon \to 0} \varepsilon\, u_\theta\left(\tfrac{1}{\varepsilon}, 0, \omega\right); \quad H^L_\beta(G)(\theta) = \liminf_{\varepsilon \to 0} \varepsilon\, u_\theta\left(\tfrac{1}{\varepsilon}, 0, \omega\right)$$

We need to show that $H^U_\beta(G) \equiv H^L_\beta(G)$.

- $H^U_\beta(G) \leq H_\beta(G^+) \wedge H_\beta(G^-)$; – uniform upper bound
- Show that $\beta \leq H^L_\beta(G)(\theta)\ \forall \theta \in \mathbb{R}$; – uniform lower bound (uses the hill condition)
- $H_\beta(G^\pm)(c_\pm) \leq \beta$; simple comparison: $\beta t + c_\pm x$ is a supersolution since $G^\pm(c_\pm) = 0$
- $H^U_\beta(G)(\theta) \leq \beta \vee G(\hat{p})$ for $c_- \leq \theta \leq c_+$ – uses the valley condition

Case $\beta > G(\hat{p})$

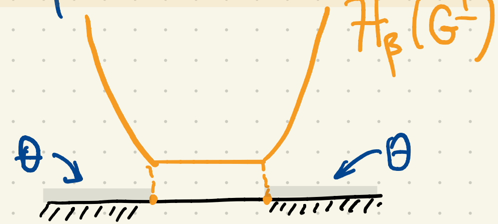


What happens in "shaded areas"?

Correctors for convex "pieces". Consider $G^{\pm}$ (convex) and check that the assumptions of Cardaliaguet and Souganidis (2017) hold. Argue that $\forall \theta$ such that $\mathcal{H}_\beta(G^{\pm})(\theta) > \beta$ and $\forall \omega \in \Omega_\theta$ of full measure there is a unique sublinear solution v_θ (corrector) of

$$(3) \quad a(x,\omega) v_\theta'' + G^{\pm}(v_\theta' + \theta) + \beta V(x,\omega) = \mathcal{H}_\beta(G^{\pm}) \text{ in } \mathbb{R}$$

satisfying $v_\theta(0,\omega) = 0 \ \forall \omega \in \Omega_\theta$.

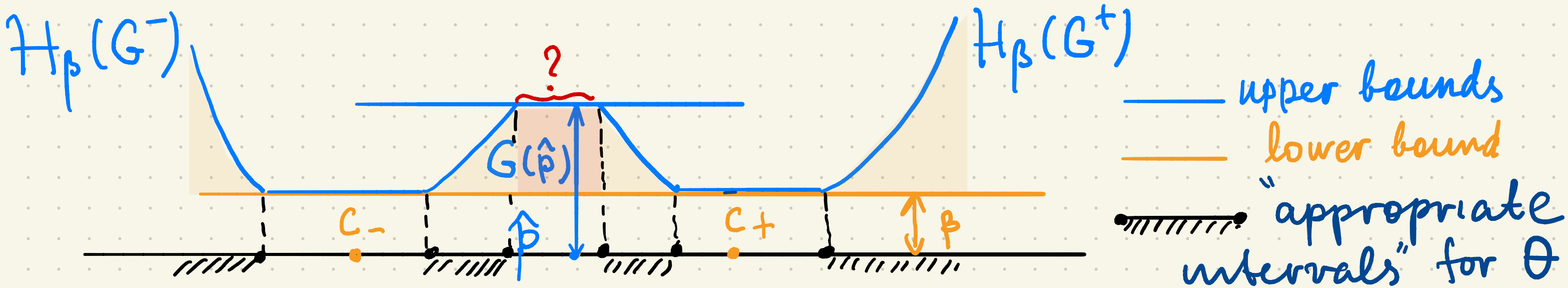


Moreover,

- Ω_θ is shift invariant;
- v_θ is $\varkappa(\theta)$-Lip, where $\varkappa$ is locally bounded;
- v_θ' is stationary (not used)

- We establish bounds on v_θ' which allow us to conclude that the same v_θ are correctors for G when θ is in "appropriate intervals".

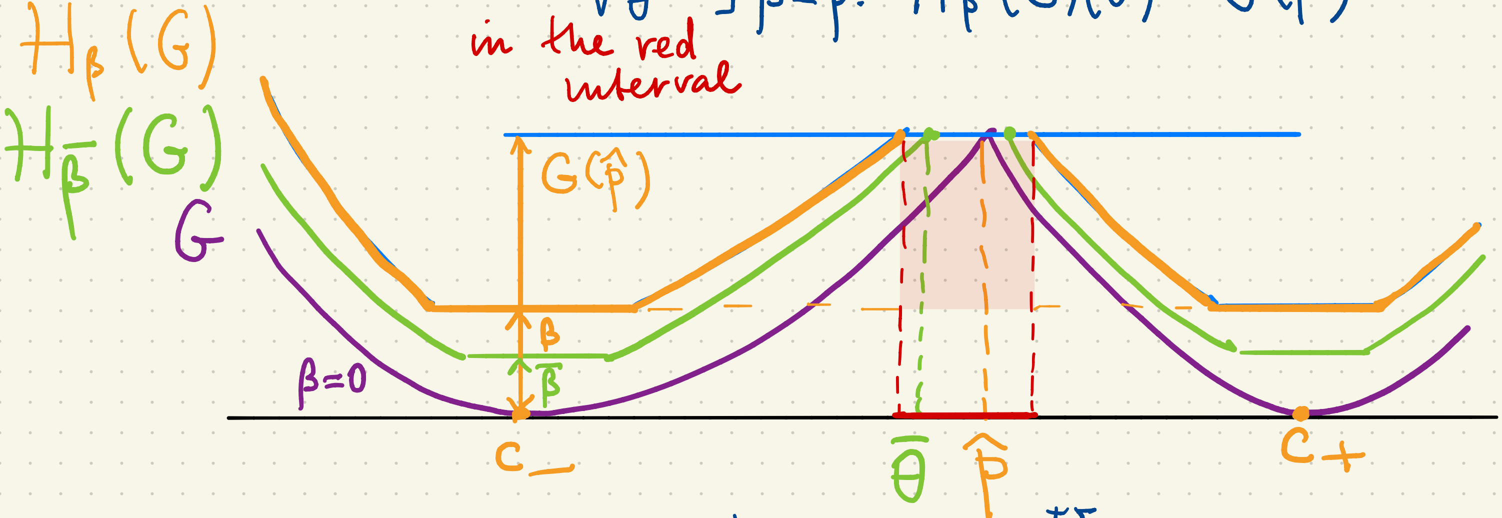
Case $\beta > G(\hat{p})$



- We conclude that on these "appropriate intervals" $H^U_\beta(G) = H^L_\beta(G) = H_\beta(G)$ and this common value coincides with $H_\beta(G^-)(\theta)$ if $\theta < \hat{p}$ and with $H_\beta(G^+)(\theta)$ if $\theta > \hat{p}$

- The "final cut" (same as in Yilmaz, Zeitouni (2019))

$$\forall \bar\theta \quad \exists \bar\beta \le \beta : \quad \mathcal{H}_{\bar\beta}(G)(\bar\theta) = G(\hat p)$$



$$G(\hat p) = \mathcal{H}_{\bar\beta}(G)(\bar\theta) \le \mathcal{H}^{L}_{\beta}(G)(\bar\theta) \le \mathcal{H}^{U}_{\beta}(G)(\bar\theta) \le G(\hat p)$$

(monotonicity in β) (upper bound for $c_- \le \bar\theta \le c_+$)

Conclusion: for θ in the red interval $\mathcal{H}_\beta(G)(\theta) = G(\hat p)$ ∎

- Broader view

$$\partial_t u^\varepsilon = \varepsilon \operatorname{tr}\left(A\left(\tfrac{x}{\varepsilon},\omega\right) D_x^2 u^\varepsilon\right) + H\left(D_x u, \tfrac{x}{\varepsilon}, \omega\right) \text{ in } [0,\infty)\times\mathbb{R}^d$$
$$u^\varepsilon\big|_{t=0} = g(x), \quad x\in\mathbb{R}^d \qquad (4)$$

When $H(p,x,\omega)$ is not convex in p there are counterexamples to homogenization for $d\geq 2$.

- Ziliotto (2017): $d=2$, $A\equiv 0$, $H(p,x,\omega) = G(p) + V(x,\omega)$
Generalization by Feldman, Souganidis (2017): $d\geq 2$.
If $G(p)$ has a strict local saddle point then it is possible to construct a random environment (very slowly mixing) and a potential V so that homogenization fails.

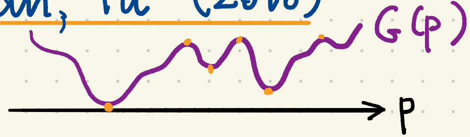
- Feldman, Fermanian, Ziliotto (2019+): $A\equiv I$, $d=2$.
$H(p,x,\omega) = G(p) + V(x,\omega)$; extension to the viscous case.

Stochastic homogenization for nonconvex Hamiltonians

Inviscid case ($A \equiv 0$)

$d=1$: Davini, Siconolfi (2009)
level set convex $H(p,x,\omega)$

Armstrong, Tran, Yu (2016)
$G(p) + V(x,\omega)$



Gao (2019), extension to $H(p,x,\omega)$

$d \geq 1$: Armstrong, Souganidis (2013)
level set convex $H(p,x,\omega)$

Viscous case ($A \geq 0$)

$d=1$ Davini, K (2017)
"pinned" Hamiltonians, such as
$\sigma(x,\omega)|p|^\gamma - b(x,\omega)|p|$, $\gamma > 1$
$\alpha_0 \leq \sigma, b \leq \frac{1}{\alpha_0}$, $\alpha_0 \in (0,1)$

Yilmaz, Zeitouni (2017); $A \equiv 1$
discrete case

$\frac{1}{2}p^2 - c|p| + \beta V(x,\omega)$ under V & H condition

K, Yilmaz, Zeitouni (2020); $A \equiv 1$
continuous case.

($d \geq 1$ will be continued on the next page)

- Open probem for $d=1$: prove homogenization for a sufficiently general class of non-convex H in the viscous case
- Homogenization for level set convex H, $d \geq 1$, in the viscous case?

Stochastic homogenization for nonconvex Hamiltonians

Inviscid case $(A \equiv 0)$

$d \geq 1$ Armstrong, Tran, Yu (2015)

$G(p) = (|p|^2 - 1)^2$

Feldman, Souganidis (2017)

H with strictly star-shaped sub-level sets.

Qian, Tran, Yu (2017)

$G(p) = |F(p)|$ where F is even, coercive, level set convex with $\min_{\mathbb{R}^d} F < 0$.

Viscous case $(A \geq 0)$

$d \geq 1$ Armstrong, Cardaliaguet (2018); finite range of dependence

$\exists \gamma > 1 : H(te, x, \omega) \equiv t^\gamma H(e, x, \omega) \ \forall t \geq 0$

$\forall \|e\| = 1$ (homogeneity)

Cardaliaguet, Souganidis (2017)

$(A(p,x,\omega), H(p,x,\omega))$ with radially symmetric law;

A is 0-homogeneous in p

$0 \leq H(\lambda p, x, \omega) \leq \lambda H(p, x, \omega)$