Mean field limits of weakly interacting diffusions and applications

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IPAM Online Workshop: Stochastic Analysis Related to Hamilton-Jacobi PDEs

22 May, 2020

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- *Convergence analysis of stochastic mirror descent: Lyapunov functions, spectral gaps and asymptotic variance* (with A. Borovykh, N. Kantas, P. Parpas). Submitted, February 2020.
- *Well-Posedness and Equilibrium Behaviour of Overdamped Dynamics Density Functional Theory* (with B.D. Goddard and R.D. Mills-Williams). Submitted, February 2020.
- *On the diffusive-mean field limit for weakly interacting diffusions exhibiting phase transitions* (with M.G. Delgadino and R. Gvalani). Submitted, January 2020.
- *A proof of the mean-field limit for λ-convex potentials by Γ-Convergence* (with J.A. Carrillo and M. Delgadino), Submitted, May 2019.
- *The sharp, the flat and the shallow: Can weakly interacting agents learn to escape bad minima?* (with N. Kantas and P. Parpas). Submitted, April 2019.
- *The Desai-Zwanzig mean field model with colored noise* (with S.N. Gomes and U. Vaes). Submitted, April (2019).
- *Phase Transitions for the Desai-Zwanzig model in multiwell and random energy landscapes* (with S.N. Gomes, S. Kalliadasis, G.A. Pavliotis, P. Yatsyshin). (Phys Rev E) (2019).
- *Long time behaviour and phase transitions for the McKean-Vlasov equation on the torus* (with R. Gvalani, J.A. Carrillo, A. Schlichting) Archive for Rational Mechanics and Analysis, 2020.
- *Mean field limits for interacting diffusions in a two-scale potential* (S.N. Gomes and G.A. Pavliotis), J. Nonlinear Sci. 28(3), pp. 905-941, (2018).
- *Mean field limits for non-Markovian interacting particles: convergence to equilibrium, GENERIC formalism, asymptotic limits and phase transitions* (with M. H. Duong). Comm. Math. Sci. (2018). `https://arxiv.org/abs/1805.04959`

Research Funded by the EPSRC, Grants EP/P031587/1, EP/L020564/1, EP/L024926/1, and by a JP Morgan Faculty award

EPSRC

- Study the mean field limit of weakly interacting diffusions:
  - The Dasai-Zwanzing model in a 2-scale potential:

$$dX_i = -V'(X_i, X_i/\varepsilon)\,dt - \theta\left(X_t^i - \frac{1}{N}\sum_{j=1}^{N} X_t^j\right)dt + \sqrt{2\beta^{-1}}\,dW_t^i.$$

  - Noisy Kuramoto oscillators:

$$\dot{x}_i = -\frac{1}{N}\sum_{j=1}^{N}\sin(x_i - x_j) + \sqrt{2\beta^{-1}}\dot{W}_i.$$

  - Models for opinion formation:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a_{ij}(|x_i - x_j|)(x_i - x_j) + \sqrt{2\beta^{-1}}\dot{W}_i.$$

- Study the mean field limits of weakly interacting diffusions:
  - Interacting non-Markovian Langevin dynamics (with H. Duong)

$$\ddot{q}_i = -\frac{\partial V}{\partial q_i} - \frac{1}{N}\sum_{j=1}^{N} U'(q_i - q_j) - \sum_{j=1}^{N}\int_0^t \gamma_{ij}(t-s)\,\dot{q}_j(s)\,ds + F_i(t), \quad i = 1, \dots N, \tag{1}$$

  - where $F(t) = (F_1(t), \dots F_N(t))$ is a mean zero, Gaussian, stationary process with autocorrelation function $E(F_i(t)\,F_j(s)) = \beta^{-1}\gamma_{ij}(t-s)$.
  - Langevin dynamics driven by colored noise (with S. Gomes and U. Vaes)

$$\dot{x}_i = -V'(x_i) - \frac{1}{N}\sum_{j=1}^{N} W'(x_i - x_j) + \eta_i, \tag{2a}$$

$$\dot{\eta}_i = -\eta_i + \sqrt{2\beta^{-1}}\,\dot{B}_i \tag{2b}$$

- applications: Models for systemic risk (Garnier, Papanicolaou...), clustering in the Hegselmann-Krause model (Chazelle, E, ...)

- These models exhibit phase transitions.
- Goal: Characterize phase transitions, estimate basins of attraction.
- Study the effect of colored, non-Gaussian, multiplicative noise.
- Solve the mean field PDE using spectral methods (with S. Gomes and U. Vaes).
- Study fluctuations around the mean field limit, in particular past the phase transition (with R. Gvalani and M. Delgadino).
- Develop optimal control strategies for the mean field dynamics–application to models for opinion formation (with B. Goddard, D. Kalise and U. Vaes)
- Applications: algorithms for sampling and optimization (with A. Boroykh N. Kantas and P. Parpas).

Elastic Stochastic Gradient Descent (Zhang et al 2015, Chaudhari et al (2016))

- Our goal is to minimize the loss function $V(x)$.
- Distributed training of deep neural networks: minimize the communication overhead between a set of workers that together optimize replicated copies of the original function V.
- Consider N distinct workers $(x_1, \dots, x_N)$ and define the average $\overline{x} = \frac{1}{N}\sum_{j=1}^{N} x_j$.
- Define the modified loss function (γ is a regularization parameter)

$$\min_x \frac{1}{N}\sum_{j=1}^{N}\Big(V(x_i) + \frac{1}{2\gamma}|x_i - \overline{x}|^2\Big).$$

- The distributed optimization algorithm corresponds to the following system of interacting agents.

$$dx_i(t) = -\nabla V(x_i(t))\,dt - \frac{1}{\gamma}\big(x_i - \overline{x}\big) + \sqrt{2\beta^{-1}}\,dW_i.$$

- Consider a system of interacting diffusions in a bistable potential:

$$dX_t^i = \left(-V'(X_t^i) - \theta\left(X_t^i - \frac{1}{N}\sum_{j=1}^{N} X_t^j\right)\right) dt + \sqrt{2\beta^{-1}}\, dB_t^i. \tag{3}$$

- The total energy (Hamiltonian) is

$$W_N(\mathbf{X}) = \sum_{\ell=1}^{N} V(X^\ell) + \frac{\theta}{4N}\sum_{n=1}^{N}\sum_{\ell=1}^{N}(X^n - X^\ell)^2. \tag{4}$$

- We can pass rigorously to the mean field limit as $N \to \infty$ using, for example, martingale techniques, (Dawson 1983, Gartner 1988, Oelschlager 1984).
- Formally, using the law of large numbers we obtain the McKean SDE

$$dX_t = -V'(X_t)\, dt - \theta(X_t - \mathbb{E}X_t)\, dt + \sqrt{2\beta^{-1}}\, dB_t. \tag{5}$$

- The Fokker-Planck equation corresponding to this SDE is the McKean-Vlasov equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(V'(x)p + \theta\left(x - \int_{\mathbb{R}} xp(x,t)\,dx\right)p + \beta^{-1}\frac{\partial p}{\partial x}\right). \tag{6}$$

- The McKean-Vlasov equation is a gradient flow, with respect to the Wasserstein metric, for the free energy functional

$$\mathcal{F}[\rho] = \beta^{-1}\int \rho\ln\rho\,dx + \int V\rho\,dx + \frac{\theta}{2}\int\int F(x-y)\rho(x)\rho(y)\,dxdy, \tag{7}$$

with $F(x) = \frac{1}{2}x^2$.

Journal of Statistical Physics, Vol. 31, No. 1, 1983

Critical Dynamics and Fluctuations for a Mean-Field Model of Cooperative Behavior

Donald A. Dawson[1,2]

Received September 20, 1982

The main objective of this paper is to examine in some detail the dynamics and fluctuations in the critical situation for a simple model exhibiting bistable macroscopic behavior. The model under consideration is a dynamic model of a collection of anharmonic oscillators in a two-well potential together with an attractive mean-field interaction. The system is studied in the limit as the number of oscillators goes to infinity. The limit is described by a nonlinear partial differential equation and the existence of a phase transition for this limiting system is established. The main result deals with the fluctuations at the critical point in the limit as the number of oscillators goes to infinity. It is established that these fluctuations are non-Gaussian and occur at a time scale slower than the noncritical fluctuations. The method used is based on the perturbation theory for Markov processes developed by Papanicolaou, Stroock, and Varadhan adapted to the context of probability-measure-valued processes.

PHYSICAL REVIEW A VOLUME 36, NUMBER 5 SEPTEMBER 1, 1987

Dynamical behavior of stochastic systems of infinitely many coupled nonlinear oscillators exhibiting phase transitions of mean-field type: H theorem on asymptotic approach to equilibrium and critical slowing down of order-parameter fluctuations

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(Received 29 September 1986)

It is shown that statistical-mechanical properties as well as irreversible phenomena of stochastic systems, which consist of infinitely many coupled nonlinear oscillators and are capable of exhibiting phase transitions of mean-field type, can be successfully explored on the basis of nonlinear Fokker-Planck equations, which are essentially nonlinear in unknown distribution functions. Results of two kinds of approaches to the study of their dynamical behavior are presented. Firstly, a problem of asymptotic approaches to stationary states of the infinite systems is treated. A method of Lyapunov functional is employed to conduct a global as well as a local stability analysis of the systems. By constructing an H functional for the nonlinear Fokker-Planck equation, an H theorem is proved, ensuring that the Helmholtz free energy for a nonequilibrium state of the system decreases monotonically until a stationary state is approached. Calculations of the second-order variation of the H functional around a stationary state yield a stability criterion for bifurcating solutions of the nonlinear Fokker-Planck equation, in terms of an inequality involving the second moment of the stationary distribution function. Secondly, the behavior of critical dynamics is studied within the framework of linear-response theory. Generalized dynamical susceptibilities are calculated rigorously from linear responses of the order parameter to externally driven fields by linearizing the nonlinear Fokker-Planck equation. Correlation functions, together with spectra of the fluctuations of the order parameter of the system, are also obtained by use of the fluctuation-dissipation theorem for stochastic systems. A critical slowing down is shown to occur in the form of the divergence of relaxation time for the fluctuations, in accordance with the divergence of the static susceptibility, as a phase transition point is approached.

I. INTRODUCTION

The study of dynamical behavior of systems exhibiting thermodynamic phase transitions has been of considerable interest for many years.[1-3] It is well known that in a thermodynamic system undergoing phase transitions critical anomaly such as critical slowing down is generally expected to occur at its transition points. Recently the con-

tained by omitting the random force $f(t)$ in Eq. (1.1) can exhibit a bifurcation at $\gamma=0$, the stochastic differential equation (1.1) has nothing to do with bifurcations nor phase transitions in that a stationary distribution for the random variable x is always uniquely determined irrespective of the values of γ and σ. This is because the corresponding linear Fokker-Planck equation

- The finite dimensional dynamics (3) is reversible with respect to the Gibbs measure

$$\mu_N(dx) = \frac{1}{Z_N} e^{-\beta W_N(x^1,\dots x^N)}\, dx^1 \dots dx^N, \quad Z_N = \int_{\mathbb{R}^N} e^{-\beta W_N(x^1,\dots x^N)}\, dx^1 \dots dx \tag{8}$$

- where $W_N(\cdot)$ is given by (4).
- the McKean dynamics (5) can have more than one invariant measures, for nonconvex confining potentials and at sufficiently low temperatures (Dawson1983, Tamura 1984, Shiino 1987, Tugaut 2014).
- The density of the invariant measure(s) for the McKean dynamics (5) satisfies the stationary nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial x}\left(V'(x)p_\infty + \theta\left(x - \int_{\mathbb{R}} x p_\infty(x)\, dx\right)p_\infty + \beta^{-1}\frac{\partial p_\infty}{\partial x}\right) = 0. \tag{9}$$

- For the quadratic interaction potential a one-parameter family of solutions to the stationary McKean-Vlasov equation (9) can be obtained:

$$
\begin{aligned}
p_\infty(x;\theta,\beta,m) &= \frac{1}{Z(\theta,\beta;m)} e^{-\beta\left(V(x)+\theta\left(\frac{1}{2}x^2 - xm\right)\right)}, && \text{(10a)} \\
Z(\theta,\beta;m) &= \int_{\mathbb{R}} e^{-\beta\left(V(x)+\theta\left(\frac{1}{2}x^2 - xm\right)\right)}\, dx. && \text{(10b)}
\end{aligned}
$$

- These solutions are subject, to the constraint that they provide us with the correct formula for the first moment:

$$
m = \int_{\mathbb{R}} x p_\infty(x;\theta,\beta,m)\, dx =: R(m;\theta,\beta). \tag{11}
$$

- This is the **selfconsistency** equation:

$$m = R(m; \theta, \beta).$$

- The critical temperature can be calculated from

$$\mathrm{Var}_{p_\infty}(x)\Big|_{m=0} = \frac{1}{\beta\theta}. \tag{12}$$

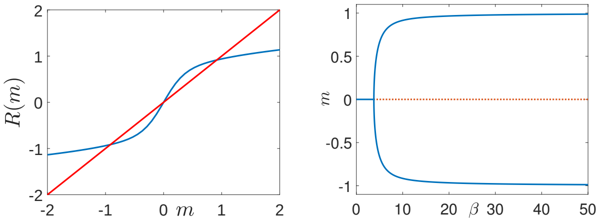

Figure: Plot of $R(m;\theta,\beta)$ and of the straight line $y = x$ for $\theta = 0.5$, $\beta = 10$, and bifurcation diagram of m as a function of β for $\theta = 0.5$ for the bistable potential $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and interaction potential $F(x) = \frac{x^2}{2}$.

The McKean-Vlasov equation on the torus

The McKean–Vlasov equation – Setup

Nonlocal parabolic PDE

$$\frac{\partial \varrho}{\partial t} = \beta^{-1}\Delta\varrho + \kappa\nabla\cdot(\varrho\nabla W \star \varrho) \qquad \text{in } \mathbb{T}_L^d \times (0,T]$$

with periodic boundary conditions, $\varrho(\cdot,0) = \varrho_0 \in \mathcal{P}(\mathbb{T}_L^d)$, $\mathbb{T}_L^d \hat{=} \left(-\frac{L}{2}, \frac{L}{2}\right)^d$

- $\varrho(\cdot,t) \in \mathcal{P}(\mathbb{T}_L^d)$ probability density of particles
- W coordinate-wise even interaction potential
- $\beta > 0$ inverse temperature (fixed)
- $\kappa > 0$ interaction strength (parameter)

Example: The noisy Kuramoto model

The Kuramoto model: $W(x) = -\sqrt{\frac{2}{L}}\cos\left(2\pi k \frac{x}{L}\right), k \in \mathbb{Z}$

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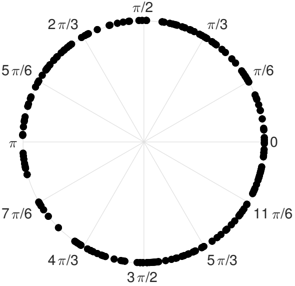
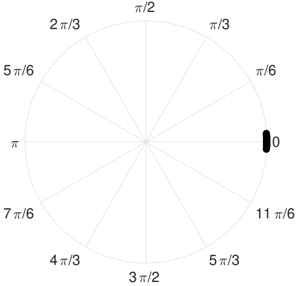


$\kappa < \kappa_c$, no phase locking



$\kappa > \kappa_c$, phase locking

Fourier representation $\widetilde{f}(k) = \langle f, w_k \rangle_{L^2(\mathbb{T}_L)}$ with $k \in \mathbb{Z}^d$

- A function $W \in L^2(\mathbb{T}_L^d)$ is H-**stable**, $W \in \mathbb{H}_s$, if

$$\widetilde{W}(k) = \langle W, w_k \rangle \geq 0, \quad \forall k \in \mathbb{Z}^d \ ,$$

- Decomposition of potential W into H-stable and H-unstable part

$$W_{\mathrm{s}}(x) = \sum_{k \in \mathbb{N}^d} \left(\langle W, w_k \rangle \right)_+ w_k(x) \qquad \text{and} \qquad W_{\mathrm{u}}(x) = W(x) - W_s(x) \ .$$

Functionals for stationary states

- Free energy functional $\mathscr{F}_\kappa$: Driving the W_2-gradient flow

$$\mathscr{F}_\kappa(\varrho) = \beta^{-1}\int_{\mathbb{T}_L^d} \varrho \log \varrho \,\mathrm{d}x + \frac{\kappa}{2}\iint_{\mathbb{T}_L^d\times\mathbb{T}_L^d} W(x-y)\varrho(x)\varrho(y)\,\mathrm{d}x\,\mathrm{d}y\ .$$

- Dissipation: $\mathscr{F}_\kappa$ is Lyapunov-function

$$\mathcal{J}_\kappa(\varrho) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\kappa(\varrho) = \int_{\mathbb{T}_L^d}\left|\nabla \log \frac{\varrho}{e^{-\beta\kappa W\star\varrho}}\right|^2 \varrho\,\mathrm{d}x\ ,$$

- Kirkwood-Monroe fixed point mapping

$$F_\kappa(\varrho) = \varrho - \mathcal{T}\varrho = \varrho - \frac{1}{Z(\varrho,\kappa)}e^{-\beta\kappa W\star\varrho}\ , \quad \text{with} \quad Z(\varrho,\kappa) = \int_{\mathbb{T}_L^d} e^{-\beta\kappa W\star\varrho}\,\mathrm{d}x\ .$$

Characterization of stationary states: The following are equivalent

- ϱ is a stationary state: $\beta^{-1}\Delta\varrho + \kappa\nabla\cdot(\varrho\nabla W \star \varrho) = 0$.
- ϱ is a root of $F_\kappa(\varrho)$.
- ϱ is a global minimizer of $\mathcal{J}_\kappa(\varrho)$.
- ϱ is a critical point of $\mathscr{F}_\kappa(\varrho)$.

$\Rightarrow \varrho_\infty \equiv L^{-d}$ is a stationary state for all $\kappa > 0$.

Existence/Uniqueness of Solutions

Theorem

Under appropriate assumptions on the potential, for $\varrho_0 \in H^{3+d}(U) \cap \mathcal{P}_{\mathrm{ac}}(U)$*, there exists a unique classical solution* ϱ *of the McKean-Vlasov equation such that* $\varrho(\cdot, t) \in \mathcal{P}_{\mathrm{ac}}(U) \cap C^2(\overline{U})$ *for all* $t > 0$*. Additionally,* $\varrho(\cdot, t)$ *is strictly positive and has finite entropy, i.e,* $\varrho(\cdot, t) > 0$ *and* $S(\varrho(\cdot, t)) < \infty$*, for all* $t > 0$*.*

Theorem

(Convergence to equilibrium) Let $\varrho(x,t)$ be a classical solution of the Mckean–Vlasov equation with smooth initial data and smooth, even, interaction potential W. Then we have:

1. *If $0 < \kappa < \frac{2\pi}{3\beta L \|\nabla W\|_\infty}$, then $\left\|\varrho - \frac{1}{L}\right\|_2 \to 0$, exponentially, as $t \to \infty$,*
2. *If $\hat{W}(k) \geq 0$ for all $k \in \mathbb{Z}$ or $0 < \kappa < \frac{2\pi^2}{\beta L^2 \|\Delta W\|_\infty}$, then $\mathcal{H}\left(\varrho | \frac{1}{L}\right) \to 0$, exponentially, as $t \to \infty$,*

where $\hat{W}(k)$ represents the Fourier transform and $\mathcal{H}\left(\varrho | \frac{1}{L}\right)$ represents the relative entropy.

J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
31 (1984), 195-221.

*On asymptotic behaviors of the solution of a non-linear diffusion equation**)

By Yozo TAMURA

(Communicated by Y. Okabe)

§ 1. Introduction

M. Kac [2] discovered the propagation of chaos for Kac's caricature of the Boltzmann equation for Maxwellian gas. In an analogy of this, H. P. McKean, Jr. [3] showed that a certain class of non-linear parabolic equations are derived from a system of n-particle diffusion processes through the propagation of chaos; if the initial distribution of the n-particle diffusion is $u_0^{\otimes n}$, then for any $m \in N$ and any $t > 0$, the m-marginal distribution of the n-particle diffusion at time t converges to m-fold direct product of $u(t)$, where $u(t)$ is a weak solution of the non-linear parabolic equation with the initial data u_0.

In this paper we consider a system of some class of nd-dimensional diffusion processes $X^{(n)}$ ($n \in N$) treated in H. P. McKean, Jr. [3]. For fixed $n \in N$, $X^{(n)}(t) = (X^{(n,1)}(t), \cdots, X^{(n,n)}(t))$ is described by the following stochastic differential equation:

Nontrivial solutions to the stationary McKean–Vlasov equation?

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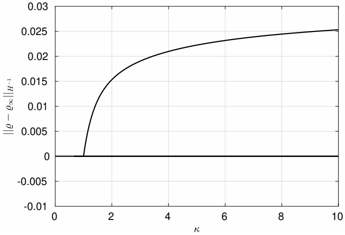
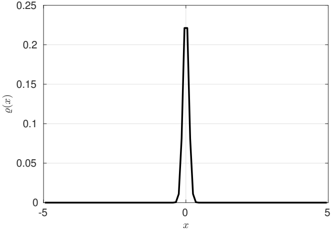
- $W \notin \mathbb{H}_s$ is a necessary condition for the existence of nontrivial steady states.
- Numerical experiments indicate one, multiple, or possibly infinite solutions
- What determines the number of nontrivial solutions?
- Birfurcation analysis of $\varrho \mapsto F_\kappa(\varrho)$.

Example: Kuramoto model: $W(x) = -\sqrt{\frac{2}{L}}\cos(2\pi x/L)$





$\Rightarrow$ 1-cluster solution and uniform state ϱ_∞.

Journal of Statistical Physics, Vol. 29, No. 3, 1982

Statistical Mechanics of the Isothermal Lane–Emden Equation

Joachim Messer[1] and Herbert Spohn[1]

Received January 4, 1982

For classical point particles in a box Λ with potential energy $H^{(N)} = N^{-1}(1/2) \sum_{i \neq j = 1}^{N} V(x_i, x_j)$ we investigate the canonical ensemble for large N. We prove that as $N \to \infty$ the correlation functions are determined by the global minima of a certain free energy functional. Locally the distribution of particles is given by a superposition of Poisson fields. We study the particular case $\Lambda = [-\pi L, \pi L]$ and $V(x, y) = -\beta \cos(x - y)$, $L > 0$, $\beta > 0$.

KEY WORDS: Classical point particles; Lane–Emden equation; canonical ensemble; instable interactions; mean field limit; equilibrium states.

1. INTRODUCTION

Let us consider a finite box Λ into which more and more classical point

Local bifurcation result

Theorem

(Local bifurcations) Let W be smooth and even and let $(1/L, \kappa)$ represent the trivial branch of solutions. Then every $k^* \in \mathbb{Z}, k > 0$ such that

1. $\mathrm{card}\Big\{k \in \mathbb{Z}, k > 0 : \hat{W}(k) = \hat{W}(k^*)\Big\} = 1$,
2. $\hat{W}(k) < 0$,

corresponds to a bifurcation point of the stationary McKean–Vlasov equation through the formula

$$\kappa_* = -\frac{\sqrt{L}}{\beta \hat{W}(k^*)} , \tag{13}$$

with $(1/L, \kappa_*)$ the bifurcation point.

J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 443–484.

Free energy and the convergence of distributions of diffusion processes of McKean type

By Yozo TAMURA*)

(Communicated by S. Kusuoka)

§ 1. Introduction.

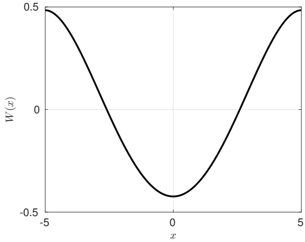
In this paper, we investigate the convergence of the probability distribution $p(t)$ of a diffusion process of McKean type at time t to an invariant probability measure as t goes to ∞ by using the free energy function. The process we consider is given by the following stochastic differential equation of McKean type on $\boldsymbol{R}^d$:

$$
(1.1)\qquad \begin{cases} dX(t) = dB(t) - \operatorname{grad}\Phi_1(X(t))dt + \operatorname{grad}\Phi_2[X(t), p(t)]dt, \\ p(t) \text{ is the probability distribution of } X(t), \\ \text{the initial distribution is } p_0, \end{cases}
$$

where $\Phi_2[x, p] = \int_{\boldsymbol{R}^d} \Phi_2(x, y)p(dy)$ for any probability measure p on $\boldsymbol{R}^d$, $\{B(t); t \geqq 0\}$ is a standard Brownian motion. We assume that the potentials Φ_1 and Φ_2 satisfy the following:

Examples of birfucation results

- Kuramoto-type of models: $W(x) = -w_k(x)$ in $d = 1$ with $\widetilde{W}(k) = -1$, satisfying both conditions. Thus we have that $\kappa_* = \frac{\sqrt{2L}}{\beta}$.
- For $W(x) = \frac{x^2}{2}$ holds $\widetilde{W}(k) = \frac{L^{5/2}\cos(\pi k)}{2\sqrt{2}\pi k^2}$ satisfying both conditions for odd values of k. Hence, every odd k is bifurcation point $\kappa_* = \frac{4k^2}{\beta L^2}$.
- $W^s(x) = -\sum_{k=1}^{\infty} \frac{1}{k^{2s+2}} w_k(x)$
  For $s \geq 1 : W^s(x) \in H^s(\mathbb{T}_L^d)$
  $\forall k > 0$: conditions (1) and (2) ok
  Infinitely many bifurcation points



Definition (Transition point [Chayes & Panferov '10])

A parameter value $\kappa_c > 0$ is said to be a transition point of $\mathscr{F}_\kappa$ if it satisfies the following conditions,

1. For $0 < \kappa < \kappa_c$: ϱ_∞ is the unique minimiser of $\mathscr{F}_\kappa(\varrho)$
2. For $\kappa = \kappa_c$: ϱ_∞ is a minimiser of $\mathscr{F}_\kappa(\varrho)$.
3. For $\kappa > \kappa_c$: $\exists \varrho_\kappa \neq \varrho_\infty$, such that ϱ_κ is a minimiser of $\mathscr{F}_\kappa(\varrho)$.

Definition (Continuous and discontinuous transition point)

A transition point $\kappa_c > 0$ is a continuous transition point of $\mathscr{F}_\kappa$ if

1. For $\kappa = \kappa_c$: ϱ_∞ is the unique minimiser of $\mathscr{F}_\kappa(\varrho)$.
2. For any family of minimizers $\{\varrho_\kappa \neq \varrho_\infty\}_{\kappa > \kappa_c}$ it holds

$$\limsup_{\kappa \downarrow \kappa_c} \|\varrho_\kappa - \varrho_\infty\|_1 = 0.$$

A transition point $\kappa_c > 0$ which is not continuous is discontinuous.

Summary of critical points:

- κ_c transition point.
- κ_* bifurcation point.
- $\kappa_\sharp$ point of linear stability, i.e., $\kappa_\sharp = -\frac{L^{\frac{d}{2}}}{\beta \min_k \widetilde{W}(k)/\Theta(k)}$ with $k_\sharp = \arg\min \widetilde{W}(k)$.

  If there is exactly one $k_\sharp$, then $\kappa_\sharp = \kappa_*$ is a bifurcation point.

Conclusion:

- To prove a discontinuous transition: Show ϱ_∞ at $\kappa_\sharp$ is no longer global minimizer.
- To prove a continuous transition:
  If $\kappa_* = \kappa_\sharp$, sufficient to show that ϱ_∞ at $\kappa_\sharp$ is the only global minimizer and investigate a resonance condition.

Theorem

(Discontinuous and continuous phase transitions) Let W be smooth and even and assume the free energy $\mathscr{F}_{\kappa,\beta}$ exhibits a transition point, $\kappa_c < \infty$. Then we have the following two scenarios:

1. *If there exist strictly positive $k^a, k^b, k^c \in \mathbb{Z}$ with $\hat{W}(k^a) = \hat{W}(k^b) = \hat{W}(k^c) = \min_k \hat{W}(k) < 0$ such that $k^a = k^b + k^c$ or $k^a = 2k^b$, then κ_c is a discontinuous transition point.*
2. *Let $k^\sharp = \arg\min_k \hat{W}(k)$ be well-defined with $\hat{W}(k^\sharp) < 0$. Let W_α denote the potential obtained by multiplying all the negative $\hat{W}(k)$ except $\hat{W}(k^\sharp)$ by some $\alpha \in (0, 1]$. Then if α is made small enough, the transition point κ_c is continuous.*

Proposition

The generalised Kuramoto model $W(x) = -w_k(x)$, for some $k \in \mathbb{N}, k \neq 0$ exhibits a continuous transition point at $\kappa_c = \kappa_\sharp$. Additionally, for $\kappa > \kappa_c$, the equation $F(\varrho, \kappa) = 0$ has only two solutions in $L^2(U)$ (up to translations). The nontrivial one, ϱ_κ minimises $\mathscr{F}_\kappa$ for $\kappa > \kappa_c$ and converges in the narrow topology as $\kappa \to \infty$ to a normalised linear sum of equally weighted Dirac measures centred at the minima of $W(x)$.

- The noisy Hegselmann–Krause system models the opinions of N interacting agents such that each agent is only influenced by the opinions of its immediate neighbours. The interaction potential is

$$W_{\mathrm{hk}}(x) = -\frac{1}{2}\left(\left(|x| - \frac{R}{2}\right)_{-}\right)^{2}$$

- for some $R > 0$. The ratio R/L measures the range of influence of an individual agent with $R/L = 1$ representing full influence.
- The Fourier transform of $W_{\mathrm{hk}}(x)$ is

$$\widetilde{W}_{\mathrm{hk}}(k) = \frac{\left(-\pi^2 k^2 R^2 + 2L^2\right)\sin\left(\frac{\pi k R}{L}\right) - 2\pi k L R \cos\left(\frac{\pi k R}{L}\right)}{4\sqrt{2}\pi^3 k^3 \sqrt{\frac{1}{L}}}, \quad k \in \mathbb{N}, k \neq 0\,. \tag{14}$$

- the model has infinitely many bifurcation points for $R/L = 1$.

- We define a rescaled version of the potential

$$W_{\mathrm{hk}}^R(x) = -\frac{1}{2R^3}\left(\left(|x| - \frac{R}{2}\right)_-\right)^2,$$

which does not lose mass as $R \to 0$.

Proposition

For R small enough, the rescaled noisy Hegselmann–Krause model possesses a discontinuous transition point.

The Onsager model for liquid crystals

- The Onsager/Maiers–Saupe model is described by the interaction potential

$$W_\ell(x) = \left|\sin\left(\frac{2\pi}{L}x\right)\right|^\ell \in L^2_s(U) \cap C^\infty(\bar{U})$$

- with $\ell \in \mathbb{N}, \ell \geq 1$, so that the Onsager and Maiers–Saupe potential correspond to the cases $\ell = 1$ and $\ell = 2$, respectively.
- The Fourier transform of $W_\ell(x)$ is

$$\widetilde{W_\ell}(k) = \frac{\sqrt{\pi}2^{\frac{1}{2}-\ell}\cos\left(\frac{\pi k}{2}\right)\Gamma(\ell+1)}{\Gamma\left(\frac{1}{2}(-k+\ell+2)\right)\Gamma\left(\frac{1}{2}(k+\ell+2)\right)}. \tag{15}$$

- Any nontrivial solutions to the stationary dynamics correspond to the so-called nematic phases of the liquid crystals.

Proposition

1. *The trivial branch of the Onsager model, $W_1(x)$, has infinitely many bifurcation points.*
2. *The trivial branch of the Maiers–Saupe model, $W_2(x)$, has exactly one bifurcation point.*
3. *The trivial branch of the model $W_\ell(x)$ for ℓ even has at least $\frac{\ell}{4}$ bifurcation points if $\frac{\ell}{2}$ is even and $\frac{\ell}{4}+\frac{1}{2}$ bifurcation points if $\frac{\ell}{2}$ is odd.*
4. *The trivial branch of the model $W_\ell(x)$ for ℓ odd has infinitely many bifurcation points if $\frac{\ell-1}{2}$ is even and at least $\frac{\ell+1}{4}$ bifurcation points if $\frac{\ell-1}{2}$ is odd.*

The Keller–Segel model for bacterial chemotaxis

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- The Keller–Segel model is used to describe the motion of a group of bacteria under the effect of the concentration gradient of a chemical stimulus, whose distribution is determined by the density of the bacteria.
- For this system, $\varrho(x,t)$ represents the particle density of the bacteria and $c(x,t)$ represents the availability of the chemical resource.
- The dynamics of the system are then described by the following system of coupled PDEs:

$$\begin{aligned} \partial_t \varrho &= \nabla\cdot\left(\beta^{-1}\nabla\varrho + \kappa\varrho\nabla c\right) && (x,t) \in U \times (0,\infty)\,, \\ -(-\Delta)^s c &= \varrho && (x,t) \in U \times [0,\infty)\,, \\ \varrho(x,0) &= \varrho_0 && x \in U \times \{0\}\,, \\ \varrho(\cdot,t) &\in C^2(\overline{U}) && t \in [0,\infty), \end{aligned} \tag{16}$$

- for $s \in (\frac{1}{2}, 1]$.

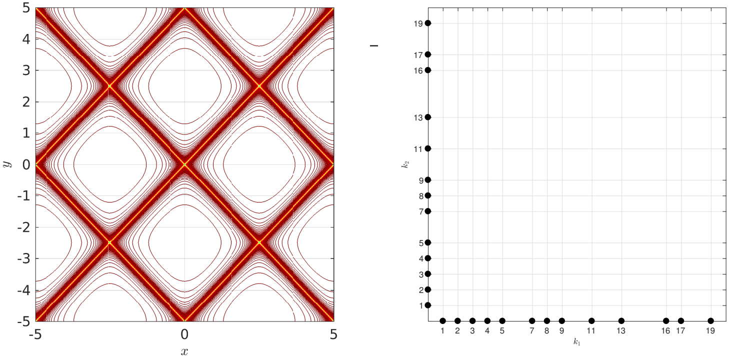
- The stationary Keller–Segel equation is given by,

$$\nabla\cdot\left(\beta^{-1}\nabla\varrho+\kappa\varrho\nabla\Phi^{s}\star\varrho\right)=0\quad x\in U\,, \tag{17}$$

- with $\varrho\in C^{2}(\bar{U})$ and where Φ^{s} is the fundamental solution of $-(-\Delta)^{s}$.

Theorem

Consider the stationary Keller–Segel equation (17). For $d\le 2$ and $s\in(\frac{1}{2},1]$, it has smooth solutions and its trivial branch (ϱ_∞,κ) has infinitely many bifurcation points.



(a)

(b)

Figure: (a). Contour plot of the Keller–Segel interaction potential Φ^s for $d = 2$ and $s = 0.51$. The orange lines indicate the positions at which the potential is singular (b). The associated wave numbers which correspond to bifurcation points of the stationary system.

Fluctuations for Noisy Kuramoto Oscillators

ELSEVIER Stochastic Processes and their Applications 71 (1997) 33–53

stochastic processes and their applications

A Hilbertian approach for fluctuations on the McKean–Vlasov model

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Received 20 March 1996; received in revised form 11 February 1997

Abstract

We consider the sequence of fluctuation processes associated with the empirical measures of the interacting particle system approximating the d-dimensional McKean–Vlasov equation and prove that they are tight as continuous processes with values in a precise weighted Sobolev space. More precisely, we prove that these fluctuations belong uniformly (with respect to the size of the system and to time) to $W_0^{-(1+D),2D}$ and converge in $C([0,T],W_0^{-(2+2D),D})$ to a Ornstein-Uhlenbeck process obtained as the solution of a Langevin equation in $W_0^{-(4+2D),D}$, where D is equal to $1+[d/2]$. It appears in the proofs that the spaces $W_0^{-(1+D),2D}$ and $W_0^{-(2+2D),D}$ are minimal Sobolev spaces in which to immerse the fluctuations, which was our aim following a physical point of view.

Keywords: Convergence of fluctuations; McKean–Vlasov equation; Weighted Sobolev spaces

1. Introduction

- The convergence of the empirical measure to the solution of the McKean-Vlasov PDE is a manifestation of the law of large numbers.
- We can study fluctuations around the (deterministic) mean field McKean-Vlasov limit.
- The fluctuations are Gaussian in the absence of phase transitions and below the phase transition for systems exhibiting phase transitions.
- The fluctuations are **non-Gaussian** at the critical temperature (Dawson 1983, Delgadino, Gvalani, GP 2019).

Theorem

Let μ_t^N denote the empirical measure and let μ_t^∞ the solution of the McKean-Vlasov PDE. Define the fluctuating field

$$Y_N(t, \cdot) := \sqrt{N}\left[\mu_t^N(\cdot) - \mu_t^\infty(\cdot)\right] \tag{18}$$

Then $Y_N(\cdot, \cdot)$ converges to $Y(\cdot, \cdot)$, in the sense of probability measures on $C([0, +\infty], \mathcal{S}')$. The generalized process $Y(\cdot, \cdot)$ is Gaussian and is given by the solution of linear stochastic PDE

$$\frac{\partial Y}{\partial t} = \mathcal{L}_t^* Y + W'(t), \tag{19}$$

where $\{W(t)\,;\, t \geq 0\}$ is a Gaussian Markov process in $\mathcal{S}'$ with covariance, for all ϕ, $\psi \in \mathcal{S}$,

$$(\langle W(t), \phi\rangle, \langle W(t), \psi\rangle) = 2\beta^{-1} \int_0^t \int \phi'(x)\psi'(x)\mu_s^\infty(x)\,dx\,ds. \tag{20}$$

- $\mathcal{L}_t^*$ is the linearized McKean-Vlasov operator around $\mu_t^\infty(\cdot)$:

$$\mathcal{L}_t^* Y = \frac{\partial}{\partial x}\left(V_\theta'(x)Y + \beta^{-1}\frac{\partial Y}{\partial x}\right) - \theta\langle y, \mu_t^\infty\rangle \frac{\partial Y}{\partial x} - \theta\langle Y, y\rangle \frac{\partial \mu_t^\infty}{\partial x}. \tag{21}$$

- Equivalently, for an arbitrary observables $\phi \in \mathcal{S}$ we have that, in distribution,

$$N^{-1/2}\sum_{j=1}^{N}\left[\phi(x_j(t)) - \int \phi(x)\mu_t^\infty(x)\right] \to \langle Y(t), \phi\rangle. \tag{22}$$

- Below the phase transition, the linearized McKean-Vlasov operator around the unique stationary state $\mathcal{L}^*$ has a one dimensional null space and all other eigenvalues are negative. The SPDE has a unique Gaussian invariant measure.
- The minimal (negative) Sobolev space in which the fluctuations live was identified by Fernandez and Meleard (1997).

- At the critical temperature, the null space of cL^* becomes two dimensional and a second eigenfunction $q_0(\cdot)$, corresponding to the 0 eigenvalue, appears.
- To study the fluctuations around the mean field limit, we need to look at longer time scales: this is an example of the **critical slowing down** phenomenon.
- This phenomenon can be studied using multiscale analysis (in the style of Papanicolaou, Stroock, Varadhan, *A martingale approach to some limit theorems*), or using linear response theory (Shiino 1987).
- The Fourier transform of the response function (the static susceptibility) and the relaxation time diverge at we approach the critical temperature (calculation of critical exponents).
- As we approach the critical temperature, the fluctuations become coherent/strongly correlated.

Theorem

Define

$$U_N(\cdot,\cdot) := N^{1/4}\left[\mu^N_{N^{1/2}t}(\cdot) - \rho_\infty(x)\,dx\right], \tag{23}$$

where $\rho_\infty(x)$ *is the stationary state with* $m = 0$. *Then as* $N \to +\infty$,

$$U_N(\cdot,\cdot) \to Z(\cdot,\cdot), \tag{24}$$

in the sense of weak convergence of probability measures on $C([0,+\infty], M(\mathbb{R}))$ *where*

$$Z(t,dx) = z(t)q_0(x)\,dx, \tag{25}$$

and

$$dz_t = cz_t^3\,dt + \sigma_*\,dw_t, \quad c,\, \sigma_* > 0. \tag{26}$$

- The process $Z(\cdot)$ has the unique equilibrium distribution

$$Z_\infty(dx) = \zeta_\infty q_0(x)\, dx,$$

where

$$\zeta_\infty \sim Z^{-1} e^{-\frac{c}{2\sigma_*^2} x^4}.$$

- The fluctuations exhibit üniversality"(we have similar results for a large class of mean field models).

- Consider again a system weakly interacting nonlinear noisy oscillators:

$$dX_t^i = -\frac{1}{N}\sum_{i\neq j}^{N} \nabla W(X_t^i - X_t^j) + \sqrt{2\beta^{-1}}\, dB_t^i$$

  with W chosen to be 1-periodic.

- Let $\rho^N = \mathrm{Law}(X_t^1, \ldots, X_t^N)$ and consider the diffusive rescaling

$$\rho^{\varepsilon,N}(x,t) := \varepsilon^{-N}\rho^N(\varepsilon^{-1}x, \varepsilon^{-2}t) \in \mathcal{P}(\mathbb{R})\,. \tag{27}$$

- Interepretation: zooming out in space and going forward in time. Can pass to the limit:
  - Bensoussan–Lions–Papanicolaou '78 (PDE approach)
  - Kipnis–Varadhan '86 (Probabilistic approach)

Theorem (The diffusive limit)

$$\rho^{N,*} = \lim_{\varepsilon \to 0} \rho^{\varepsilon,N}$$

exists (with convergence in weak-$\star$) and satisfies

$$\partial_t \rho^{N,*} = \nabla \cdot (A^{\mathrm{eff},N} \nabla \rho^{N,*}),$$

where the diffusion matrix is given by

$$A^{\mathrm{eff},N} = \int_{\mathbb{T}^N} (I + \nabla \Psi^N(y))\, M_N(y)\, dy$$

with M_N the invariant measure of the quotiented process on the torus, and Ψ^N the solution of the corrector problem

$$\nabla \cdot (\nabla \Psi^N M^N) = -\nabla M^N, \quad x \in \mathbb{T}^N.$$

The diffusive limit is affected by the problem on the torus, that exhibits phase transitions in the mean field limit.

Question: $\lim_{N \to \infty} \rho^{N,*} = ?$.

Another interpretation of the mean-filed limit:

$$\frac{1}{N} d_2^2(\rho^N, \rho^{\otimes N}) \to 0, \quad N \to \infty\,,$$

where ρ solves the McKean–Vlasov equation. Another question : $\lim_{N\to\infty} \rho^{\varepsilon,N} \approx \rho^{\varepsilon,\otimes N} \stackrel{\varepsilon\to 0}{\to} ?.$

Theorem (Delgadino–Gvalani–P. '19)

Let $\mathcal{F}_\beta$ be the free energy on the torus and assume it exhibits a phase transition at some $\beta = \beta_c$. Then for $\beta < \beta_c$

$$\lim_{N\to\infty} \lim_{\varepsilon\to 0} \rho^{\varepsilon,N} = \lim_{\varepsilon\to 0} \lim_{N\to\infty} \rho^{\varepsilon,N} .$$

On the other hand if $\beta > \beta_c$, there exists initial data $\rho_0^{\otimes N}$ such that

$$\lim_{N\to\infty} \lim_{\varepsilon\to 0} \rho^{\varepsilon,N} \neq \lim_{\varepsilon\to 0} \lim_{N\to\infty} \rho^{\varepsilon,N} .$$

Conclusions

- Complete analysis of local and global bifurcations for the McKean-Vlasov equation on the torus.
- Study of fluctuations and of the combined mean field/homogenization limit for Kuramoto oscillators.
- Study the effect of memory, colored noise/non-gradient structure, hypoellipticity etc.
- Study dynamical metastability phenomena.
- Predicting phase transitions, linear response theory, optimal control.