Mean-Field-Type Games

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Acknowledgments

- Collaborators:
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  Foundations of mean-field-type game theory

Outline

1 Introduction

2 MFTG problem

3 COVID-19 and Spread of SARS-COV-2

Mean-Field Games: some references

- Infinite number of agents: Borel 1921, Volterra'26, Hotelling'29,von Neumann'44, Nash'51, Wardrop'52, Aumann'64, Selten'70, Schmeidler'73, Dubey et al.'80-, . . .
- Discrete [time/state] mean-field games:
  - Jovanovic'82, Jovanovic & Rosenthal'88, Bergins & Bernhardt'92, Weibull& Benaïm'03-, Weintraub, Benkard, Van Roy'05-, Sandholm '06-, Adlaska, Johari, Goldsmith'08-, Benaïm & Le Boudec'08-, Gast & Gaujal'09, Bardenave'09-, Gomes, Mohr & Souza'10-, Borkar & Sundaresan'12, Elliott'12-,Bayraktar, Budhiraja, Cohen'17- . . .
- Continuous-time mean-field games
  - Krusell & Smith'98, Benamou & Brenier'00-, Huang, Caines, Malhame'03-, Lasry & Lions'06-, Kotelenez & Kurtz'07-, Li & Zhang'08-, Buckdahn, Djehiche, Li and Peng'09-, Gueant'09-, Gomes et al.'09-, Yin, Mehta, Meyn, and Shanbhag'10, Djehiche et al' 10, Feng et al.'10-, Dogbe'10-, Achdou et al.'10-, LaChapelle'10-, Zhu, Başar'11, Bardi'12, Bensoussan,Sung,Yam,Yung'12-, Kolokoltsov'12-, Carmona & Delarue'12-,Yong'13-,Gangbo & Swiech'14-, Pham'16-,Fischer'17-,Nuno'17-, . . .

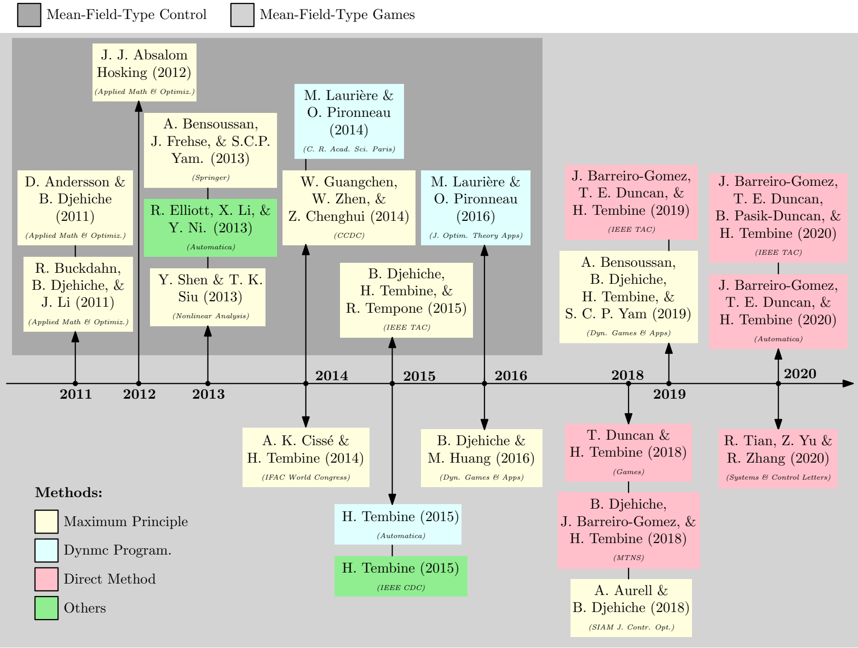
Risk-Sensitive Mean-Field Games

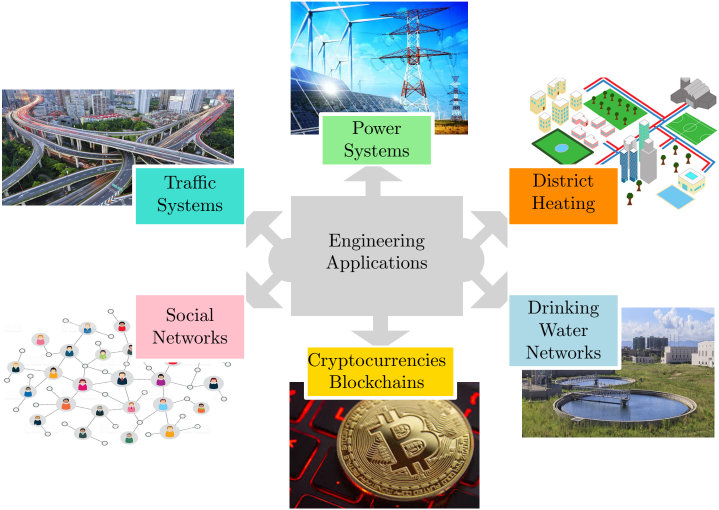
- risk-sensitive cost functional $\frac{1}{\gamma_i}\log\left(\mathbb{E}e^{\gamma_i C_i}\right)$
- small risk-sensitive index: the performance includes the expected expected cost, **the variance of cost and higher moments**
- different behaviors: *risk-seeking, risk-averse, risk-neutral, mixture*

Refs: Zhu,Tașar, Moon,Saldi, Raginsky, Djehiche, ...

Mean-Field-Type Games: some references

- One decision-maker or mean-field-type control:
  - Buckdahn, Djehiche, Li and Peng'09-, Andersson& Djehiche et al' 11-, Buckdahn, B. Djehiche, J. Li'11-, Bensoussan, Frehse, Yam '12-, Carmona & Delarue'12-, Elliott, Li, Ni'13-, Yong'13-, Wang,Zhang'14-, Lauriere, Pironneau'14-, Bayraktar, Cosso and Pham'16-, - . . .
- Cooperative mean-field-type games
  - Cisse et al. 2014-, Djehiche et al' 17-,Barreiro-Gomez et al. . . .
- Mean-field-type games
  - Ruimin'12-, Hosking'12-,Chen & Zhu'14-, Tcheukam et al.'16-, Duncan et al.'18-, Barreiro-Gomez et al.'18-, Aurell'18-, Choutri'18-, . . .


Mean-Field-Type Control
Mean-Field-Type Games
J. J. Absalom Hosking (2012) (Applied Math & Optimiz.)
A. Bensoussan, J. Frehse, & S.C.P. Yam. (2013) (Springer)
M. Laurière & O. Pironneau (2014) (C. R. Acad. Sci. Paris)
D. Andersson & B. Djehiche (2011) (Applied Math & Optimiz.)
R. Elliott, X. Li, & Y. Ni. (2013) (Automatica)
W. Guangchen, W. Zhen, & Z. Chenghui (2014) (CCDC)
M. Laurière & O. Pironneau (2016) (J. Optim. Theory Apps)
R. Buckdahn, B. Djehiche, & J. Li (2011) (Applied Math & Optimiz.)
Y. Shen & T. K. Siu (2013) (Nonlinear Analysis)
B. Djehiche, H. Tembine, & R. Tempone (2015) (IEEE TAC)
J. Barreiro-Gomez, T. E. Duncan, & H. Tembine (2019) (IEEE TAC)
A. Bensoussan, B. Djehiche, H. Tembine, & S. C. P. Yam (2019) (Dyn. Games & Apps)
J. Barreiro-Gomez, T. E. Duncan, B. Pasik-Duncan, & H. Tembine (2020) (IEEE TAC)
J. Barreiro-Gomez, T. E. Duncan, & H. Tembine (2020) (Automatica)
2011
2012
2013
2014
2015
2016
2018
2019
2020
A. K. Cissé & H. Tembine (2014) (IFAC World Congress)
B. Djehiche & M. Huang (2016) (Dyn. Games & Apps)
T. Duncan & H. Tembine (2018) (Games)
R. Tian, Z. Yu & R. Zhang (2020) (Systems & Control Letters)
H. Tembine (2015) (Automatica)
H. Tembine (2015) (IEEE CDC)
B. Djehiche, J. Barreiro-Gomez, & H. Tembine (2018) (MTNS)
A. Aurell & B. Djehiche (2018) (SIAM J. Contr. Opt.)
Methods:
Maximum Principle
Dynmc Program.
Direct Method
Others


Power
Systems
Traffic
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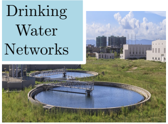
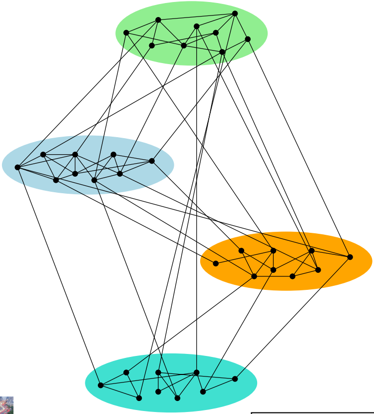
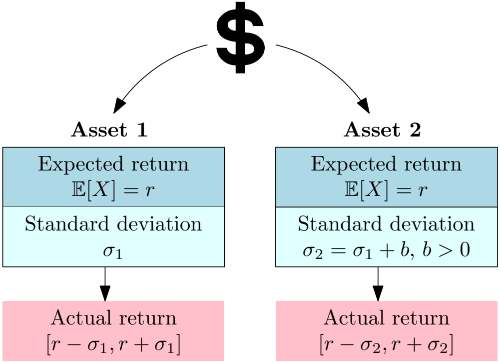
District
Heating
Engineering
Applications
Social
Networks
Drinking
Water
Networks
Cryptocurrencies
Blockchains





Interdependence!
System of Systems
Network of Networks

Mean-Variance Paradigm (Portfolio Problem)



Asset 2 can give a higher return
but it is more risky!

Diversification Problem

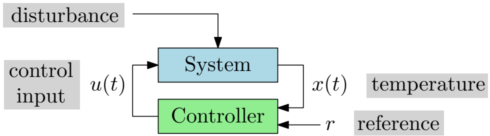
A way to invest optimally
while reducing risk

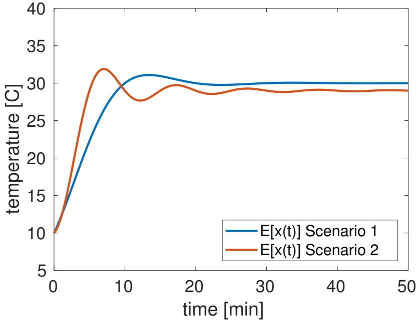
$$\max \left(\mathbb{E}[X] - \sqrt{\mathrm{var}(X)} \right)$$

Nobel Memorial Prize in Economic Sciences 1990.

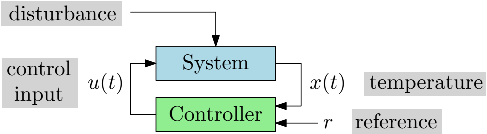
H. Markowitz. Portfolio selection. *The Journal of Finance*, 7:77–91, 1952.

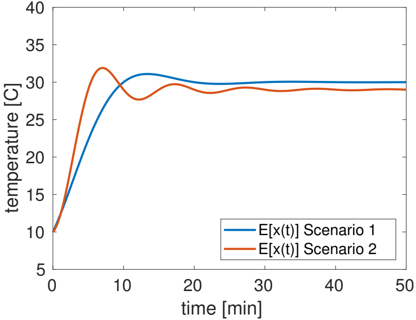
Variance-Awareness

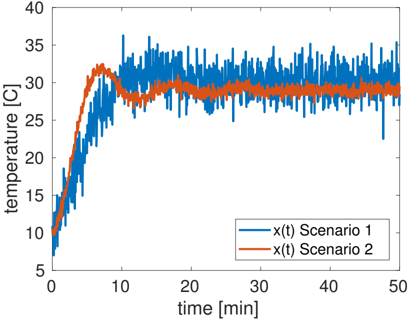




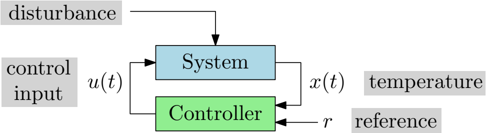
Variance-Awareness

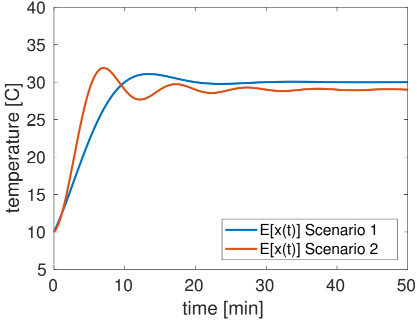


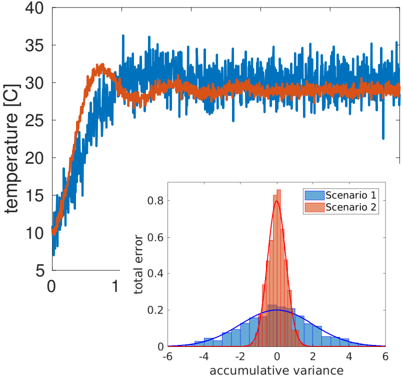




Variance-Awareness







Variance-awareness: stylized case

Let $\epsilon \in (0,1),\ 0 < t_0 < t_1$.

Variance-aware control problem

$$\inf_a \int \|s\|^2 \mu(t_1, ds) - (1-\epsilon)\| \int y \mu(t_1, dy)\|^2$$
$$+\frac{1}{4}\int_{t_0}^{t_1} [\int \|a(t,s)\|^2 \mu(t,ds) + \frac{1}{\epsilon}\| \int a(t,y)\mu(t,dy)\|^2] dt$$
subject to
$$\mu_t(t,s) = -\mathrm{div}_s(a(t,s)\mu(t,s)) + \Delta_s \mu(t,s),$$
$$\mu(t_0, ds) = \mu_0(ds)$$

Legendre-Fenchel Duality: $\lambda l(x) \mapsto \lambda l^*(\frac{x}{\lambda})$ for $\lambda > 0$

$\lambda \frac{x^2}{2} \mapsto \frac{p^2}{2\lambda}$

Optimal Cost

Optimal Cost

$$\hat{V}(t,\mu) = \inf_a \int \|s\|^2 \tilde{\mu}(t_1, ds) - (1-\epsilon)\| \int y \tilde{\mu}(t_1, dy)\|^2 + \frac{1}{4}\int_t^{t_1} [\int \|a(t,s)\|^2 \tilde{\mu}(t,ds) + \frac{1}{\epsilon}\| \int a(t,y)\tilde{\mu}(t,dy)\|^2] dt$$

subject to

$$\tilde{\mu}_t(t,s) = -\text{div}_s(a(t,s)\tilde{\mu}(t,s)) + \Delta_s \tilde{\mu}(t,s),$$
$$\tilde{\mu}(t,ds) = \mu(ds)$$

Wasserstein gradient

$\hat{V}_\mu(t,s,\mu) := \nabla_s([\frac{\delta \hat{V}}{\delta \mu}(t,\mu)](s)).$

Quantities-of-interest: $(\hat{V}, a^*)$

Variance-awareness: stylized case

Consider the following HJB equation in $(t_0, t_1) \times \mathcal{P}_2(\mathbb{R}^d)$:

Bellman

$$\hat{V}_t - \int \|\hat{V}_\mu\|^2 \mu(ds) + (1-\epsilon)\| \int \hat{V}_\mu \mu(dy)\|^2 + \int div_s(\hat{V}_\mu)\mu(ds) = 0$$
$$\hat{V}(t_1, \mu) = \int \|s\|^2 \mu(ds) - (1-\epsilon)\| \int y\mu(dy)\|^2,$$

$$a^*(t,s) = -2\hat{V}_\mu(t,s,\mu) + 2(1-\epsilon)\int \hat{V}_\mu(t,y,\mu)\mu(dy),$$

Semi-explicit solution

$$\hat{V}(t,\mu) = \alpha(t)\int \|s\|^2\mu(ds) + (\beta(t)-\alpha(t))\| \int y\mu(dy)\|^2 + \delta(t)\int \mu(ds),$$

$$\dot{\alpha} - 4\alpha^2 = 0, \quad \alpha(t_1) = 1,$$
$$\dot{\beta} - 4\epsilon\beta^2 = 0, \quad \beta(t_1) = \epsilon,$$
$$\delta(t) = 2\int_t^{t_1} \alpha(t')dt'.$$

Explicit solution

$$\alpha(t) = \frac{1}{1+4(t_1-t)}$$
$$\beta(t) = \frac{1}{\frac{1}{\epsilon}+4\epsilon(t_1-t)}$$
$$\delta(t) = 2\int_t^{t_1} \alpha(t')dt' = \frac{1}{2}\log\left(1+\frac{4(t_1-t)}{1+4(t_1-t)}\right)$$

As we can see, there is no singularity formation within (t_0, t_1) for any $\epsilon \in (0, 1)$.
The optimal cost of the variance-aware control problem is

$$\hat{V}(t_0, \mu_0) = \alpha(t_0)\int \|s\|^2\mu_0(ds)$$
$$+(\beta(t_0)-\alpha(t_0))\|\int y\mu_0(dy)\|^2 + \delta(t_0)\int \mu_0(ds)$$

$$= \frac{1}{1+4(t_1-t_0)}\int \|s\|^2\mu_0(ds)$$
$$+\left(\frac{1}{\frac{1}{\epsilon}+4\epsilon(t_1-t_0)} - \frac{1}{1+4(t_1-t_0)}\right)\|\int y\mu_0(dy)\|^2$$
$$+\frac{1}{2}\log\left(1+\frac{4(t_1-t_0)}{1+4(t_1-t_0)}\right)\int \mu_0(dy)$$

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1 Introduction

2 MFTG problem

3 COVID-19 and Spread of SARS-COV-2

Semi-explicitly solvable mean-field-type games

Let $\epsilon_i \in (0,1)$. $I = 2$ decision-makers.

Variance-aware game problem

$$\inf_{a_i} \int \|s\|^2 \mu(t_1, ds) - (1-\epsilon_i)\| \int y\mu(t_1, dy)\|^2$$
$$+\frac{1}{2}\int_{t_0}^{t_1} [\frac{1}{2}\int \|a_i(t,s)\|^2 \mu(t,ds) + \frac{1}{2\epsilon_i}\| \int a_i(t,y)\mu(t,dy)\|^2] dt$$
subject to
$$\mu_t(t,s) = -\text{div}_s(\textstyle\sum_{j=1}^2 a_j(t,s)\mu(t,s)) + \Delta_s \mu(t,s),$$
$$\mu(t_0, ds) = \mu_0(ds)$$

This is not an exact potential game. (Rosenthal'73, Monderer & Shapley'96)

Quantities-of-interest: $(\hat{V}_i, a_i^*)_{i\in\mathcal{I}}$

Semi-explicitly solvable mean-field-type game

Consider the HJB system in $(t_0, t_1) \times \mathcal{P}_2(\mathbb{R}^d)$:

Bellman system

$$i \in \{1, 2\},$$
$$\hat{V}_{i,t} - \int \|\hat{V}_{i,\mu}\|^2 \mu(ds) + (1-\epsilon_i)\| \int \hat{V}_{i,\mu}\mu(ds)\|^2 + \int div_s(\hat{V}_{i,\mu})\mu(ds)$$
$$-2\int \langle \hat{V}_{-i,\mu}, \hat{V}_{i,\mu}\rangle \mu(ds) + 2(1-\epsilon_i)\langle \int \hat{V}_{-i,\mu}\mu(ds), \int \hat{V}_{i,\mu}\mu(dy)\rangle = 0,$$

$$\hat{V}_i(t_1, \mu) = \int \|s\|^2 \mu(ds) - (1-\epsilon_i)\| \int y\mu(dy)\|^2$$

Idea of the Proof

Coupled ODEs.

$$\hat{V}_i(t, \mu) = \alpha_i(t) \int \|s\|^2 \mu(ds) + (\beta_i(t) - \alpha_i(t))\| \int y\mu(dy)\|^2 + \delta_i(t) \int \mu(ds),$$

A Class of MFTG: finitely many agents

$$
\begin{aligned}
&\inf_{a_i\in\mathcal{A}_i} \int h_i(y,\mu(t_1))\mu(t_1,dy) + \int_{t_0}^{t_1}\int l_i(t,s,\mu(t),a(t))\mu(t,ds)dt,\\
&\quad\text{subject to}\\
&\mu_t(t,s) = -\mathrm{div}_s(b(t,s,\mu(t),a(t))\mu(t,s)) + \epsilon\Delta_s\mu(t,s), \qquad (t_0,t_1)\times\mathcal{S}\\
&\mu(t_0,ds) \triangleq \mu_0(ds), \qquad \{t_0\}\times\mathcal{S}\\
&\quad i\in\mathcal{I},\\
&\quad a(t) = (a_i(t))_{i\in\mathcal{I}}
\end{aligned}
$$

$\mu(t,dy) = \mu^{\mu_0,a}(t,dy)$ a measure of the state
Quantities-of-interest: $(\hat{V}_i, a_i^*)_{i\in\mathcal{I}}$

Bellman system

HJB system

$$\begin{cases} \text{Domain: } (t_0, t_1) \times \mathcal{P}_2(\mathcal{S}), \\ \hat{V}_{i,t}(t,\mu) + \int [\hat{H}_i(t,s,\mu,\hat{V}_\mu) - \epsilon \langle \hat{V}_{i,\mu}, \nabla_s (\log \mu) \rangle] \mu(ds) = 0, \\ \hat{V}_i(t_1,\mu) = \int h_i(s,\mu)\mu(ds), \\ i \in \mathcal{I}. \end{cases}$$

Integrand Hamiltonian

$$\hat{H}_i(t,s,\mu,\hat{V}_\mu) = \inf_{a_i \in A_i} \{ l_i + \langle b, \hat{V}_{i,\mu} \rangle \}$$

The HJB system $\{\hat{V}_i\}_i$ does provide an equilibrium value (if any) in state-and-mean-field-type feedback form

MASS

$\hat{H}_i(t, s, \mu, \hat{U}_s) = \inf_{a_i \in A_i}\{l_i + \langle b, \hat{U}_{i,s}\rangle\}$

MASS: Master Adjoint System

$$\begin{cases} \text{Domain: } (t_0, t_1) \times \mathcal{S} \times \mathcal{P}_2(\mathcal{S}), \\ \hat{U}_{i,t}(t, s, \mu) + \hat{H}_i + \epsilon \Delta_s \hat{U}_i \\ + \int \frac{\delta}{\delta\mu}[\hat{H}_i - \epsilon\langle \hat{U}_{i,s}, \nabla_s(\log\mu)\rangle](s')\mu(ds') = 0, \\ \hat{U}_i(t_1, s, \mu) = h_i(s, \mu) + \int \frac{\delta}{\delta\mu}[h_i(s, \mu)](s')\mu(ds'), \\ i \in \mathcal{I}. \end{cases}$$

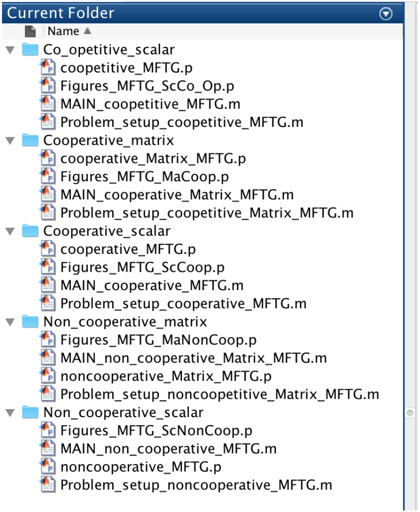
MASS $\{\hat{U}_i\}_i$ is not necessarily an equilibrium value of the original game

Solvability of MASS: LQ-MFTG case

$$h_i = \frac{1}{2}q_{i,t_1}\mathrm{var}(s(t_1)) + \frac{1}{2}\bar{q}_{i,t_1}\bar{s}^2(t_1) + \varepsilon_{i3}(t_1)\bar{s}(t_1)$$
$$l_i = \left(q_i\mathrm{var}(s) + \bar{q}_i\bar{s}^2 + r_i\mathrm{var}(a_i) + \bar{r}_i\bar{a}_i^2\right)$$
$$+\left(\varepsilon_{i1}\mathrm{cov}(s,a_i) + \bar{\varepsilon}_{i1}\bar{s}\bar{a}_i + \varepsilon_{i2}\bar{a}_i + \varepsilon_{i3}\bar{s} + \sum_{k\neq i}\varepsilon_{i4k}\mathrm{cov}(a_i,a_k) + \bar{\varepsilon}_{i4k}\bar{a}_i\bar{a}_k\right),$$
$$b = b_0 + b_1 s + \bar{b}_1\bar{s} + \sum_{j=1}^{I} b_{2j}a_j + \sum_{j=1}^{I}\bar{b}_{2j}\bar{a}_j$$
$$\sigma = \sigma_0 + \sigma_1 s + \bar{\sigma}_1\bar{s} + \sum_{j=1}^{I}\sigma_{2j}a_j + \sum_{j=1}^{I}\bar{\sigma}_{2j}\bar{a}_j$$
$$\gamma = \gamma_0 + \gamma_1 s + \bar{\gamma}_1\bar{s} + \sum_{j=1}^{I}\gamma_{2j}a_j + \sum_{j=1}^{I}\bar{\gamma}_{2j}\bar{a}_j,$$
$$\tilde{s}(t) \sim \tilde{q}_{\tilde{s}\tilde{s}'}, \qquad s(0) \perp \{B, N, \tilde{s}_0\},$$
$$b_k(t,\tilde{s}); \sigma_k(t,\tilde{s}); \bar{b}_k(t,\tilde{s}); \bar{\sigma}_k(t,\tilde{s}) \in \mathbb{R},\ \gamma_k(t,\tilde{s},\theta) \in \mathbb{R},\ \tilde{s} \in \tilde{\mathcal{S}},$$
$$\bar{X} = \mathbb{E}[X|\mathcal{F}_t^{\tilde{s}}],$$

Başar. Stochastic linear quadratic nonzero-sum differential games with delayed information patterns. Proc. 7th IFAC World Congress, pp. 1025–1032, June 1978.

MATLAB Toolbox



J. Barreiro-Gomez and H. Tembine. A MATLAB-based Mean-Field-Type Games Toolbox: Continuous-Time version. IEEE Access, vol 7, pp. 126500-126514, 2020.

Solvability of MASS: Beyond LQ-MFTG

State	Cost	Noise
Drift: 0 $s(t) \in \tilde{\mathcal{S}}$ $\mathcal{I} = \{1, \ldots, I\}$	$r_i(a_i - \bar{a}_i)^2 + \bar{r}_i \bar{a}_i^2 + \epsilon_i \bar{a}_i$	Switching: $\tilde{q}_{\tilde{s}\tilde{s}'}(a)$ $= \sum_j b_{2j\tilde{s}\tilde{s}'}(a_j - \bar{a}_j)^2 + \bar{b}_{2j\tilde{s}\tilde{s}'}\bar{a}_j^2$ $+ \sum_j b_{1j\tilde{s}\tilde{s}'}(a_j - \bar{a}_j) + \bar{b}_{1j\tilde{s}\tilde{s}'}\bar{a}_j$ $+ \sum_j \bar{b}_{oj\tilde{s}\tilde{s}'}$,
Drift: $\sum_{j\in\mathcal{I}}[q_j(a_j - \bar{a}_j)^2 + \bar{q}_j \bar{a}_j^2$ $+\epsilon_{1j}(a_j - \bar{a}_j) + \bar{\epsilon}_{1j}\bar{a}_j]$ $\mathcal{I} = \{1, \ldots, I\}$	$r_i(a_i - \bar{a}_i)^2 + \bar{r}_i \bar{a}_i^2 + \bar{\epsilon}_{2i}\bar{a}_i$	Brownian: σdB Jump: $\int_\Theta \gamma d\tilde{N}$
Drift: $\frac{1}{2} cot(\frac{s-\bar{s}}{2}) + \sum_j b_{2j}(a_j - \bar{a}_j)$ $+\frac{1}{2} cot(\frac{\bar{s}}{2}) + \sum_j \bar{b}_{2j}\bar{a}_j$ $\mathcal{I} = \{1, \ldots, I\}$	$((a_i - \bar{a}_i)^2 - q_i)\cos^2(\frac{s-\bar{s}}{4}) + q_i$ $+(\bar{a}_i^2 - \bar{q}_i)\cos^2(\frac{\bar{s}}{4}) + \bar{q}_i$	Brownian: σdB Switching: $\tilde{q}_{\tilde{s}\tilde{s}'}$
Drift: $\frac{1}{2} coth(\frac{s-\bar{s}}{2}) + \sum_j b_{2j}(a_j - \bar{a}_j)$ $+\frac{1}{2} coth(\frac{\bar{s}}{2}) + \sum_j \bar{b}_{2j}\bar{a}_j$ $\mathcal{I} = \{1, \ldots, I\}$	$((a_i - \bar{a}_i)^2 + q_i)\cosh^2(\frac{s-\bar{s}}{4}) - q_i$ $+(\bar{a}_i^2 + \bar{q}_i)\cosh^2(\frac{\bar{s}}{4}) - \bar{q}_i$	Brownian: σdB Switching: $\tilde{q}_{\tilde{s}\tilde{s}'}$
Drift: $-(a_1 - \bar{a}_1) + b_1(s - \bar{s})$ $+b_2\epsilon(s - \bar{s})(a_2 - \bar{a}_2)$ $-\bar{a}_1 + \bar{b}_{11}\bar{s} + \bar{b}_{12}\bar{y} + \bar{b}_{13}\bar{z} + \bar{b}_2\bar{a}_2\bar{s}$ $\mathcal{I} = \{1, \ldots, I\}$	$-q\ var(x) - r_1\ var(a_1)$ $+\bar{r}_1 \frac{\bar{a}_1^{\rho}}{\rho}$	Brownian: $\sigma(s - \bar{s})(a_2 - \bar{a}_2)dB$ Common noise: $\bar{\sigma}\bar{s}\bar{a}_2 dB_o$ Switching: $\tilde{q}_{\tilde{s}\tilde{s}'}$
Drift: $b_1(s - \bar{s}) + \sum_j b_{2j}(a_j - \bar{a}_j)$ $+\bar{b}_1\bar{s} + \sum_j \bar{b}_{2j}\bar{a}_j$ $\mathcal{I} = \{1, \ldots, I\}$	$q_i \frac{(s-\bar{s})^{2k_i}}{2k_i} + r_i \frac{(a_i - \bar{a}_i)^{2k_i}}{2k_i}$ $+\bar{q}_i \frac{\bar{s}^{2\bar{k}_i}}{2\bar{k}_i} + \bar{r}_i \frac{\bar{a}_i^{2\bar{k}_i}}{2\bar{k}_i}$	Brownian: $(s - \bar{s})\sigma dB$ Jump: $(s - \bar{s})\int_\Theta \gamma d\tilde{N}$ Switching: $\tilde{q}_{\tilde{s}\tilde{s}'}$ Gauss-Volterra: $(s - \bar{s})\sigma_{gv} dB_{gv}$

J. Barreiro-Gomez, T. E. Duncan, B. Pasik-Duncan, and H. Tembine. Semi-Explicit Solutions to some *Non-Linear Non-Quadratic* Mean-Field-Type Games: A Direct Method. IEEE Transactions on Automatic Control, 2020.

Solvability: MFTG with polynomial cost

$$L_i(s,a) = q_{i,t_1}\frac{(s(t_1)-\bar{s}(t_1))^{2k_i}}{2k_i} + \bar{q}_{i,t_1}\frac{(\bar{s}(t_1))^{2\bar{k}_i}}{2\bar{k}_i}$$
$$+\int_{t_0}^{t_1} q_i\frac{(s-\bar{s})^{2k_i}}{2k_i} + r_i\frac{(a_i-\bar{a}_i)^{2k_i}}{2k_i} + \bar{q}_i\frac{\bar{s}^{2\bar{k}_i}}{2\bar{k}_i} + \bar{r}_i\frac{\bar{a}_i^{2\bar{k}_i}}{2\bar{k}_i}\,dt,$$

$\inf_{a_i} \mathbb{E}[L_i(s,a)]$ subject to

$$ds = [b_1(s-\bar{s}) + \sum_j b_{2j}(a_j - \bar{a}_j) + \bar{b}_1\bar{s} + \sum_j \bar{b}_{2j}\bar{a}_j]dt$$
$$+(s-\bar{s})[\sigma dB + \int_\Theta \gamma d\tilde{N}],$$
$$s(0) = s_0,$$
$$\mathbb{P}(\tilde{s}(t+\epsilon) = \tilde{s}'|\tilde{s},u) = \int_t^{t+\epsilon}\tilde{q}_{\tilde{s}\tilde{s}'}dt' + o(\epsilon),\ \tilde{s}'\neq\tilde{s}$$
$$\tilde{s}(0) = \tilde{s}_0,$$

where $k_i \geq 1, \bar{k}_i \geq 1$ are natural numbers, the coefficients are time and switching dependent,

Semi-Explicit Equilibrium strategies

$$a_i^* = \sum_{\tilde{s}\in\tilde{\mathcal{S}}} \mathbb{1}_{\{\tilde{s}(t)=\tilde{s}\}} \left[-\left(\frac{b_{2i}\alpha_i}{r_i}\right)^{\frac{1}{2k_i-1}} (s-\bar{s}) - \left(\frac{\bar{b}_{2i}\bar{\alpha}_i}{\bar{r}_i}\right)^{\frac{1}{2\bar{k}_i-1}} \bar{s} \right],$$

Semi-Explicit Equilibrium costs

$$cost_i^* = \sum_{\tilde{s}_0} \int_{s_0} \mu_0(ds_0, \tilde{s}_0)[\alpha_i(t_0, \tilde{s}_0)\frac{(s_0-\bar{s}_0)^{2k_i}}{2k_i} + \bar{\alpha}_i(t_0, \tilde{s}_0)\frac{\bar{s}_0^{2\bar{k}_i}}{2\bar{k}_i}],$$

$$
\begin{aligned}
&\dot{\alpha}_i + q_i + 2k_i\alpha_i b_1 + \alpha_i k_i(2k_i-1)\sigma^2 \\
&+\alpha_i \int_\theta [(1+\gamma)^{2k_i} - 1 - 2k_i\gamma]\nu(d\theta) + \sum_{\tilde{s}'}[\alpha_i(t,\tilde{s}') - \alpha_i(t,\tilde{s})]\tilde{q}_{\tilde{s}\tilde{s}'} \\
&-(2k_i-1)r_i\left(\frac{b_{2i}\alpha_i}{r_i}\right)^{\frac{2k_i}{2k_i-1}} - 2k_i\alpha_i\left[\sum_{j\neq i} b_{2j}\left(\frac{b_{2j}\alpha_j}{r_j}\right)^{\frac{1}{2k_j-1}}\right] = 0, \\
&\alpha_i(t_1,\tilde{s}) = q_i(t_1,\tilde{s}),\ \tilde{s}\in\tilde{\mathcal{S}}
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\bar{\alpha}}_i + \bar{q}_i + 2\bar{k}_i\bar{\alpha}_i\bar{b}_1 + \sum_{\tilde{s}'}[\bar{\alpha}_i(t,\tilde{s}') - \bar{\alpha}_i(t,\tilde{s})]\tilde{q}_{\tilde{s}\tilde{s}'} \\
&-(2\bar{k}_i-1)\bar{r}_i\left(\frac{\bar{b}_{2i}\bar{\alpha}_i}{\bar{r}_i}\right)^{\frac{2\bar{k}_i}{2\bar{k}_i-1}} - 2\bar{k}_i\bar{\alpha}_i\left[\sum_{j\neq i} \bar{b}_{2j}\left(\frac{\bar{b}_{2j}\bar{\alpha}_j}{\bar{r}_j}\right)^{\frac{1}{2\bar{k}_j-1}}\right] = 0, \\
&\bar{\alpha}_i(t_1,\tilde{s}) = \bar{q}_i(t_1,\tilde{s}),\ \tilde{s}\in\tilde{\mathcal{S}}
\end{aligned}
$$

Outline

1 Introduction

2 MFTG problem

3 COVID-19 and Spread of SARS-COV-2

COVID-19 and Spread of SARS-COV-2

Virus: Severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2)
Disease: coronavirus disease 2019 (COVID-19)

Context-awareness

- Epidemic
  - mobility map/pattern
  - locality, position, infection status, age, gender, family size
  - pre-existing health conditions (per locality)
  - probability of being exposed (coughing, sneezing, surface contact)
  - hospital capacity, testing capacity (per week)
  - pandemic-related decision-making for authorities, firms and individuals
- Economic
  - decision-making for authorities, firms and individuals related to the economics of the Covid-19 pandemic
  - Individuals: Consumption (demand), Firms: Production (supply)
  - Authorities: Budget allocation

Authorities

The authority in locality l decides on

- Migration rules (lockdown, confinement, isolation, curfew)
- Budget allocation and incentives

Multiple objectives for authority in locality l :

- reduce the number of deaths,
- reduce the number of infected,
- reduce economic losses
- maximize the number of recovered

$$\begin{cases} \text{Long-term objectives subject to} \\ \text{constraints} \\ \text{dynamics} \end{cases}$$

Consumption goods firms

Firms (producing essential, moderate-essential, less-essential goods) in locality l decide on

- Production, Total working hours
- Budget constraint

Multiple objectives for j :

- Reduce the number of infected employees ,
- Maximize profit

$$\begin{cases} \text{Long-term objectives subject to} \\ \text{constraints} \\ \text{dynamics} \end{cases}$$

Each individual's risk-awareness

An individual in locality l decides on

- Meeting rate, local movement, consumption

Multiple objectives for an individual

- reduce the risk of being infected,
- maximize the probability of being recovered (once infected)
- reduce the risk of exposing the others (if co-opetitive),
- reduce economic losses

$$\begin{cases} \text{Long-term objectives subject to} \\ \text{constraints} \\ \text{dynamics} \\ \text{(locality, position, infection status, age,} \\ \quad \text{pre-existing health, gender, height, family size)} \end{cases}$$

for susceptible: economic: $\log(\frac{w \ a_1 + ls_g}{1+tr_g}) + b_1(v, spot) - \frac{\hat{\lambda}_1}{2} a_1^2 - \hat{\lambda}_2 \ var_\mu(a_1)$

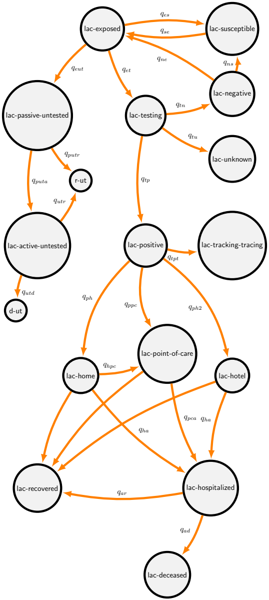
pandemic: $s - q \ var_\mu(a_2) - \bar{q}\bar{a}_2^2 - h_0(n_{infectious}(\mathcal{B}_\epsilon(x)))\|v\|^2 - \hat{\lambda}_3 \ var_\mu(v)$

Example of state dynamics

$m(t, l, x, s, z, c)$ be a measure of people in

- $t=$ time
- $l = (l_1, l_2, l_3)$ be a locality of a specific area l_3 of the city l_2 of region/country l_1
- (x_1, x_2, h) a geographical position of an individual inside the area l_3.
- s=infection status= (susceptible, exposed, testing, testing+, testing-, testing-unknown, isolation-home, isolation-hotel, isolation-point-of-care, hospitalized (active), recovered, dead)
- z=age
- c=pre-existing conditions ($\emptyset$, diabetes(type 2), hypertension, diabetes+high blood pressure), gender, height, family size

Dynamics at a given location with testing



Inflow-Outflow of infection status

$$
\left(\begin{array}{ll}
s_1 = s: & -s_l\lambda_{se} + e_l\lambda_{es} + t_{-,l}\delta_l\lambda_{t_-s} + t_{u,l}\delta_l\lambda_{t_u s} + tr_l\delta_l\lambda_{tr,s} \\
s_2 := e: & s_l\lambda_{se} + p_l(1-\delta_l)\lambda_{pe} + a_l(1-\delta_l)\lambda_{ae} \\
 & +t_{-,l}\delta_l\lambda_{t_-e} + t_{u,l}\delta_l\lambda_{t_u e} - \delta_l\lambda_{et}(l)e_l - e_l\lambda_{es} - e_l(1-\delta_l)\lambda_{ea} \\
 & -e_l(a_{4l}(o|s,z)(1-\delta_l)\lambda_{Sm} + a_{2l}(m|p,z)(1-\delta_l)\lambda_{ep,l}p_l) + r_l(1-\delta_l)\lambda_{re} \\
s_3 := p: & -p_l(1-\delta_l)\lambda_{pe} - 2p_l(a_{2l}(m|p,z)\bar{a}_{2l}(m|p))(1-\delta_l)\lambda_{pa}\frac{np_l-1}{n} + a_l(1-\delta_l)\lambda_{ap}\frac{p_l}{h_l+p_l} \\
 & +e_l(a_{4l}(o|e,z)(1-\delta_l)\lambda_{Sm} + a_{2l}(m|p,z)(1-\delta_l)\lambda_{ep,l}p_l) + r_l(1-\delta_l)\lambda_{rp} \\
s_4 := a: & 2p_l(a_{2l}(m|p,z)\bar{a}_{2l}(m|p))(1-\delta_l)\lambda_{pa}\frac{np_l-1}{n} - a_l(1-\delta_l)\lambda_{ae} - a_l(1-\delta_l)\lambda_{ap}\frac{p_l}{h_l+p_l} \\
 & +e_l(1-\delta_l)\lambda_{ea} - a_l(1-\delta_l)\lambda_{ad}(l,z,c) - a_l(1-\delta_l)\lambda_{ar}(l,z,c) \\
s_5 := r: & (1-\delta_l)\lambda_{ar}(l,z,c)a_l - r_l(1-\delta_l)\lambda_{rp} - r_l(1-\delta_l)\lambda_{rs} \\
s_6 := d: & (1-\delta_l)\lambda_{ad}(l,z,c)a_l \\
s_7 := t: & \delta_l\lambda_{tt,t}(l,c)tt_l + \delta_l\lambda_{et}(l)e_l - t_l\delta_l\lambda_{t,t_+} - t_l\delta_l\lambda_{t,t_-} - t_l\delta_l\lambda_{t,t_u} \\
s_8 := t_-: & t_l\delta_l\lambda_{t,t_-} - t_{-,l}\delta_l\lambda_{t_-s} - t_{-,l}\delta_l\lambda_{t_-e} \\
s_9 := t_u: & t_l\delta_l\lambda_{t,t_u} - t_{u,l}\delta_l\lambda_{t_u s} - t_{u,l}\delta_l\lambda_{t_u e} \\
s_10 := t_+: & t_l\delta_l\lambda_{t,t_+} - t_{+,l}\delta_l\lambda_{t_+,tt} - t_{+,l}\delta_l\lambda_{t_+,th_1} - t_{+,l}\delta_l\lambda_{t_+,tpc} - t_{+,l}\delta_l\lambda_{t_+,th_2} \\
s_{11} := tt: & \delta_l \text{ contacting, verifying, making a list, suggesting test} \\
s_{12} := th_1: & t_{+,l}\delta_l\lambda_{t_+,th_1} - th_{1,l}\delta_l\lambda_{th_1,thos}a_{t1}\bar{a}_{t1}(\bar{th}_{1,l}-\frac{1}{n}) - th_{1,l}\delta_l\lambda_{th_1,tr} - th_{1,l}\delta_l\lambda_{th_1,tpc} \\
s_{13} := tpc: & t_{+,l}\delta_l\lambda_{t_+,tpc} - tpc_l\delta_l\lambda_{tpc,thos}a_{t3}\bar{a}_{t3}(t\bar{p}c_l-\frac{1}{n}) - tpc_l\delta_l\lambda_{tpc,tr} \\
s_{14} := th_2: & t_{+,l}\delta_l\lambda_{t_+,th_2} - th_{2,l}\delta_l\lambda_{th_2,thos}a_{t2}\bar{a}_{t2}(\bar{th}_{2,l}-\frac{1}{n}) - th_{2,l}\delta_l\lambda_{th2,tr} \\
s_{15} := thos: & th_{1,l}\delta_l\lambda_{th_1,thos}a_{t1}\bar{a}_{t1}(\bar{th}_{1,l}-\frac{1}{n}) + tpc_l\delta_l\lambda_{tpc,thos}a_{t3}\bar{a}_{t3}(t\bar{p}c_l-\frac{1}{n}) \\
 & +th_{2,l}\delta_l\lambda_{th_2,thos}a_{t2}\bar{a}_{t2}(\bar{th}_{2,l}-\frac{1}{n}) - thos_l\delta_l\lambda_{thos,tr} - thos_l\delta_l\lambda_{thos,td} \\
s_{16} := tr: & thos_l\delta_l\lambda_{thos,tr} + th_{1,l}\delta_l\lambda_{th_1,tr} + tpc_l\delta_l\lambda_{tpc,tr} + th_{2,l}\delta_l\lambda_{th_2,tr} - tr_l\delta_l\lambda_{tr,s} \\
s_{17} := td: & thos_l\delta_l\lambda_{thos,td}
\end{array}\right)
$$

Dynamics

$$\begin{cases} m_t = \text{local (spatial) mobility} \\ \text{local spread of the disease: coughing and sneezing} \\ \text{local spread of the disease via surface contact} \\ \text{intra-city mobility in } l_3 \\ \text{inter-city mobility from/to } l_2 \\ \text{international connectivity from/to } l_1 \\ \text{aging} \\ \text{population death} \\ \text{new borns with age 0,} \\ \text{number of hospitalized patients vs hospital capacity in } l_3 \\ \text{number of tests vs (weekly) test capacity in } l_3 \\ \text{initial population distribution} \\ \text{city architecture constraint} \\ \text{map constraint} \\ \text{Allowed/feasible mobility areas/exits} \end{cases}$$

Kolmogorov equation

$$
\begin{aligned}
m_t &= -div_x(vm) + \frac{1}{2} trace[(\sigma^*\sigma m)_{xx}] + \int_\Theta [m(l, x-\gamma, s, z, t) - m(l, x, s, z, t) + \langle m_x, \gamma\rangle]\nu(d\theta) \\
&+ \int_{\mathcal{B}_v(x)} m(l, dx', s, z, t) q(x, s, z; (l, x', s, z, t, m, v)) - m \int_{\mathcal{B}_v(x)} q(dx', s, z; (l, x, s, z, t, m, v)) \\
&+ \sum_{s'\neq s} \int m(l, dx', s', dz', t) \tilde{q}(x, s, z; (l, x', s', z', t, m, \lambda, a, \delta)) - m \sum_{s'\neq s} \int \tilde{q}(dx', s', dz'; (l, x, s, z, t, m, \lambda, a, \delta)) \\
&+ \sum_{l'\neq l_3} \int m(l', dx', s', z, t) \eta_3(l, x, s, z; (l', x', s', z, t, m)) - m \sum_{l'\neq l_3} \int \eta_3(l', dx', s', z; (l, x, s, z, t, m)) \\
&+ \sum_{l'\neq l_2} \int m(l', dx', s', z, t) \eta_2(l, x, s, z; (l', x', s', z, t, m)) - m \sum_{l'\neq l_2} \int \eta_2(l', dx', s', z; (l, x, s, z, t, m)) \\
&+ \sum_{l'\neq l_1} \int m(l', dx', s', z, t) \eta_1(l, x, s, z; (l', x', s', z, t, m)) - m \sum_{l'\neq l_1} \int \eta_1(l', dx', s', z; (l, x, s, z, t, m)) \\
&- \tilde{v} m_z - d(l, z, m) \\
& m(l, x, s, z, c, 0) = m_0(l, x, s, z, c) \\
& m(l, x, s, 0, c, t) = \int \tilde{b}(l, x, z, t, m) m(l, x, s, dz, c, t), \\
& \int \sum_c m(l, dx, hospitalized, dz, c, t) \leq \bar{c}_l(t) \\
& \int \sum_c m(l, dx, testing, dz, c, t) \leq \bar{t}_l(t) \\
& x \in \mathcal{D},
\end{aligned} \tag{1}
$$

Alain Bensoussan, Boualem Djehiche, Hamidou Tembine, Sheung Chi Phillip Yam: Mean-Field-Type Games with Jump and Regime Switching. Dynamic Games and Applications 10(1): 19-57 (2020)

Interaction term:

- at work
- shopping areas
- home

Transition from susceptible to exposed:

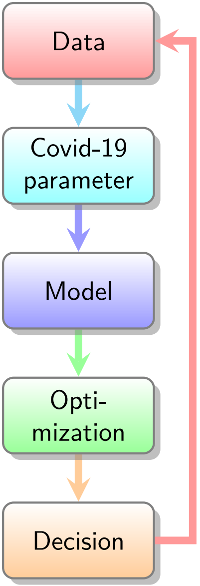
$$m(I, x, s, z, c, t) \int \beta(x', x) \text{ infectious}_I(t, dx')$$

$\beta(x', x)$: physical distancing

$$\beta(x', x) = \mathbb{1}_{\{d_2(x,x') \leq \epsilon\}}$$

infectious states: $\{p, a, t_+, th_1, tpc, th_2, hosp\}$

Integrated model



Model calibration, verification and validation

Data at: CSSE **https://github.com/CSSEGISandData/COVID-19/**

- Measurement data set of $y(l_1, s, t_k)$ for $\int \sum_{l_2,l_3} \sum_c \sum_c m(l, dx, s, dz, c, t_k)$. The data y can be noisy/inaccurate. Measurement : $y(., t_k) \in \mathbb{R}^{5\times 195}$
- optimization step with the measurement $\hat{y}(l_1, t_k, s)$

$$\inf_{\lambda, m_0} \sum_s \int_0^{t_k} \| \int_x \int_z \sum_c m(l, dx, s, dz, c, t) - \hat{m}_l(s,t)\|_2^2 dt,$$

  where
  $\hat{m}_l(s,t) = \hat{m}_l(s, t_{i-1}) + \frac{t-t_{i-1}}{(t_i - t_{i-1})}\mathbb{I}_{(t_{i-1},t_i)}(t)(y(l,s,t_i) - \hat{m}_l(s,t_{i-1}))$ for $t \in (t_{i-1}, t_i)$.
- Trajectory with data-dependent parameters $m_t(l,x,s,z,c,t) = f(l,x,s,z,c,t,\lambda_{data}, a, v, \eta, \delta, m)$
- We repeat the procedure as new data comes

This leads to a data-driven MFTG model

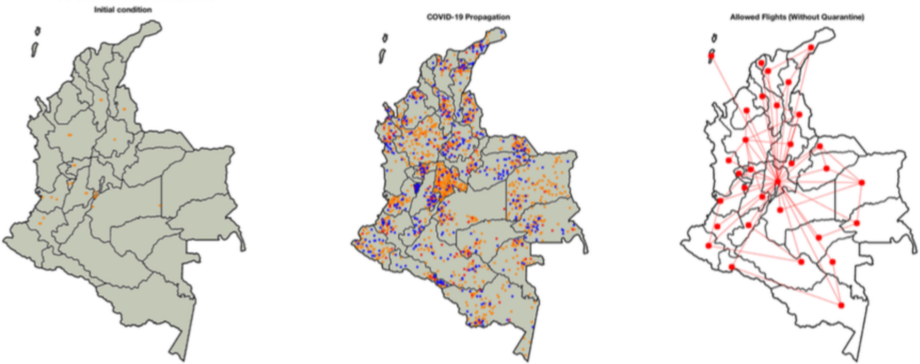
Implementation setup

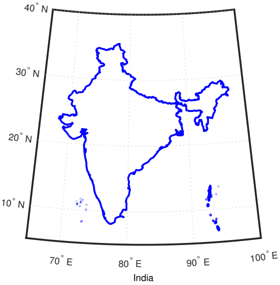
Grid on the OpenStreet map

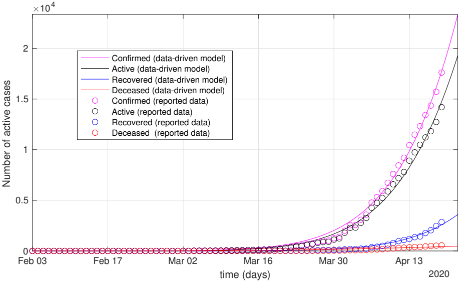
- $l_1 \in \{1, \ldots, 195\}$ countries
- $n(t_0) = \sum_l n_l(t_0) = 7.8\ 10^9$ people
- Local mobility on OpenStreet map complemented (with Facebook, Google, Telecom, local transportation data)
- IMF, WB, WHO, CDC data for Covid-19 comorbidities, economic policy per country
- Example of prevalent conditions: diabetes, hypertension, obesity.

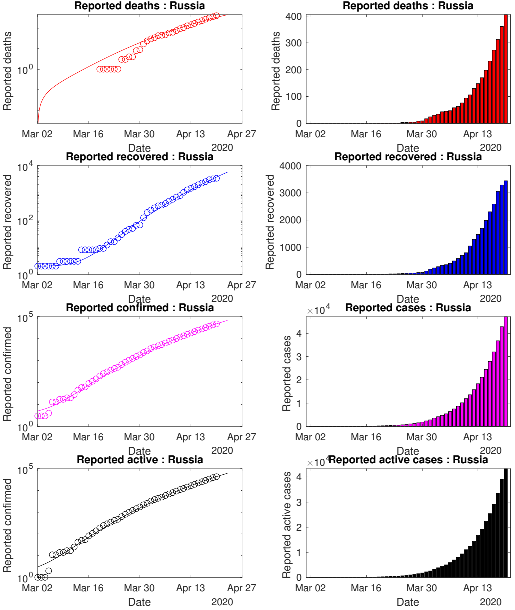
COVID-19 Data as of March 30

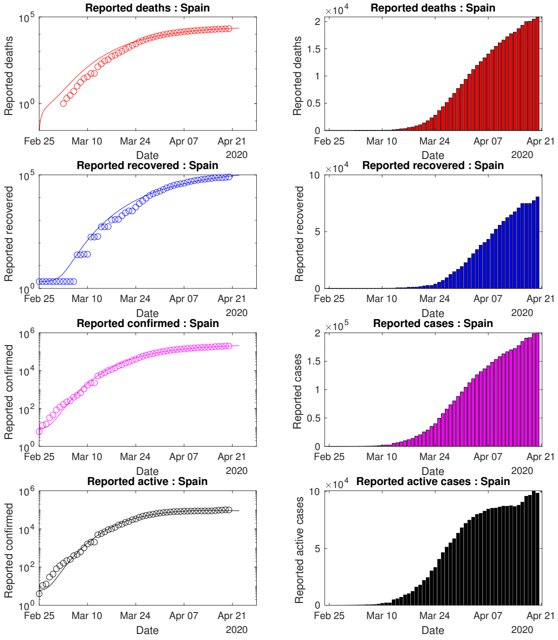
Effect of Local Mobility on Covid-19 Spread



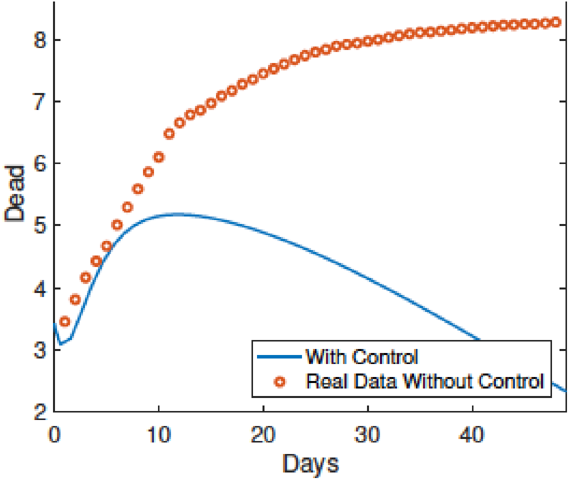

40° N
30° N
20° N
10° N
70° E
80° E
90° E
100° E
India


×10^4
Number of active cases
time (days)
Feb 03
Feb 17
Mar 02
Mar 16
Mar 30
Apr 13
2020
Confirmed (data-driven model)
Active (data-driven model)
Recovered (data-driven model)
Deceased (data-driven model)
Confirmed (reported data)
Active (reported data)
Recovered (reported data)
Deceased (reported data)


Reported deaths : Russia
Reported deaths
10^0
Mar 02
Mar 16
Mar 30
Apr 13
Apr 27
Date
2020
Reported deaths : Russia
Reported deaths
400
300
200
100
0
Mar 02
Mar 16
Mar 30
Apr 13
Date
2020
Reported recovered : Russia
Reported recovered
10^4
10^2
10^0
Mar 02
Mar 16
Mar 30
Apr 13
Apr 27
Date
2020
Reported recovered : Russia
Reported recovered
4000
3000
2000
1000
0
Mar 02
Mar 16
Mar 30
Apr 13
Date
2020
Reported confirmed : Russia
Reported confirmed
10^5
10^0
Mar 02
Mar 16
Mar 30
Apr 13
Apr 27
Date
2020
$\times 10^4$
Reported cases : Russia
Reported cases
4
3
2
1
0
Mar 02
Mar 16
Mar 30
Apr 13
Date
2020
Reported active : Russia
Reported confirmed
10^5
10^0
Mar 02
Mar 16
Mar 30
Apr 13
Apr 27
Date
2020
$\times 10^4$
Reported active cases : Russia
Reported active cases
4
3
2
1
0
Mar 02
Mar 16
Mar 30
Apr 13
Date
2020


Reported deaths : Spain
Reported deaths
Date
Reported recovered : Spain
Reported recovered
Reported confirmed : Spain
Reported confirmed
Reported cases : Spain
Reported cases
Reported active : Spain
Reported active cases : Spain
Reported active cases
Feb 25
Mar 10
Mar 24
Apr 07
Apr 21
2020

Data-driven model fixed parameter vs linear approximated strategies



Summary

- semi-explicit solution to some linear-quadratic MFTGs with $s, a_i, \int y\mu(t, dy), \int a_i(t, s)\mu(t, ds)$,
- semi-explicit solution to some nonlinear and/or non-quadratic MFTGs with
  - $s, a_i, \int y\mu(dy), \int a_i(t, s)\mu(ds)$,
  - cos, cosh, coth
  - Noise modelling: Brownian, Poisson, Fractional Brownian, Gauss-Volterra
- HJB system for basic MFTGs
- Master adjoint system (MASS) for basic MFTGs

THANK YOU