Numerical methods for non-linear Fokker Planck equations and applications to Mean Field Games

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joint work with
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High Dimensional Hamilton-Jacobi PDEs
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Outline

A nonlinear Fokker-Planck equation

The nonlinear FP equation

$$\begin{cases} \partial_t m - \frac{1}{2}\sum_{i,j}\partial_{x_i x_j}\left(a_{ij}[m](x,t)m\right) + \mathrm{div}(b[m](x,t)\,m) = 0 & \mathbb{R}^d \times \mathbb{R}^+ \\ m(\cdot,0) = m_0(\cdot) & \mathbb{R}^d \end{cases}$$

- b is a given vector field depending on m, non locally in space and possibly non locally in time
- $(a_{i,j}(m,x,t))$ is a given diffusion matrix (possible degenerate) depending on m, non locally in space and possibly non locally in time; such that

$$a_{i,j}(m,x,t) = \sum_{p=1}^{r} \sigma_{i,p}\cdot\sigma_{j,p} = (\sigma(\sigma)^\top)_{ij},$$

  for all $i,\ j = 1,\dots d$, where $r \in \mathbb{N}\setminus\{0\}$, and for all $p = 1,\dots,r$.
- the density of the initial law is given by m_0:

  $m_0 \geq 0$ and $\int_{\mathbb{R}^d} m_0(x)\mathrm{d}x = 1.$

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Some applications

- Non local interactions due to collective phenomena (biophysics, social behavior)
- Hughes model $b[m](x,t) = -f^2(m(x,t))Dv[m](x,t)$ where $v[m]$ is the solution of a stationary HJB

$$|Dv| = \frac{1}{f(m(x,t))}$$

- Mean Filed Games: $b[m](x,t) = -DH(Dv[m](x,t))$ where $v[m]$ is the solution of a backward HJB

$$\begin{cases} -\partial_t v - \frac{\sigma^2}{2}\Delta v + H(Dv) = f(x,m(t)) \\ v(x,T) = g(x,m(T)). \end{cases}$$

Probabilistic interpretation

(FPK) describes the evolution of the law of the diffusion processes $X(t) \in \mathbb{R}^d$

$$\begin{cases} \mathrm{d}X(t) & = & b(m, X(t), t)\mathrm{d}t + \sigma(m, X(t), t)\mathrm{d}W(t) \quad t \in [0, T], \\ X(0) & = & X_0, \end{cases}$$

where the $r-$dimensional Brownian motion $\{W\}$ independent of X_0, the distribution of X_0 is given by m_0.

- $b(m, x, t) = b(m(t), x, t)$ and $\sigma_{i,j}(m, x, t) = \sigma_{i,j}(m(t), x, t)$, the FPK equation is called McKean-Vlasov equation well-posedness (T. Funaki '84,S. Méléard '96).
- existence in the general case: first order (V.I. Bogachev, M. Rockner, and S. V. Shaposhnikov 2009), second order case (O.A. Manita and S.V. Shaposhnikov 2013)

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Representation formula for the Fokker Planck equation

(Linear case)$b[m](x,t) = b(x,t), \quad \sigma[m](x,t) = \sigma(x,t)$

(**Lip**) b and σ Lipschitz w.r. to x, uniformly in $t \in [0,T]$

Let Φ be the solution of

$$\mathrm{d}X(t) = b(X(t),t)\mathrm{d}t + \sigma(X(t),t)\mathrm{d}W(t), \;\; X(0) = X_0,$$

$$\Phi(\omega,x,0,t) = x + \int_0^t b(\Phi(\omega,x,0,s),s)\mathrm{d}s + \int_0^t \sigma(\Phi(\omega,x,0,s),t)\mathrm{d}W(s),$$

then

$$\boxed{m(t)(A) := \mathbb{E}\left(\Phi(\cdot,0,t)\sharp\bar{m}_0(A)\right)} \quad \forall\, A \in \mathcal{B}(\mathbb{R}^d), \quad t \in [0,T].$$

Analogously, we have that for each $h > 0$

$$m(t+h)(A) = \mathbb{E}\left(\Phi(\cdot,t,t+h)\sharp m(t)(A)\right) \quad \forall\, A \in \mathcal{B}(\mathbb{R}).$$

or equivalently, for $\phi \in \mathcal{C}_c^0(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \phi(x)\mathrm{d}\left(m(t+h)\right)(x) = \int_{\mathbb{R}^d} \mathbb{E}\left[\phi\left(\Phi(x,t,t+h)\right)\right]\mathrm{d}\left(m(t)\right)(x).$$

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Semi-discretization in time $d = 1$

Given $h > 0$, we set $t_k = kh$ for $k = 0, \dots, N_T$.

A random walk discretization of the Brownian motion $W(\cdot)$:

Weak Euler in dimensione $d = 1$:

$$\Phi_h^+(x, t_k) := x + hb(x, t_k) + \sigma(x, t_k)\sqrt{h},$$
$$\Phi_h^-(x, t_k) := x + hb(x, t_k) - \sigma(x, t_k)\sqrt{h}.$$

$$m(t_{k+1})(A) = \frac{1}{2}\left(\Phi^+(\cdot, t_k)\sharp m(t_k)(A)\right) + \frac{1}{2}\left(\Phi^-(\cdot, t_k)\sharp m(t_k)(A)\right)$$

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Fully-discrete scheme, $d = 1$

Given $\Delta x > 0$, we set $\mathcal{G}_{\Delta x} := \{x_i = i\Delta x, i \in \mathbb{Z}\}$ and $\mathcal{S}^{\Delta x,h} := \{(\mu_{i,k})_{i\in\mathbb{Z},\, k=0,\dots,N} \;;\; \mu_{i,k} \geq 0,\;\; \sum_{i\in\mathbb{Z}} \mu_{i,k} = 1\}$,

Discrete measure:

$m_k = \sum_{j\in\mathbb{Z}} m_{j,k}\delta_{x_j} \quad \forall\, k = 0,\dots,N-1.$

$$\sum_j \phi(x_j) m_{j,k+1} = \frac{1}{2}\sum_j \phi\left(\Phi^+(x_j,t_k)\right) m_{j,k} + \frac{1}{2}\sum_j \phi\left(\Phi^-(x_j,t_k)\right) m_{j,k}.$$

$\mathbb{P}_1$-projection : $\{\beta_i\}$ are $\mathbb{P}_1$-basis function, $\phi(x) = \beta_i(x)$.

$$m_{i,k+1} = \frac{1}{2}\sum_j \beta_i\left(\Phi^+_{j,k}\right) m_{j,k} + \frac{1}{2}\sum_j \beta_i\left(\Phi^-_{j,k}\right) m_{j,k}.$$

where

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Fully-discrete scheme for the non linear case

Given $\mu \in \mathcal{S}^{\Delta x,h}$

Non linear discrete characteristics

$$\Phi^{+}_{i,k}[\mu] := x_i + hb[\mu](x_i,t_k) + \sqrt{h}\sigma[\mu](x_i,t_k),$$
$$\Phi^{-}_{i,k}[\mu] := x_i + hb[\mu](x_i,t_k) - \sqrt{h}\sigma[\mu](x_i,t_k).$$

The discretization of (FPK) we propose is:

find $m \in \mathcal{S}^{\Delta x,h}$ such that

$$(S)\begin{cases} m_{i,0} = \bar{m}_0(E_i) \\ m_{i,k+1} = \frac{1}{2}\sum_{j\in\mathbb{Z}}\left[\beta_i(\Phi^{+}_{j,k}[m]) + \beta_i(\Phi^{-}_{j,k}[m])\right] m_{j,k} \\ \forall\, i \in \mathbb{Z}^d, \quad k = 0,\dots,N-1. \end{cases}$$

where $E_i = [x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}]$.

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$$(S)\begin{cases} m_{i,0} = \bar{m}_0(E_i) \\ m_{i,k+1} = \frac{1}{2}\sum_{j\in\mathbb{Z}}\left[\beta_i(\Phi^{+}_{j,k}[m]) + \beta_i(\Phi^{-}_{j,k}[m])\right] m_{j,k} \\ \forall\, i \in \mathbb{Z}^d, \quad k = 0,\dots,N-1. \end{cases}$$

where $E_i = [x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}]$.

Main properties

- Non-negative : $m_{i,k} \geq 0$ for $k = 0, \ldots, N-1, i \in \mathbb{Z}$
- Mass conservative : $\sum_i m_{i,k} = 1$ for $k = 0, \ldots, N-1$
- Generalizable to any dimension
- Generalizable to handle Dirichlet and Neumann Boundary conditions
- Generalizable to handle degeneracy of the diffusion matrix
- Large time steps are allowed: inverse CFL type condition

$$\frac{(\Delta x)^2}{h} \to 0$$

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Dual Problem

Kolmogorov forward equation (FP)

$$\begin{cases} \partial_t m = \frac{1}{2}\sum_{i,j} \partial_{x_i x_j}\left(a_{ij}(x)m\right) - \mathsf{div}\left(b(x)m\right) & \mathbb{R}^d \times (0,T] \\ m(\cdot,0) = m_0 \end{cases}$$

Kolmogorov backward equation(KB):

$$\begin{cases} -\partial_t u = \frac{1}{2}\sum_{i,j} a_{ij}(x)\partial_{x_i x_j} u + b(x)^\top Du & \mathbb{R}^d \times (0,T] \\ u(\cdot,T) = u_T \end{cases} \tag{1}$$

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L^* is the dual of L with respect to the L_2 inner product:

$$\int L(f) g \mathrm{d}x = \int L^*(g) f \mathrm{d}x$$

Dual Schemes $d = 1$

The SL scheme for FP can be written in vectorial form as

$$\boxed{\mu_{k+1} := B^* \mu_k}$$

where, $\mu_k = (\mu_{j,k})_k$ and $(B^*)_{i,j} = \frac{1}{2}\left(\beta_i\left(\Phi_{j,+}\right) + \beta_i\left(\Phi_{j,-}\right)\right)$.
The SL scheme for KB

$$v_{i,k} = \frac{1}{2}\left(I[v_{k+1}](\Phi_{i,+}) + I[v_{k+1}](\Phi_{i,-})\right) = \frac{1}{2}\sum_{j\in\mathbb{Z}}\left[\beta_j\left(\Phi_{i,+}\right) + \beta_j\left(\Phi_{i,-}\right)\right] v_{j,k+1}$$

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Main assumptions

(H)

- $\bar{m}_0 \in \mathcal{P}_2(\mathbb{R}^d)$.
- The maps b and σ are continuous.
- There exists $C > 0$ such that

$$|b(m,x,t)| + |\sigma(m,x,t)| \leq C(1+|x|) \ \forall\, m, \ \ x \in \mathbb{R}^{d_\ell}, \ t \in [0,T].$$

(Lip)

- b and σ are Lipschitz w.r. to x, uniformly in $t \in [0,T]$

Well posedeness

Proposition

Under assumption **(H)**, *there exists at least one solution* $m_{i,k} \in \mathcal{S}_h^{\Delta x}$ *of* (S).

Given $m_{i,k} \in \mathcal{S}^{\Delta x,h}$, we define its extension $m_{\Delta x}(t) \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$

$$m_{\Delta x}(t) := \left(\frac{t-t_k}{h}\right) \sum_{i\in\mathbb{Z}^d} m_{i,k+1}\delta_{x_i} + \left(\frac{t_{k+1}-t}{h}\right) \sum_{i\in\mathbb{Z}^d} m_{i,k}\delta_{x_i}$$

for $t \in [t_k, t_{k+1}[$ and $k = 0, \dots, N-1$.
Let us denote the Wasserstein distance by

$$d_1(\mu_1,\mu_2) = \sup\left\{ \int_{\mathbb{R}^d} f(x)\mathrm{d}(\mu_1-\mu_2)(x) \ ; \ f \in \mathsf{Lip}_1(\mathbb{R}^d) \right\}.$$

Convergence

Theorem

Under assumptions **(H)** *and* **(Lip)**, *and* $\frac{(\Delta x)^2}{h} \to 0$, *we have that as* $(\Delta x) \to 0$

$$(m_{\Delta x,h}) \to (m)$$

in $C([0,T],\mathcal{P}_1)$, *where* m *is solution of* (FPK) *(there exists at least one) and* $m_{\Delta x,h}$, *is solution of* (S).

Remark: The result holds in any dimension $d \geq 1$

Convergence non regular case

If **(Lip)** does not hold, and b, σ verifiy only **(H)**, it is necessary to regularize them, by using mollifiers.

$$b^\varepsilon[m](x,t) := \varphi_\varepsilon \star b[m](x,t), \quad \sigma^\varepsilon[m](x,t) := \varphi_\varepsilon \star \sigma[m](x,t)$$

We will apply this technique to approximate the solution of Mean Field Game Problem.

Find $m^\varepsilon \in \mathcal{S}^{\Delta x, h}$ such that

$$(S^\varepsilon) \begin{cases} m^\varepsilon_{i,0} = \bar{m}_0(E_i) \\ m^\varepsilon_{i,k+1} = \frac{1}{2} \sum_{j \in \mathbb{Z}} \left[\beta_i(\Phi^{+,\varepsilon}_{j,k}[m^\varepsilon]) + \beta_i(\Phi^{-,\varepsilon}_{j,k}[m^\varepsilon]) \right] m^\varepsilon_{j,k} \\ \forall\, i \in \mathbb{Z}^d, \quad k = 0, \dots, N-1. \end{cases}$$

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Under assumptions **(H)**, *and* $\frac{(\Delta x)^2}{h} \to 0, \frac{h}{\varepsilon^2} \to 0$ *we have that as* $(\Delta x, h, \varepsilon) \to 0$

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In the uniform elliptic case, the assumption $h = o(\varepsilon)$

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In the degenerate elliptic case, we need to construct approximations which are absolutely continuous w.r. to the Lebesgue measure.
If $d = 1$, uniform bound in L^∞ is shown for the approximated densitiy and a convergence result is proved.

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Mean Field Game

In this case the velocity field in the FP is

$$b[m](x,t) = Dv[m](x,t)$$

where $v[m]$ is the solution of the first equation in the following system:

$$(\text{MFG}) \quad \begin{cases} -\partial_t v - \sigma\Delta v + \frac{1}{2}|Dv|^2 = F(x, m(t)), & \text{in } \mathbb{R} \times (0,T), \\ \partial_t m - \sigma\Delta m - \text{div}\big(Dvm\big) = 0, & \text{in } \mathbb{R} \times (0,T), \\ v(x,T) = G(x, m(T)) & \text{in } \mathbb{R} \times \{T\} \\ m(0) = m_0. & \text{in } \mathbb{R} \times \{0\} \end{cases}$$

Model introduced independently by Huang-Malhamé-Caines and, independently, by Lasry-Lions in 2006.

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(H1) F and G are continuous.

(H2) There exists a constant $c_0 > 0$ such that for any $m \in \mathcal{P}_1$

$$\|F(\cdot, m)\|_{C^2} + \|G(\cdot, m)\|_{C^2} \leq c_0,$$

where $\|f(\cdot)\|_{C^2} := \sup_{x \in \mathbb{R}^d}\{|f(x)| + |Df(x)| + |D^2 f(x)|\}$.

(H3) The initial condition $m_0 \in \mathcal{P}_1$ is absolutely continuous w. r. to the Lebesgue measure, with density m_0 s.t. $\text{supp}(m_0) \subset B(0, c)$ and $\|m_0\|_\infty \leq c$, for $c > 0$.

(H4) The following *monotonicity* conditions hold true

$$\int_{\mathbb{R}^d} [F(x, m_1) - F(x, m_2)] \, \mathrm{d}[m_1 - m_2](x) \geq 0 \quad \text{for all} \quad m_1, m_2 \in \mathcal{P}_1$$

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Some references of Numerical Approximation of MFG

- Second order problem ($\sigma \neq 0$)
  - Y.Achdou, I.Capuzzo-Dolcetta ('10),
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  - A.Lachapelle, M.-T.Wolframm (Steepest descente approach for the optimal control problem),
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  - F.Camilli, F.J.Silva ('12) (semi-discrete Semi-Lagrangian scheme)
  - S. Hadikhanloo, F.J.Silva ('19) (Semi-Lagrangian type with no interpolation)
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SL scheme for HJB

- We use a Semi-Lagrangian scheme to approximate v.
- We call $v_{\Delta x}$ the resulting interpolated discrete value functions
- We regularize them by using space convolution

$$v^{\varepsilon}_{\Delta x,h}[m](\cdot,t) := \phi_{\varepsilon} * v_{\Delta x}[m](\cdot,t) \quad \forall\, t \in [0,T],$$

- We approximate the drift by

$$b[m](x,t) := -Dv^{\varepsilon}_{\Delta x}[m](x,t).$$

- If $\sigma = 0$ then $Dv^{\varepsilon_n}_{\Delta x_n}[m_n] \to Dv[m]$ a.e., the convergence result has been proved only for the case $d = 1$
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- If $\sigma = 0$ then $Dv^{\varepsilon_n}_{\Delta x_n}[m_n] \to Dv[m]$ a.e., the convergence result has been proved only for the case $d = 1$
- If $\sigma \neq 0$ then $Dv^{\varepsilon_n}_{\Delta x_n}[m_n] \to Dv[m]$ uniformly , the convergence is proved in general dimension.

Numerical test First order MFG

Domain $\Omega \times (0,T) = (-3,3) \times (0,5)$.
Running cost

$$F(x,t,m(t)) = d(x,\mathcal{D})^2 V_\delta(x,m(t)),$$

$V_\delta(x,m) = (\phi_\delta \star (\phi_\delta \star m))(x)$, $\phi_\delta(x) := \frac{1}{\delta\sqrt{2\pi}} \exp\left(-x^2/(2\delta^2)\right)$, $\delta = 0.01$
$d(x,\mathcal{D})$ is the distance function from the set $\mathcal{D} := [1,1.5] \cup [-2,-2.5]$.
Final cost: $G(x,T,m(T)) = F(x,T,m(T))$
Initial mass distribution:

$$m_0(x) = \frac{\nu(x)}{\int_\Omega \nu(x)dx} \quad \text{with} \quad \nu(x) = e^{-x^2/0.2}$$

Regularizing kernel $\phi_\varepsilon(x)$, with $\varepsilon = 0.15$.
Diffusion term $\sigma = 0$, first order MFG system
Discretization step $\Delta x = h = 0.02$.
Fix point: computed by a learning procedure as proposed by Cardaliaguet and Hadikhanloo.

Numerical test First order MFG

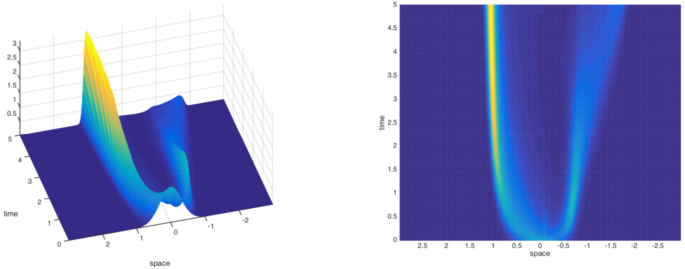

Figure: **Density evolution 3d and 2d view in the (x, t) domain**

Numerical test First order MFG

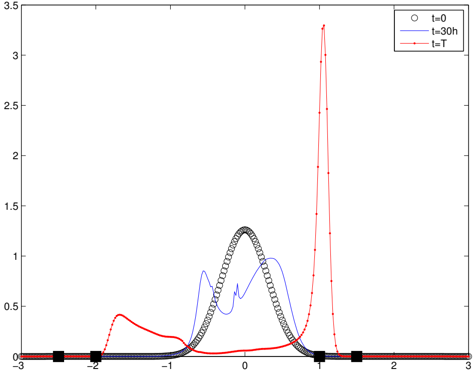

Figure: **Density at time** $t = 0, 0.6, T$ (black squares on the x axis represents the 'meeting areas')

A new Hughes type model

In this model the velocity field in the FP is

$$b[m](x,t) = Dv[m](x,t)$$

where $v[m]$ is the solution of the first equation in the system.

$$\begin{cases} -\partial_s v(x,s) + \frac{1}{2}|Dv(x,s)|^2 = F(x,s,m(t))) & \text{in } \mathbb{R}\times(t,T), \\ \partial_t m - \mathrm{div}(Dvm) = 0 & \text{in } \mathbb{R}\times(0,T), \\ v(x,T) = G(x,m(t)) & \text{for } x\in\mathbb{R}, \\ m(\cdot,0) = m_0(\cdot) \end{cases} \tag{2}$$

where

$$F(x,s,m(t)) = d(x,\mathcal{P})^2 V_\delta(x,m(t)).$$

Since $b[m](x,t)$ depends on $m(s)$ only at time past t we get an Explicit scheme

Numerical test new Hughes type model

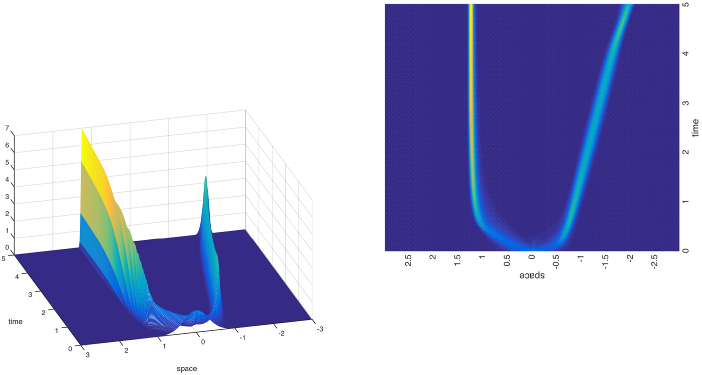

Figure: **Density evolution 3d and 2d view in the (x, t) domain**

Numerical test new Hughes type mode

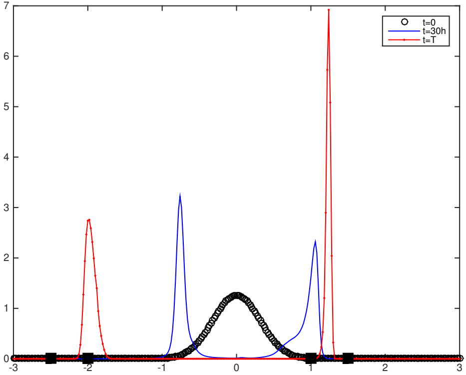

Figure: **Density at time** $t = 0, 30h, T$ (black squares on the x axis represents the 'meeting areas')

Numerical test new Hughes type mode

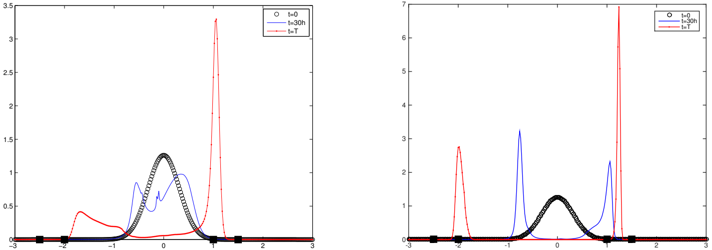

Figure: **MFG(left) vs Hughes type model (right)**

Table of Contents

A Lagrange Galerkin scheme for the continuity equation

$$(CE)\begin{cases}\partial_t m + \mathrm{div}\,(b(x,t)m) = 0 & (0,T)\times\mathbb{R}^d,\\ m(0,\cdot) = m_0(\cdot) & \mathbb{R}^d\end{cases}$$

where

- **(A1)** Let us suppose $b(x,t)\in L^\infty(0,T;(W^{1,\infty}(\mathbb{R}^d))^d)$
- **(A2)** $m_0(\cdot)\in L^2(\mathbb{R}^d)$ with compact support

Representation formula: for any $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\phi(x)m(x,t_{k+1})\mathrm{d}x = \int_{\mathbb{R}^d}\left[\phi\left(\Phi(x,t_k)\right)\right]\ m(x,t_k)\mathrm{d}x$$

where $\Phi(x,t_k)$ are the forward characteristics, solving

$$\begin{cases}\dot X(s) = b(X(s),s), & s\in[0,h],\\ X(t_k)=x,\end{cases}$$

A Lagrange Galerkin scheme for the continuity equation

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A Lagrange Galerkin scheme for the continuity equation

- Set

$$\Phi_h(x,t_k) := x + hb(x,t_k)$$

Semi-discrete scheme

$$\int_{\mathbb{R}^d} \phi(x)m(x,t_{k+1})\mathrm{d}x \quad = \int_{\mathbb{R}^d} \left[\phi\left(\Phi_h(x,t_k)\right)\right] \; m(x,t_k)\mathrm{d}x$$

- Structured mesh $\mathcal{G}_{\Delta x} := \{x_i = i\Delta x; i \in \mathbb{Z}^d\}$ with $\Delta x > 0$ a given space step,
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$$m_{\Delta x}(x,t_k) := \sum_{i\in\mathbb{Z}^d} m_{i,k}\beta_i(x) \qquad \forall\, x \in \mathbb{R}^d,$$

for some weights $\{m_{i,k} \mid k = 0,\dots,n, \;\; i \in \mathbb{Z}^d\} \subseteq \mathbb{R}$

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A Lagrange Galerkin scheme for the continuity equation

We project the semi discrete scheme

$$\int_{\mathbb{R}^d} \phi(x) m(x, t_{k+1}) \mathrm{d}x \quad = \int_{\mathbb{R}^d} \left[\phi\left(\Phi_h(x, t_k)\right)\right] \; m(x, t_k) \mathrm{d}x$$

in $V_{\Delta x}$: find $m_{i,k}$, with $i \in \mathbb{Z}^d$ and $k = 0, ..., N$

$$(LG) \begin{cases} \sum_{i \in \mathbb{Z}^d} m_{i,k+1} \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\mathrm{d}x = \sum_{i \in \mathbb{Z}^d} m_{i,k} \int_{\mathbb{R}^d} \beta_j(\Phi_h(t_k, x))\beta_i(x)\mathrm{d}x \\ \sum_{i \in \mathbb{Z}^d} m_{i,0} \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\mathrm{d}x = \int_{\mathbb{R}^d} m_0(x)\beta_j(x)\mathrm{d}x. \end{cases}$$

The (LG) scheme for (CE) can be written in vectorial form as

$$\boxed{M m_{k+1} := B m_k}$$

where, $m_k = (m_{j,k})_j$,

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Ref. Morton, Priestley, Suli ('88)

A Lagrange Galerkin scheme for the continuity equation

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$$\int_{\mathbb{R}^d} \phi(x) m(x, t_{k+1}) \mathrm{d}x \quad = \int_{\mathbb{R}^d} [\phi(\Phi_h(x, t_k))] \; m(x, t_k) \mathrm{d}x$$

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A Lagrange Galerkin scheme for the continuity equation

Proposition

Under assumption **(A1)-(A2)**, *the following assertions hold true:*

(i) *Well-posedness* *There exists a unique solution to (LG).*

(ii) *Non-negativity* *If* $\{\beta_i\}_{i\in\mathbb{Z}^d}$ *is s.t.* $\int_{\mathbb{R}^d}\beta_i(x)\beta_j(x)=\delta_{i,j}$ *and* $\beta_i(x)\geq 0$, *we have* $m_{i,k}\geq 0$

(iii) *Mass conservation* $\int_{\mathbb{R}^d} m_{\Delta x}(t_k,x)\mathrm{d}x=1$

(iv) *L^2-stability* *If* h *is sufficiently small, there exists* $C>0$, *s.t.*

$$\|m_{\Delta x}(t_k,\cdot)\|_{L^2}\leq C\|m_0\|_{L^2}$$

(v) *Equi-continuity* *Suppose* $(\Delta x)^2=O(h)$, *for all* $t_1,t_2\in[0,T]$, *we have that*

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First order MFG

We consider the first order case $\sigma = 0$ and the particular case of a quadratic Hamiltonian:

$$
\text{(MFG)} \begin{cases} -\partial_t v(x,t) + \frac{1}{2}|Dv(x,t)|^2 = F(x, m(t)) & \mathbb{R}^d \times (0,T) \\ v(x,T) = G(x, m(T)) & \mathbb{R}^d \\ \partial_t m(x,t) - \operatorname{div}(Dv(x,t)m(x,t)) = 0 & \mathbb{R}^d \times (0,T) \\ m(0) = m_0 & \mathbb{R}^d \end{cases}
$$

SL scheme for HJB

- We use a Semi-Lagrangian scheme to approximate $v[m]$.
- We call $v_{\Delta x}[m]$ the resulting interpolated discrete value functions
- We regularize them by using space convolution

$$v^{\varepsilon}_{\Delta x}[m](\cdot,t) := \phi_{\varepsilon} * v_{\Delta x}[m](\cdot,t) \quad \forall\, t \in [0,T],$$

Lemma

For every $t \in [0,T]$, the following assertions hold true:

(i) *Lipschitz property The function $v^{\varepsilon}_{\Delta x}[\mu](\cdot,t)$ is Lipschitz with constant d_0 independent of $(\Delta x, h, \mu, t)$.*

(i) *Semiconcavity There exists $d_1 > 0$ independent of $(\Delta x, h, \varepsilon, \mu, t)$, such that*

$$\langle D^2 v^{\varepsilon}_{\Delta x}[\mu](x,t)y, y\rangle \leq d_1 \left(1 + \frac{\Delta x^2}{\varepsilon^4}\right)|y|^2 \qquad \forall\, x, y \in \mathbb{R}^d. \qquad (3)$$

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A Lagrange Galerkin scheme for deterministic MFG

Given $\mu \in C([0,T];\mathcal{P}_1)$ and $\varepsilon > 0$ let us define

$$\Phi_h^\varepsilon[\mu](x,t_k) := x - hDv_{\Delta x}^\varepsilon[\mu](x,t_k)$$

We propose the following scheme for (MFG):

$$\boxed{\text{Find } \mu = (\mu_i^k) \text{ such that } \quad \mu_{i,k} = m_{i,k}^\varepsilon[\mu]}$$

where $m_{i,k}^\varepsilon[\mu]$ is defined as

$$\begin{cases} \sum_{i\in\mathbb{Z}^d} m_{i,k+1}^\varepsilon \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\mathrm{d}x = \sum_{i\in\mathbb{Z}^d} m_{i,k}^\varepsilon \int_{\mathbb{R}^d} \beta_j(\Phi_h^\varepsilon[\mu](t_k,x))\beta_i(x)\mathrm{d}x \\ \sum_{i\in\mathbb{Z}^d} m_{i,0}^\varepsilon \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\mathrm{d}x = \int_{\mathbb{R}^d} m_0(x)\beta_j(x)\mathrm{d}x \end{cases}$$

The (LG) scheme for (MFG) can be written in vectorial form as

$$\boxed{Mm_{k+1}^\varepsilon := B^\varepsilon m_k^\varepsilon}$$

where, $m_k^\varepsilon = (m_{j,k}^\varepsilon)_j$, $(B^\varepsilon)_{i,j} = \int_{\mathbb{R}^d} \beta_j(\Phi_h^\varepsilon[\mu](t_k,x))\beta_i(x)\mathrm{d}x$.

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Find $\mu = (\mu_i^k)$ such that $\mu_{i,k} = m_{i,k}^\varepsilon[\mu]$

where $m_{i,k}^\varepsilon[\mu]$ is defined as

$$\begin{cases} \sum_{i\in\mathbb{Z}^d} m_{i,k+1}^\varepsilon \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\mathrm{d}x = \sum_{i\in\mathbb{Z}^d} m_{i,k}^\varepsilon \int_{\mathbb{R}^d} \beta_j(\Phi_h^\varepsilon[\mu](t_k,x))\beta_i(x)\mathrm{d}x \\ \sum_{i\in\mathbb{Z}^d} m_{i,0}^\varepsilon \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\mathrm{d}x = \int_{\mathbb{R}^d} m_0(x)\beta_j(x)\mathrm{d}x \end{cases}$$

The (LG) scheme for (MFG) can be written in vectorial form as

$$Mm_{k+1}^\varepsilon := B^\varepsilon m_k^\varepsilon$$

where, $m_k^\varepsilon = (m_{j,k}^\varepsilon)_j$, $(B^\varepsilon)_{i,j} = \int_{\mathbb{R}^d} \beta_j(\Phi_h^\varepsilon[\mu](t_k,x))\beta_i(x)\mathrm{d}x$.

A Lagrange Galerkin scheme for deterministic MFG

Given $\mu \in C([0,T];\mathcal{P}_1)$ and $\varepsilon > 0$ let us define

$$\Phi_h^\varepsilon[\mu](x,t_k) := x - hDv_{\Delta x}^\varepsilon[\mu](x,t_k)$$

We propose the following scheme for (MFG):

$$\boxed{\text{Find } \mu = (\mu_i^k) \text{ such that } \quad \mu_{i,k} = m_{i,k}^\varepsilon[\mu]}$$

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Convergence analysis LG for MFG

Key Property: Semiconcavity of $v^{\varepsilon}_{\Delta x}$

Proposition

Under assumptions **(H1)-(H2)-(H3)**, *the following assertions hold true:*

(i) *L^2-stability If h is sufficiently small, there exists a constant $C > 0$, such that*

$$\|m^{\varepsilon}_{\Delta x}(t_k, \cdot)\|_{L^2} \leq c\|m_0\|_{L^2}$$

(i) *Equicontinuity Suppose $(\Delta x)^2 = O(h)$, for all $t_1, t_2 \in [0, T]$, we have that*

$$\mathbf{d}_1(m^{\varepsilon}_{\Delta x}(t_1), m_{\Delta x}(t_2)) \leq C|t_1 - t_2|.$$

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Convergence analysis LG for MFG

Theorem

Under assumptions **(H1)-(H2)-(H3)**, *consider a sequence of positive numbers* $\Delta x_n, h_n, \varepsilon_n$ *satisfying that*

$$\Delta x_n/\varepsilon_n^2 \leq C \quad (\Delta x_n)^2/h_n \to 0, \quad h_n/\varepsilon_n \to 0$$

as $\varepsilon_n \downarrow 0$. *Let* $\{m_{\Delta x_n}^{\varepsilon_n}\}_{n\in\mathbb{N}}$ *be a sequence of solutions of (LG) for the corresponding parameters* $\Delta x_n, h_n, \varepsilon_n$.
Then every limit point in $C([0,T];\mathcal{P}_1)$ *and in* $L^2\left(\mathbb{R}^d \times [0,T]\right)$*-weak of* $m_{\Delta x_n}^{\varepsilon_n}$ *(there exists at least one) solves (MFG).*
In particular, if **(H4)** *holds we have that* $m_{\Delta x_n}^{\varepsilon_n} \to m$ *(the unique solution of (MFG)) in* $C([0,T];\mathcal{P}_1)$ *and in* $L^2\left(\mathbb{R}^d \times [0,T]\right)$*-weak.*

Possible Setting of parameters

$$h = \Delta x, \quad \varepsilon = \sqrt{\Delta x}$$

LG+area-weighting

In general, the integral $\int_{\mathbb{R}^d} \beta_j(\Phi_h^\varepsilon(t_k, x))\beta_i(x)\mathrm{d}x$ can not be exactly computed.

- inexact integration: quadrature formulae
- area-weighting: approximate the trajectories neglecting the deformation caused by advection and compute exact integration

Using area-weighting + basis $\beta_i^0 \in \mathbb{P}_0$

$$(M)_{i,j} = \int_{\mathbb{R}^d} \beta_i^0(x)\beta_j^0(x)\mathrm{d}x = \delta_{i,j}$$

$$\begin{aligned}(B_{aw}^\varepsilon)_{i,j} = \quad & \int_{\mathbb{R}^d} \beta_j^0(x - x_i + \Phi_h^\varepsilon(t_k, x_i))\beta_i^0(x)\mathrm{d}x = \\ & \int_{\mathbb{R}^d} \beta_j^0(x - hDv_{\Delta x}^\varepsilon[\mu](t_k, x_i))\beta_i^0(x)\mathrm{d}x = \\ & \beta_j^1(x_i - hDv_{\Delta x}^\varepsilon[\mu](t_k, x_i)) = (B^*)_{i,j}.\end{aligned}$$

Ref. Morton, Priestley, Suli ('88), Ferretti ('12)

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Conclusions and Future Works

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- we have proposed a scheme for non-linear non-local FP
- it allows large time steps and is explicit
- it applies to get existence and numerical approximation of a new Hughes model
- it applies to approximate second order possibly degenerate MFG
- we have proposed a LG scheme for MFG first order getting convergence in arbitrary dimension

Future works

- Extension to non-linear non-local FP with general Neumann condition and application on MFG (with E. Calzola and F.J.Silva)
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References

- F.Camilli, F.J.Silva, *A semi-discrete in time approximation for a model first order-finite horizon mean field game problem*, Network & Heterogeneous Media, 7 (2012), 263-277.
- E.Carlini, F.J.Silva, *A fully discrete semi–Lagrangian scheme for a first order mean field games problem*, SIAM Journal on Numerical Analysis, 52 (1), pp. 45-67, 2014.
- E.Carlini, F.J.Silva, *A Semi-Lagrangian scheme for a degenerate second order Mean Field Game system*, DCDS-A, 2015.
- E.Carlini, F.J.Silva, *On the discretization of some nonlinear Fokker-Planck-Kolmogorov equations and applications*, SIAM Journal on Numerical Analysis.SIAM J. Numer. Anal., 56(4), 2148–2177, 2018
- E.Carlini, F.J.Silva *A Fully-Discrete Scheme for Systems of Nonlinear Fokker-Planck-Kolmogorov Equations*. PDE Models for Multi-Agent Phenomena. Springer INdAM Series, 2018