Graphon Mean Field Games and the GMFG Equations: A Dynamical Equilibrium Theory for Large Populations on Large Networks

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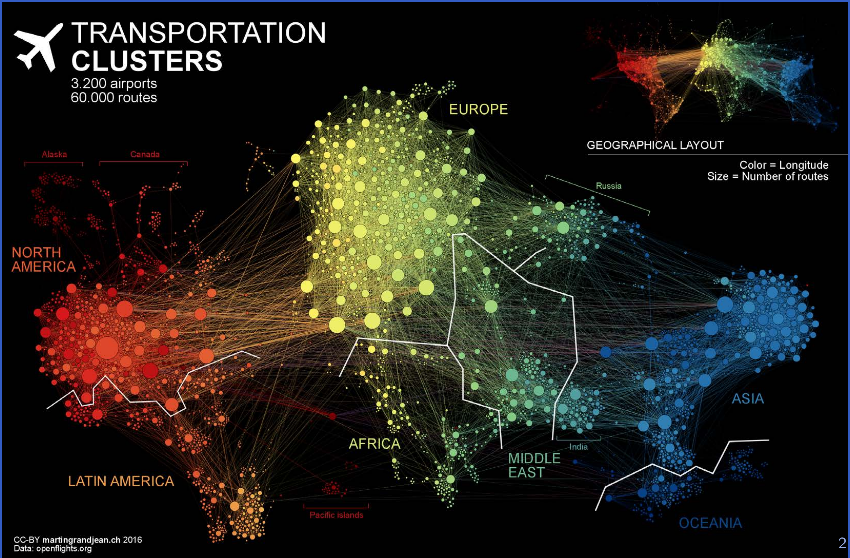
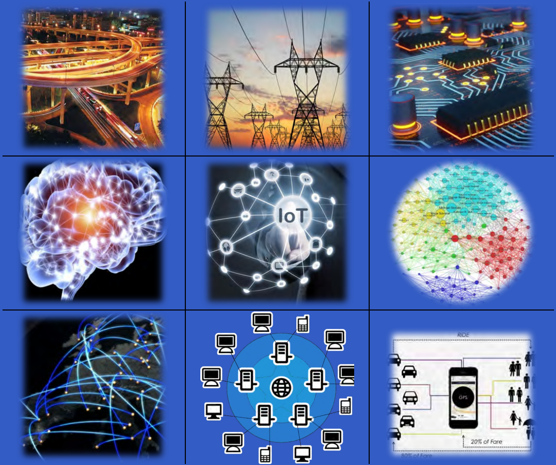
Graphon Mean Field Games: Motivation

Non-uniform Connections - Relatively Dense Population Clusters



Motivation for a Graphon Theory of Systems and MFG

In the contemporary designed environment and in nature networks are ubiquitous. Moreover artificial networks are growing in size and complexity. Consider: Online Social Networks, Brain Networks, Power Grid Networks, Transportation Networks, IoT, etc.



Motivation for a Graphon Theory of Systems and MFG

The Standard MFG Model:

States, costs, vector fields, disturbances, etc, are simply averaged when, as a mass, they play a role in the behaviour of a large population system.

This is equivlant to an implicit assumption that the individual agents are distributed over the nodes of a very large scale network (VLSN) which is completely connected (an edge for each node pair) and all edges have equal weight (i.e. a uniform graph).

Assumption Often Does Not Hold.

None of the network examples depicted earlier satisfy this assumption globally, but some do so (approximately) locally.

So standard MFG theory is often applicable in large networks locally (i.e. cluster by cluster), but is not necessaril valid globally.

Motivation for a Graphon Theory of Systems and MFG

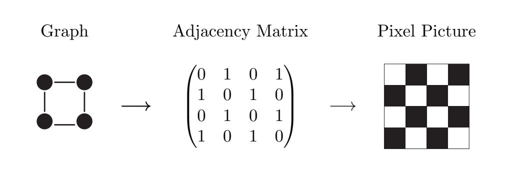
Focus of Interest:

Complex networks characterized by:

- Large number of nodes (millions, billions, etc)
- Complex connections, typically non-uniform and locally asymptotically dense. Density assumption adopted for current work, but most natural and artificial systems are sparse.
- Intrinsically capable of growth in size

The recently developed mathematical theory of graphons provides a methodology for analyzing arbitrarily complex networks.

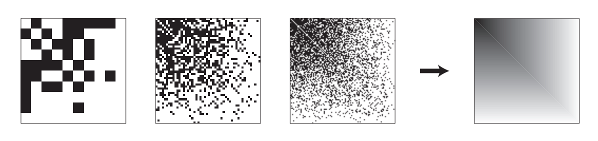
Locally dense theory well established. Locally sparse theory is in development in terms of basic theory and applications.



Graph, Adjacency Matrix, Pixel Picture

Pixel picture for N node graph on the unit square $[0,1] \times [0,1]$.

Square elements have sides of length $\frac{1}{N}$.



Uniform Attachment Graph Sequence (Each Cycle: $N-1$ node graph; new node: attached with prob. $1/N$ to each old $N-1$ node, and for all old unattached pairs, attach them with prob. $1/N$.) Convergence to a Limit Graphon.

Graphons (Lovasz, AMS 2012) : bounded symmetric Lebesgue measurable functions

$$\mathbf{W} : [0,1]^2 \to [0,1].$$

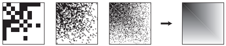
Graphons may be interpreted as weighted edge graphs on the vertex set $[0,1]$.

Classes of Graphon Spaces $\mathcal{W}_0 := \{\mathbf{W} : [0,1]^2 \to [0,1]\}$

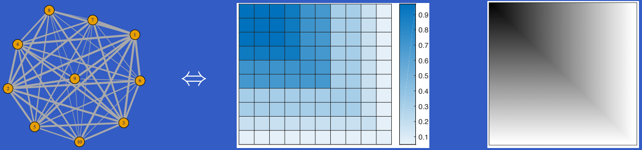
$$\mathcal{W}_I := \{\mathbf{W} : [0,1]^2 \to I\}$$

Introduction to Graphons

Example': Uniform Attachment Graphon



Uniform Attachment Graphon: $\mathbf{U}(x, y) = 1 - \max(x, y)$, $x, y \in [0, 1]$.



(i) N - node Weighted Graph G_N Generated from $\mathbf{U}$ by sampling (i,j) locations for the weights, (ii) Corresponding Stepfunction, (iii) Graphon Limit $\mathbf{U}$

Cut norm

$$\|\mathbf{W}\|_\square := \sup_{M,T\subset[0,1]} \left| \int_{M\times T} \mathbf{W}(x,y)dxdy \right| \tag{1}$$

Cut metric

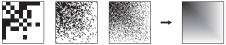
$$d_\square(\mathbf{W},\mathbf{V}) := \inf_{\phi} \|\mathbf{W}^\phi - \mathbf{V}\|_\square \tag{2}$$

d_{L^2} metric

$$d_{L^2}(\mathbf{W},\mathbf{V}) := \inf_{\phi} \|\mathbf{W}^\phi - \mathbf{V}\|_2 \tag{3}$$

where $\mathbf{W}^\phi(x,y) = \mathbf{W}(\phi(x),\phi(y))$.

Since $\|\mathbf{W}\|_\square \leq \|\mathbf{W}\|_{L^2}$ for any graphon $\mathbf{W}$, convergence in d_{L^2} implies convergence in $d_\square$. Converse holds subject to symmetry and common normed limit conditions.



Theorem (Lovasz and Szegedy, 2006; LL AMS2012)

The graphon spaces $(\mathcal{W}_I, d_\square)$*,* I *any closed interval in* $\mathbb{R}$*, are compact.*

Previous and Related Work

Related Literature

- Huang M.Y. , P.E. Caines & R.P. Malhamé, "The NCE (Mean Field) Principle with Locality Dependent Cost Interactions." *IEEE Trans. on Automatic Control,* Vol. 55, No. 12, December, 2010, pp. 2799 - 2805.
- Guant, O. (2015). Existence and uniqueness result for mean field games with congestion effect on graphs. Applied Mathematics & Optimization, 72(2), 291-303.
- Delarue, F. (2017). Mean field games: A toy model on an Erdös-Rényi graph. ESAIM: Proceedings and Surveys, 60, 1-26.
- Parise, F., & Ozdaglar, A. (2018). Graphon games. arXiv preprint arXiv:1802.00080.
- Caines P. E. & Huang, M.Y. "Graphon Mean Field Games and the GMFG Equations." Proc. 57th IEEE Conference on Decision and Control, Miami Beach, FL, USA, December, 2018, pp 4129- 4134.
- Caines P. E. & Huang, M.Y. "Graphon Mean Field Games and the GMFG Equations: $epsilon$-Nash Equilibria". Proc. 58th IEEE Conference on Decision and Control, Nice, France, December, 2019. Published as Cahier du GERAD: G-2019-81, November, 2019, pp. 286-292.
- Carmona, R., Cooney, D., Graves, C., & Lauriere, M. (2019). Stochastic Graphon Games: I. The Static Case. arXiv preprint arXiv:1911.10664.
- Bayraktar, E., Chakraborty, S., & Wu, R. (2020). Graphon mean field systems. arXiv preprint arXiv:2003.13180.

Basic Formulation of Nonlinear MFG Systems

Basic Problem Formulation:

- Notation: Integer valued subscript for minor agents $\{\mathcal{A}_i : 1 \leq i \leq N\}$.
- $\mathcal{A}_i$: State : $x_i^N \in \mathbb{R}^n$ Control: $u_i^N \in \mathbb{R}^m$,

Dynamics of each Minor agent $\mathcal{A}_i$ is averaged over all other agents' states:

$$dx_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} f(t, x_i^N(t), u_i^N(t), x_j^N(t))dt + \frac{1}{N}\sum_{j=1}^{N} \sigma(t, x_i^N(t), x_j^N(t))dw_i(t), \qquad 1 \leq i \leq N.$$

MFG Nonlinear Formulation

Performance Functions Minor Agents:

$$J_i^N(u_i^N; u_{-i}^N) := \mathbb{E}\int_0^T \Big(\frac{1}{N}\sum_{j=1}^N l[t, x_i^N(t), x_0^N(t), u_i^N(t), x_j^N(t)]\Big)dt.$$

- If there exists a major agent it would have non-negligible influence on the mean field (mass) behaviour of the minor agents and so the mean field would no longer be a deterministic function of time. Omitted in dynamics and performance functions in current work.

Information and Control Patterns: Complete Decentralization

Local to Agent i:$1 \leq i \leq N$

Controls are only functions of $\mathcal{A}_i's$ Observations and $\mathcal{A}_i$ can only observe its own state history.

Formally: A_i's set of control inputs $\mathcal{U}_i$ consists of all feedback controls $u_i(t)$ taking values in $\mathcal{U}_{loc,i}$: $U \subset\subset \mathbb{R}^m$ and
Adapted to : $\mathcal{F}_i := \sigma(x_i(\tau); \tau \leq t)$,$1 \leq i \leq N$

Underlying sample space:

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t^N\}_{t\geq 0}, \mathbb{P})$: a complete filtered probability space
- $\mathcal{F}_t^N := \sigma\{x_j(0), w_j(s) : 1 \leq j \leq N, 1 \leq s \leq t\}$.

Basic Formulation MFG Theory

Controlled McKean-Vlasov Equations
(when the limits exist):

Infinite popn. dynamics: x state of generic agent

$$dx_t = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(t, x_t, u_t, x_j^N(t))dt + \sigma dw_t$$

$$= f[x_t, u_t, \mu_t]dt + \sigma dw_t$$

$$f[x, u, \mu_t] := \mathbb{E} f[x_t, u_t, y_t] = \int_{\mathbb{R}} f(x, u, y)\mu_t(dy)$$

$\mu_t(\cdot) \quad = \quad$ measure $=$ probability distribution at t of the state of a generic member of the infinite limit population.

Note: Markovian jointly in (x, μ)

Basic Formulation of Nonlinear MFG Systems

Summary: Controlled MKV Dynamics and MKV Performance Function for a Generic Agent:

- (If the limits exist) Infinite population limit dynamics (Markovian jointly in (x, μ)):

$$
\begin{aligned}
dx_t &= f[x_t, u_t, \mu_t]dt + \sigma dw_t \quad \mu_0 = \mathcal{L}(x_0) \\
f[x, u, \mu_t] &\triangleq \int_{\mathbb{R}} f(x, u, y)\mu_t(dy)
\end{aligned}
$$

- (If the limits exist) Infinite population limit cost:

$$
J(u, \mu) \triangleq \mathbb{E} \int_0^T l[x_t, u_t, \mu_t]dt
$$

where $\mu_t(\cdot) \quad = \quad$ measure $=$ probability distribution at t of the state of a generic member of the population.

Mean Field Game Equations

Assume:
(a) for any control law u for a generic agent the infinite popn. limits exist for the given dynamics and performance functions,
(b) there exists an infinite popn. Nash equilibrium.
Then:
(i) the generic agent best response is generated by an HJB eqn, and
(ii) the corresponding generic agent state distribution (measure μ, density p assumed to exist) is generated by an FPK equation:

[MF-HJB] $$-\frac{\partial V}{\partial t} = \inf_{u \in U} \left\{ f[x,u,\mu] \frac{\partial V}{\partial x} + l[x,u,\mu] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}$$
$$V(T,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}$$

[MF-FPK] $$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial \{ f[x,u,\mu] p(t,x) \}}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p(t,x)}{\partial x^2}$$

[MF-BR] $$u_t = \varphi(t, x_t | \mu_t), \quad (t,x) \in [0,T] \times \mathbb{R}$$

Basic Mean Field Game MV HJB-FPK Theory

Theorem : Existence, Uniqueness and $\epsilon-$Nash Property

Subject to technical conditions:
(i) (HMC 06 LL 06) The MKV MFG Equations have a unique solution with the best response control generating a unique Nash equilibrium given by

$$u^0(t) = \varphi(t, x_t|\mu_t).$$

(ii) Furthermore (HMC 06),
$\forall \epsilon > 0 \ \exists N(\epsilon)$ s.t. $\forall N \geq N(\epsilon)$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}} J_i^N(u_i, u_{-i}^0) \leq J_i^N(u_i^0, u_{-i}^0),$$

for all $1 \leq i \leq N < \infty$, where $u_i \in \mathcal{U}$ is adapted to $\mathcal{F}^N := \{\sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)\}$.

From Mean Field Game MV HJB-FPK Theory to Graphon MFG Theory

Basic Modelling Hypotheses for Graphon Mean Field Game Theory

(1) At every node in an asymptotically infinite network system sequence $\{G^N\}$:

(2) An asymptotically infinite population of symmetric agents interacts locally as in classical MFG.

(3) The local averaged dynamics and costs are averaged globally over all network nodes.

(4) Averaging weights in finite graph case given by: $\{G^N; 1 \leq N < \infty\}$.
Asymptotically by the limit graphon G^∞.

The Graphon Mean Field Game Equations (i)

$$\text{[HJB]}(\alpha) \quad -\frac{\partial V_\alpha(t,x)}{\partial t} = \inf_{u \in U} \Big\{ \widetilde{f}[x,u,\mu_G;g_\alpha] \frac{\partial V_\alpha(t,x)}{\partial x} + \widetilde{l}[x,u,\mu_G;g_\alpha] \Big\} + \frac{\sigma^2}{2} \frac{\partial^2 V_\alpha(t,x)}{\partial x^2},$$

$$V_\alpha(T,x) = 0, \qquad (t,x) \in [0,T] \times \mathbb{R}^n, \quad \alpha \in [0,1],$$

$$\text{[FPK]}(\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} = -\frac{\partial \{ \widetilde{f}[x, u^0(x_\alpha,\mu_G;g_\alpha) p_\alpha(t,x) \}}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p_\alpha(t,x)}{\partial x^2}, \qquad \mu_\alpha(0,dx) = \mu_0(dx),$$

$$\text{[BR]}(\alpha) \quad u^0(x_\alpha,\mu_G;g_\alpha) = \arg\inf_u H(x_\alpha,u,\mu_G;g_\alpha) =: \varphi(t, x_\alpha | \mu_G; g_\alpha)$$

The Graphon Mean Field Game Equations (ii)

The *graphon local mean field* μ_α, the corresponding set of all the *local mean fields* $\mu_G = \{\mu_\beta; 0 \leq \beta \leq 1\}$, and the graphon function $g_\alpha = \{g(\alpha, \beta); 0 \leq \beta \leq 1\}$ are inter-related by the FPK and the defining integral relation

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] := \int_{[0,1]} \int_{\mathbb{R}} f(x_\alpha, u_\alpha, x_\beta) g(\alpha, \beta) \mu_\beta(dx_\beta) d\beta$$

which gives the complete graphon mean field dynamics via the sum

$$\tilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] := f_0(x_\alpha, u_\alpha) + f[x_\alpha, u_\alpha, \mu_G; g_\alpha].$$

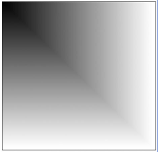
The graphon mean field cost functions $\widetilde{l}[x, u, \mu_G; g_\alpha]$ are defined similarly.

Graphon to Classical Mean Field Games : GMFG to MFG

We retrieve the simple standard MFG framework when the agents' dynamics and costs are uniform, and, further, the network is totally connected with uniform link weights giving the uniform graphon $\{g(\alpha, \beta) = 1; 0 \leq \alpha, \beta \leq 1\}$.
Since then GMFG equations equations have the unique solution (see below) with all local graphon mean fields equal, i.e. $\mu_{t,\alpha} =: \mu_t$, for all α.

Image of a non-uniform graphon with function

$$g(\alpha, \beta) = 1 - \max(\alpha, \beta),$$
$$\alpha, \beta \in [0, 1]$$



Graphon Mean Field Games : GMFG Conditions I

We need to restrict $\mu_G(\cdot)$ to a specific class.
$\{\mu_G(t), 0 \leq t \leq T\}$ is from the admissible set $\mathcal{M}_{[0,T]}$ if
(M1) for each fixed t, $\int_B \mu_\beta(t, dy)$ is a Lebesgue measurable function of β;
(M2) there exists $\eta \in (0, 1]$ such that for any bounded and Lipschitz continuous function ϕ on $\mathbb{R}$,

$$\sup_{\beta \in [0,1]} \left| \int_{\mathbb{R}} \phi(y)\mu_\beta(t_1, dy) - \int_{\mathbb{R}} \phi(y)\mu_\beta(t_2, dy) \right| \leq C_h |t_1 - t_2|^\eta$$

where C_h may be selected to depend only on the Lipschitz constant Lip(ϕ) for ϕ.
Condition (M1) ensures integrations with respect to $d\beta$ are well defined. By condition (M2), the drift term in the HJB equation has a certain time continuity, which facilitates the existence analysis of the best response.

Graphon Mean Field Games : GMFG Conditions II

(H1) U is a compact set.
(H2) $f(x,u,y)$ and $l(x,u,y)$ ($f_0(x,u)$ and $l_0(x,u)$, resp.) are continuous and bounded functions on $\mathbb{R} \times U \times \mathbb{R}$ ($\mathbb{R} \times U$, resp.), and are Lipschitz continuous in (x,y) (in x, resp.) unif. in u.
(H3) For f_0, f and l_0, l, their first and second derivatives with respect to x are all uniformly continuous and bounded.
(H4) $f(x,u,y)$ ($f_0(x,u)$, resp.) is Lipschitz continuous in u, uniformly with respect to (x,y) (to x, resp.).
(H5) For any $q \in \mathbb{R}$, $\alpha \in [0,1]$ and any probability measure ensemble μ_G satisfying (M1), the set

$$S(x,q) = \arg\min_u [q(\tilde{f}[x,u,\mu_G; g_\alpha]) + \widetilde{l}[x,u,\mu_G; g_\alpha]]$$
$$= \arg\min_u [q(f_0(x,u) + f[x,u,\mu_G; g_\alpha]) + \widetilde{l}[x,u,\mu_G; g_\alpha]]$$

is a singleton, and the resulting u as a function of (x,q), is Lipschitz continuous in (x,q), uniformly wrt μ_G and α.

Graphon Mean Field Games : GMFG Conditions III

Assumptions on the graphon function $g(\alpha, \beta)$.

(H6) For any bounded and measurable function $h(\beta)$, the function $\int_0^1 g(\alpha, \beta) h(\beta) d\beta$ is continuous in $\alpha \in [0, 1]$.

(H7) The best response as a bounded and continuous function of (t, x) depends continuously on $\alpha \in [0, 1]$ (by the norm of $C([0, T] \times \mathbb{R}; U)$), where x stands for the state of the α-agent.

(H8) All agents have i.i.d. initial states with distribution $\mu_0(dx)$

Convergence of the finite graph sequence

(H9)

$$\lim_{k \to \infty} \sum_j \left| \frac{1}{M_k} g^k_{C_i, C_j} - \int_{\beta \in I_j} g_{I_i^*, \beta} d\beta \right| = 0,$$

where I_i^* means the midpoint of the interval I_i.

Graphon Mean Field Games : GMFG Thm 1

Theorem: Existence and Uniqueness of Solutions to the GMFG Equation Systems (PEC, Huang, CDC 2018)

Subject to conditions (H1)-(H5), and a parameter dependent contraction gain condition, there exists a unique solution to the graphon dynamical GMFG equations which

(i) gives the feedback control best response (BR) strategy $\varphi(t, x_t|\mu_G; g_\alpha)$ *depending only upon the agent's state and the graphon local mean fields* (i.e. $(x_t, \mu_G; g_\alpha)$), *and*

(ii) *generates a Nash equilibrium.*

Graphon Mean Field Games : GMFG Thm 2

Theorem: ϵ-Nash Equilibria for GMFG System (PEC, Huang, CDC 2019)

Let the conditions of Theorem 1 hold together with conditions (H6) - (H9).

Assume the "mid-point rule" is used to generate finite graph-finite population control laws from the GMFG infinite graph-infinite population solutions.

Then the joint strategy $\{u_i^o(t) = \varphi(t, x_t | \mu_G; g_\alpha)\}$ *yields an ϵ-Nash equilibrium for all ϵ, namely,*

$\forall \epsilon > 0 \ \exists N(\epsilon), K(\epsilon)$ s.t. $\forall N \geq N(\epsilon), k \geq K(\epsilon)$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}} J_i^N(u_i, u_{-i}^0) \leq J_i^N(u_i^0, u_{-i}^0),$$

where $u_i \in \mathcal{U}$ *is adapted to* $\mathcal{F}^N := \{\sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)\}$.

LQG-GMFG Example - Finite Population (1)

Linear Quadratic Gaussian - GMFG Systems: Example

Individual Agent's Dynamics:

$$dx_i = (Ax_i + Dz_i + Bu_i)dt + \Sigma dw_i, \quad 1 \leq i \leq N.$$

- x_i: state of the ith agent
- u_i: control
- w_i: disturbance (standard Wiener process)
- $\mathcal{V}_k$: set of vertices $\{1, ..., N_k\}$
- C_ℓ: set of agents in the ℓth cluster

For $i \in C_q$ and adjacency matrix $M = [m_{q\ell}]$:

$$z_i = \frac{1}{|\mathcal{V}_k|} \sum_{\ell \in \mathcal{V}_k} m_{q\ell} \frac{1}{|C_\ell|} \sum_{j \in C_\ell} x_j$$

LQG-GMFG Example - Finite Population (2)

Individual Agent's Cost:

$$J_i(u_i,\nu_i) \triangleq \frac{1}{2}\mathbb{E}\int_0^T \Big[[(x_i-\nu_i)^\intercal Q(x_i-\nu_i)+u_i^\intercal R u_i]dt$$
$$+(x_i(T)-\nu_i(T))^\intercal Q_T(x_i(T)-\nu_i(T))\Big], \quad 1\le i\le N,$$

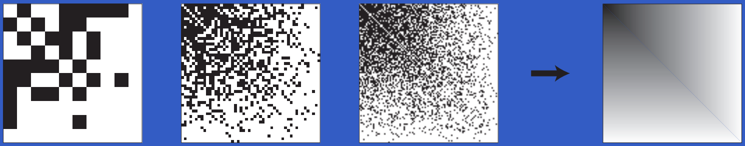
where $Q, Q_T \ge 0, R > 0$, and $\nu_i \triangleq \gamma(z_i+\eta)$ is the tracked process.

Main features in finite population finite graph case:

- Agents are coupled via dynamics and costs over a finite bidirectional graph of clusters
- Agent i's tracked process ν_i:
  - i stochastic
  - ii depends on other agents' control laws
  - iii depends on the location in the graph of x_i's cluster

LQG-GMFG Example - Infinite Population (1)

The sequence and limit of the underlying graph sequence chosen for this example: The Uniform Attachement Graph (LL2012)



Mean field coupling at agent α in the limit:

$$z_\alpha = \int_{[0,1]} [M(\alpha,\beta) \int_{R^n} x_\beta \mu_\beta(dx_\beta)] d\beta, \qquad \alpha,\beta \in [0,1]$$

LQG-GMFG Example - Infinite Population (2)

Assume the LGQ GMFG Equations have a unique solution. [Need contraction method modified from general nonlinear GMFG case.]

Individual Agent's Dynamics:

$$dx_\alpha = (Ax_\alpha + Dz_\alpha + Bu_\alpha)dt + \Sigma dw_\alpha, \quad \alpha \in [0,1].$$

Individual Agent agent α Cost:

$$J_\alpha(u_\alpha, \nu_\alpha) \triangleq \frac{1}{2}\mathbb{E}\int_0^T \Big[[(x_\alpha - \nu_\alpha)^\intercal Q(x_\alpha - \nu_\alpha) + u_\alpha^\intercal R u_\alpha]dt \\ +(x_\alpha(T) - \nu_\alpha(T))^\intercal Q_T(x_\alpha(T) - \nu_\alpha(T))\Big]$$

where $Q, Q_T \geq 0, R > 0$ and $\nu_\alpha \triangleq \gamma(z_\alpha + \eta)$.

Graphon local mean field for UA Graphon (UA soln. assumed):

$$z_\alpha = \int_{[0,1]} \Big[(1 - \max(\alpha,\beta)) \int_{R^n} x_\beta \mu_\beta(dx_\beta)\Big] d\beta, \qquad \alpha, \beta \in [0,1].$$

LQG-GMFG Example - Infinite Population Summary (3)

Optimal tracking (BR) control for any agent in cluster C_α:

$$
\begin{aligned}
u_\alpha(t) &= -R^{-1}B^\intercal[\Pi_t x_\alpha(t) + s_\alpha(t)] \\
-\dot{\Pi}_t &= A^\intercal \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^\intercal \Pi_t + Q, \quad \Pi_T = Q_T \\
-\dot{s}_\alpha(t) &= \left(A - BR^{-1}B^\intercal \Pi_t\right)^\intercal s_\alpha(t) + \Pi_t D z_\alpha(t) - Q\nu_\alpha(t), \\
s_\alpha(T) &= -Q_T \nu_\alpha(T)
\end{aligned}
$$

Graphon local mean field (mean) and tracked process (cost coupling)

$$
\nu_\alpha \triangleq \gamma(z_\alpha + \eta), \quad \alpha \in [0,1] \quad z_\alpha = \int_{[0,1]} M(\alpha,\beta)\bar{x}_\beta d\beta
$$

$$
\bar{x}_\beta \triangleq \lim_{|C_\beta|\to\infty} \frac{1}{|C_\beta|} \sum_{j\in C_\beta} x_j = \int_{R^n} x_\beta \mu_\beta(dx_\beta)
$$

The GMFG scheme closes with the local mean state process of x_α

$$
\dot{\bar{x}}_\alpha = (A - BR^{-1}B^\intercal \Pi_t)\bar{x}_\alpha + D z_\alpha - BR^{-1}B^\intercal s_\alpha, \quad \alpha \in [0,1].
$$

First Steps Towards the Master Equation for LQ-GMFGs

Individual Agent's Dynamics: In dimension $n = 1$, $\forall t \in [0, T]$

$$dx_\alpha(t) = \big(ax_\alpha(t) + bu_\alpha(t) + cz_\alpha(t)\big)dt, \quad \epsilon^\alpha \sim \mathcal{N}(0,1), \quad \alpha \in [0,1].$$

Individual Agent's Cost: let $q, \bar{q} \geq 0, r > 0$

$$J_\alpha(u_\alpha, z_\alpha) \triangleq \mathbb{E}\Big[\int_0^T \big[\frac{q}{2}(x_\alpha(t) - z_\alpha(t))^2 + \frac{r}{2}|u_\alpha(t)|^2\big]dt + \frac{\bar{q}}{2}(x_\alpha(T) - z_\alpha(T))^2\Big]$$

Graphon local mean field (mean) at agent α for a graph $g(\alpha, \beta)$:

$$z_\alpha = \int_{[0,1]} \big[g(\alpha,\beta) \int_R x_\beta \mu_\beta(dx_\beta)\big] d\beta, \qquad \alpha \in [0,1].$$

First Steps Towards the Master Equation for LQ-GMFGs

Theorem: Existence of Solutions to the LQ-GMFGs (Foguen-Tcheundom, PEC, Huang)

There are solutions $(u_{\alpha,o}(t), \mu_{\alpha,o}(t))_{(\alpha,t)\in[0,1]\times[0,T]}$ to LQ-GMFGs, if and only if, there are solutions $(x_{\alpha,o}(t), y_{\alpha,o}(t))_{(\alpha,t)\in[0,1]\times[0,T]}$ to the following Forward Backward Stochastic Differential Equation (FBSDEs) :

$$dx_{\alpha,o}(t) = \Big(ax_{\alpha,o}(t) - \frac{b^2}{r}y_{\alpha,o}(t) + c\mathbb{E}\big[\bar{x}_{\alpha,o,g}(t)\big]\Big)dt, \tag{4}$$

$$dy_{\alpha,o}(t) = -\Big(ay_{\alpha,o}(t) + qx_{\alpha,o}(t) - q\mathbb{E}\big[\bar{x}_{\alpha,o,g}(t)\big]\Big)dt \tag{5}$$

$$y_{\alpha,o}(T) = \bar{q}x_{\alpha,o}(T) - \bar{q}\mathbb{E}\big[\bar{x}_{\alpha,o,g}(T)\big] \quad x_{\alpha,o}(0) = \xi^\alpha$$

where, for all $(\alpha, t) \in [0,1] \times [0,T]$,

$$\bar{x}_{\alpha,o,g}(t) = \int_0^1 g(\alpha,\beta)x_{\beta,o}(t)d\beta, \qquad \text{and}$$

$$\mu_{\alpha,o}(t) = \mathcal{L}\big(x_{\alpha,o}(t)\big), \quad u_{\alpha,o}(t) = -\frac{b}{r}y_{\alpha,o}(t).$$

First Steps Towards the Master Equation for LQ-GMFGs

Theorem: Existence of Solutions to the Master Equation for LQ-GMFGs (Foguen-Tcheundom, PEC, Huang)

Let $(x_{\alpha,o}(t), y_{\alpha,o}(t))_{(\alpha,t)\in[0,1]\times[0,T]}$ be the unique solution to the above FBSDEs, then there exists a unique Master Field

$$\begin{aligned} \mathbf{U} : [0,1] \times [0,T] \times \mathbb{R} \times \mathcal{P}_2\big(\mathbb{R}\big) &\longrightarrow \mathbb{R} \\ (\alpha, t, x, m) &\mapsto \mathbf{U}(\alpha, t, x, m) \end{aligned} \tag{6}$$

such that $\forall(\alpha, t) \in [0,1] \times [0,T]$,

$$y_{\alpha,o}(t) = \mathbf{U}\big(\alpha, t, x_{\alpha,o}(t), \mathcal{L}\big(\bar{x}_{\alpha,o,g}(t)\big)\big). \tag{7}$$

First Steps Towards the ME for LQ-GMFGs (cont'd)

Theorem (cont'd): Existence of Solutions to the ME LQ-GMFGs: Form of the Master Equation

The Master Field $\mathbf{U}$ satisfies the LQ-GMFG Master Equation:

$$\partial_t \mathbf{U}(\alpha,t,x,m) + \left(a\mathbf{U}(\alpha,t,x,m) + qx - q\int_{\mathbb{R}} v dm(v) \right) + \left(ax - \frac{b^2}{r}\mathbf{U}(\alpha,t,x,m) + c\int_{\mathbb{R}} v dm(v) \right) \partial_x \mathbf{U}(\alpha,t,x,m) + \int_{\mathbb{R}} \left(aw - \frac{b^2}{r}\int_0^1 g(\alpha,\beta)\mathbf{U}(\beta,t,x,m)d\beta + c\int_0^1 g(\alpha,\beta) \int_{\mathbb{R}} v dm(v) d\beta \right) \partial_\mu \mathbf{U}(\alpha,t,x,m)(w) dm(w) = 0, \quad (8)$$

with the terminal condition,

$$\mathbf{U}(\alpha,T,x,m) = \bar{q}x - \bar{q}\int_{\mathbb{R}} v dm(v).$$

Future Directions and Open Problems

- Directed Graphons and Sparse Graphons of infinite networks: graphenes and graphomes.
- The Master Equation formulation of GMFG theory.
  LQG GMFG case: Foguen et al.
- Estimation methodology for VLSNs and graphon limits: especially for epidemiology (initiation: FLUTE travel data for epidemiology, Salii et al.), power grids and finance.
- Implications for GMFG properties of Global and Local geometry and topology, and of standard classes of finite graphs and their limits.
- Solvability: role of spectral and eigenspace information, low rank graphon operators following (linear) Graphon Control theory work (Gao et al).