Resolvent Degree and the Search for Lower Bounds

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Braids, Polynomials and Hilbert's 13th Problem
IPAM
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Ongoing joint work with Benson Farb and Mark Kisin.

Topology of Algebraic Functions

(video thanks to Sean Howe and Amie Wilkinson)

Fundamental problem

Find and understand roots of a polynomial

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

Algebraic Functions

18th/19th century perspective

View

$$(a_1, \ldots, a_n) \mapsto \{z \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0\}$$

as a (multi-valued) function in the a_is.

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U_n is the *universal n-valued algebraic function*.

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More generally: $a_i \in \mathbb{C}(X)$.

When nature hands you an algebraic function . . .

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You want to do two things:

2. Prove no simpler formula is possible.(**Focus today.**)

Algebraic Functions

Modern definition

Alg. function Φ on X,

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$$\{(x,z) \in X \times \mathbb{P}^1 \mid z \in \Phi(x)\}$$

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Shorthand: $E_\Phi \xrightarrow{\Phi} X$

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A **formula** *for* Φ *in functions* $\{\Psi_1, \ldots, \Psi_r\}$ *is*

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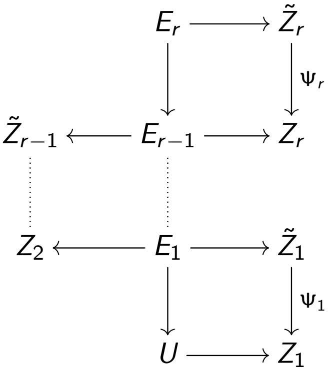

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$$
\begin{array}{ccccc}
 & & E_r & \longrightarrow & \tilde{Z}_r \\
 & & \downarrow & & \downarrow \Psi_r \\
\tilde{Z}_{r-1} & \longleftarrow & E_{r-1} & \longrightarrow & Z_r \\
\vdots & & \vdots & & \\
Z_2 & \longleftarrow & E_1 & \longrightarrow & \tilde{Z}_1 \\
 & & \downarrow & & \downarrow \Psi_1 \\
 & & U & \longrightarrow & Z_1
\end{array}
$$

and a factoring of $E_r \longrightarrow U \subset X$ *through a dominant map* $E_r \longrightarrow E_\Phi|_U$.

Essential Dimension

Let Φ be an algebraic function with cover $E_\Phi \xrightarrow{\Phi} X$.

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Definition

The **essential dimension** $\mathrm{ed}(\Phi)$ *is the least d for which there exists a dense Zariski open $U \subset X$ and a pullback square*

$$\begin{array}{ccc} E|_U & \longrightarrow & \tilde{Z} \\ {\scriptstyle \Phi|_U}\downarrow & & \downarrow \\ U & \longrightarrow & Z \end{array}$$

with $\dim(Z) = d$.

Essential Dimension at a prime

Let Φ be an algebraic function with cover $E_\Phi \xrightarrow{\Phi} X$.

Definition

Let p be a prime. Define the **essential dimension at** *p by*

$$\mathrm{ed}_p(\Phi) := \min_{\Psi \quad \textit{with } p \nmid \deg(\Psi)} \mathrm{ed}(\Phi|_{E_\Psi}).$$

Resolvent Degree

Let Φ be an algebraic function with cover $E_\Phi \longrightarrow X$.

Definition

The **resolvent degree** $\mathrm{RD}(\Phi)$ *is the least* d *for which there exists a formula for* Φ *in algebraic functions* $\{\Psi_1, \ldots, \Psi_r\}$ *with* $\mathrm{ed}(\Psi_i) \leq d$ *for all* i*.*

Natural v. accessory irrationalities

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2. *an* accessory irrationality *for Φ if*

$$\pi_1(E_\Psi) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Mon}(E_\Phi \longrightarrow X)$$

is surjective.

Example

Theorem (Kronecker, 1861)

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(**Simplest 1-variable formula possible!**)

Efficacy of accessory irrationalities?

"We do not know anything general about the efficacy of this accessory irrationality. Rather, we are dependent on tentative experiments in individual cases."

(Felix Klein, 1905)

Accessory Irrationalities

Proposition

Let Φ *be an algebraic function with simple monodromy group* G*. Then any formula for* Φ *in functions* $\{\Psi_1, \ldots, \Psi_r\}$ *can be replaced with a formula in functions* $\{\Psi_1, \ldots, \Psi_{s-1}, \Psi'_s\}$ *for* $s \leq r$ *such that*

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3. Ψ'_s *has monodromy G.*

Accessory Irrationalities

Corollary

Let Φ *be an algebraic function. Then*

$$\mathrm{RD}(\Phi) \geq \min_{acc.\ irrat.\ \Psi} \mathrm{ed}(\Phi|_{E_\Psi}).$$

Many constructions of formulas

Where to find obstructions?

1895: Poincaré's Answer: Topology!

JOURNAL

DE

L'ÉCOLE POLYTECHNIQUE.

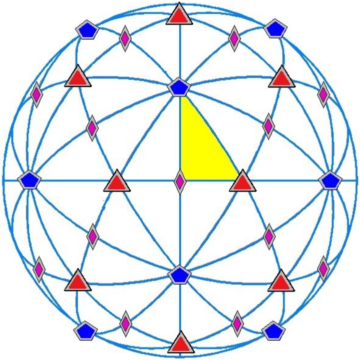
ANALYSIS SITUS;

Par M. H. POINCARÉ.

INTRODUCTION.

La Géométrie à n dimensions a un objet réel; personne n'en doute aujourd'hui. Les êtres de l'hyperespace sont susceptibles de définitions précises comme ceux de l'espace ordinaire, et si nous ne pouvons nous les représenter, nous pouvons les concevoir et les étudier. Si donc, par exemple, la Mécanique à plus de trois dimensions doit être condamnée

Poincaré gave three intended applications:



Extending Klein's classification of finite Möbius groups to higher dimensions.

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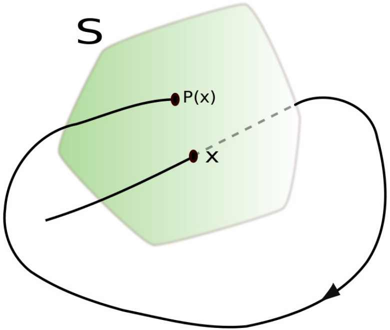

Differential equations and celestial mechanics (dynamical systems)

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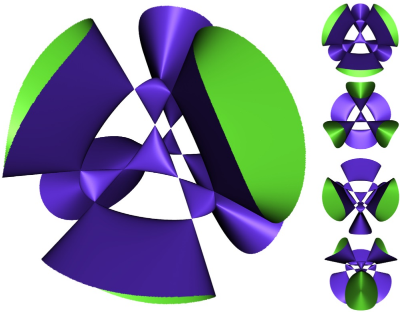
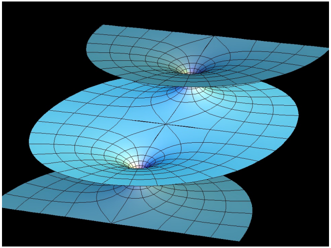


Algebraic functions of two variables.

Topology of algebraic functions

19th Century Examples



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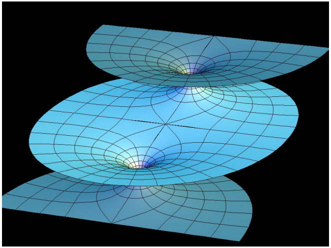

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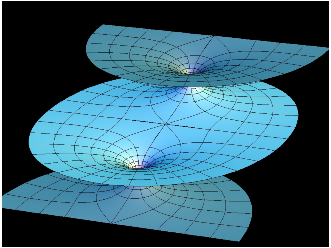

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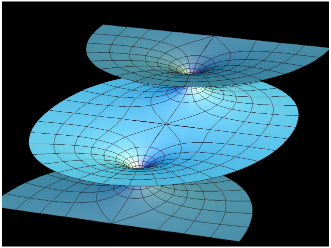

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3. $H_i(X)$ - identifies Betti numbers with a count of geometric "cycles"

1900: Hilbert's Challenge

SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL (1).

Qui ne soulèverait volontiers le voile qui nous cache l'avenir afin de jeter un coup d'œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront

1900: Hilbert's Challenge

Problem (Hilbert's 13th)

"Prove that the equation of the seventh degree $f^7 + xf^3 + yf^2 + zf + 1 = 0$ *is not solvable with the help of any continuous functions of only two arguments."*

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> *"the* genuine *Hilbert problem."* *(Arnold, 2006)*

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Observation (Riemann, Poincare)

global structure of alg. functions determined by topology!

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Hilbert: give *topological* obstruction!

Hilbert's 13th Problem

"At the present time, there appear very many topological papers, but when it comes to fundamental problems, we have hardly gone beyond Poincaré, or strictly speaking, Riemann, this in spite of the fact that such progress would be of great significance, for among other things, the theory of algebraic functions of two variables."

(Max Dehn, 1928)

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Theorem (Khovanskii)

There is no formula for the general quintic in 1-variable functions which does not use division.

Fundamental obstacles to H13

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1. Arbitrary accessory irrationalities (Buhler-Reichstein/Burda)
2. Birationality of the problem (Khovanskii)

We need more tools!

Hints from Arnold

"There's perhaps some kind of a mixed Hodge structure whose weight filtration provides the information."

(Vladimir Arnold, 2006)

Hodge Theory and ed, RD

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   2.2 is a complex orbifold,
   2.3 is a quasi-projective algebraic variety, and
   2.4 comes equipped with "congruence covers"

$$\mathcal{A}_{g,n} \longrightarrow \mathcal{A}_g$$

   where $\mathcal{A}_{g,n}$ is the moduli of pairs $(A, \mathcal{B})$ with $A \in \mathcal{A}_g$ and $\mathcal{B}$ a basis for $H^1(A; \mathbb{Z}/n\mathbb{Z})$.

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Theorem (Farb-Kisin-W)

There exists a family of representations $A_n \longrightarrow Sp_{2g}(\mathbb{F}_2)$ *such that*

$$\mathrm{ed}_2(\mathcal{A}_{g,2} \longrightarrow \mathcal{A}_{g,2}/A_n) \geq 2\lfloor \frac{n}{4} \rfloor =: e(n).$$

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n	6	7	8	9
H(n)	2	3	4	4
e(n)	2	2	4	4

Conjecture

For every 2-power cover $h\colon E \longrightarrow \mathcal{A}_{g,2}/A_n$,

$$\mathrm{ed}(h^*\mathcal{A}_{g,2} \longrightarrow E) \geq e(n).$$

Hopefully, more soon from this conference!

Thank you!