Resolvent Degree and Classical Solutions

Jesse Wolfson
University of California, Irvine

Braids, Polynomials and Hilbert's 13th Problem
IPAM
February 19, 2019

Joint work with Benson Farb

(Within broader joint project with Benson Farb and Mark Kisin)

Fundamental problem

Find and understand roots of a polynomial

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

Algebraic Functions

18th/19th century perspective

View

$$(a_1, \ldots, a_n) \mapsto \{z \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0\}$$

as a (multi-valued) function in the a_is.

Algebraic Functions

18th/19th century perspective

View

$$(a_1, \dots, a_n) \mapsto \{z \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0\}$$

as a (multi-valued) function in the a_is.

$$U_n \colon \mathbb{C}^n \longrightarrow \operatorname{Sym}^n \mathbb{C} := \mathbb{C}^n / S_n$$
$$(a_1, \dots, a_n) \mapsto \{z \in \mathbb{C} \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0\}.$$

Algebraic Functions

18th/19th century perspective

View

$$(a_1, \ldots, a_n) \mapsto \{z \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0\}$$

as a (multi-valued) function in the a_is.

$$\begin{aligned} U_n \colon \mathbb{C}^n &\longrightarrow \operatorname{Sym}^n \mathbb{C} := \mathbb{C}^n / S_n \\ (a_1, \ldots, a_n) &\mapsto \{z \in \mathbb{C} \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0\}. \end{aligned}$$

U_n is the *universal n-valued algebraic function*.

Algebraic Functions

18th/19th century perspective

View

$$(a_1, \dots, a_n) \mapsto \{z \mid z^n + a_1 z^{n-1} + \dots + a_n = 0\}$$

as a (multi-valued) function in the a_is.

$$U_n \colon \mathbb{C}^n \longrightarrow \operatorname{Sym}^n \mathbb{C} := \mathbb{C}^n / S_n$$
$$(a_1, \dots, a_n) \mapsto \{z \in \mathbb{C} \mid z^n + a_1 z^{n-1} + \dots + a_n = 0\}.$$

More generally: $a_i \in \mathbb{C}(X)$.

When nature hands you an algebraic function . . .

You want to do two things:

When nature hands you an algebraic function . . .

You want to do two things:

1. Construct the simplest formula you can.

When nature hands you an algebraic function . . .

You want to do two things:

1. Construct the simplest formula you can.
2. Prove no simpler formula is possible.

When nature hands you an algebraic function . . .

You want to do two things:

1. Construct the simplest formula you can. (**Focus today.**)

Algebraic Functions

Modern definition

Alg. function Φ on X $\rightsquigarrow$

Algebraic Functions

Modern definition

Alg. function Φ on X $\rightsquigarrow$

$$\{(x,z) \in X \times \mathbb{P}^1 \mid z^n + a_1(x)z^{n-1} + \ldots + a_n(x) = 0\}$$

Algebraic Functions

Modern definition

Alg. function Φ on X $\rightsquigarrow$

$$E_\Phi := \overline{\{(x,z) \in X \times \mathbb{P}^1 \mid z^n + a_1(x)z^{n-1} + \ldots + a_n(x) = 0\}}$$

Algebraic Functions
Modern definition

Alg. function Φ on $X \rightsquigarrow$

$$E_\Phi := \overline{\{(x,z) \in X \times \mathbb{P}^1 \mid z^n + a_1(x)z^{n-1} + \ldots + a_n(x) = 0\}}$$

$$\begin{array}{ccc} E_\Phi & \xrightarrow{(x,z)\mapsto z} & \mathbb{P}^1 \\ {\scriptstyle (x,z)\mapsto x}\downarrow & & \\ X & & \end{array}$$

Algebraic Functions

Modern definition

Alg. function Φ on X $\rightsquigarrow$

$$E_\Phi := \overline{\{(x,z) \in X \times \mathbb{P}^1 \mid z^n + a_1(x)z^{n-1} + \ldots + a_n(x) = 0\}}$$

$$\begin{array}{ccc} E_\Phi & \xrightarrow{(x,z)\mapsto z} & \mathbb{P}^1 \\ {\scriptstyle (x,z)\mapsto x}\big\downarrow & & \\ X & & \end{array}$$

Shorthand: $E_\Phi \xrightarrow{\Phi} X$

Algebraic Functions

Example

Consider the universal quadratic

$$U_2(b, c) = \{z \mid z^2 + bz + c = 0\}.$$

Then: $X = \mathbb{C}^2$,

Algebraic Functions

Example

Consider the universal quadratic

$$U_2(b, c) = \{z \mid z^2 + bz + c = 0\}.$$

Then: $X = \mathbb{C}^2$,
$E_{U_2} = \mathbb{C}^2$,

Algebraic Functions

Example

Consider the universal quadratic

$$U_2(b, c) = \{z \mid z^2 + bz + c = 0\}.$$

Then: $X = \mathbb{C}^2$,
$E_{U_2} = \mathbb{C}^2$,

$$\begin{array}{ccc} E_{U_2} = \mathbb{C}^2 & \xrightarrow{(z_1, z_2) \mapsto z_1} & \mathbb{P}^1 \\ \Big\downarrow {\scriptstyle (z_1, z_2) \mapsto (-z_1 - z_2, z_1 z_2)} & & \\ \mathbb{C}^2 & & \end{array}$$

Algebraic Formulas
Example 1

Recall that $\sqrt[d]{-}$ is algebraic function with

Algebraic Formulas

Example 1

Recall that $\sqrt[d]{-}$ is algebraic function with

$X = \mathbb{P}^1$,

Algebraic Formulas
Example 1

Recall that $\sqrt[d]{-}$ is algebraic function with

$X = \mathbb{P}^1,$

$E_{\sqrt[d]{-}} = \mathbb{P}^1,$

Algebraic Formulas

Example 1

Recall that $\sqrt[d]{-}$ is algebraic function with

$X = \mathbb{P}^1$,

$E_{\sqrt[d]{-}} = \mathbb{P}^1$,

$$\begin{array}{ccc} E_{\sqrt[d]{-}} = \mathbb{P}^1 & \xrightarrow{z \mapsto z} & \mathbb{P}^1 \\ {\scriptstyle z \mapsto z^d} \big\downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

Algebraic Formulas

Example 1

Theorem (Babylonians)

$U_2(b,c) = \frac{-b+\sqrt{b^2-4c}}{2}$.

Algebraic Formulas

Example 1

Theorem (Babylonians)

$U_2(b, c) = \frac{-b+\sqrt{b^2-4c}}{2}$.
Equivalently, the pullback square

$$\begin{array}{ccc} E_1 & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow \sqrt{-} \\ \mathbb{C}^2 & \xrightarrow[(b,c)\mapsto b^2-4c]{\Delta_2} & \mathbb{P}^1 \end{array}$$

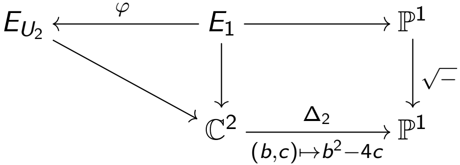
Algebraic Formulas

Example 1

Theorem (Babylonians)

$U_2(b, c) = \frac{-b+\sqrt{b^2-4c}}{2}$.

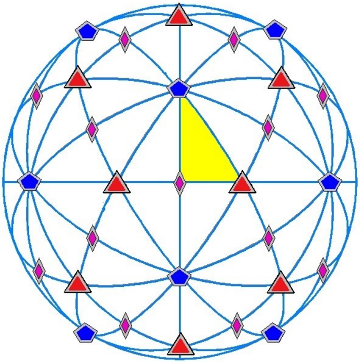
Equivalently, the pullback square sits in a commuting diagram



where $\varphi(b, c, \delta) = (\frac{-b+\delta}{2}, \frac{-b-\delta}{2})$.

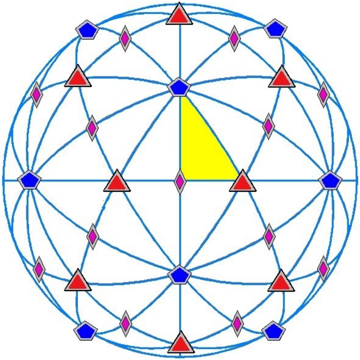
Algebraic formulas

Example 2



Algebraic formulas

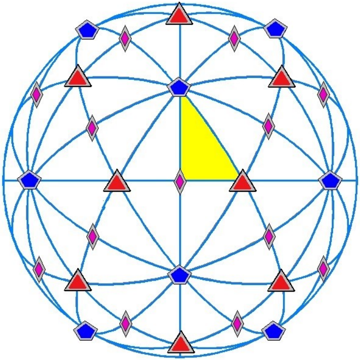
Example 2



$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ - rational map with zeroes the vertices.

Algebraic formulas

Example 2



$f\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ - rational map with zeroes the vertices.
$H\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ - rational map with zeroes the centers of the faces.

Algebraic formulas

Example 2

The *Icosahedral* function $\mathcal{I}$ is the algebraic function with

Algebraic formulas

Example 2

The *Icosahedral* function $\mathcal{I}$ is the algebraic function with

$X = \mathbb{P}^1$,

Algebraic formulas
Example 2

The *Icosahedral* function $\mathcal{I}$ is the algebraic function with

$X = \mathbb{P}^1$,

$E_{\mathcal{I}} = \mathbb{P}^1$,

Algebraic formulas

Example 2

The *Icosahedral* function $\mathcal{I}$ is the algebraic function with

$$X = \mathbb{P}^1,$$

$$E_{\mathcal{I}} = \mathbb{P}^1,$$

$$\begin{array}{ccc} E_{\mathcal{I}} = \mathbb{P}^1 & \xrightarrow{z \mapsto z} & \mathbb{P}^1 \\ {\scriptstyle z \mapsto \frac{H(z)^3}{1728 f(z)^5}} \Big\downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

Algebraic formulas

Example 2

Theorem (Klein)

Let $U_5(\mathbf{a})$ denote the universal quintic. There exist rational functions Δ_5, A and φ such that

$$U_5(\mathbf{a}) = \mathcal{I}(\varphi(\mathbf{a}, \sqrt{\Delta_5(\mathbf{a})}, \sqrt{A(\mathbf{a})})).$$

Algebraic formulas

Example 2

Theorem (Klein)

Equivalently, there is a tower of pullback squares

$$
\begin{array}{ccccc}
 & & E_3 & \longrightarrow & \mathbb{P}^1 \\
 & & \big\downarrow & & \big\downarrow \mathcal{I} \\
\mathbb{P}^1 & \longleftarrow & E_2 & \xrightarrow{\varphi} & \mathbb{P}^1 \\
\sqrt{-} \big\downarrow & & \big\downarrow & & \\
\mathbb{P}^1 & \underset{\Delta_2}{\longleftarrow} & E_1 & \longrightarrow & \mathbb{P}^1 \\
 & & \big\downarrow & & \big\downarrow \sqrt{-} \\
 & & \mathbb{C}^5 & \xrightarrow{A} & \mathbb{P}^1
\end{array}
$$

such that $E_3 \longrightarrow \mathbb{C}^5$ *factors through* E_{U_5}.

Algebraic formulas

Definition

A **formula** *for* Φ *in functions* $\{\Psi_1, \ldots, \Psi_r\}$ *is*

Definition

A **formula** *for* Φ *in functions* $\{\Psi_1, \ldots, \Psi_r\}$ *is a (finite) tower of pullback squares*

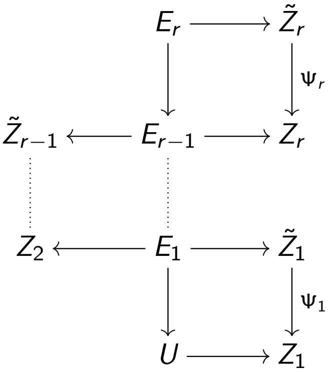
$$
\begin{array}{ccccc}
 & & E_r & \longrightarrow & \tilde{Z}_r \\
 & & \downarrow & & \downarrow \Psi_r \\
\tilde{Z}_{r-1} & \longleftarrow & E_{r-1} & \longrightarrow & Z_r \\
\vdots & & \vdots & & \\
Z_2 & \longleftarrow & E_1 & \longrightarrow & \tilde{Z}_1 \\
 & & \downarrow & & \downarrow \Psi_1 \\
 & & U & \longrightarrow & Z_1
\end{array}
$$

Algebraic formulas

Definitions

Definition

A **formula** *for* Φ *in functions* $\{\Psi_1, \ldots, \Psi_r\}$ *is a (finite) tower of pullback squares*



and a factoring of $E_r \longrightarrow U \subset X$ *through a dominant map* $E_r \longrightarrow E_\Phi|_U$.

Algebraic formulas

Example 2

Theorem (Klein)

Let $U_5(\mathbf{a})$ denote the universal quintic. There is a tower of pullback squares

$$
\begin{array}{ccccc}
 & & E_3 & \longrightarrow & \mathbb{P}^1 \\
 & & \big\downarrow & & \big\downarrow \mathcal{I} \\
\mathbb{P}^1 & \longleftarrow & E_2 & \xrightarrow{\varphi} & \mathbb{P}^1 \\
\sqrt{-} \big\downarrow & & \big\downarrow & & \\
\mathbb{P}^1 & \underset{\Delta_2}{\longleftarrow} & E_1 & \longrightarrow & \mathbb{P}^1 \\
 & & \big\downarrow & & \big\downarrow \sqrt{-} \\
 & & \mathbb{C}^5 & \xrightarrow{A} & \mathbb{P}^1
\end{array}
$$

such that $E_3 \longrightarrow \mathbb{C}^5$ factors through E_{U_5}.

Algebraic formulas

Example 2

Theorem (Klein)

There is a formula for the general quintic in the functions $\sqrt{-}$ *and* $\mathcal{I}$.

Algebraic Formulas

Example 3

Theorem (Jordan-Klein(?))

Given one line on a smooth cubic surface, there is a formula for the remaining 26 using only $\sqrt[d]{-}$ *and* $\mathcal{I}$.

Proof (video by Steve Trettel)

Algebraic Formulas

Example 4

A - principally polarized abelian surface

Algebraic Formulas

Example 4

A - principally polarized abelian surface

$A[3] \subset A$ - set of 3-torsion points

Algebraic Formulas

Example 4

A - principally polarized abelian surface

$A[3] \subset A$ - set of 3-torsion points

$\Phi_{2,3} : A \mapsto p \in A[3]$

Algebraic Formulas

Example 4

A - principally polarized abelian surface

$A[3] \subset A$ - set of 3-torsion points

$\Phi_{2,3} : A \mapsto p \in A[3]$

Theorem (Klein-Burkhardt)

There exists a formula for a line on a smooth cubic surface in $\sqrt[d]{-}$ *and* $\Phi_{2,3}$.

Simplest algebraic formulas?

Question (Klein, 1888)

What is the simplest formula for $\Phi_{2,3}$*?*

Simplest algebraic formulas?

Question (Klein, 1888)

What is the simplest formula for $\Phi_{2,3}$*? With/Without accessory irrationalities?*

Simplest algebraic formulas?

"We consider as the simplest the one that has the least number of variables." *(Felix Klein, 1893)*

Essential Dimension

Let Φ be an algebraic function with cover $E_\Phi \xrightarrow{\Phi} X$.

Essential Dimension

Let Φ be an algebraic function with cover $E_\Phi \xrightarrow{\Phi} X$.

Definition

The **essential dimension** $\mathrm{ed}(\Phi)$ *is the least d for which there exists a dense Zariski open $U \subset X$ and a pullback square*

$$\begin{array}{ccc} E|_U & \longrightarrow & \tilde{Z} \\ {\scriptstyle \Phi|_U}\downarrow & & \downarrow \\ U & \longrightarrow & Z \end{array}$$

with $\dim(Z) = d$.

Essential Dimension

Let Φ be an algebraic function with cover $E_\Phi \xrightarrow{\Phi} X$.

Definition

The **essential dimension** $\mathrm{ed}(\Phi)$ *is the least d for which there exists a dense Zariski open $U \subset X$ and a pullback square*

$$\begin{array}{ccc} E|_U & \longrightarrow & \tilde{Z} \\ {\scriptstyle \Phi|_U}\big\downarrow & & \big\downarrow \\ U & \longrightarrow & Z \end{array}$$

with $\dim(Z) = d$*. (Φ becomes a function of d-***variables after a rational transformation.)**

Resolvent Degree

Let Φ be an algebraic function with cover $E_\Phi \longrightarrow X$.

Definition

The **resolvent degree** $\mathrm{RD}(\Phi)$ *is the least* d *for which there exists a formula for* Φ *in algebraic functions* $\{\Psi_1, \ldots, \Psi_r\}$ *with* $\mathrm{ed}(\Psi_i) \leq d$ *for all* i*.*

Resolvent Degree

Let Φ be an algebraic function with cover $E_\Phi \longrightarrow X$.

Definition

The **resolvent degree** $\mathrm{RD}(\Phi)$ *is the least* d *for which there exists a formula for* Φ *in algebraic functions* $\{\Psi_1, \ldots, \Psi_r\}$ *with* $\mathrm{ed}(\Psi_i) \leq d$ *for all* i*.*

$$\mathrm{RD}(n) := \mathrm{RD}(U_n).$$

Resolvent Degree

Let Φ be an algebraic function with cover $E_\Phi \longrightarrow X$.

Definition

The **resolvent degree** $\mathrm{RD}(\Phi)$ *is the least* d *for which there exists a formula for* Φ *in algebraic functions* $\{\Psi_1, \ldots, \Psi_r\}$ *with* $\mathrm{ed}(\Psi_i) \leq d$ *for all* i*.*

$$\mathrm{RD}(n) := \mathrm{RD}(U_n).$$

Obs: if you require $r \equiv 1$ above, recover ed.

RD v. ed

ed - no irrationalities

RD v. ed

ed - no irrationalities

RD - all accessory irrationalities

RD v. ed

ed - no irrationalities

RD - all accessory irrationalities

- $\mathrm{ed}(-) \geq \mathrm{RD}(-)$.

RD v. ed

ed - no irrationalities

RD - all accessory irrationalities

- $\mathrm{ed}(-) \geq \mathrm{RD}(-)$.
- Difference can be arbitrarily large:

RD v. ed

ed - no irrationalities

RD - all accessory irrationalities

- ed$(-) \geq$ RD$(-)$.
- Difference can be arbitrarily large:

  Consider the function

$$(a_1, \ldots, a_n) \mapsto \sqrt{a_1} + \ldots + \sqrt{a_n}.$$

RD v. ed

ed - no irrationalities

RD - all accessory irrationalities

- ed$(-) \geq$ RD$(-)$.
- Difference can be arbitrarily large:

  Consider the function

$$(a_1, \ldots, a_n) \mapsto \sqrt{a_1} + \ldots + \sqrt{a_n}.$$

$$\mathrm{ed}(\sqrt{a_1} + \ldots + \sqrt{a_n}) = n > 1 = \mathrm{RD}(\sqrt{a_1} + \ldots + \sqrt{a_n}).$$

RD v. ed

ed - no irrationalities

RD - all accessory irrationalities

- $\mathrm{ed}(-) \geq \mathrm{RD}(-)$.
- Difference can be arbitrarily large:

  Consider the function

$$(a_1, \ldots, a_n) \mapsto \sqrt{a_1} + \ldots + \sqrt{a_n}.$$

$$\mathrm{ed}(\sqrt{a_1} + \ldots + \sqrt{a_n}) = n > 1 = \mathrm{RD}(\sqrt{a_1} + \ldots + \sqrt{a_n}).$$

- Many lower bounds on ed.

RD v. ed

ed - no irrationalities

RD - all accessory irrationalities

- ed$(-) \geq$ RD$(-)$.
- Difference can be arbitrarily large:

  Consider the function

$$(a_1, \dots, a_n) \mapsto \sqrt{a_1} + \dots + \sqrt{a_n}.$$

$$\mathrm{ed}(\sqrt{a_1} + \dots + \sqrt{a_n}) = n > 1 = \mathrm{RD}(\sqrt{a_1} + \dots + \sqrt{a_n}).$$

- Many lower bounds on ed.
- **No nontrivial lower bounds for** RD**!**

Resolvent Degree

Classical results

Theorem (Babylonians, Khayyam, Tartaglia, Cardano, Ferrari)

$\mathrm{RD}(n) = 1$ *for* $n \leq 4$.

Resolvent Degree

Classical results

Theorem (Babylonians, Khayyam, Tartaglia, Cardano, Ferrari)

$\mathrm{RD}(n) = 1$ *for* $n \leq 4$.

Theorem (Bring, Klein)

$\mathrm{RD}(5) = 1$.

Theorem (Hamilton)

There exists an unbounded monotone function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that for $n \geq H(r)$,

$$\mathrm{RD}(n) \leq n - r.$$

Theorem (Hamilton)

There exists an unbounded monotone function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n \geq H(r)$,

$$\mathrm{RD}(n) \leq n - r.$$

In particular,

$$\mathrm{RD}(6) \leq 2,$$

Theorem (Hamilton)

There exists an unbounded monotone function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n \geq H(r)$,

$$\mathrm{RD}(n) \leq n - r.$$

In particular,

$$\mathrm{RD}(6) \leq 2,$$

$$\mathrm{RD}(7) \leq 3,$$

Theorem (Hamilton)

There exists an unbounded monotone function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that for $n \geq H(r)$,

$$\mathrm{RD}(n) \leq n - r.$$

In particular,

$$\mathrm{RD}(6) \leq 2,$$

$$\mathrm{RD}(7) \leq 3,$$

$$\mathrm{RD}(8) \leq 4,$$

Theorem (Hamilton)

There exists an unbounded monotone function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n \geq H(r)$,

$$\mathrm{RD}(n) \leq n - r.$$

In particular,

$$\mathrm{RD}(6) \leq 2,$$
$$\mathrm{RD}(7) \leq 3,$$
$$\mathrm{RD}(8) \leq 4,$$
$$\mathrm{RD}(9) \leq 5.$$

Theorem (Hamilton)

There exists an unbounded monotone function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n \geq H(r)$,

$$\mathrm{RD}(n) \leq n - r.$$

In particular,

$$\mathrm{RD}(6) \leq 2,$$
$$\mathrm{RD}(7) \leq 3,$$
$$\mathrm{RD}(8) \leq 4,$$
$$\mathrm{RD}(9) \leq 5.$$

Sylvester–Hammond, 1887 - generating function for $H(r)$

RD and Roots of Polynomials

"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date." *(Sylvester, Hammond 1887)*

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to the present, add:

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to the present, add:

- Klein's school 1870s - 1900s

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to the present, add:

- Klein's school 1870s - 1900s
- Wiman 1927,

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to the present, add:

- Klein's school 1870s - 1900s
- Wiman 1927,
- Hilbert 1927,

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to the present, add:

- Klein's school 1870s - 1900s
- Wiman 1927,
- Hilbert 1927,
- Segre 1947, 1955,

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* (Sylvester, Hammond 1887)

To bring this up to the present, add:

- Klein's school 1870s - 1900s
- Wiman 1927,
- Hilbert 1927,
- Segre 1947, 1955,
- Brauer 1975.

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to the present, add:

- Klein's school 1870s - 1900s
- Wiman 1927,
- Hilbert 1927,
- Segre 1947, 1955,
- Brauer 1975. (**0 citations!**)

That's it!

What's Known

Theorem (Hamilton, 1836)

There exists a monotone unbounded function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

What's Known

Theorem (Hamilton, 1836)

There exists a monotone unbounded function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Hamilton computed the initial values of H:

r	4	5	6	7	8	9
H(r)	5	11	47	923	409,619	83,763,206,255

Theorem (Klein, 1884)

The symmetries of the icosahedron imply $\mathrm{RD}(5) = 1$.

Theorem (Klein, 1884)

The symmetries of the icosahedron imply $\mathrm{RD}(5) = 1$.

Theorem (Klein?)

The exceptional isomorphism $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \cong \mathfrak{so}_4$ *implies* $\mathrm{RD}(5) = 1$.

Hamilton's Results

Klein's Challenge

Theorem (Klein, 1884)

The symmetries of the icosahedron imply $\mathrm{RD}(5) = 1$.

Theorem (Klein?)

The exceptional isomorphism $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \cong \mathfrak{so}_4$ *implies* $\mathrm{RD}(5) = 1$.

Theorem (Klein, 1886)

The exceptional isomorphism $\mathfrak{sl}_4 \cong \mathfrak{so}_6$ *implies* $\mathrm{RD}(7) \leq 3$.

Hamilton's Results

Klein's Challenge

Theorem (Klein, 1884)

The symmetries of the icosahedron imply $\mathrm{RD}(5) = 1$*.*

Theorem (Klein?)

The exceptional isomorphism $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \cong \mathfrak{so}_4$ *implies* $\mathrm{RD}(5) = 1$*.*

Theorem (Klein, 1886)

The exceptional isomorphism $\mathfrak{sl}_4 \cong \mathfrak{so}_6$ *implies* $\mathrm{RD}(7) \leq 3$*.*

Theorem (Klein, 1905)

The Valentiner representation $A_6 \circlearrowright \mathbb{P}^2$ *implies* $\mathrm{RD}(6) \leq 2$*.*

Theorem (Wiman)

For $n > 7$, there is no action of A_n on $\mathbb{P}^m$ for $m < n - 2$.

Theorem (Wiman)

For $n > 7$, there is no action of A_n on $\mathbb{P}^m$ for $m < n - 2$.

$\therefore$ Projective actions cannot recover Hamilton's bounds for $n \geq 8$.

Problem

For $n \geq 8$, find and understand a Kleinian solution of the general degree n polynomial.

Problem
For $n \geq 8$, find and understand a Kleinian solution of the general degree n polynomial.

What geometry replaces $A_n \circlearrowright \mathbb{P}^n$?

Hilbert's Idea

Recall: Hamilton showed $\mathrm{RD}(9) \leq 5$.

Hilbert's Idea

Recall: Hamilton showed $\mathrm{RD}(9) \leq 5$.

Theorem (Hilbert, 1927)

The 27 lines on a cubic surface imply that $\mathrm{RD}(9) \leq 4$.

Hilbert's Idea

Recall: Hamilton showed $\mathrm{RD}(9) \leq 5$.

Theorem-Sketch (Hilbert, 1927)

The 27 lines on a cubic surface should *imply that* $\mathrm{RD}(9) \leq 4$.

Hilbert's Idea

Recall: Hamilton showed $\mathrm{RD}(9) \leq 5$.

Theorem-Sketch (Hilbert, 1927)

The 27 lines on a cubic surface should *imply that* $\mathrm{RD}(9) \leq 4$.

Theorem (Segre, 1947)

$\mathrm{RD}(9) \leq 4$

Hilbert's Idea

Recall: Hamilton showed $\mathrm{RD}(9) \leq 5$.

Theorem-Sketch (Hilbert, 1927)

The 27 lines on a cubic surface should *imply that* $\mathrm{RD}(9) \leq 4$.

Theorem (Segre, 1947)

$\mathrm{RD}(9) \leq 4$

(completely different proof.)

Up to the present

By mid-20th century, Hamilton's Theorem appears to have been forgotten:

Up to the present

By mid-20th century, Hamilton's Theorem appears to have been forgotten:

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Up to the present

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Up to the present

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Segre and Brauer reproved soon after, but no formula.

Up to the present

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}(n) \leq n - r$.

Segre and Brauer reproved soon after, but no formula.

Theorem (Brauer, 1975)

Let $B(r) := (r-1)!$ *For* $n > B(r)$, $\mathrm{RD}(n) \leq n - r$.

Improving on Hamilton, Brauer

Use geometry!

Improving on Hamilton, Brauer

Use geometry!

Bring/Hamilton: ruling of quadric surfaces $\Rightarrow \mathrm{RD}(5) = 1$.

Improving on Hamilton, Brauer

Use geometry!

- Bring/Hamilton: ruling of quadric surfaces $\Rightarrow \mathrm{RD}(5) = 1$.
- Hilbert: lines on cubic surfaces $\Rightarrow \mathrm{RD}(9) \leq 4$.

Improving on Hamilton, Brauer

Use geometry!

- Bring/Hamilton: ruling of quadric surfaces $\Rightarrow \mathrm{RD}(5) = 1$.
- Hilbert: lines on cubic surfaces $\Rightarrow \mathrm{RD}(9) \leq 4$.
- General method: use linear subspaces on higher dimensional hypersurfaces!

Improving on Hamilton, Brauer

Use geometry!

- Bring/Hamilton: ruling of quadric surfaces $\Rightarrow \mathrm{RD}(5) = 1$.
- Hilbert: lines on cubic surfaces $\Rightarrow \mathrm{RD}(9) \leq 4$.
- General method: use linear subspaces on higher dimensional hypersurfaces!

Theorem (Farb–W)

There exist monotone increasing functions $FW, \varphi \colon \mathbb{N} \longrightarrow \mathbb{N}$ *s.t.*

Improving on Hamilton, Brauer

Use geometry!

- Bring/Hamilton: ruling of quadric surfaces $\Rightarrow$ $\mathrm{RD}(5) = 1$.
- Hilbert: lines on cubic surfaces $\Rightarrow$ $\mathrm{RD}(9) \leq 4$.
- General method: use linear subspaces on higher dimensional hypersurfaces!

Theorem (Farb–W)

There exist monotone increasing functions $FW, \varphi \colon \mathbb{N} \longrightarrow \mathbb{N}$ s.t.

- *For $n > FW(r)$, $\mathrm{RD}(n) \leq n - r$,*

Improving on Hamilton, Brauer

Use geometry!

Bring/Hamilton: ruling of quadric surfaces $\Rightarrow$ $\mathrm{RD}(5) = 1$.

Hilbert: lines on cubic surfaces $\Rightarrow$ $\mathrm{RD}(9) \leq 4$.

General method: use linear subspaces on higher dimensional hypersurfaces!

Theorem (Farb–W)

There exist monotone increasing functions $FW, \varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ s.t.

- *For $n > FW(r)$, $\mathrm{RD}(n) \leq n - r$,*
- *For all $d \geq 0$, $r \geq \varphi(d)$, then $B(r)/FW(r) \geq d!$*

Improving on Hamilton, Brauer

Theorem (Farb–W)

There exist monotone increasing functions $FW, \varphi \colon \mathbb{N} \longrightarrow \mathbb{N}$ *s.t.*

- *For* $n > FW(r)$, $\mathrm{RD}(n) \leq n - r$,
- *For all* $d \geq 0$, $r \geq \varphi(d)$, *then* $B(r)/FW(r) \geq d!$

$FW(5) = 9$

Improving on Hamilton, Brauer

Theorem (Farb–W)

There exist monotone increasing functions $FW, \varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ s.t.

- *For $n > FW(r)$, $\mathrm{RD}(n) \leq n - r$,*
- *For all $d \geq 0$, $r \geq \varphi(d)$, then $B(r)/FW(r) \geq d!$*

$FW(5) = 9$

We don't expect our bounds are sharp.

Improving on Hamilton, Brauer

Theorem (Farb–W)

There exist monotone increasing functions $FW, \varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ s.t.

- *For $n > FW(r)$, $\mathrm{RD}(n) \leq n - r$,*
- *For all $d \geq 0$, $r \geq \varphi(d)$, then $B(r)/FW(r) \geq d!$*

$FW(5) = 9$

We don't expect our bounds are sharp.

No improvement to Hamilton/Hilbert in low degrees ($n < 36$).

Improving on Hamilton, Brauer

Theorem (Farb–W)

There exist monotone increasing functions $FW, \varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ s.t.

- *For $n > FW(r)$, $\mathrm{RD}(n) \leq n - r$,*
- *For all $d \geq 0$, $r \geq \varphi(d)$, then $B(r)/FW(r) \geq d!$*

$FW(5) = 9$

We don't expect our bounds are sharp.

No improvement to Hamilton/Hilbert in low degrees ($n < 36$).

Kleinian approach totally open!

Tschirnhaus transformations

Hamilton/Hilbert method

$U_n(\mathbf{a})$ - polynomial with roots x_i.

Tschirnhaus transformations

Hamilton/Hilbert method

$U_n(\mathbf{a})$ - polynomial with roots x_i.
Key tool: *Tschirnhaus transformation*

$$T_{\mathbf{b}}(x_i) := b_0 + b_1 x_i + \ldots + b_{n-1} x_i^{n-1}$$

Tschirnhaus transformations

Hamilton/Hilbert method

$U_n(\mathbf{a})$ - polynomial with roots x_i.
Key tool: *Tschirnhaus transformation*

$$T_{\mathbf{b}}(x_i) := b_0 + b_1 x_i + \ldots + b_{n-1} x_i^{n-1}$$

$x_i \mapsto T_{\mathbf{b}}(x_i)$ symmetric in roots

Tschirnhaus transformations

Hamilton/Hilbert method

$U_n(\mathbf{a})$ - polynomial with roots x_i.

Key tool: *Tschirnhaus transformation*

$$T_{\mathbf{b}}(x_i) := b_0 + b_1 x_i + \ldots + b_{n-1} x_i^{n-1}$$

$x_i \mapsto T_{\mathbf{b}}(x_i)$ symmetric in roots $\rightsquigarrow$

$$\mathbf{a} \mapsto T_{\mathbf{b}}(\mathbf{a}).$$

Tschirnhaus transformations

Hamilton/Hilbert method

$U_n(\mathbf{a})$ - polynomial with roots x_i.

Key tool: *Tschirnhaus transformation*

$$T_{\mathbf{b}}(x_i) := b_0 + b_1 x_i + \ldots + b_{n-1} x_i^{n-1}$$

$x_i \mapsto T_{\mathbf{b}}(x_i)$ symmetric in roots $\rightsquigarrow$

$$\mathbf{a} \mapsto T_{\mathbf{b}}(\mathbf{a}).$$

$\mathbb{C}^n_{\mathbf{b}}$ - space of Tschirnhaus transformations

Tschirnhaus transformations

Hamilton/Hilbert method

$U_n(\mathbf{a})$ - polynomial with roots x_i.

Key tool: *Tschirnhaus transformation*

$$T_{\mathbf{b}}(x_i) := b_0 + b_1 x_i + \ldots + b_{n-1} x_i^{n-1}$$

$x_i \mapsto T_{\mathbf{b}}(x_i)$ symmetric in roots $\rightsquigarrow$

$$\mathbf{a} \mapsto T_{\mathbf{b}}(\mathbf{a}).$$

$\mathbb{C}^n_{\mathbf{b}}$ - space of Tschirnhaus transformations

$(\mathbf{a}, \mathbf{b}) \mapsto T_{\mathbf{b}}(\mathbf{a}) \rightsquigarrow$

$$\mathbb{C}^n_{\mathbf{a}} \times \mathbb{C}^n_{\mathbf{b}} \xrightarrow{\text{ev}} \mathbb{C}^n_{\mathbf{a}}.$$

Tschirnhaus transformations

Hamilton/Hilbert method

$U_n(\mathbf{a})$ - polynomial with roots x_i.
Key tool: *Tschirnhaus transformation*

$$\mathbb{C}^n_{\mathbf{a}} \times \mathbb{C}^n_{\mathbf{b}} \xrightarrow{\text{ev}} \mathbb{C}^n_{\mathbf{a}}.$$

$$T_i := \text{ev}^{-1}(a_i = 0)/\mathbb{C}^\times \text{ - family of degree } i \text{ hypersurfaces } /\mathbb{C}^n_{\mathbf{a}}$$

$$\begin{array}{ccc} T_i & \hookrightarrow & \mathbb{C}^n_{\mathbf{a}} \times \mathbb{P}^{n-1} \\ \downarrow & \swarrow & \\ \mathbb{C}^n_{\mathbf{a}} & & \end{array}$$

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Consider

$$\begin{array}{c} \bigcap_{j=1}^{n-d} T_{i_j} \\ \big\downarrow \\ \mathbb{C}^n \end{array}$$

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Construct a (rational) multi-section

$$\begin{array}{ccc} & & \bigcap_{j=1}^{n-d} T_{i_j} \\ & \nearrow & \downarrow \\ E & \longrightarrow & \mathbb{C}^n \end{array}$$

with $\mathrm{RD}(E \longrightarrow \mathbb{C}^n) \leq d$.

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Observation (Brauer)

$\bigcap_{i=1}^{r} T_i|_{\mathbb{C}(\mathbf{a})}$ has a dense collection of points of height $r!$ over $\mathbb{C}(\mathbf{a})$.

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Observation (Brauer)

$\bigcap_{i=1}^{r} T_i|_{\mathbb{C}(\mathbf{a})}$ *has a dense collection of points of height* $r!$ *over* $\mathbb{C}(\mathbf{a})$.

$\Rightarrow$ Brauer's bounds.

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Observation (Hamilton)

Quadrics have large linear subspaces (with formulas in $\sqrt{-}$).

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Observation (Hamilton)

Quadrics have large linear subspaces (with formulas in $\sqrt{-}$).

$\Rightarrow$ can halve Brauer's bounds.

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Observation (Hilbert)

Cubic surfaces have lines and smooth *cubic surfaces form a 4-dimensional moduli space.*

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Observation (Farb–W)

Arbitrary degree i hypersurfaces in $\mathbb{P}^N$ have k-planes if some $\delta(i, k, N) \geq 0$.

Sketch proof

To show $\mathrm{RD}(n) \leq d$:

Observation (Farb–W)

Arbitrary degree i hypersurfaces in $\mathbb{P}^N$ have k-planes if some $\delta(i, k, N) \geq 0$.

Hilbert idea + generic smoothness of T_i + Merkurjev–Suslin $\Rightarrow$ FW-bounds.

Conclusion

"The study of [resolvent degree], far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians."

(Sylvester, Hammond 1887)

Conclusion

"The study of [resolvent degree], far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians."

(Sylvester, Hammond 1887)

Sylvester and Hammond's words apply just as much today!