Quantum Field Theory of Exotic Systems

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There will be very few references. Most of the material and many references can be found in [Gorantla, Lam, NS, Shao, 2103.01257, 2108.00020]

Goal

Fractons is a general name for a number of theoretical puzzling phases of matter. They have a lattice formulation, but they do not seem to fit the general, standard framework of continuum quantum field theory.

I was asked to talk about "QFT for Fractons." But I would like to frame it in a larger context – the relation between lattice theories and continuum QFT.

Zhenghan Wang: "the continuum is so slippery."

We will start with a review of known systems, partly from a new perspective.

Then, we will focus on some exotic systems, which are a good introduction to systems with fractons, because they exhibit subsystem global symmetries and their associated subtleties.

The next step will be to gauge these subsystem symmetries.

Lattice vs. continuum QFT

QFT is enormously successful. Yet, it is not mathematically rigorous.

One approach is to regularize it by placing it on a lattice. It can be a spatial lattice, when we use a Hamiltonian formalism, or it can be a Euclidean spacetime lattice for a Lagrangian formulation.

- Then, the system is well defined.
- Continuum limit: introduce a lattice spacing a, take $a \to 0$ and the number of sites to infinity holding all the physical lengths fixed. (Can also scale the lattice parameters to special values.)
  - Compute correlation functions at positions $a \ll x$.

Here, the lattice gives a definition of the continuum QFT. It is its UV version.

Lattice vs. continuum QFT

In condensed matter physics, the problem is defined on a lattice and the goal is to find its low-energy/long-distance limit.

- It is expected to be described by an effective continuum field theory.
- Unlike the lattice model, the continuum field theory depends on a finite number of parameters – universality – and hence, it is more effective.
- Often, it has new properties, not present on the lattice, e.g., emergent symmetries and new dualities.
- This use of QFT leads to a powerful description of possible phases and the transitions between them.

Here, the continuum QFT is the answer to a question – the IR limit of a given problem.

Challenges in using a lattice to define a given continuum QFT

- Does the limit exist and is it independent of the details of the lattice theory?
- Some continuum theories depend on the topology of field space, which relies on continuity of the fields. How is this captured by the lattice theory?
  - This issue affects various topological terms in the action, certain global symmetries, anomalies, etc. (More below.)
- Some QFTs (e.g., theories with self-dual forms or chiral fermions) do not admit a suitable Euclidean lattice action.
- Some QFTs do not even have a continuum Lagrangian, let alone a lattice version of it.

Challenges in finding a continuum low-energy QFT of a given lattice model

Exotic models, e.g., XY-plaquette model [Paramekanti, Balents, Fisher; …] (see below), fracton models [Chamon; Haah; Vijay, Haah, and Fu; …] (see below), do not have a standard continuum limit.

In order to understand that, we should first review and extend the discussion of other situations we do understand.

Canonical example of lattice vs. continuum: XY-model in 1+1d

[…; Jose, Kadanoff, Kirkpatrick, Nelson; …]

One of the most studied quantum field theories.

Use a Euclidean formulation.

Start with a 1+1d Euclidean, periodic lattice with sites labeled by $(\hat{\tau} = 1,2, \dots , L_\tau,\ \hat{x} = 1,2, \dots , L_x)$.

The degrees of freedom are phases $e^{i\phi}$ at the sites.

The Euclidean action is

$$S = -\beta \sum_{links} \cos\left(\Delta_\mu \phi\right)$$

Global $U(1)$ symmetry (momentum)

$$\phi(\hat{x}, \hat{\tau}) \to \phi(\hat{x}, \hat{\tau}) + \alpha$$

Canonical example of lattice vs. continuum: XY-model in 1+1d

[…; Jose, Kadanoff, Kirkpatrick, Nelson; …]

$$S = -\beta \sum_{links} \cos(\Delta_\mu \phi)$$

The continuum theory, is obtained by introducing a lattice spacing a and taking $L_x, L_\tau \to \infty$ and $a \to 0$ holding the physical lengths $\ell_x = L_x a$ and $\ell_\tau = L_\tau a$ fixed.

For large β, we expect ϕ to become smooth (discontinuities are suppressed) and then the action becomes

$$S = \frac{\beta}{2} \int d\tau dx (\partial_\mu \phi)^2 \qquad \phi \sim \phi + 2\pi$$

In high-energy physics it is common to use $R = \sqrt{\pi\beta}$.

XY-model in 1+1d – the continuum theory – $c = 1$ compact boson

$$S = \frac{\beta}{2}\int d\tau dx \left(\partial_\mu \phi\right)^2$$

- Free (quadratic action).
- Global symmetries $\partial_\mu j_\mu = 0$

$$Q = \oint dx\, j_\tau$$

  - $U(1)^m$ momentum (the charge) $j_\mu^m = -i\beta\partial_\mu\phi$

$$\phi(x,\tau) \to \phi(x,\tau) + \alpha$$

  - $U(1)^w$ winding (vorticity), emergent (not present on the lattice) $j_\mu^w = \frac{\epsilon_{\mu\nu}}{2\pi}\partial_\nu\phi$

    No action on the fields.

$c = 1$ compact boson

- Spectrum: quantize using Lorentzian signature time t. The finite circumference of space is ℓ

$$\phi(x,t) = \phi_0(t) + 2\pi W \frac{x}{\ell} + \sum_{k\in\mathbb{Z}_{\neq 0}} a_k(t) e^{2\pi i \frac{kx}{\ell}}$$

  - Plane waves (oscillators) with energy of order $E \sim \frac{1}{\ell}$
  - States charged under the momentum symmetry: the charge is the momentum $p_0 \in \mathbb{Z}$ conjugate to ϕ_0 and the energy is obtained from $L = \frac{\beta\ell}{2}(\partial_t\phi_0)^2 \to H = \frac{1}{2\beta\ell}p_0^2$ $E \sim \frac{1}{\beta\ell}$
  - States charged under the winding symmetry: $W \in \mathbb{Z}$ and the energy is obtained by substituting in the action $E \sim \frac{\beta}{\ell}$

$c = 1$ compact boson

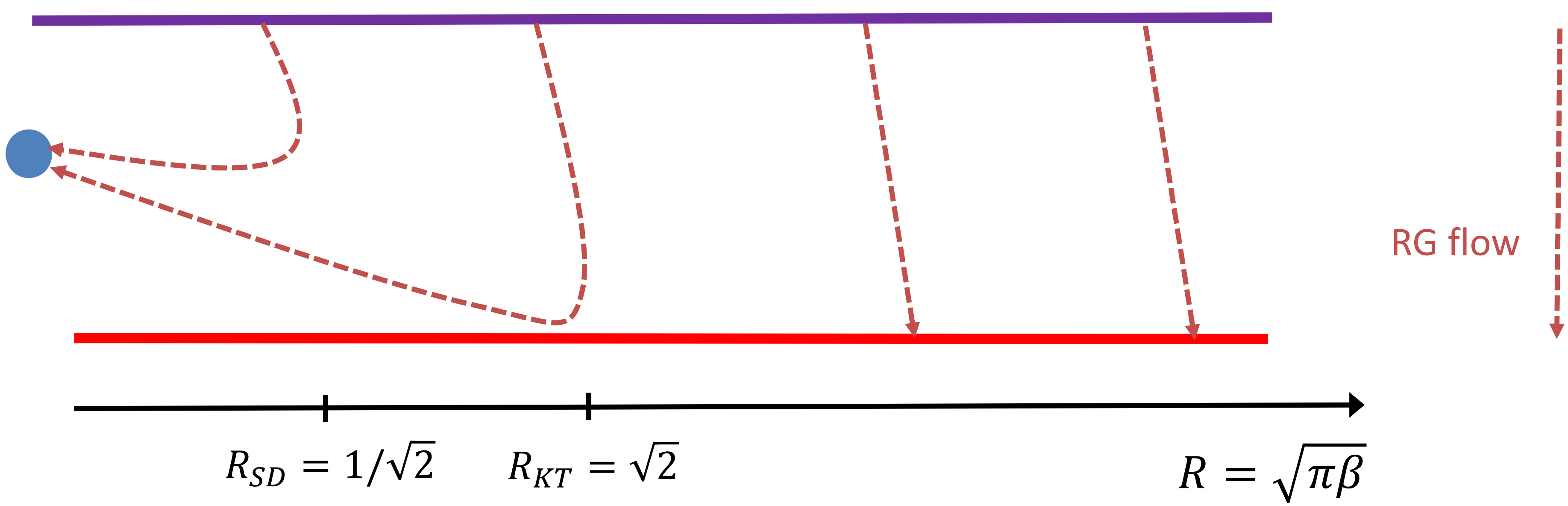
- The spectrum is gapless – the energies of all these states vanish as $\ell \to \infty$.
- The energies of all these states are of order $\frac{1}{\ell}$ and hence they are equally important.

$c = 1$ compact boson

- Exact self-duality (T-duality): exchanging $\beta \leftrightarrow \frac{1}{(2\pi)^2\beta}$ (in terms of $R = \sqrt{\pi\beta}$, it is $R \leftrightarrow \frac{1}{2R}$) and $U(1)^m \leftrightarrow U(1)^w$. Not present on the lattice.
- This continuum model exists for all β, while the original lattice model (with cosine) has a BKT transition at $\beta = \frac{2}{\pi}$ ($R = \sqrt{2}$) and is gapped for smaller β. (For small β, the value of β in the continuum and lattice models are not the same.)
  - For smaller β, the operators charged under the winding symmetry are relevant and cannot be ignored. (This is related to their dimension $\sim \beta$.) They gap the system.
  - This is not the case for large β, where they are irrelevant, and then the winding symmetry is an emergent symmetry.

Lattice XY-model in 1+1d – RG flow

$$S = -\beta \sum_{links} \cos(\Delta_\mu \phi)$$



$$S = \frac{\beta}{2} \int d\tau dx \left(\partial_\mu \phi\right)^2$$

$c = 1$ compact boson – ‘t Hooft anomaly

Couple the system to background gauge fields for these symmetries

$$S = \int d\tau dx \left(\frac{\beta}{2} \left(\partial_\mu \phi - A_\mu \right)^2 + \frac{i}{2\pi} \left(\tilde{A}_x \partial_\tau \phi - \tilde{A}_\tau \partial_x \phi \right) \right)$$

$$\phi \to \phi + \alpha(x, \tau)$$

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$$\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu \tilde{\alpha}$$

$$S \to S + \frac{i}{2\pi} \int d\tau dx \, \alpha \left(\partial_x \tilde{A}_\tau - \partial_\tau \tilde{A}_x \right)$$

No problem to leading order in A and $\tilde{A}$. Hence, there is no problem with the global symmetry. (And of course, there is no inconsistency.)

But for nonzero gauge fields, we cannot preserve both the $U(1)^m$ and the $U(1)^w$ gauge symmetries. This cannot be fixed by adding any local term in 1+1d. (Can cancel it using a local term in a 2+1d bulk.)

Lattice XY-model in 1+1d vs. $c = 1$ compact boson

- The spectrum of plane waves and momentum states is easy to derive on the lattice at weak coupling (large β) – same as in the continuum.
- No winding symmetry on the lattice, but at weak coupling, there is an approximate winding symmetry with the same winding states as in the continuum.
- No self-duality on the lattice (relates strong to weak coupling).
- Since there is no winding symmetry on the lattice, we cannot discuss its ‘t Hooft anomaly.

Can we find another lattice model, with all these properties for all β?

XY-model in 1+1d – modify the lattice theory

Use the Villain formulation – replace $\phi \in S^1$ with $\phi \in \mathbb{R}$ coupled to a gauge field $n_\mu \in \mathbb{Z}$ on the links (with gauge symmetry $k \in \mathbb{Z}$)

$$S_{Villain} = \frac{\beta}{2} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2$$

$$\phi \sim \phi + 2\pi\, k$$

$$n_\mu \sim n_\mu + \Delta_\mu k$$

This action is free (quadratic). Its physics is similar to that of the original, nonlinear action with cosines.

Following [...; Gross, Klebanov; ...; Sachdev, Park; ...], "suppress the vortices" on the lattice. Can do it by adding

$$\kappa \sum_{plaq} (\Delta_\tau n_x - \Delta_x n_\tau)^2$$

XY-model in 1+1d – getting closer to the continuum

$$\frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + \kappa\sum_{plaq}(\Delta_\tau n_x - \Delta_x n_\tau)^2$$

For $\kappa \to \infty$, the field strength (curvature) of the $\mathbb{Z}$ gauge field, $\Delta_\tau n_x - \Delta_x n_\tau$ vanishes – the gauge field is flat.

We can replace the action by the modified Villain action [Gorantla, Lam, NS, Shao]

$$S_{mod.Villain} = \frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i\sum_{plaq}\tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

with a Lagrange multiplier field $\tilde{\phi} \sim \tilde{\phi} + 2\pi$ on the plaquettes.

This lattice theory is similar to the continuum theory…

XY-model in 1+1d – modified Villain action

$$S_{mod.Villain} = \frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i\sum_{plaq}\tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

- Free
- Exact global symmetries
  - $U(1)^m$ momentum, $\phi \to \phi + \alpha$, $j_\mu^m = -i\,\beta(\Delta_\mu\phi - 2\pi n_\mu)$
  - $U(1)^w$ winding, $\tilde{\phi} \to \tilde{\phi} + \tilde{\alpha}$, $j_\mu^w = \frac{\epsilon_{\mu\nu}}{2\pi}(\Delta_\nu\phi - 2\pi n_\nu)$
  - 't Hooft anomaly. Essentially as in the continuum. These global symmetries act locally. But the Lagrangian density is not invariant; only e^{-S} is invariant.
  - Hence it is gapless for all β.

XY-model in 1+1d – modified Villain action

$$S_{mod.Villain} = \frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i\sum_{plaq}\tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

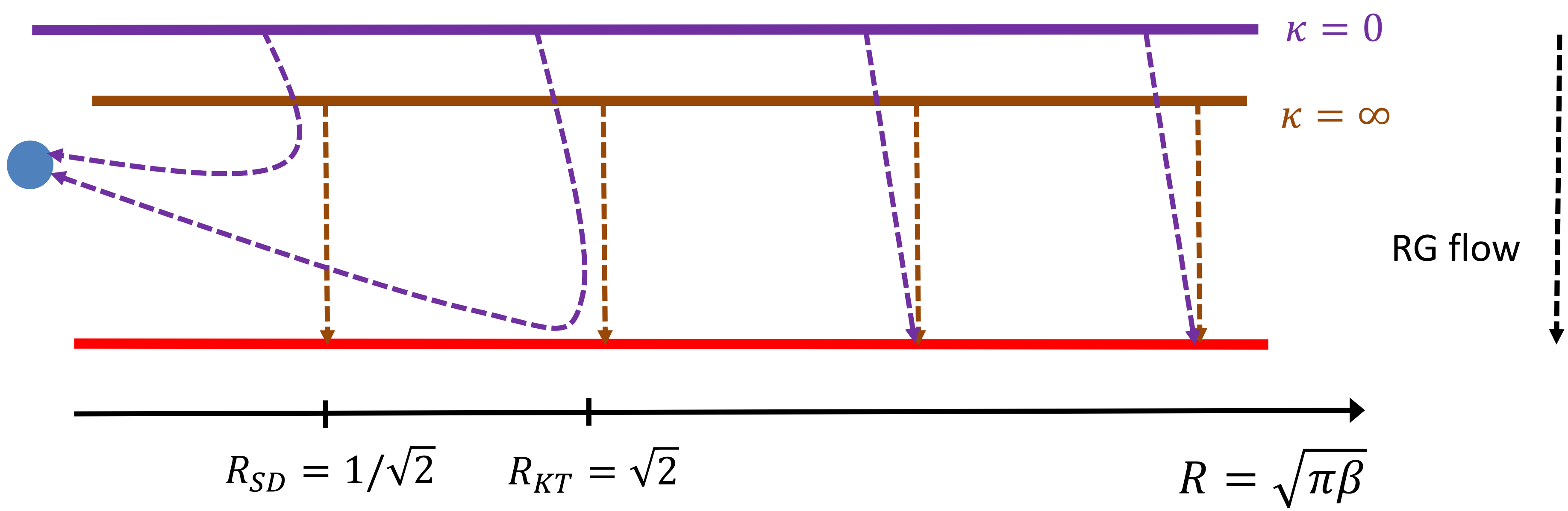
- Using Poisson resummation, self-duality: $\phi \leftrightarrow \tilde{\phi}, \;\; \beta \leftrightarrow \frac{1}{(2\pi)^2\beta}$.

  This self-duality is exact on the lattice. It is related, but not exactly the same, as the standard duality of the XY lattice model.
- For small a and large L with fixed $\ell = La$, the same spectrum and the same anomaly as in the continuum theory for all β.

This lattice model is very close to the continuum theory and provides a good regularization of it.

XY-model in 1+1d vs. the modified Villain model vs. the continuum theory

$$\frac{\beta}{2}\sum_{links}\left(\Delta_\mu \phi - 2\pi n_\mu\right)^2 + \kappa \sum_{plaq}(\Delta_\tau n_x - \Delta_x n_\tau)^2$$



An exotic theory: XY-plaquette model in 2+1d [Paramekanti, Balents, Fisher; …]

We will use a Euclidean-time, Lagrangian formulation.

On the lattice, phases $e^{i\phi}$ at the sites with the action

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos(\Delta_x \Delta_y \phi)$$

Global $U(1)$ subsystem (momentum) symmetry

$$\phi(\hat{x}, \hat{y}, \hat{\tau}) \to \phi(\hat{x}, \hat{y}, \hat{\tau}) + \alpha_x(\hat{x}) + \alpha_y(\hat{y})$$

$L_x + L_y - 1$ elements. For simplicity, we will set $L_x = L_y = L$.

Continuum limit: $a_\tau \sim a^2 \to 0$, with $\ell_\tau = L_\tau a_\tau,\ \ell = aL$ fixed.

$\mu_0 = \beta_0 \frac{a_\tau}{a^2}, \quad \mu = \frac{1}{\beta} \frac{a_\tau}{a^2}$

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi\right)^2 \right) \qquad \phi \sim \phi + 2\pi$$

XY-plaquette model in 2+1d – the continuum limit – ϕ-theory

- Free
- Because of the derivative structure, some discontinuous (in x, y) field configurations $\phi = \phi_x(x,\tau) + \phi_y(y,\tau)$ are not suppressed
- Subsystem global symmetries $\partial_\tau j_\tau = \partial_x \partial_y j_{xy}$

$$Q^x(x) = \oint dy\, j_\tau \quad , \quad Q^y(y) = \oint dx\, j_\tau$$

  – $U(1)^m$ momentum $j_\tau^m = i\mu_0 \partial_\tau \phi \,, \;\; j_{xy}^m = \frac{i}{\mu} \partial_x \partial_y \phi$

  $\phi(x,y,\tau) \to \phi(x,y,\tau) + \alpha_x(x) + \alpha_y(y)$

  $\alpha_x(x), \alpha_y(y)$ can be discontinuous.

  – $U(1)^w$ winding (vorticity), emergent

$$j_\tau^w = \frac{1}{2\pi} \partial_x \partial_y \phi \,, \;\; j_{xy}^w = \frac{1}{2\pi} \partial_\tau \phi$$

ϕ-theory in 2+1d

‘t Hooft anomaly between them [Gorantla, Lam, NS, Shao]

Couple the system to background gauge fields for these subsystem symmetries (such tensor gauge fields will appear below):

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2} (\partial_\tau \phi - A_\tau)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi - A_{xy} \right)^2 + \frac{i}{2\pi} \left(\tilde{A}_\tau \partial_x \partial_y \phi + \tilde{A}_{xy} \partial_\tau \phi \right) \right)$$

$$\phi \to \phi + \alpha(x, y, \tau)$$

$$A_\tau \to A_\tau + \partial_\tau \alpha \quad , \qquad A_{xy} \to A_{xy} + \partial_x \partial_y \alpha$$

$$\tilde{A}_\tau \to \tilde{A}_\tau + \partial_\tau \tilde{\alpha} \quad , \qquad \tilde{A}_{xy} \to \tilde{A}_{xy} + \partial_x \partial_y \tilde{\alpha}$$

$$S \to S + \frac{i}{2\pi} \int d\tau dx \, \alpha \left(\partial_x \partial_y \tilde{A}_\tau - \partial_\tau \tilde{A}_{xy} \right)$$

ϕ-theory in 2+1d – spectrum [NS, Shao]

- Quantize using Lorentzian signature time t. The finite circumference of space is ℓ (for simplicity, $\ell_x = \ell_y = \ell$)

$$\phi(x,y,t) = \phi_x(x,t) + \phi_y(y,t) + \sum_{k_x,k_y \in \mathbb{Z}_{\neq 0}} a_{(k_x,k_y)}(t) e^{2\pi i \left(\frac{k_x x}{\ell} + \frac{k_y y}{\ell}\right)}$$

  - Plane waves (oscillators) with $\omega^2 = (2\pi)^2 \frac{k_x^2 k_y^2}{\mu\mu_0 \ell^4}$. Because of this dispersion relation:
    - $\omega \sim \frac{1}{\ell^2}$ (and not $\frac{1}{\ell}$, as in more standard systems)
    - For large ℓ, can have low ω with large $p_x = \frac{k_x}{\ell}$, provided $p_y = \frac{k_y}{\ell}$ is sufficiently small – high momentum with low energy. This leads to UV/IR mixing. (More below.)

ϕ-theory in 2+1d – spectrum

$$\phi(x,y,t) = \phi_x(x,t) + \phi_y(y,t) + \sum_{k_x,k_y\in\mathbb{Z}_{\neq 0}} a_{(k_x,k_y)}(t)e^{2\pi i\left(\frac{k_x x}{\ell}+\frac{k_y y}{\ell}\right)}$$

– States charged under the momentum symmetry:
  - The modes $\phi_x(x,t)$, $\phi_y(y,t)$ can be thought of as associated with the spontaneous breaking of the momentum symmetry. We will soon see that this is not the case.
  - They include the standard winding modes $\phi = \frac{2\pi}{\ell}\left(W_x x + W_y y\right)$ and hence these should not be considered separately.

ϕ-theory in 2+1d – spectrum

- For simplicity, ignore the facts that $\phi_x(x,t)$ and $\phi_y(y,t)$ share their zero modes and that this zero mode couples to them. Then, $\phi_x(x,t)$ and $\phi_y(y,t)$ are independent rotors at different positions:

$$S = \frac{\ell\mu_0}{2}\int d\tau \left(\oint dx\,\left(\partial_t\phi_x(x,t)\right)^2 + \oint dy\left(\partial_t\phi_y(y,t)\right)^2\right)$$

  Restoring the lattice spacing a,

$$H = \frac{1}{2\ell\mu_0\, a}\left(\sum_{\hat{x}} n_x(\hat{x})^2 + \sum_{\hat{y}} n_y(\hat{y})^2\right)\ , \qquad n_x(\hat{x}), n_y(\hat{y}) \in \mathbb{Z}$$

  Their energies diverge $\sim \frac{1}{\mu_0\ell a} \to \infty$.

- The momentum subsystem symmetry was spontaneously broken in the classical theory, but it is restored in the quantum theory.

ϕ-theory in 2+1d – spectrum

What about states charged under the winding subsystem symmetry? To be periodic modulo 2π and carry charge, they must be of the form

$$\phi = \frac{2\pi}{\ell}\left(x\Theta(y-y_0) + y\Theta(x-x_0) - \frac{xy}{\ell}\right) \qquad 0 \leq x, y < \ell$$

$$j_\tau^w = \frac{1}{2\pi}\partial_x\partial_y\phi = \frac{1}{\ell}\left(\delta(y-y_0) + \delta(x-x_0) - \frac{1}{\ell}\right)$$

$$Q^x(x) = \oint dy\, j_\tau^w = \delta(x-x_0),$$

$$Q^y(y) = \oint dy\, j_\tau^w = \delta(y-y_0)$$

These configurations have infinite energy. Restoring the lattice spacing a, their energy is $\sim \frac{(2\pi)^2}{\mu\ell a}$. (They will be important later.)

ϕ-theory in 2+1d – spectrum

To summarize:

- Plane waves (oscillators) with energy of order $E \sim \frac{1}{\ell^2}$
- States charged under the momentum symmetry $E^m \sim \frac{1}{\mu_0 \ell a}$
- States charged under the winding symmetry $E^w \sim \frac{(2\pi)^2}{\mu \ell a}$

Only the plane waves are in the spectrum of the continuum theory.

The momentum and winding states exist on the lattice, but they are not dynamical excitations in the continuum theory.

Since they carry conserved charges, they are defects that can be added to the continuum theory.

They are similar to the charged states in the toric code, which are represented by defects in the continuum TQFT.

ϕ-theory in 2+1d

- Exact self-duality – T-duality (not present on the lattice)
  - $\mu_0 \leftrightarrow \frac{\mu}{(2\pi)^2}$
  - $U(1)^m \leftrightarrow U(1)^w$
- The defects charged under these symmetries are exchanged. Note that their (divergent) energies are exchanged:

  $$E^m \sim \frac{1}{\mu_0 \ell a} \leftrightarrow \quad E^w \sim \frac{(2\pi)^2}{\mu \ell a}$$

Many questions:

- Should we trust this continuum analysis with divergent energies? Make the treatment more rigorous.
- How much of that depends on the continuum theory? Can we find these phenomena (winding subsystem symmetry, 't Hooft anomaly, self-duality, etc.) on the lattice?

XY-plaquette model in 2+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

Repeat the discussion of the 1+1d XY-model for this model.

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos\left(\Delta_x \Delta_y \phi\right)$$

Use the Villain form

$$S_{Villain} = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2$$

Here $\phi \in \mathbb{R}, n_\tau, n_{xy} \in \mathbb{Z}$. $\mathbb{Z}$ tensor gauge symmetry with $k \in \mathbb{Z}$ (similar to the tensor gauge symmetry above and below)

$$\phi \sim \phi + 2\pi k$$
$$n_\tau \sim n_\tau + \Delta_\tau k$$
$$n_{xy} \sim n_{xy} + \Delta_x \Delta_y k$$

This gauge symmetry makes ϕ effectively circle valued.

XY-plaquette model in 2+1d – getting closer to the continuum

- Add to the action the gauge invariant term

$$\kappa \sum_{cubes} f^2 \qquad\qquad f = \Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau$$

- For $\kappa \to \infty$ the field strength ("curvature") of the $\mathbb{Z}$ tensor gauge field (n_τ, n_{xy}), f vanishes. Then, we can replace the action by the modified Villain action

$$S_{mod.Villain} = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2 + i \sum_{cubes} \tilde{\phi}(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)$$

with a Lagrange multiplier $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.

XY-plaquette model in 2+1d – getting closer to the continuum

$$S_{mod.Villain} = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2 + i \sum_{cubes} \tilde{\phi}(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)$$

Similar to the continuum version of the XY-plaquette model:

- Free
- Exact subsystem global symmetries
  - $U(1)^m$ momentum $\phi(\hat{x},\hat{y},\hat{\tau}) \to \phi(\hat{x},\hat{y},\hat{\tau}) + \alpha_x(\hat{x}) + \alpha_y(\hat{y})$
  - $U(1)^w$ winding $\tilde{\phi}(\hat{x},\hat{y},\hat{\tau}) \to \tilde{\phi}(\hat{x},\hat{y},\hat{\tau}) + \tilde{\alpha}_x(\hat{x}) + \tilde{\alpha}_y(\hat{y})$

XY-plaquette model in 2+1d – getting closer to the continuum

- ‘t Hooft anomaly between these two global symmetries – essentially as in the continuum analysis.
- The spectrum is as in the continuum analysis: light plane waves and heavy momentum and winding states.
- Using Poisson resummation, exact self-duality:
  - $\phi \leftrightarrow \tilde{\phi}$
  - $\beta_0 \leftrightarrow \frac{1}{(2\pi)^2 \beta}$
  - $U(1)^m \leftrightarrow U(1)^w$

This lattice model is completely rigorous. It exhibits all the features we found the in continuum model.

For $\beta, \beta_0 \gg 1$, it is the same as the nonlinear model with cosines.

ϕ-theory in 2+1d – Robustness

The main surprising result of the analysis of the spectrum is that the states charged under the momentum and winding subsystem symmetries have high energy – infinite in the continuum limit.

The operators that create these states, $\exp(i\phi)$ and $\exp\left(i\tilde{\phi}\right)$, exist on the lattice, but vanish in the continuum theory. These operators have infinite dimension in the continuum limit – infinitely irrelevant.

Consequently, the continuum theory is robust under small deformations violating the momentum and winding symmetries.

This is similar to the issue of robustness of the 1+1d compact boson above or below the BKT point. The main differences are that here, both the momentum and winding symmetries are robust, and that they are infinitely robust.

ϕ-theory in 2+1d – UV/IR mixing

Plane waves $E \sim \frac{1}{\ell^2} = \frac{1}{L^2 a^2}$

The charged states $E \sim \frac{1}{\ell a} = \frac{1}{L a^2}$

We are interested in $L \to \infty$.

Above, we took $a \to 0$ with fixed $\ell = La$. This kept the plane waves and pushed the charged states to infinity.

Alternatively, if we hold a fixed, i.e., $\ell \to \infty$, all these states have zero energy.

We see that

$$[\ell \to \infty \,, a \to 0] \neq 0$$

UV/IR mixing.

ϕ-theory in 2+1d – UV/IR mixing

For $\ell \to \infty$ (set $\mu = \mu_0 = 1$ and drop constants)

$$\langle \partial_\tau \phi(0,0,0) \partial_\tau \phi(x,y,\tau) \rangle \sim \begin{cases} -\dfrac{1}{(xy)^2} & |\tau| \ll |xy| \\ -\dfrac{1}{\tau^2} \log \dfrac{|\tau|}{|xy|} & |\tau| \gg |xy| \end{cases}$$

Singular as $xy \to 0$. This seems like a UV divergence. As $x \to 0$, it is associated with large momenta p_x. (And similarly for $y \to 0$.)

Because of the dispersion relation $\omega^2 \sim \left(p_x p_y\right)^2$, we can have large p_x with finite ω, provided p_y is small enough. Regularize the IR by setting finite ℓ, then, $|p_y| \geq \frac{1}{\ell}$ and the singularity becomes $-\frac{1}{\tau^2} \log \frac{\ell}{|y|}$.

Again, UV/IR mixing.

2+1d tensor gauge theory – A-theory

Above, we discussed systems with a global subsystem symmetry with current conservation $\partial_\tau j_\tau = \partial_x \partial_y j_{xy}$.

We also coupled them to background gauge fields $A_\tau j_\tau + A_{xy} j_{xy}$ with the gauge symmetry

$$A_\tau \sim A_\tau + \partial_\tau \alpha$$
$$A_{xy} \sim A_{xy} + \partial_x \partial_y \alpha$$

Now, we will study this gauge theory for dynamical (not background) fields and will focus on the pure gauge theory – no matter fields.

This and related theories on the lattice and in the continuum were discussed in [Xu, Wu; Slagle, Kim; Bulmash, Barkeshli; Ma, Hermele, Chen; Pretko; You, Burnell, Hughes; …].

2+1d tensor gauge theory – A-theory

The gauge invariant continuum action is

$$S = \int d\tau dx dy \left(\frac{1}{2g^2} E_{xy}^2 + \frac{i\theta}{2\pi} E_{xy} \right)$$
$$E_{xy} = \partial_\tau A_{xy} - \partial_x \partial_y A_\tau$$

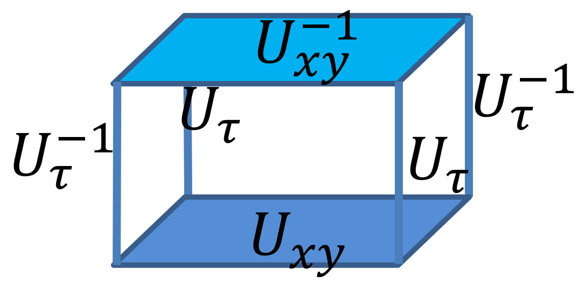
No magnetic field in 2+1d.

Here, we included also a θ-term. It will soon be clear why it is nontrivial (not a total derivative) and why the physics is periodic in θ – flux quantization.

2+1d tensor gauge theory – A-theory

On a Euclidean lattice, place phases U_τ on the time links and phases U_{xy} on the spatial plaquettes.

The interaction is associated with cubes. It is a product of two U_{xy} on the spatial faces and four U_τ on the time links.



The gauge parameters are phases $e^{i\alpha}$ at the sites. Each phase multiplies the link and plaquette variables that touch it.

2+1d tensor gauge theory – A-theory

Alternatively, we can use a Villain formulation

$$S = \sum_{cubes} \left(\frac{\beta}{2} \left(\Delta_\tau A_{xy} - \Delta_x \Delta_y A_\tau - 2\pi\, n_{\tau xy} \right)^2 + i\theta n_{\tau xy} \right)$$

Here, $A_\tau, A_{xy} \in \mathbb{R}$, $n_{\tau xy} \in \mathbb{Z}$ and we have the gauge symmetry

$$A_\tau \sim A_\tau + \Delta_\tau \alpha + 2\pi k_\tau$$
$$A_{xy} \sim A_{xy} + \Delta_x \Delta_y \alpha + 2\pi k_{xy}$$
$$n_{\tau xy} \sim n_{\tau xy} + \Delta_\tau k_{xy} - \Delta_x \Delta_y k_\tau$$

with $\alpha \in \mathbb{R}$, $k_\tau, k_{xy} \in \mathbb{Z}$.

There is no need to modify the Villain theory by constraining the field strength (curvature) of $n_{\tau xy}$ – there is no such field strength.

This formulation of the gauge theory is free (quadratic), the gauge group is $U(1)$ rather than $\mathbb{R}$, and it is easy to add the θ-term, which is manifestly nontrivial and periodic in θ.

2+1d tensor gauge theory – A-theory

Note, the continuum A_{xy} and A_τ differ from their lattice counterparts by factors of a^2 and a_τ respectively.

Hence, $\beta = \frac{1}{g^2 a^2 a_\tau}$.

Therefore, we can consider a continuum limit $a, a_\tau \to 0$

- with fixed lattice coupling β and then $g \to \infty$
- or with fixed continuum coupling g, and then $\beta \to \infty$.

2+1d tensor gauge theory – A-theory

Next, we will discuss

- Global aspects of the gauge group (flux quantization)
- Defects and operators
- The spectrum

We can use either the continuum or the lattice (in its Villain form) formulations. We'll write it in continuum language.

As in the discussion of the matter theory, the ϕ-theory, this Villain formulation allows us to justify the treatment of the continuum formulation.

2+1d A-theory – flux quantization

Globally, α is subject to the identifications

$$\alpha \sim \alpha + 2\pi\left(w_x(x) + w_y(y)\right) \quad , \qquad w_x(x), w_y(y) \in \mathbb{Z}$$

Therefore, we can have large gauge transformations like

$$\alpha = \frac{2\pi}{\ell}\left(x\,\Theta(y - y_0) + y\,\Theta(x - x_0) - \frac{xy}{\ell}\right)$$

In a Euclidean box, with such a transition function around the Euclidean time circle, we find quantized fluxes

$$\oint d\tau\, dx\, E_{xy} = 2\pi\delta(y - y_0)$$

$$\oint d\tau\, dy\, E_{xy} = 2\pi\delta(x - x_0)$$

$$\int d\tau dx dy\, E_{xy} \in 2\pi\mathbb{Z}$$

2+1d A-theory – defects

The simplest defect (in Lorentzian signature) is

$$\exp\left(i\int_{-\infty}^{\infty} dt\ A_0\right)$$

represents the world-line of a probe particle at fixed position. Gauge invariance prevents it from moving. It is a fracton.

A dipole of probe particles at x_1, x_2 is represented by the gauge invariant defect (similar dipole at y_1, y_2)

$$\exp\left(i\int_{x_1}^{x_2} dx\ \int_{\mathcal{C}}\left(dt\ \partial_x A_0 + dy\ A_{xy}\right)\right)$$

$\mathcal{C}$ is a curve in the (y, t) plane. The dipole is restricted to a line. It is a lineon.

This restricted mobility follows from the gauge symmetry.

2+1d A-theory – operators

Defects at fixed time are "Wilson strips." Along y (and similarly along x)

$$W_x(x_1, x_2) = \exp\left(i \int_{x_1}^{x_2} dx \oint dy\, A_{xy} \right)$$

Point operators E_{xy}

Since Gauss law sets $\partial_x \partial_y E_{xy} = 0$,

$$E_{xy} = e_x(x) + e_y(y)$$

They satisfy

$$[W, E_{xy}] = \begin{cases} -g^2 W & \text{when they touch} \\ 0 & \text{when they do not touch} \end{cases}$$

2+1d A-theory – spectrum

The Lorentzian signature Lagrangian is

$$\mathcal{L} = \frac{1}{2g^2} E_{xy}^2 + \frac{\theta}{2\pi} E_{xy}$$

Because of the flux quantization, $\theta \sim \theta + 2\pi$.

Choose $A_0 = 0$, then Gauss law sets

$$\partial_x \partial_y E_{xy} = 0$$

Up to a gauge transformation

$$A_{xy} = \frac{1}{\ell} f_x(x,t) + \frac{1}{\ell} f_y(y,t)$$

No local excitations.

Reminiscent of ordinary $U(1)$ gauge theory in 1+1d.

2+1d A-theory – spectrum

$$A_{xy} = \frac{1}{\ell} f_x(x,t) + \frac{1}{\ell} f_y(y,t)$$

Ignoring the (common) zero mode and focusing on one of them

$$L = \frac{1}{2g^2\ell} \oint dx\, (\partial_0 f_x)^2 + \frac{\theta}{2\pi} \oint dx \partial_0 f_x$$

The large gauge transformation above means that

$$f_x(x,t) \sim f_x(x,t) + 2\pi\delta(x - x_0)$$

i.e., here the radius of the rotor is infinite.

We restore the spatial lattice with lattice spacing a. Then, the periodicity of f_x is $2\pi/a$ and the energies are small – vanish in the continuum limit. Infinite ground state degeneracy in that limit

$$\frac{g^2 a\, \ell}{2} \sum_{\hat{x}} \left(n(\hat{x}) - \frac{\theta}{2\pi} \right)^2 \qquad , \qquad n(\hat{x}) \in \mathbb{Z}$$

2+1d A-theory – spectrum

$$\frac{g^2 a\,\ell}{2}\sum_{\hat{x}}\left(n(\hat{x})-\frac{\theta}{2\pi}\right)^2 \quad , \qquad n(\hat{x})\in\mathbb{Z}$$

If instead, we take the continuum limit with fixed lattice coupling $\beta=\frac{1}{g^2a^2a_\tau}$, then the energies diverge as

$$\frac{\beta\ell}{2a\,a_\tau}\sum_{\hat{x}}\left(n(\hat{x})-\frac{\theta}{2\pi}\right)^2$$

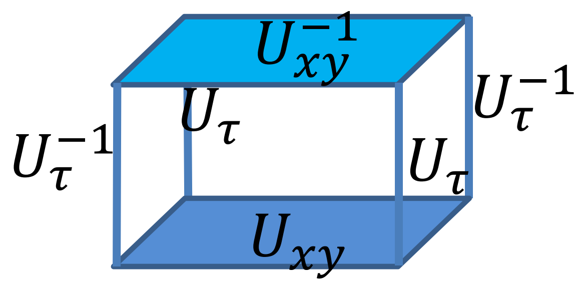
and we are left only with the ground state $n(\hat{x})=0$.

All these results about the continuum theory, using discontinuous fields and gauge parameters (flux quantization, defects and operators, the spectrum) can be justified using the Villain formulation of this theory.

2+1d $\mathbb{Z}_N$ tensor gauge theory on the lattice

On a Euclidean lattice, place $\mathbb{Z}_N$ phases U_τ on the time links and $\mathbb{Z}_N$ phases U_{xy} the spatial plaquettes.

The interaction is associated with cubes. It is a product of two U_{xy} on the spatial faces and four U_τ on the time links.



The gauge parameters are $\mathbb{Z}_N$ phases $e^{i\alpha}$ at the sites. Each phase multiplies the link and plaquette variables that touch it.

2+1d $\mathbb{Z}_N$ tensor gauge theory on the lattice

Alternatively, we can use a modified Villain formulation. This is a $\mathbb{Z}_N$ tensor gauge theory with vanishing field strength (curvature).

We place integers n_τ on the Euclidean time links, n_{xy} on the spatial plaquettes, and $\tilde{n}$ on the cubes, and write the action

$$S = \frac{2\pi i}{N} \sum_{cubes} \tilde{n}\left(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau\right)$$

It has the gauge symmetry $k, k_\tau, k_{xy}, \tilde{k} \in \mathbb{Z}$

$$n_\tau \sim n_\tau + \Delta_\tau k + N\, k_\tau$$
$$n_{xy} \sim n_{xy} + \Delta_x \Delta_y k + N\, k_{xy}$$
$$\tilde{n} \sim \tilde{n} + N\, \tilde{k}$$

(n_τ, n_{xy}) are the same as the integer tensor gauge fields of the Villain formulation of the XY-plaquette model.

2+1d $\mathbb{Z}_N$ tensor gauge theory in the continuum

Couple the ϕ-theory to the A-theory, Higgsing $U(1) \to \mathbb{Z}_N$. In Euclidean signature,

$$\mathcal{L} = ib\,(\partial_\tau \phi - NA_\tau) + ie\,\left(\partial_x \partial_y \phi - NA_{xy}\right)$$
$$A_\tau \to A_\tau + \partial_\tau \alpha$$
$$A_{xy} \to A_{xy} + \partial_x \partial_y \alpha$$
$$\phi \to \phi + N\,\alpha$$

b and e act as Lagrange multipliers, forcing the Higgsing.

We can dualize ϕ to $\tilde{\phi}$, as above, and find a BF-like description

$$\mathcal{L} = i\frac{N}{2\pi}\tilde{\phi}\,E_{xy} \qquad , \qquad \tilde{\phi} \sim \tilde{\phi} + 2\pi$$

2+1d $\mathbb{Z}_N$ tensor gauge theory – defects

The gauge invariant defects in Lorentzian signature are as in the A-theory

- fracton at a fixed position

$$\exp\left(i\int_{-\infty}^{\infty} dt\, A_0\right)$$

- dipole lineon, which can move in y, but not in x (similarly with $x \leftrightarrow y$)

$$\exp\left(i\int_{x_1}^{x_2} dx\, \int_{\mathcal{C}} \left(dt\, \partial_x A_0 + dy\, A_{xy}\right)\right)$$

The only difference is that their N'th powers are trivial defects.

2+1d $\mathbb{Z}_N$ tensor gauge theory – operators

The gauge invariant operators are:

- Electric "Wilson strips" along y (similarly along x)

$$W_x(x_1, x_2) = exp\left(i \int_{x_1}^{x_2} dx \oint dy\, A_{xy}\right) \quad , \qquad W_x^N = 1$$

- Magnetic point operators $\mathcal{O} = \exp\left(i\tilde{\phi}\right) \quad , \qquad \mathcal{O}^N = 1$

  Gauss law sets $\mathcal{O} = \mathcal{O}_x(x)\mathcal{O}_y(y)$

They satisfy $W\mathcal{O} = e^{-\frac{2\pi i}{N}}\mathcal{O}\, W$ when they touch

$W\mathcal{O} = \mathcal{O}\, W$ when they do not touch

2+1d $\mathbb{Z}_N$ tensor gauge theory – spectrum

In Lorentzian signature,

$$\mathcal{L} = b\,(\partial_0\phi - NA_0) + e\left(\partial_x\partial_y\phi - NA_{xy}\right)$$

Solve $A_0 = \frac{1}{N}\partial_0\phi\,,\;\; A_{xy} = \frac{1}{N}\partial_x\partial_y\phi.$

Then, the ground states are $\{\phi\}/\phi \sim \phi + N\alpha$

These are generated by

$$\phi = 2\pi\left(\frac{x}{\ell^x}\,\Theta(y-y_0) + \frac{y}{\ell^y}\,\Theta(x-x_0) - \frac{xy}{\ell^x\ell^y}\right)$$

For every value of x_0 and for every value of y_0 we have an integer modulo N. Accounting for the common zero mode and placing on a lattice, we have

$$N^{L^x+L^y-1}$$

states.

2+1d $\mathbb{Z}_N$ tensor gauge theory

- This entire discussion of the continuum theory can be phrased using the modified Villain version of the model.
- Gapped. Only zero energy states (infinitely many in the continuum limit)
- The spectrum is in the simplest representation of the algebra of operators.
- The local operators $\mathcal{O} = \exp\left(i\tilde{\phi}\right)$ generate an electric symmetry. It is a symmetry of the nonlinear lattice model.
- The strip operators generate a magnetic symmetry. It exists in the continuum and in the modified Villain lattice version of the model, but not in the original non-linear lattice model.

2+1d $\mathbb{Z}_N$ tensor gauge theory – robustness

- $\mathcal{O} = \exp\left(i\tilde{\phi}\right)$ preserves all the symmetries of the lattice model, but not the magnetic symmetry, which is absent in the nonlinear lattice model.
  - Starting with the nonlinear lattice model, $\mathcal{O} = \exp\left(i\tilde{\phi}\right)$ can appear in the continuum Lagrangian and destabilize it – lift the degeneracy. The system is not robust!
  - The nonlinear lattice model does not exhibit the ground state degeneracy.
  - This is not the case for its modified Villain version.

If we want to have fractons and robustness (no point operators acting in the space of ground states), we have to go up in dimensions.

Reminiscent of the ordinary $\mathbb{Z}_N$ gauge theory in 1+1d

- It has a continuum description based on Higgsing
$\mathcal{L} = ib\,(\partial_\tau \phi - NA_\tau) + e\,(\partial_x \phi - NA_x)$
- It has a dual BF-description $\mathrm{i}\,\frac{N}{2\pi}\tilde{\phi}\,E$.
- The holonomy of the $\mathbb{Z}_N$ gauge theory generates a magnetic $\mathbb{Z}_N$ global symmetry.
- It has no bulk excitations.
- On a circle, it has N degenerate states labeled by the holonomy.
- $\mathcal{O} = \exp(i\tilde{\phi})$ breaks that magnetic symmetry and makes it not robust. Starting with a lattice theory without this symmetry, the N states are not degenerate.

3+1d models

Repeat the discussion in 2+1d:

- Here, we preserve the cubic group $S_4 \subset SO(3)$.
- Lattice
  - nonlinear model
  - modified Villain model with more symmetries, anomalies, and exact dualities
- Different ways of taking the low-energy limit
  - Fixed lattice coupling constants
  - Fixed continuum coupling constants – low energy combined with a limit of the lattice coupling. This continuum theory has similar symmetries, duality, etc. to the modified Villain model.
- Reminiscent of ordinary field theories in 2+1d.

3+1d models

Gapless theories

- ϕ-theory: similar to the discussion in 2+1d, except that it is dual to an exotic $U(1)$ gauge theory, $\hat{A}$-theory.
- A-theory: similar to the discussion in 2+1d, except that it is dual to a non-gauge theory, $\hat{\phi}$-theory.
- Several other systems associated with other $U(1)$ subsystem symmetries, e.g., a $U(1)$ version of the checkerboard model.

Gapped theories are obtained by Higgsing these gauge theories $U(1) \to \mathbb{Z}_N$.

3+1d models

- 3 dual exotic $\mathbb{Z}_N$ gauge theories (similar to the discussion in 2+1d): Higgs the A-theory by ϕ, Higgs the $\hat{A}$-theory by $\hat{\phi}$, a BF-type theory involving A and $\hat{A}$ (related to [Slagle, Kim]).
  - They describe the long-distance behavior of one of the most celebrated fracton models, the X-cube model [Vijay, Haah, Fu].
  - Unlike the 2+1d exotic $\mathbb{Z}_N$ theory, it does not have local operators. Hence, it is robust!
  - Similar to ordinary $\mathbb{Z}_N$ gauge theory in 2+1d (and the related toric code)

Summary

- It is common to use a lattice theory to define a continuum quantum field theory.
- The low-energy limit of a lattice theory is expected to be a continuum quantum field theory.
- Exotic lattice models are challenging counter examples, primarily because of their UV/IR mixing:
  - Subsystem global symmetry
  - Large ground state degeneracy (infinite in the continuum limit)
  - Discontinuous and even singular observables in the continuum limit
- These seem incompatible with the standard framework of continuum QFT.

Summary

- The continuum field theory descriptions of these exotic models necessarily involve discontinuous fields.
- The modified Villain versions of these models provide a rigorous justification for the analysis of these continuum field theories.

Thank you

Stay healthy