LIMIT SHAPES OF NON-INTERSECTING BROWNIAN BRIDGES VIA HARMONIC ENVELOPES

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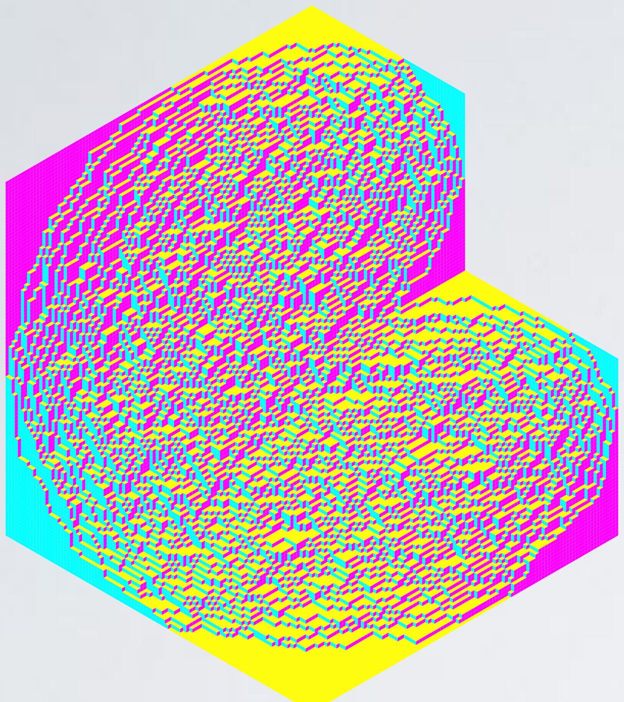
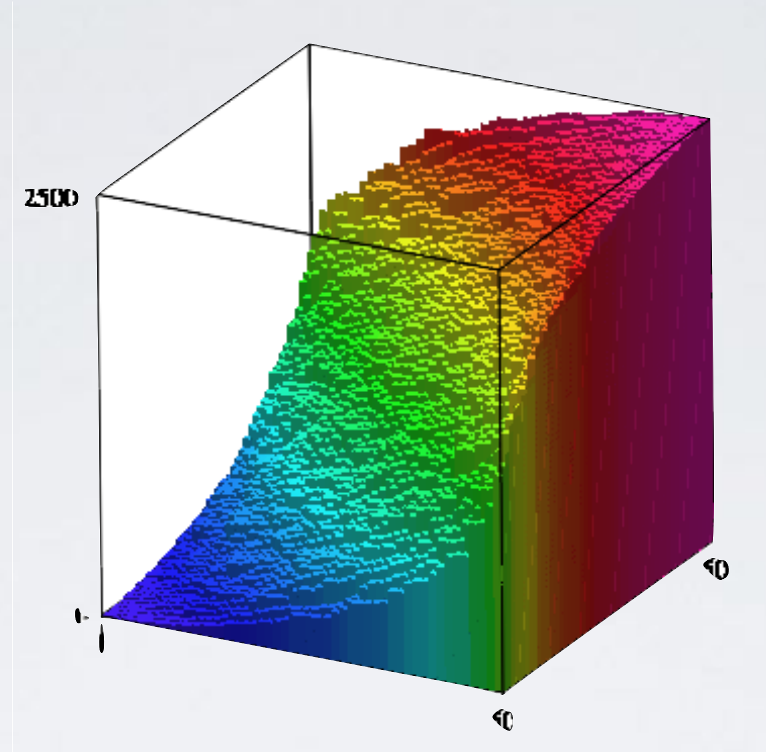
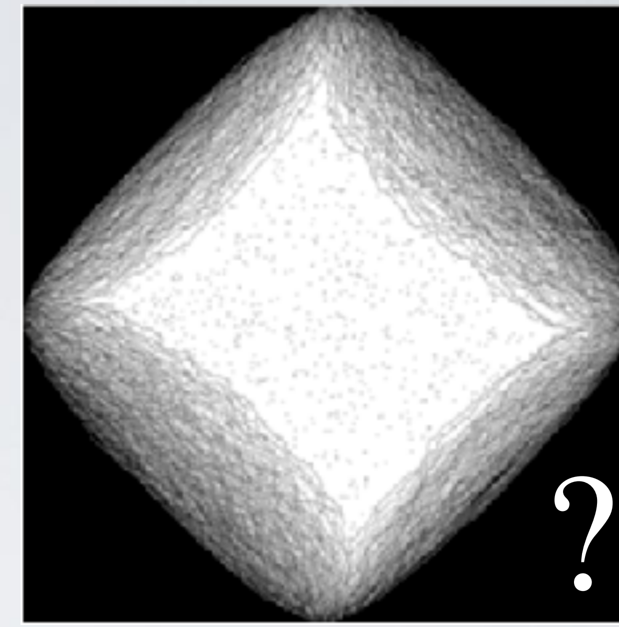
IPAM, Los Angeles - May 2024

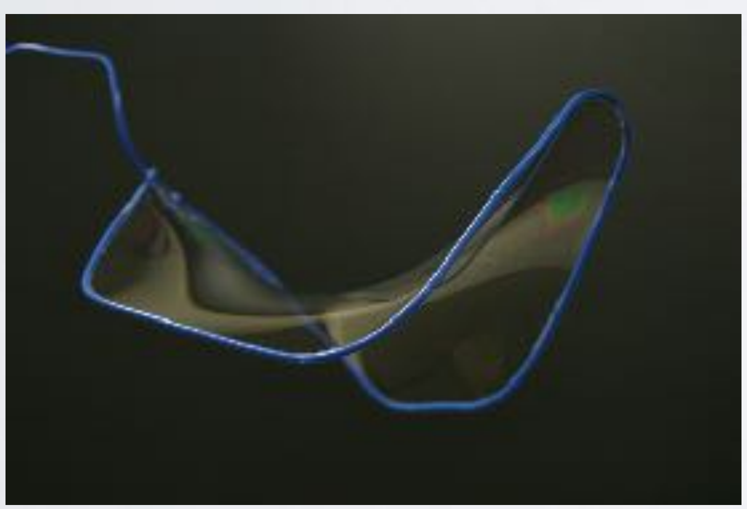
Picture credits: Borodin, Chelkak, Colomo-Sportiello, Grela-Majumdar-Schehr, Huang, Kenyon, Reshetikhin, Wikipedia

ZOO OF LIMIT SHAPES

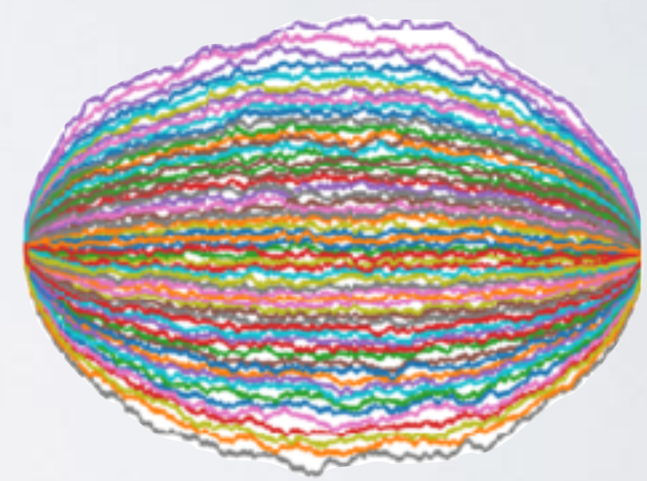








$$\min_{h|_{\partial\Omega}=h_0} \int_{\Omega} \sigma(\nabla h)$$

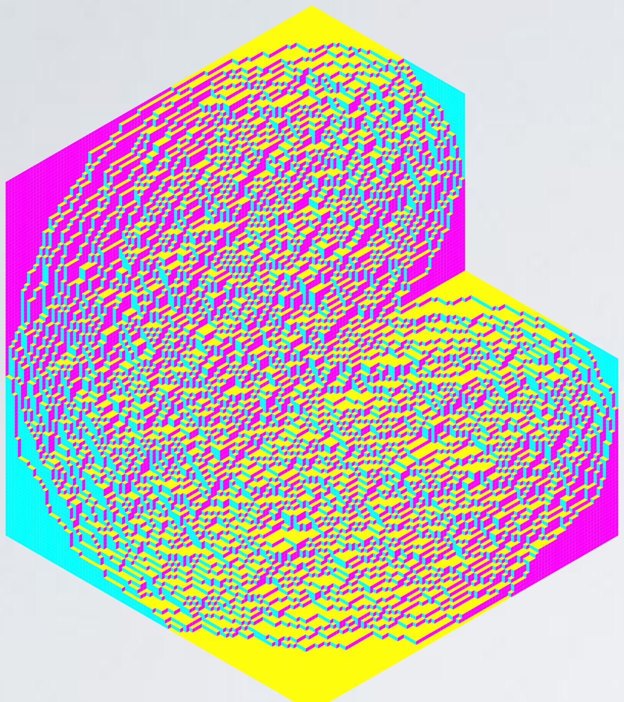


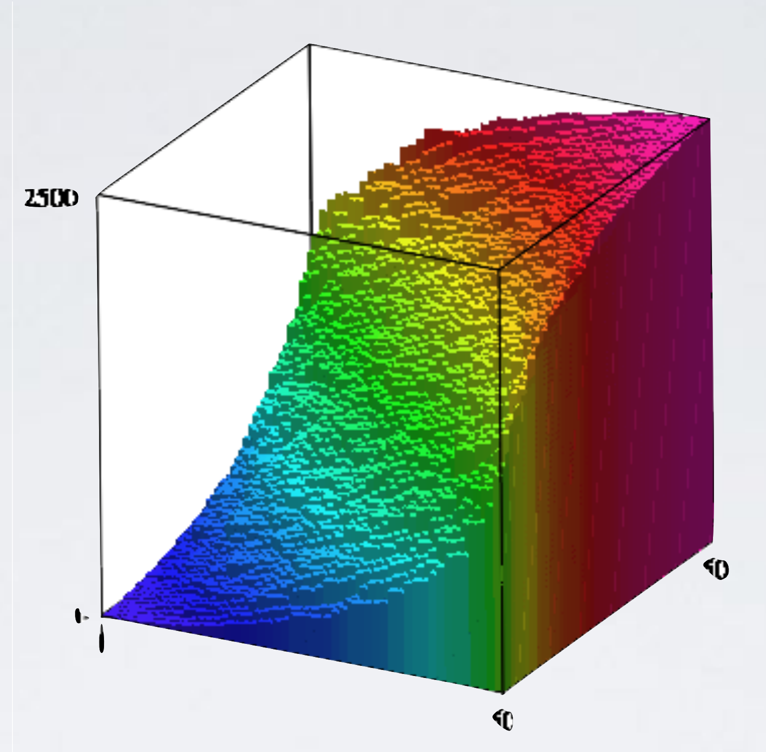
variational approach

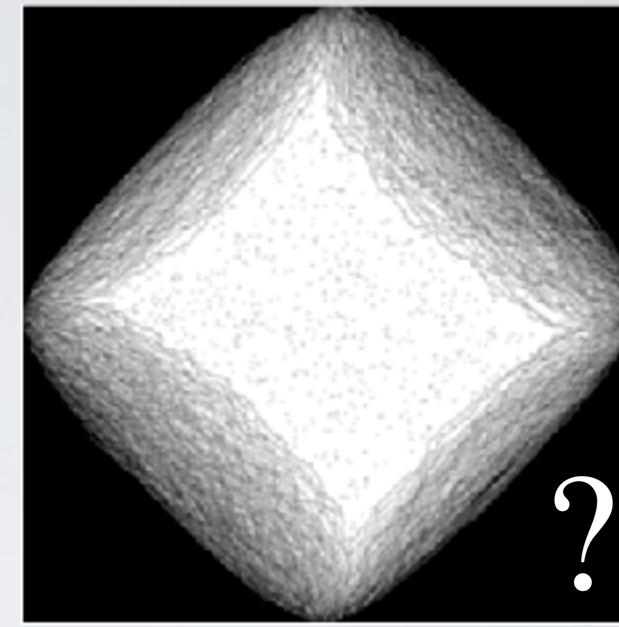
unifies

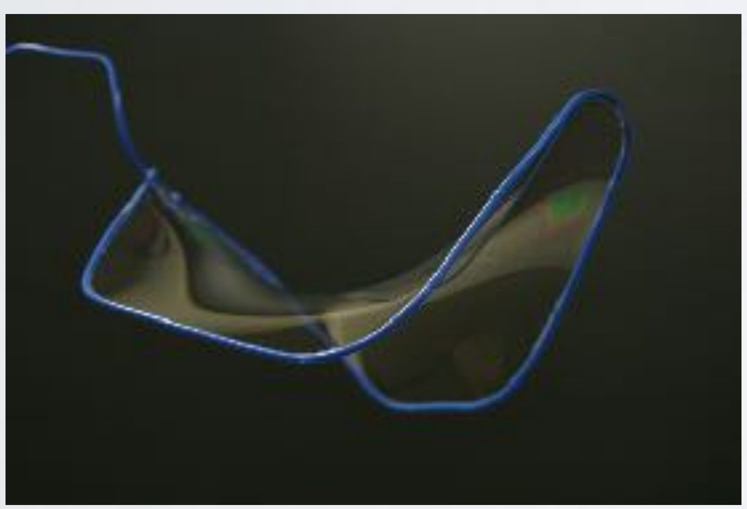
general boundary conditions & a variety of models

ZOO OF LIMIT SHAPES









$$\min_{h|_{\partial\Omega}=h_0} \int_\Omega \sigma(\nabla h)$$

variational approach

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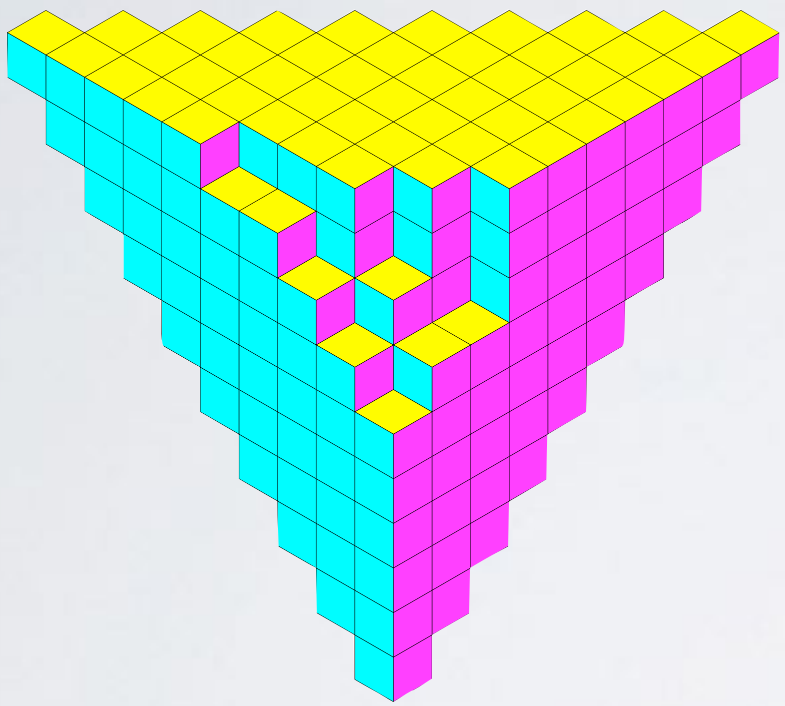
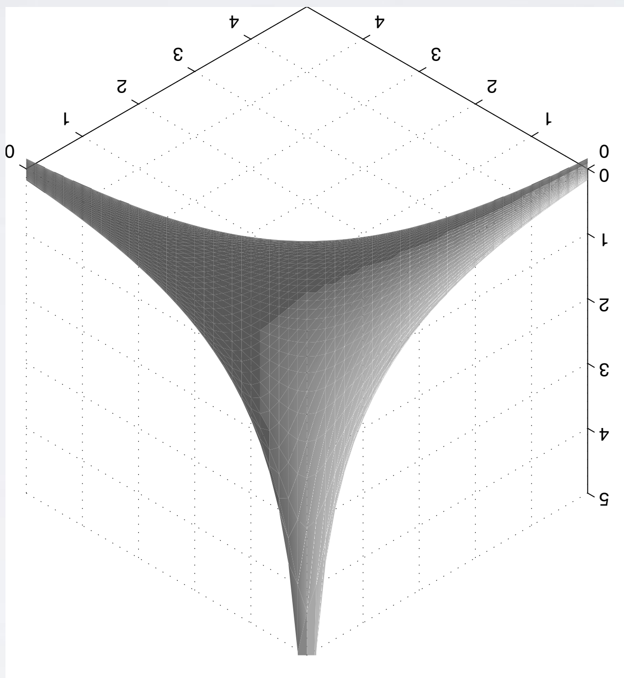
WULFF SHAPE

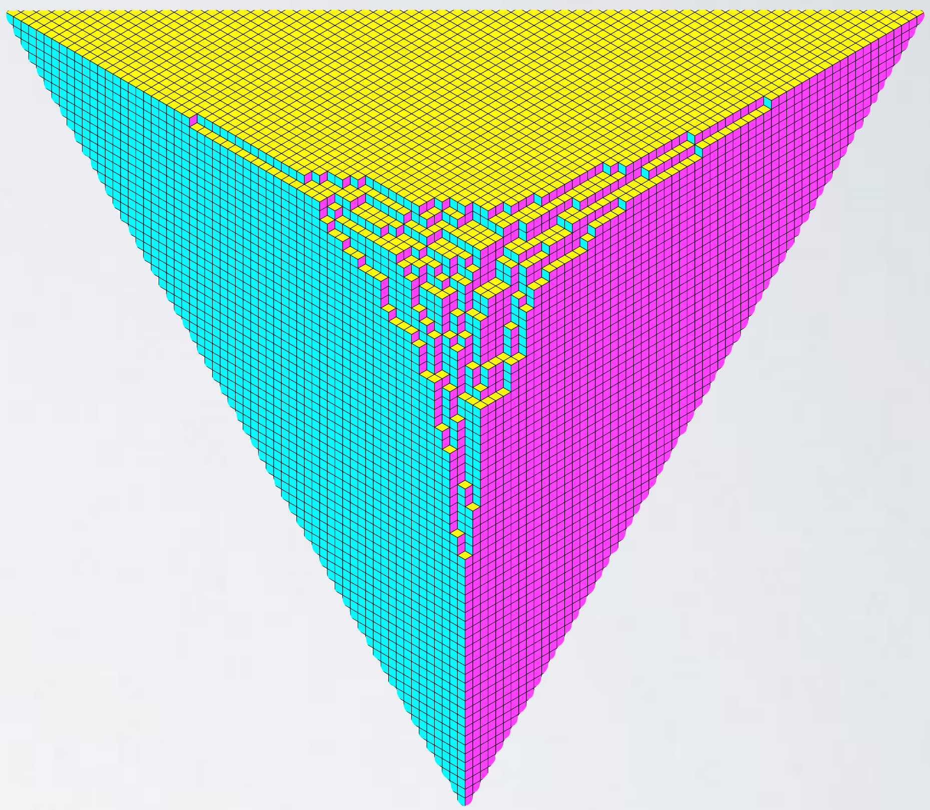
Wulff shape - Legendre dual of surface tension

itself a limit shape

(lozenge tilings $\leftrightarrow$ 3D Young diagram)







"fundamental solution"

(facets, facet-rough transition, phases, algebraic boundary etc)

Kenyon-Okounkov-Sheffield

$\det D^2\sigma \equiv \pi^2$ for the dimer model ("free fermions")

LIMIT SHAPES IN THE DIMER MODEL

polygonal boundary (extremal boundary)

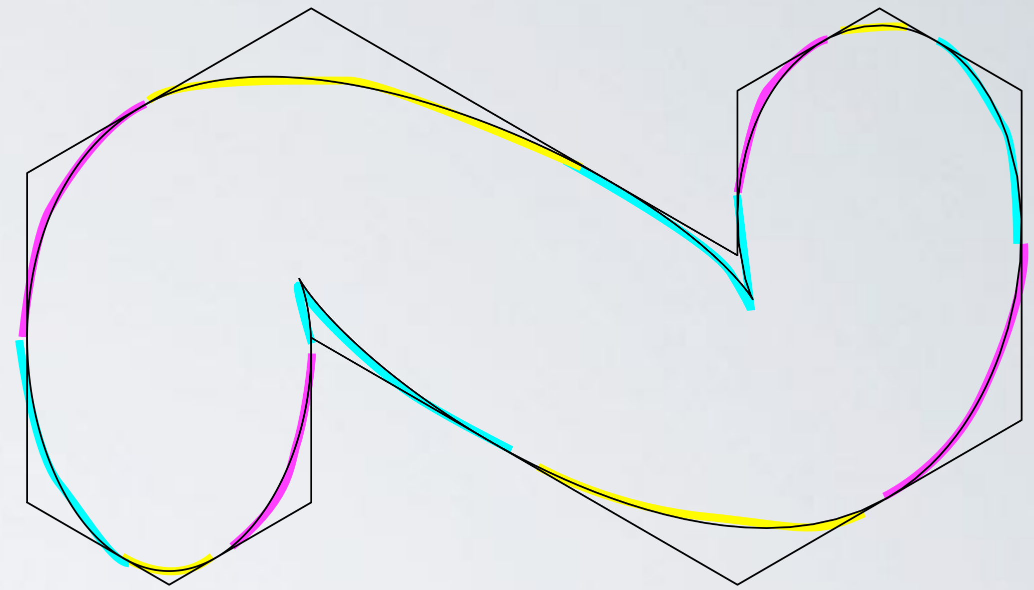
Kenyon-Okounkov

"match a limit shape to boundary"

lozenges

&

dominos



Petrov, Bufetov-Gorin, Bufetov-Knizel

special polygons - also fluctuations

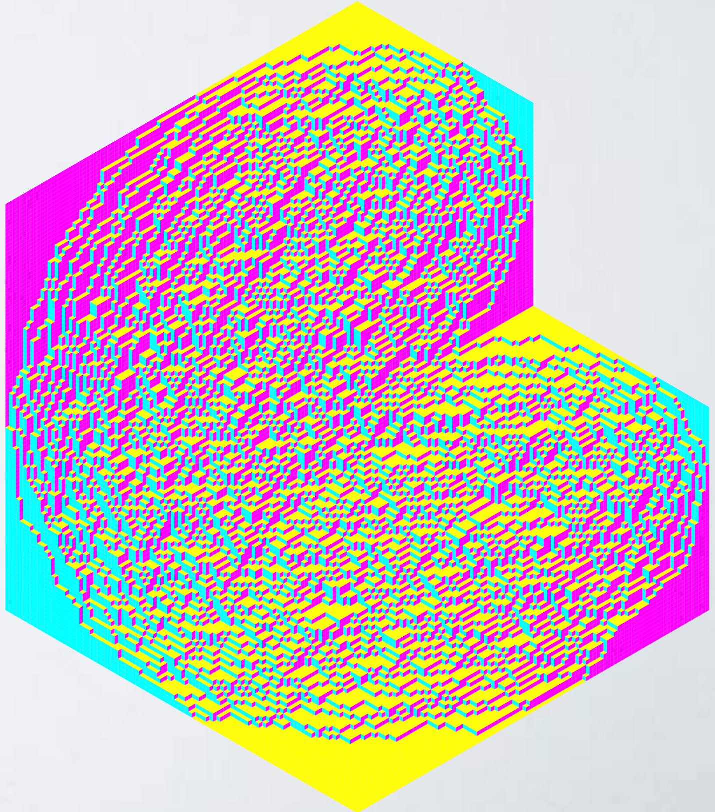
algebraic arctic curve

cloud curves

Astala-Duse-P-Zhong

"prove things about the minimizer"

general dimer model



frozen boundaries are universal

key property $\det D^2\sigma \equiv \pi^2$, "free fermions"

More on dimer limit shapes: talk by Alexei Borodin

WHY ALGEBRAIC BOUNDARIES?

polygonal boundaries $\longrightarrow$ complex structure degenerates *everywhere* on the boundary $\longrightarrow$ double compactification *algebraic curve*

Issue: lack of continuity up to the boundary

deSilva-Savin **properness**

$$\nabla h(x,y) \to \partial N$$

$$(x,y) \to \partial \mathscr{L}$$

a priori only $\dfrac{\bar\partial z(x,y)}{\partial z(x,y)} \to \partial \mathbb{D}$

properness+holomorphicity $\longrightarrow$ **weak holomorphicity across reflection**

ISOTHERMAL COORDINATE

Riemannian metric associated to the Wulff shape

$$\gamma := \frac{-\sigma_{ss} - i\sqrt{\sigma_{ss}\sigma_{tt} - \sigma_{st}^2}}{\sigma_{ss}}$$

(analogue of logarithmic Gauss map)

$$\frac{z_s}{z_t} = -\frac{1}{\bar{\gamma}}$$

$$(s, t) \in \mathscr{N} \leftrightarrow z \in \mathbb{C}$$

$$(x, y) \in \mathscr{L} \mapsto \nabla h(x, y) \mapsto z(x, y)$$

Euler-Lagrange $\frac{z_x}{z_y} = \gamma(z)$ (analogue of complex Burgers)

z is the conformal coordinate of the model

"Write everything in terms of z"

κ-HARMONIC ENVELOPE

$\sigma(s,t)$
$(s,t) \in \mathcal{N}$

Kenyon-P

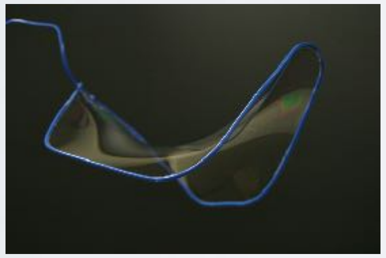
$\kappa(z) = \sqrt{\det D^2\sigma}$ as a function of $z \in \mathbb{H}$

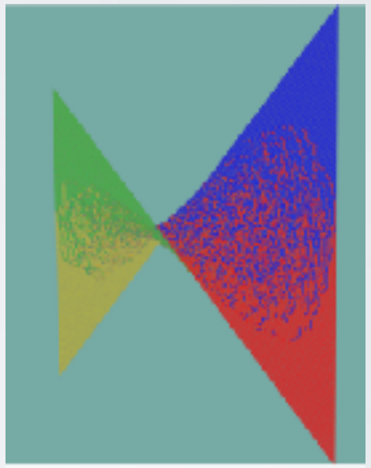
$$\nabla \cdot \kappa \nabla u = 0$$

Thm: s, t and h-$(sx+ty)$ are all κ-harmonic(z) in the liquid region

(multi-valued in z)

Corollary: Free-fermionic limit shapes are *envelopes* of harmonically moving planes in R³





minimal surfaces

$(x, y, h(x,y))$

$\gamma_z = 0$

$\longleftrightarrow$

dimer limit shapes

$(h_x, h_y, h - \nabla h \cdot (x,y))$

$\gamma_{\bar{z}} = 0$

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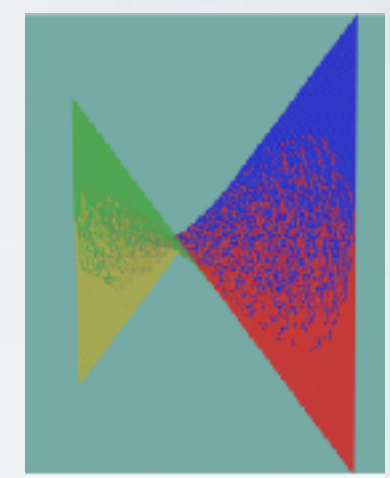

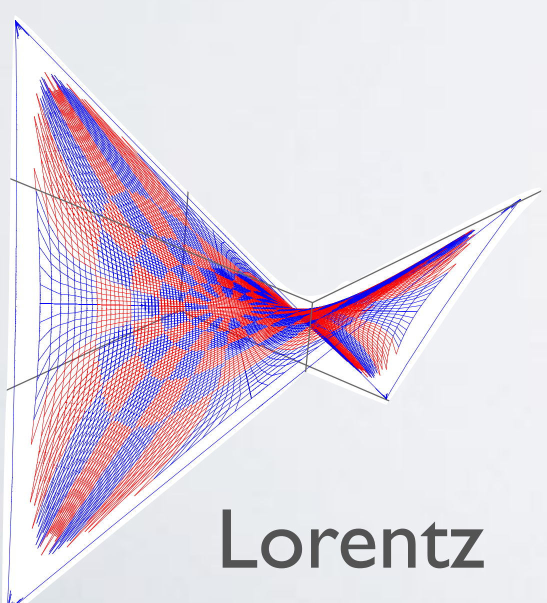

Lorentz maximal surfaces

? ⟷

Marianna Russkikh's talk

dimer limit shapes

$$(h_x, h_y, h - \nabla h \cdot (x,y))$$

$$\gamma_{\bar{z}} = 0$$

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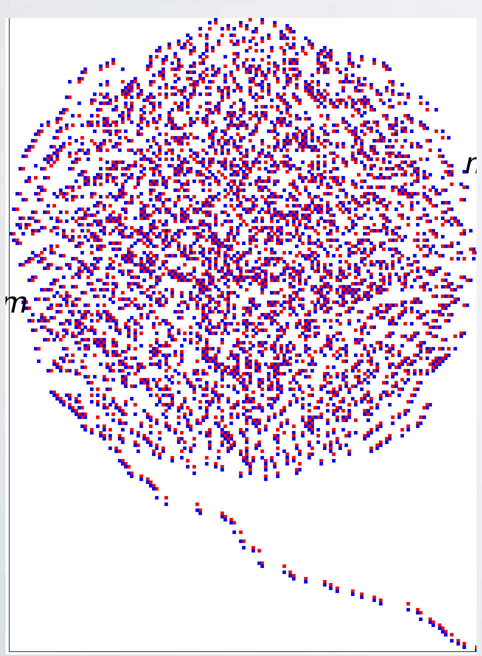
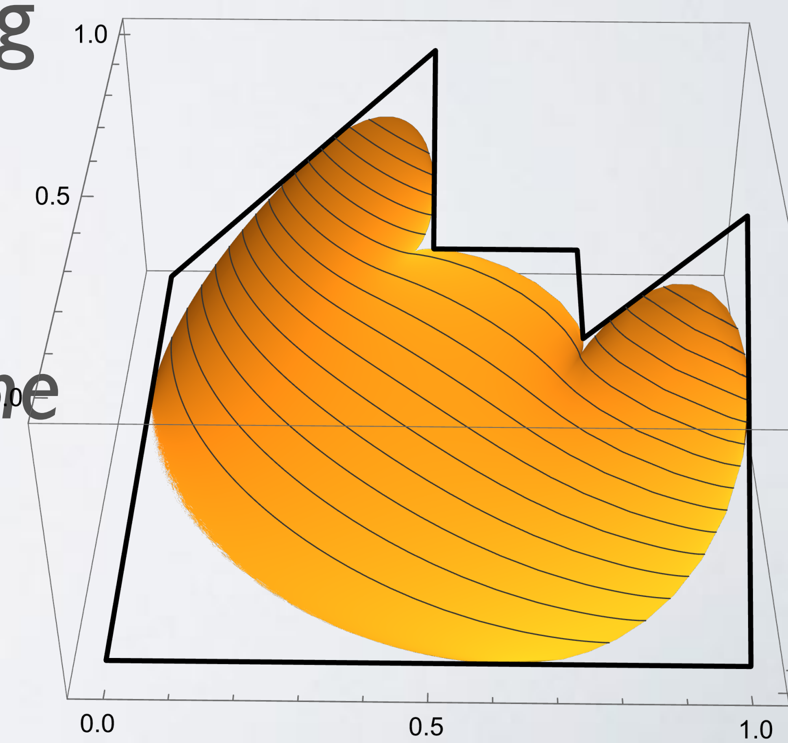


tangent method
Colomo-Sportiello

"tangent plane method"



κ-HARMONIC ENVELOPE

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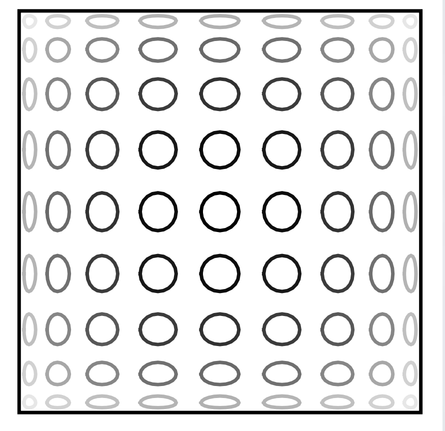
Thm: s, t and h-$(sx+ty)$ are all κ-harmonic(z) in the liquid region

(multi-valued in z)

Corollary: Free-fermionic limit shapes are *envelopes* of harmonically moving planes in R^3



Kenyon-Wolfram
multinomial dimers
(Workshop I)

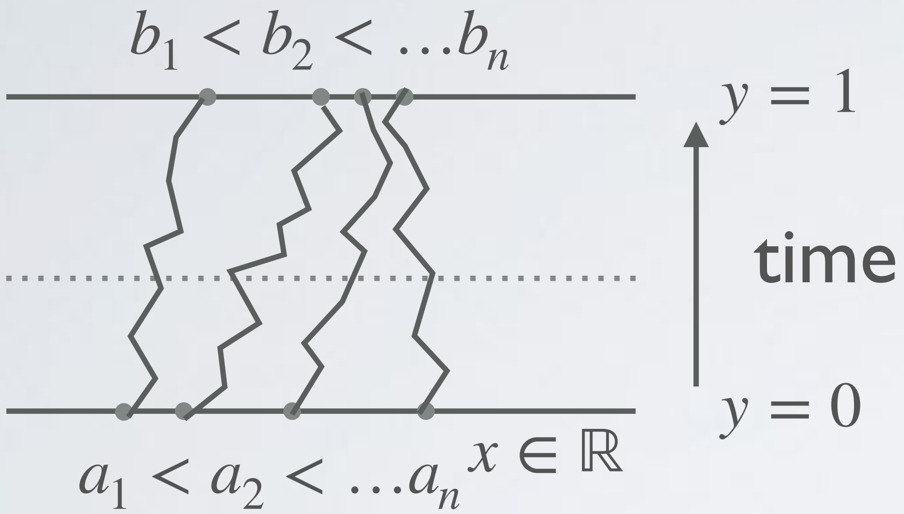
fluctuations: $\kappa - GFF$ in the complex structure of z

(physics expectation)

Brun-Dubail

VARIATIONAL PRINCIPLE FOR NON-INTERSECTING BROWNIAN BRIDGES

Guionnet, Matystin, Guionnet-Zeitouni
Belinschi-Guionnet-Huang



$$\frac{1}{n}\sum \delta_{b_i} \to \mu_1$$

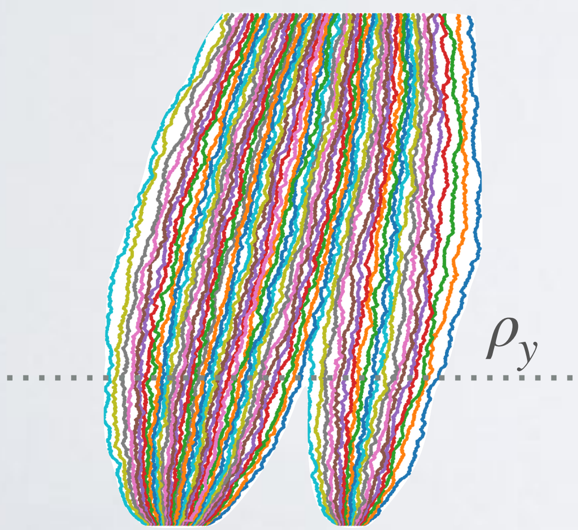
$$\frac{1}{n}\sum \delta_{a_i} \to \mu_0$$

$\Rightarrow$

$$\frac{1}{n}\sum \delta_{x_i(y)} \to \rho_y(x)dx$$

$$\min_{\{(\rho_y, v_y):\ 0<y<1\}} \int_0^1 \int_{\mathbb{R}} \left(\rho_y v_y^2 + \frac{\pi^2}{3}\rho_y^3 \right) dxdy$$

$$\partial_y \rho_y + \partial_x(\rho_y v_y) = 0$$



$$h(x, y) := \int_{-\infty}^{x} \rho_y$$

$$f(x, y) := v_y(x) + i\pi\rho_y(x)$$

$$\partial_y f + f\partial f_x = 0$$

complex Burgers

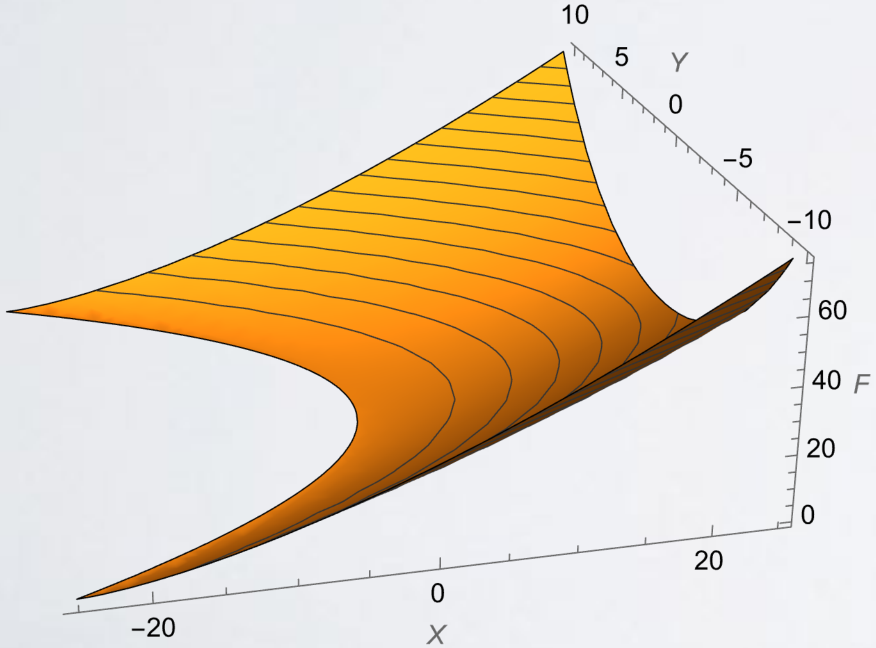
SURFACE TENSION

$$\sigma(s,t) = \frac{\pi^2}{3}s^3 + \frac{t^2}{s} \qquad \det H_\sigma = 4\pi^2 \qquad (s,t) = \nabla h \in [0,\infty)\times\mathbb{R}$$

$$z = -\frac{t}{s} + i\pi s \in \mathbb{H}$$

$$X = \sigma_s = \pi^2 s - \frac{t^2}{s^2} \qquad Y = \sigma_t = \frac{2t}{s}$$

harmonic conjugates



$$F(X,Y) = \frac{2}{3\pi}\left(X + \frac{Y^2}{4}\right)^{3/2}$$

$$(s,t) = \left(\frac{1}{\pi}\operatorname{Im} z, -\frac{1}{2\pi}\operatorname{Im} z^2\right)$$

$$(\sigma_s, \sigma_t) = (-\operatorname{Re} z^2, -2\operatorname{Re} z)$$

$$\sigma_s + i\,2\pi t = -z^2$$

$$\sigma_t - i\,2\pi s = -2z$$

COMPLEX BURGERS EQUATION

$$\sigma_{ss}h_{xx} + 2\sigma_{st}h_{xy} + \sigma_{tt}h_{yy} = 0$$

$$(x, y) \in \mathscr{L} \mapsto \nabla h = (h_x, h_y) = (s, t) \mapsto z(x, y)$$

$$(\sigma_s(h_x, h_y) + i\,2\pi h_y)_x + (\sigma_t(h_x, h_y) - i\,2\pi h_x)_y = 0$$

$$(-z^2)_x + (-z)_y = -\,2zz_x - 2z_y = 0$$

$$\frac{z_x}{z_y} = -\frac{1}{z}$$

$$(h - sx - ty)_{z\bar{z}} = -\,(s_{z\bar{z}}x + t_{z\bar{z}}y) - (s_z x_{\bar{z}} + t_z y_{\bar{z}}) = \frac{i}{2\pi}\left(x_{\bar{z}} - zy_{\bar{z}}\right)$$

$$= \frac{i}{2\pi}\frac{x_{\bar{z}}z_x + y_{\bar{z}}z_y}{z_x} = \frac{i}{2\pi}\frac{z(x, y)_{\bar{z}}}{z_x} = 0$$

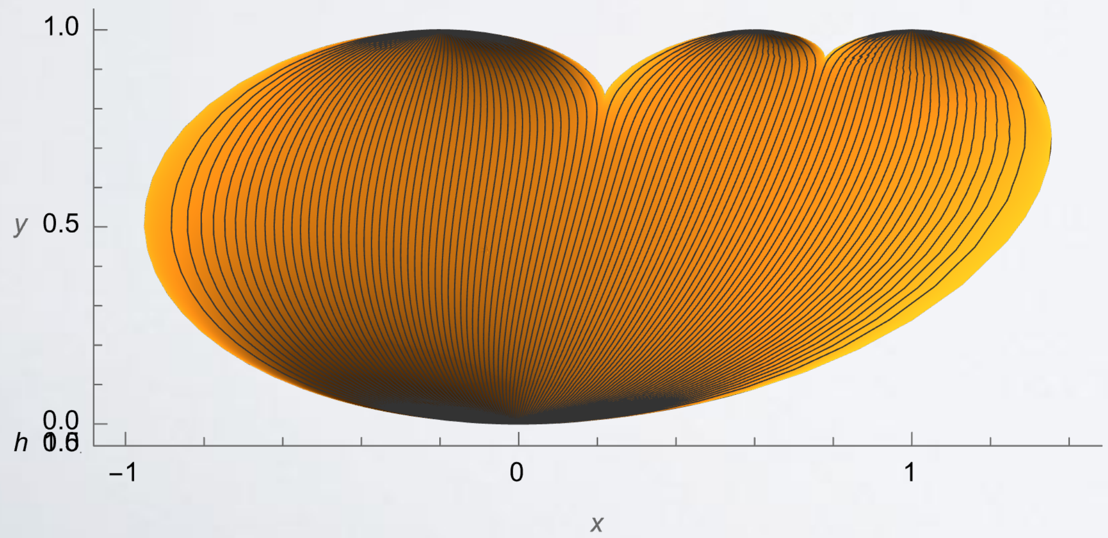
ONE-SIDED BOUNDARY CONDITION

$$\delta_0 \to \mu$$

$$\mu_y = \mu \boxplus \mu_{sc}^{(y)}$$

Thm (P): $z = u - \mathscr{C}_\mu(u)$ $\quad x = \dfrac{\operatorname{Im} u/z}{\operatorname{Im} 1/z}, \quad y = 1 - \dfrac{\operatorname{Im} u}{\operatorname{Im} z}$ $\quad u \in \mathbb{H}$

$$h_{BB} = 1 + \frac{1}{\pi}\left(\operatorname{Im} u \, \operatorname{Re}(u - z) - \int_{\mathbb{R}} \arg(u - x)\, d\mu(x) \right)$$

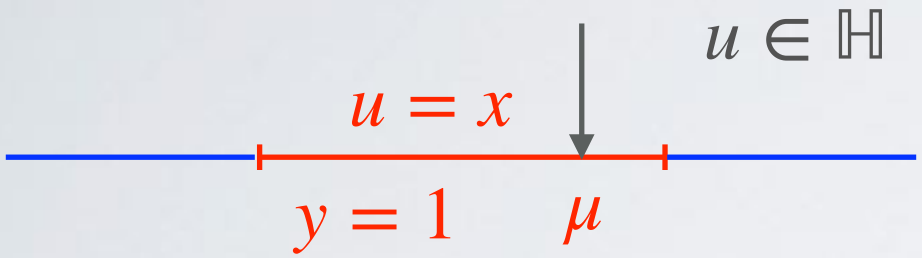


PROOF

$$s_z x + t_z y + c_z = 0 \qquad \textit{(envelope equation)}$$

$$x + \frac{t_z}{s_z}(y-1) + \frac{(c+t)_z}{s_z} = 0 \qquad z \leftrightarrow (x,y)?$$

$$x - z(u)(y-1) - u = 0$$



$$\frac{1}{\pi}\mathrm{Im}\, z(u)dx \to \mu \qquad u \to \partial\mathbb{H}$$

$$z(u) = u - \mathscr{C}_\mu(u)$$

$$u \in \mathbb{H}$$

$$(c+t)(u) := 1 - \frac{1}{\pi}\int \arg(u-x)d\mu(x) + \frac{1}{\pi}\mathrm{Im}\,(\frac{u^2}{2} - uz)$$

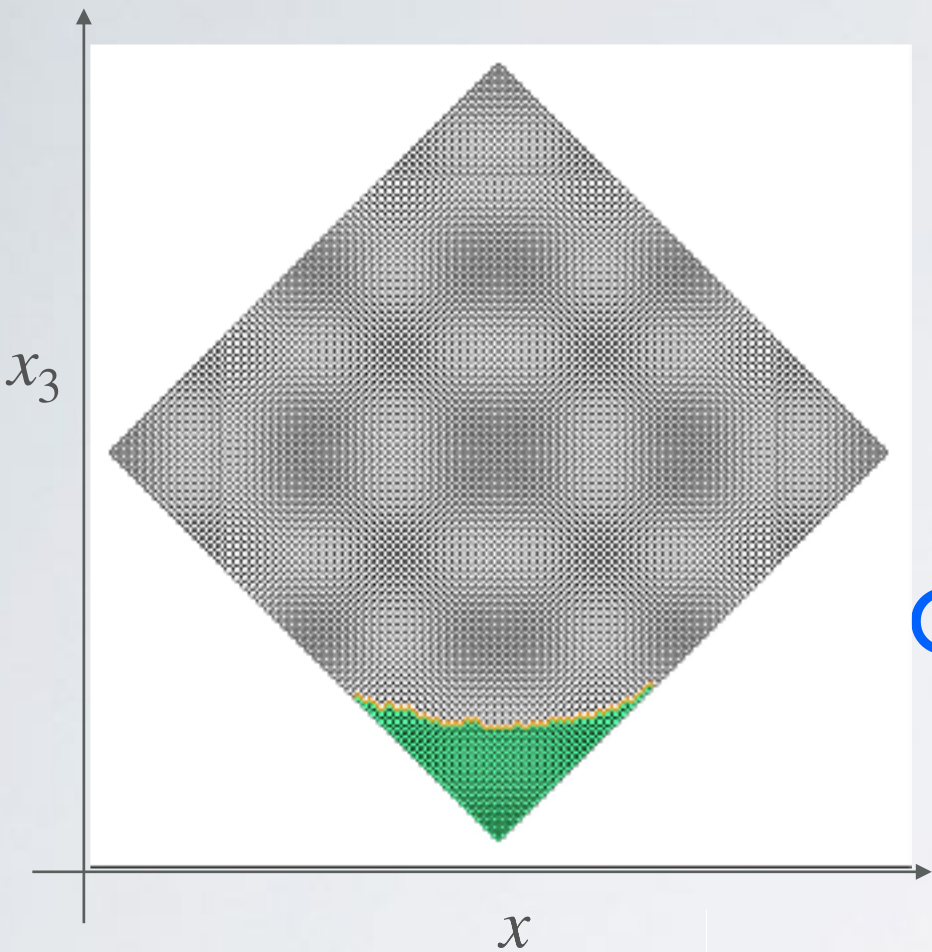
$$(c+t)_u = \frac{i}{2\pi}(\mathscr{C}_\mu(u) - u + uz' - z) = \frac{i}{2\pi}uz \qquad -\frac{(c+t)_u}{s_u} = u$$

$$h = sx + t(y-1) + (c+t) = \frac{1}{\pi}\left(-\mathrm{Im}\,(u\bar{z}) + \mathrm{Re}\, z\, \mathrm{Im}\, u\right) + c + t =$$

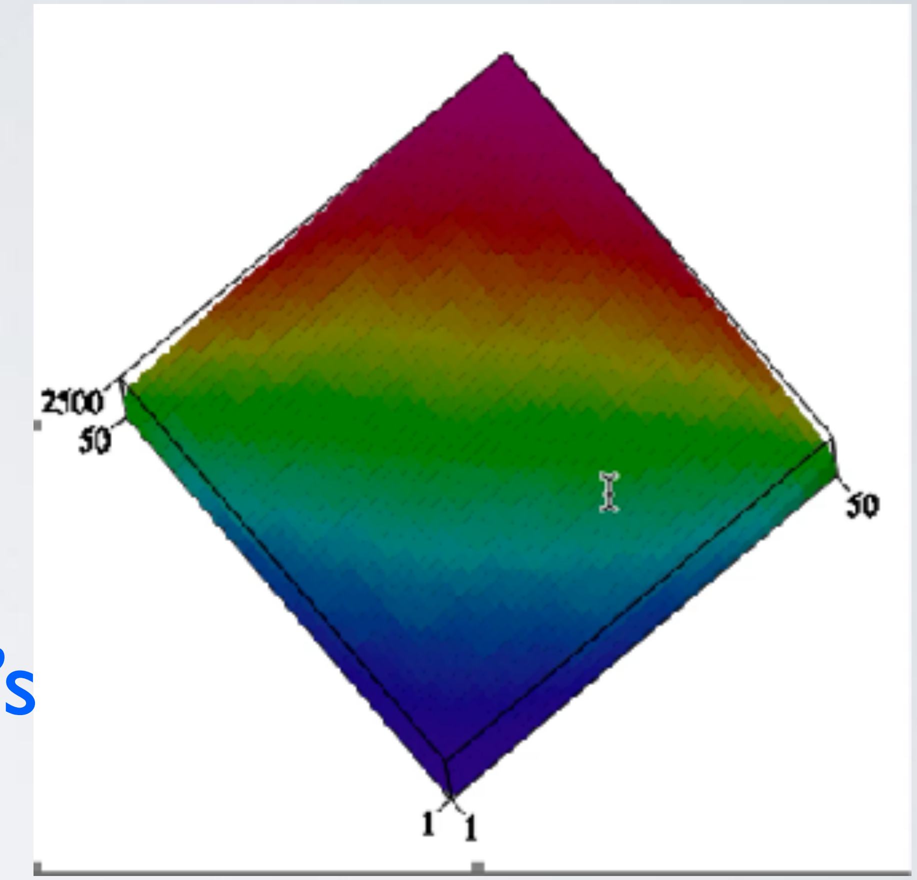
$$1 + \frac{1}{\pi}\left(\mathrm{Im}\, u\, \mathrm{Re}\,(u-z) - \int_{\mathbb{R}} \arg(u-x)\, d\mu(x)\right)$$

YOUNG TABLEAUX LIMIT SHAPES

square diagram

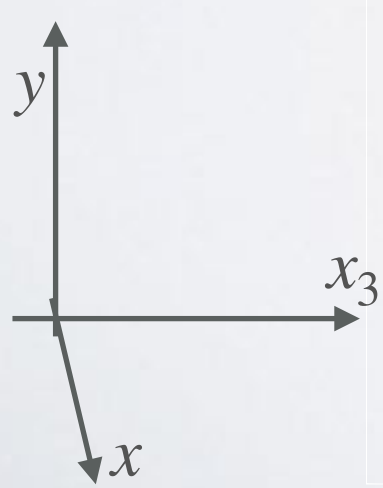


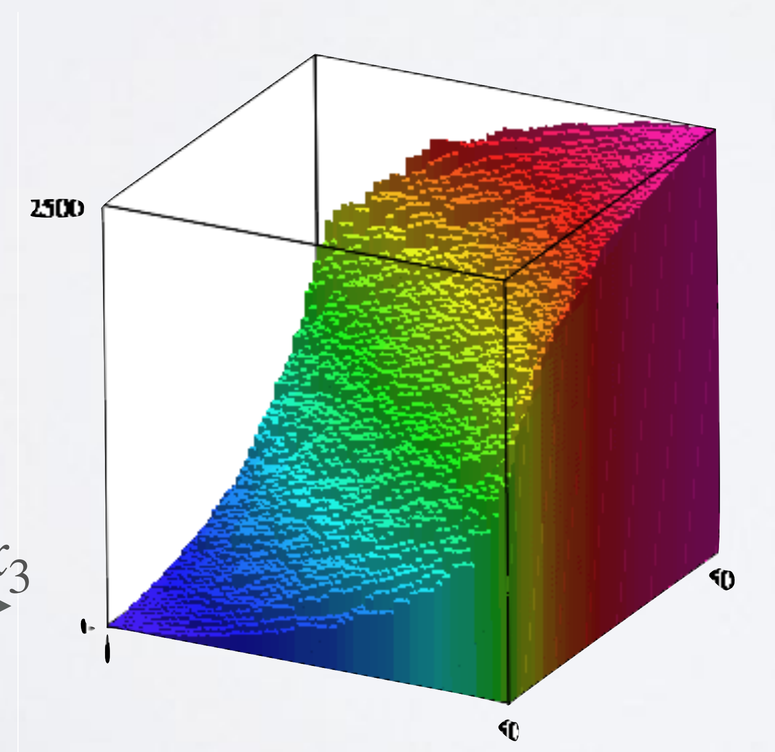
Dan Romik's
MacTableaux

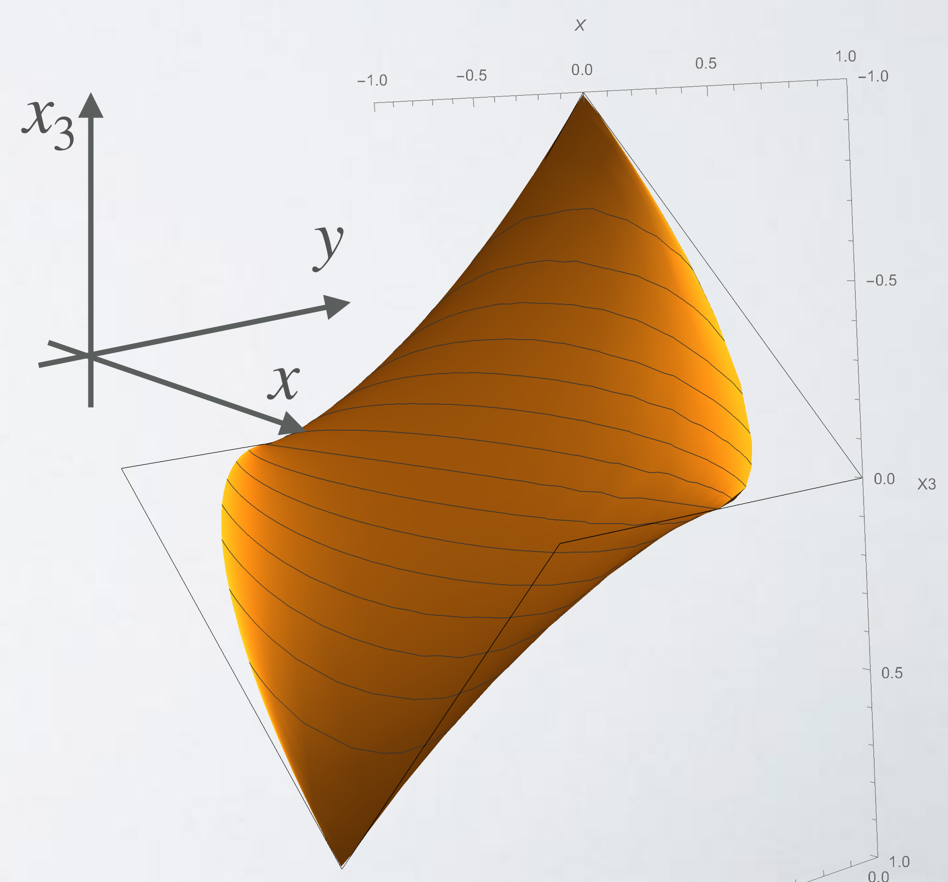


Cyril Banderier et al.'s
YoungPackage

$y \in [0,1]$ "time"







Young Tableaux vs Brownian Bridges

W. Sun, A. Gordenko

$$(s,t) = \nabla h \in [-1,1] \times [0,\infty)$$

$$\sigma_{YT}(s,t) = -\left(1 + \log \frac{\cos \frac{\pi}{2} s}{\frac{\pi}{2} t}\right) t$$

$$\sigma_{BB}(s,t) = \frac{\pi^2}{3} s^3 + \frac{t^2}{s}$$

$$(s,t) = (1 - \frac{2}{\pi} \arg z, \frac{2}{\pi} \mathrm{Im}\, z)$$

$$(s,t) = (\frac{1}{\pi}\, \mathrm{Im}\, z, -\frac{1}{2\pi}\, \mathrm{Im}\, z^2)$$

$$(\sigma_s, \sigma_t) = (\mathrm{Re}\, z, \log |z|)$$

$$\gamma(z) = \frac{s_z}{t_z} = -1/z$$

$$\frac{z_x}{z_y} = -1/z$$

YT LIMIT SHAPE RESULTS

Biane (1998)

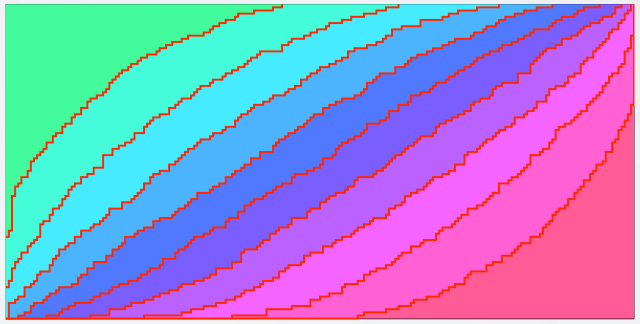
general profile

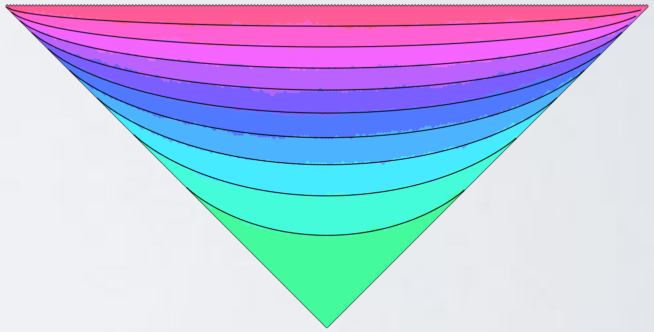
implicit limit shapes
in terms of *free cummulants*

Pittel-Romik (2007)
Angel-Holroyd-Romik-Virág

handful of examples
(square, rectangle, staircase)

explicit limit shapes





Borga-Boutillier-Féray-Méliot (2023)

arbitrary rectangular profile

bead process

$$x_3'(x) = \frac{2}{\pi} \arctan \frac{x(1-2y)}{\sqrt{4y(1-y)-x^2}}, \quad |x| \leq \sqrt{4y(1-y)}$$

family of *one-dimensional*
variational problems

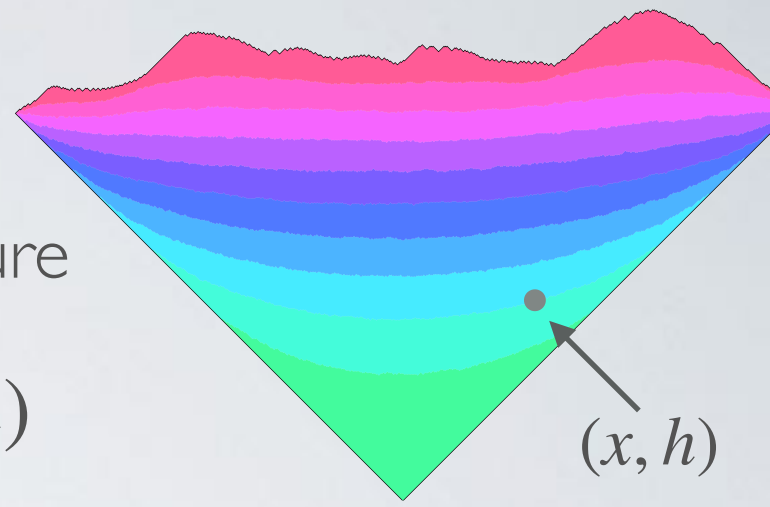
Jacopo Borga's talk Workshop II

YOUNG TABLEAUX LIMIT SHAPES

$$\tau = \frac{1}{2}\omega'', \quad G_\omega(u) = \exp \int_{\mathbb{R}} \log \frac{1}{u-x} d\tau(x) = \mathscr{C}_\mu(u)$$

signed measure   Kerov   (co)transition measure

$$1/G_\omega(u) = u - \frac{1}{2} A_\omega \mathscr{C}_\nu(u)$$



asymptotic value of the tableau

Thm (P): $z = 1/G_\omega(u), \quad x - z(u)\,(y-1) - u = 0, \quad u \in \mathbb{H}$

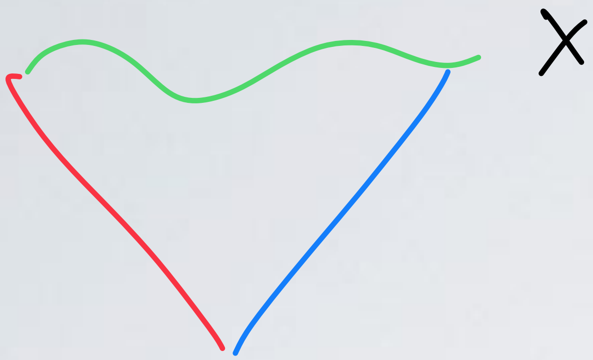
Limit surface

$u \in \mathbb{H}$

$$x = \frac{\operatorname{Im} u/z}{\operatorname{Im} 1/z}, \quad y = 1 - \frac{\operatorname{Im} u}{\operatorname{Im} z} \quad \nabla h = (s(z), t(z))$$

$$h_{YT} = \left(1 - \frac{2}{\pi}\arg(z(u))\right)\frac{\operatorname{Im} u/z}{\operatorname{Im} 1/z} + \frac{2}{\pi}\int_{\mathbb{R}} x \arg(u-x)\, d\tau(x)$$

PROOF

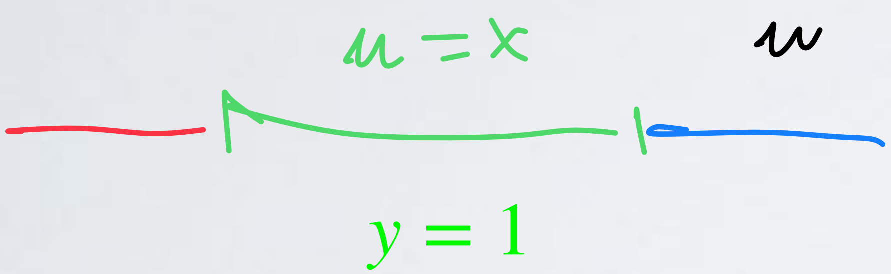


$$s_z x + t_z y + c_z = 0 \quad \textit{(envelope equation)}$$

$$x + \frac{t_z}{s_z}(y - 1) + \frac{(c+t)_z}{s_z} = 0 \qquad z \leftrightarrow (x, y)?$$

$$x - z(u)(y-1) - u = 0$$

$$-\arg z(u) = \frac{\pi}{2}(\omega'(u) - 1) \qquad u \in \partial\mathbb{H}$$

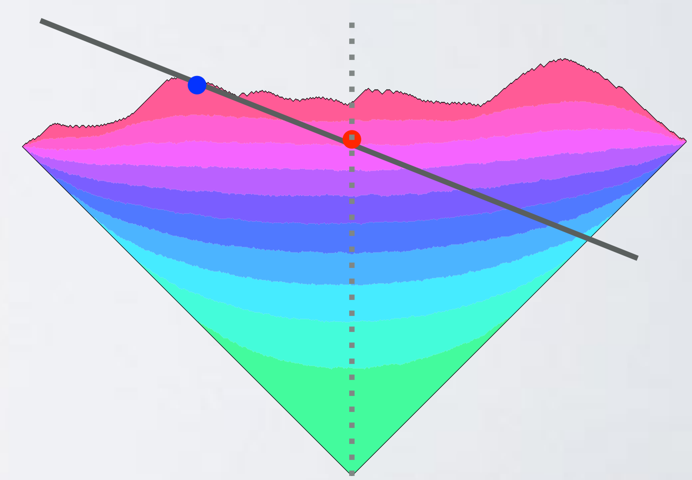


$$z(u) = \exp\left(\int_{\mathbb{R}} \log(u-x)\,d\tau(x)\right) =: 1/G_\omega(u) \qquad u \in \mathbb{H}$$

$$h = sx + (h - sx) = sx + t(y-1) + c + t = sx + \mathscr{P}\tilde{c}(u)$$

$$-\frac{2}{\pi}\operatorname{Im} u$$

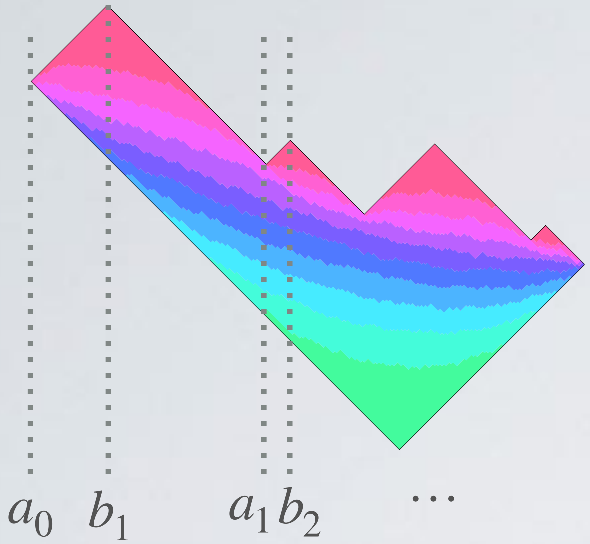
$$\mathscr{P}\tilde{c}(u) = \frac{2}{\pi}\int_{\mathbb{R}} x \arg(u-x)\,d\tau(x)$$

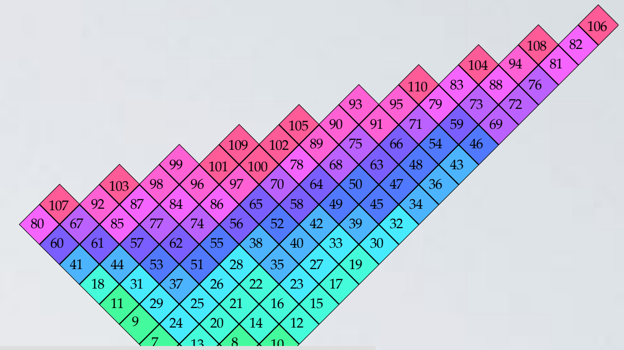


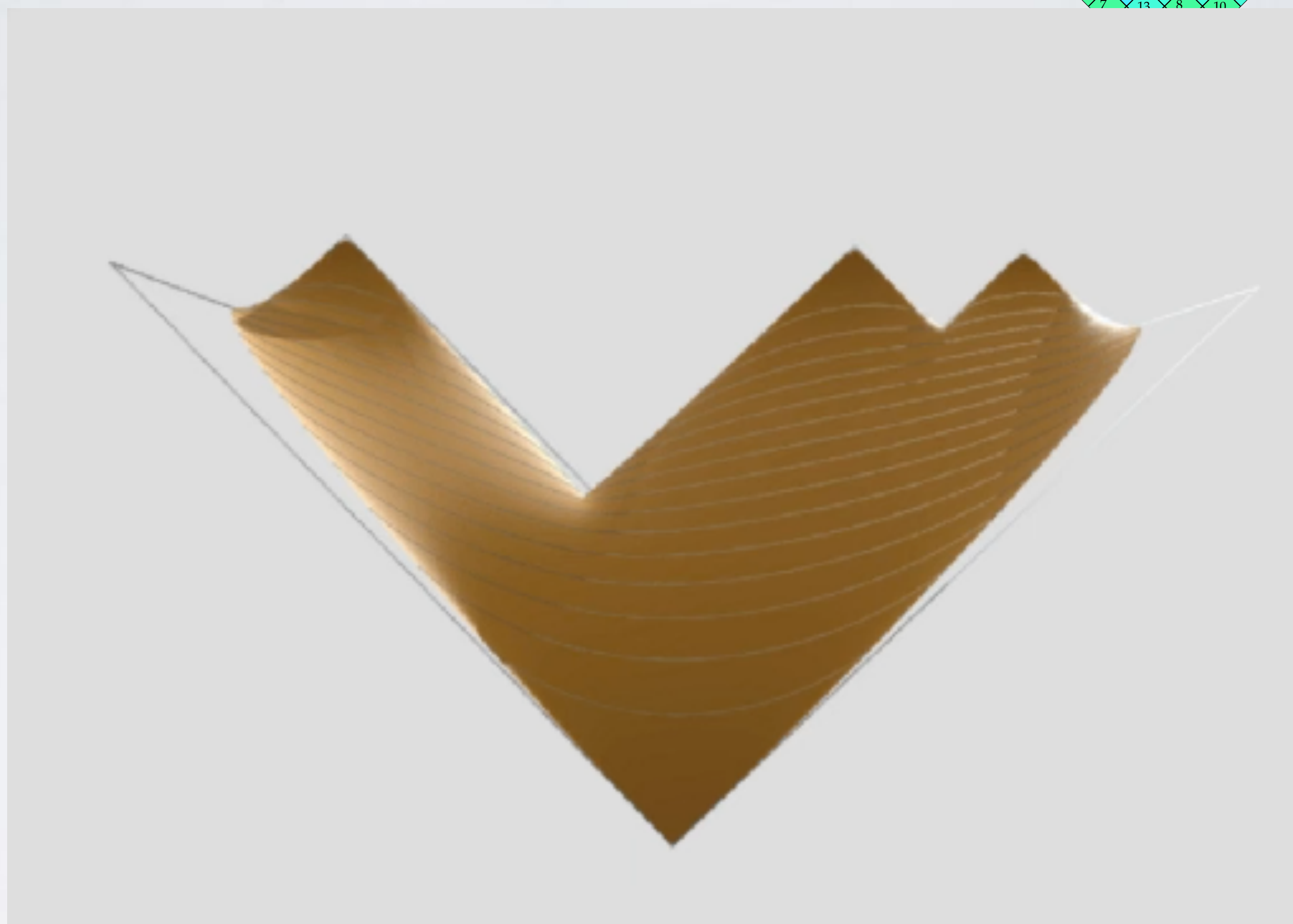
intercept function

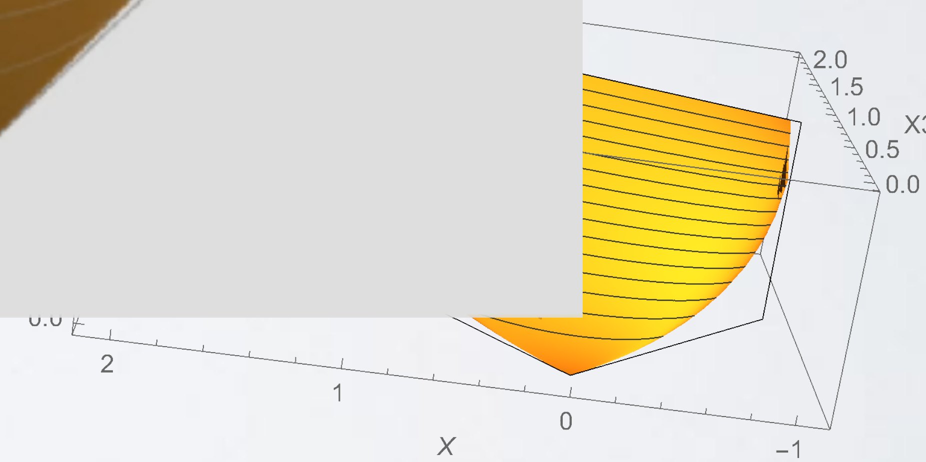
$$\tilde{c}(x) = \omega(x) - \omega'(x)x$$

EXAMPLES

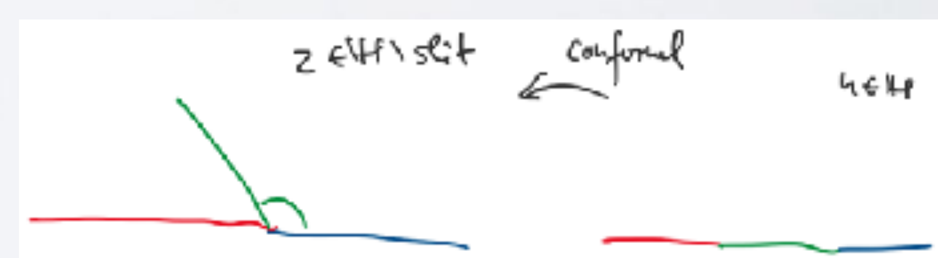








$$G_\omega(u) = \frac{\prod (u - b_i)}{\prod (u - a_i)}$$

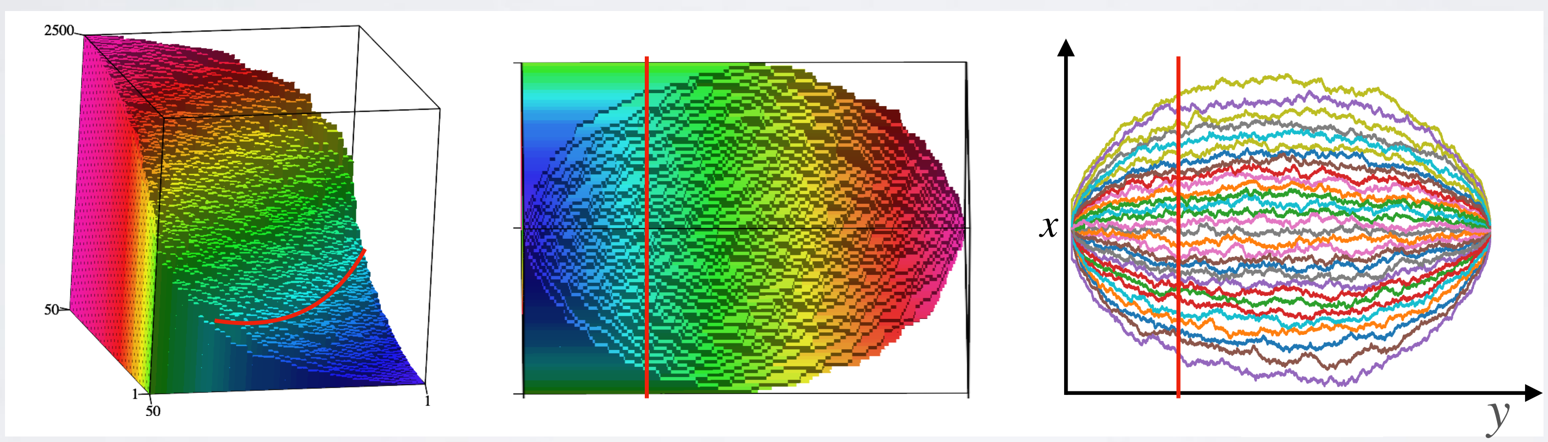


DUALITY

$$\gamma(z) = -1/z$$

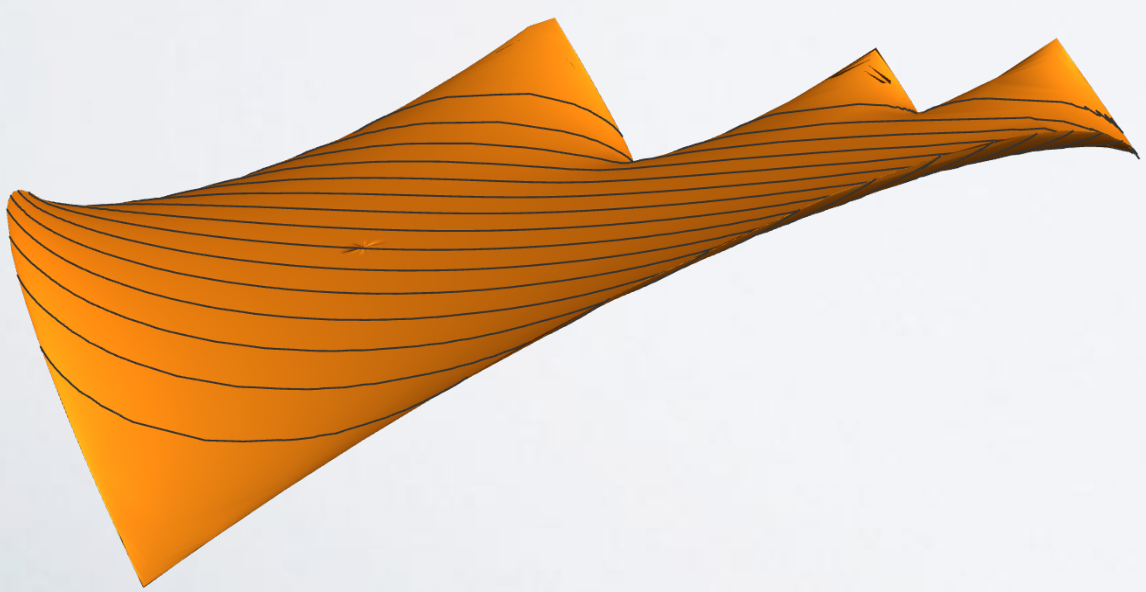
$$t_{YT}(z) = \frac{2}{\pi}\mathrm{Im}\, z = 2s_{BB}(z)$$

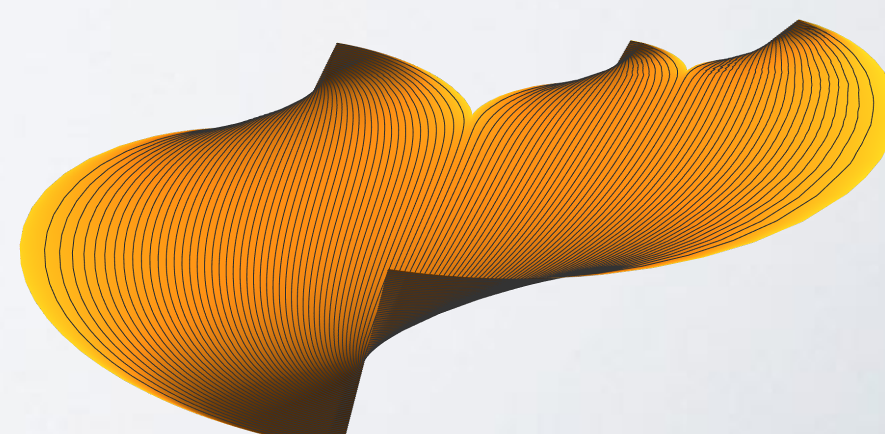
duality via co-transition measures



$$\mu_y = \mathrm{cotrans}\,(\omega_y)$$

$$\partial_y h_{YT} = 2\partial_x h_{BB}$$





FREE COMPRESSION

R-transform $\frac{1}{\mathscr{C}_\mu(u)} + R_\mu(\mathscr{C}_\mu(u)) = u$

free convolution

free compression $R_{\mu^\alpha}(w) = R_\mu(\alpha w), \quad \alpha \leqslant 1$

$R_{\mu \boxplus \nu}(w) = R_\mu(w) + R_\nu(w)$

Nica-Speicher

Biane: transition measure of Young tableaux contours

variational formulation
(minors for random matrices)

Shlyakhtenko-Tao, Johnston

$$\sigma(s,t) = -\log t - \log(\sin(\pi \frac{s}{t}))$$

$$\det H_\sigma = \frac{\pi^2}{t^4}$$

(fake) trivial potential

$$(s,t) = \left(\frac{\arg z}{\operatorname{Im} z}, \frac{\pi}{\operatorname{Im} z}\right)$$

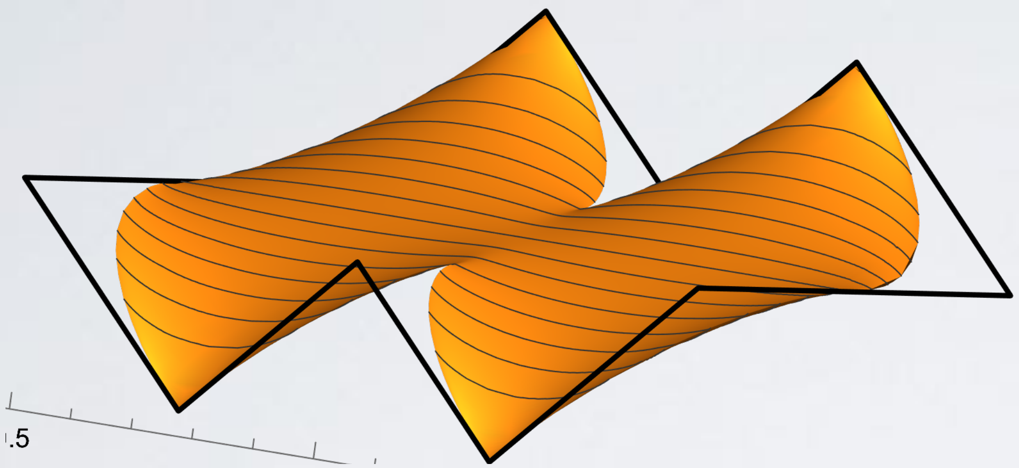
$\det H_\sigma = (\text{harmonic}(z))^4$

$$\mu^\alpha((-\infty, \lambda]) = x$$

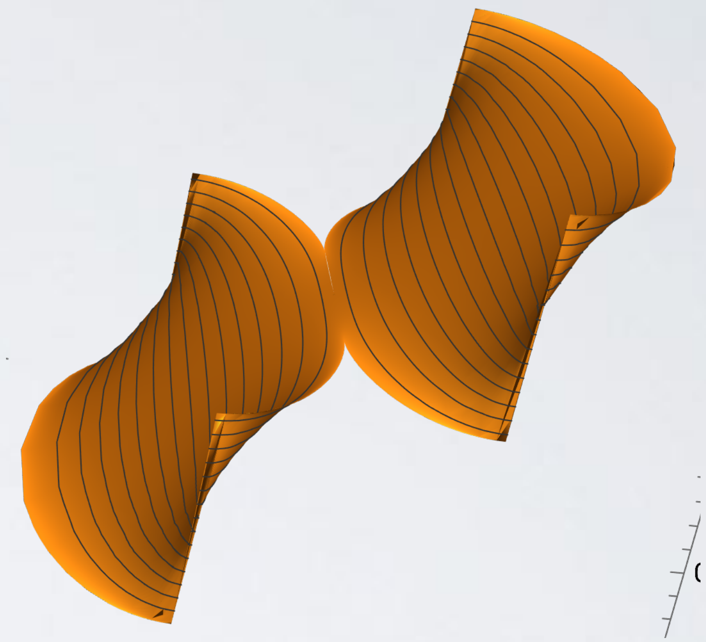
Corollary (P): explicit uniform Gelfand-Tsetlin surfaces (α, x, λ)

Metcalfe

SKEW SHAPES/ TWO-SIDED BOUNDARY?



rectangular boundary



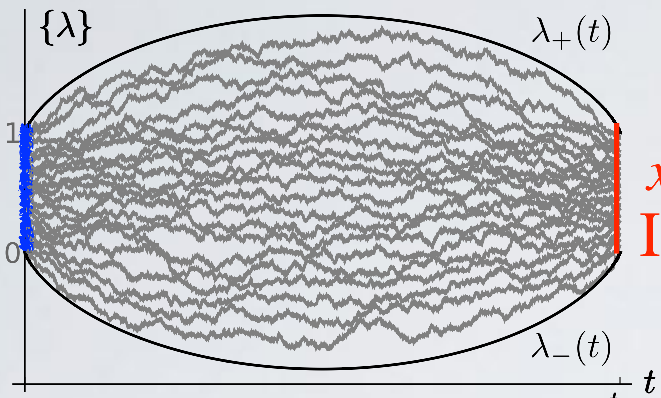
only atoms

Family of rational solutions
non-trivial to match to boundary conditions

"FLAT-TO-FLAT" GEOMETRY

Grela-Majumdar-Schehr

adapted complex Burgers from Langevin equation in a non-standard form



$x = u$ $\quad$ $x = ?$ $\quad$ $\mathrm{Im}\, z = \pi$

$y = 0$ $\quad$ $y = 1$

$$\lambda_{[0,1]} \rightarrow \lambda_{[0,1]}$$

$$\mu_0 \rightarrow \lambda_{[0,1]}$$

$$x - z(u)y - u = 0$$

$$z = G - G_0^{-1}(G)$$

$$G\left(\frac{x}{1-y}, \frac{y}{1-y}\right)$$

μ_0 $\quad$ 0

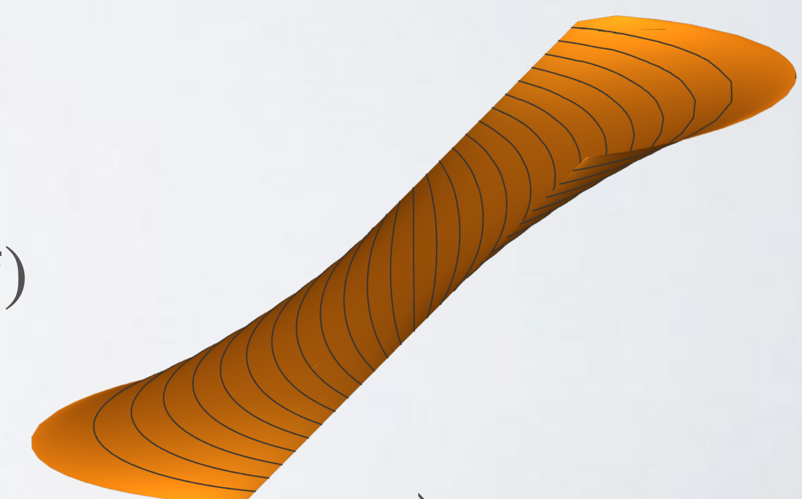
$u \in S$

$-i\pi$ $\quad$ $\lambda_{[0,1]}$

$$G_0(u) = e^u \int \frac{1}{e^u - e^x} d\mu_0(x) \qquad z = G_0(u) - u$$

$$\mu_0 = \lambda_{[0,1]} \qquad G_0(w) = \log\left(\frac{1-e^u}{1-e^{u-1}}\right)$$

$$x = \frac{\mathrm{Im}\, u/z}{\mathrm{Im}\, 1/z}, \quad y = -\frac{\mathrm{Im}\, u}{\mathrm{Im}\, z} \qquad h = \frac{1}{\pi}\Big(\mathrm{Re}\, z\, \mathrm{Im}\, u - \mathrm{Im}\,(u\bar{z})$$



$$-\mathrm{Im}\left(\mathrm{Li}_2\left(e^{1-u}\right) - \mathrm{Li}_2\left(e^{-u}\right) - \frac{1}{2}u\left(u + 2\log\left(1-e^{1-u}\right) - 2\log\left(1-e^{-u}\right)\right)\right)\Big)$$

THANK YOU!