Open vs Closed Toda from Q-systems

Rinat Kedem (Illinois)

Joint with P. Di Francesco,
+ Schrader, Shapiro,
+ Shiraishi, Bershtein, Bourgine

Open & Closed Toda from cluster Algebras

Open:

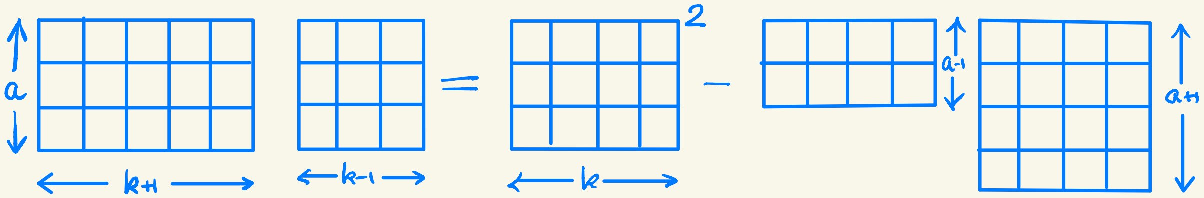
- Classical Q-systems and Toda Hamiltonians
- Cluster algebra structure and quantization
- Time translation operator, Macdonald theory and q-Toda
- Universal Harish-Chandra series

Closed:

- $\hat{A}_{N-1}$ - Q-systems and cluster algebra structure
- Time translation operators
- Closed Toda Hamiltonians
- Universal series, affine Macdonald operators, affine characters...

sl_N Q-systems:

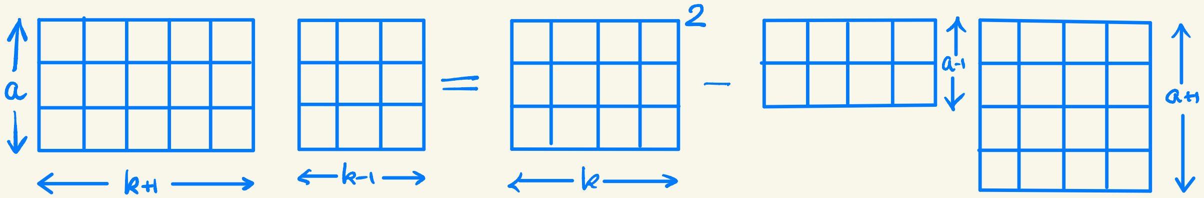
- Generators $\{Q_{a,k} \mid a\in[1,N-1],\ k\in\mathbb{Z}\}$
- Relations $\boxed{Q_{a,k+1}Q_{a,k-1} = Q_{a,k}^2 - Q_{a+1,k}Q_{a-1,k}}$ $\quad Q_{0,k}=Q_{N,k}=1 \ (k\in\mathbb{Z})$



- sl_2 : $Q_{k+1}Q_{k-1}=Q_k^2-1$
- <u>Discrete evolution in k</u> $\quad T: Q_{a,k}\mapsto Q_{a,k+1}$

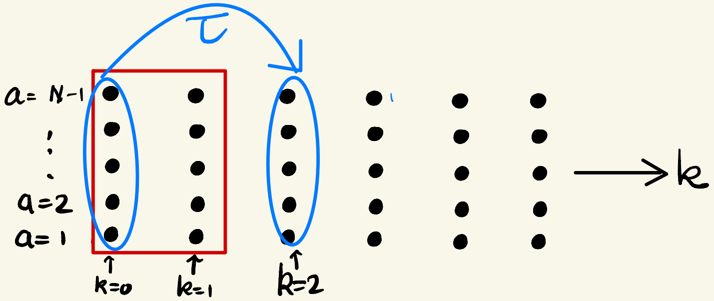
$\mathfrak{sl}_N$ Q-systems:

- Generators $\{ Q_{a,k} \mid a \in [1, N-1],\ k \in \mathbb{Z} \}$
- Relations $\boxed{Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - Q_{a+1,k} Q_{a-1,k}}$ $\quad Q_{0,k} = Q_{N,k} = 1 \ (k \in \mathbb{Z})$



- $\mathfrak{sl}_2$: $Q_{k+1} Q_{k-1} = Q_k^2 - 1$
- Discrete evolution in k $\quad \tau : Q_{a,k} \mapsto Q_{a,k+1}$

  - Initial data $\{ Q_{a,0} ; Q_{a,1} \}_{a=1}^{N-1}$ + Q-sys determine $\{ Q_{a,k} \}_{a \in 1, \ldots, N-1}^{k \in \mathbb{Z}}$



- If $Q_{a,0} = 1$, $Q_{a,1} = \underset{\text{elementary}}{e_a(z_1, \ldots, z_N)} = S_{\left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}\right\}a}(\underline{z})$ $\Rightarrow$ $Q_{a,k} = \underset{\text{Schur}}{S_{k\omega_a}}(\underline{z}) = S_{\underbrace{\left.\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}\right\}a}_{k}}$

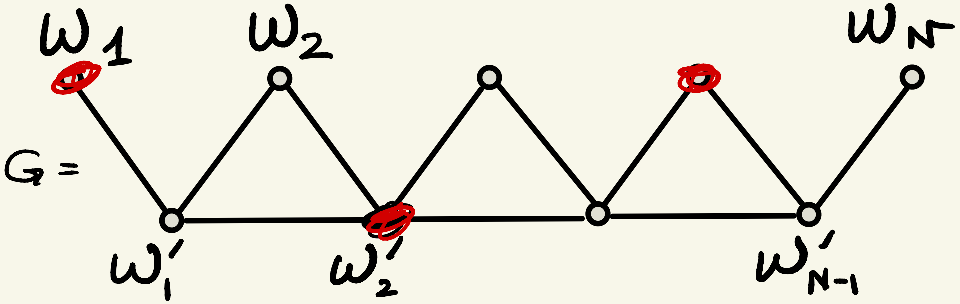
$\Rightarrow$ Characters of $\mathfrak{sl}_N$ - mods $V(k\omega_a)$ satisfy the Q-system.

Integrability: conserved quantities $\{H_i\}_{i\in 1,\dots,N}$

- $\{H_i\}_{i=1}^{N}$: Laurent Polynomials in $\{Q_{a,k}\}$

[DFK09] H_i = partition function of hard particles on G

G =



$$w_a = \frac{Q_{a-1,0}\,Q_{a,1}}{Q_{a,0}\,Q_{a-1,1}}$$

$$w'_a = -\frac{Q_{a-1,0}\,Q_{a+1,1}}{Q_{a,0}\,Q_{a,1}}$$

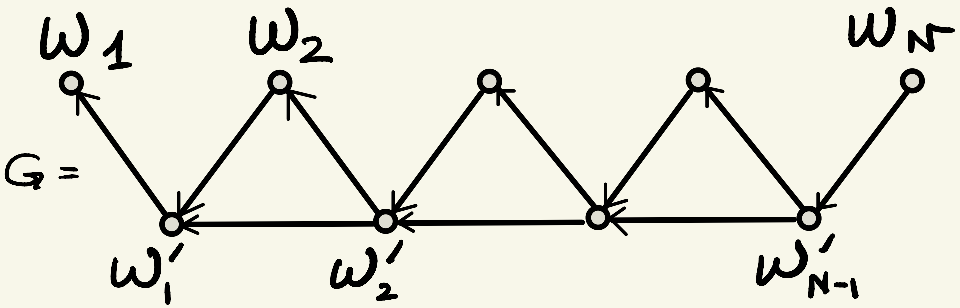
- Translation invariance: $\mathcal{T}(H_i) = H_i$

$$H_1 = \sum_a (w_a + w'_a)$$

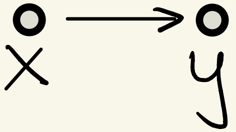
Coxeter Toda Hamiltonians (Reshetikhin et al)

Poisson structure:

G =



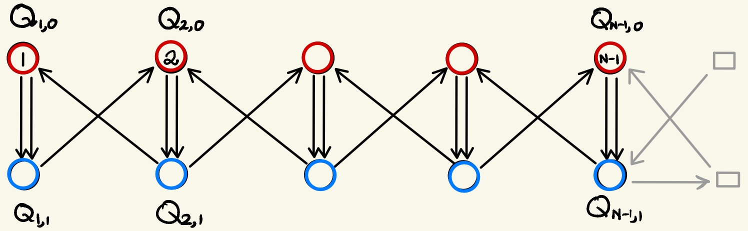
where



$\Rightarrow \{x,y\} = xy$

Hamiltonians Poisson-commute $\{H_a, H_b\} = 0$

Cluster structure



- $\{Q_{a,k}\}$ = cluster A-variables for Cluster Algebra

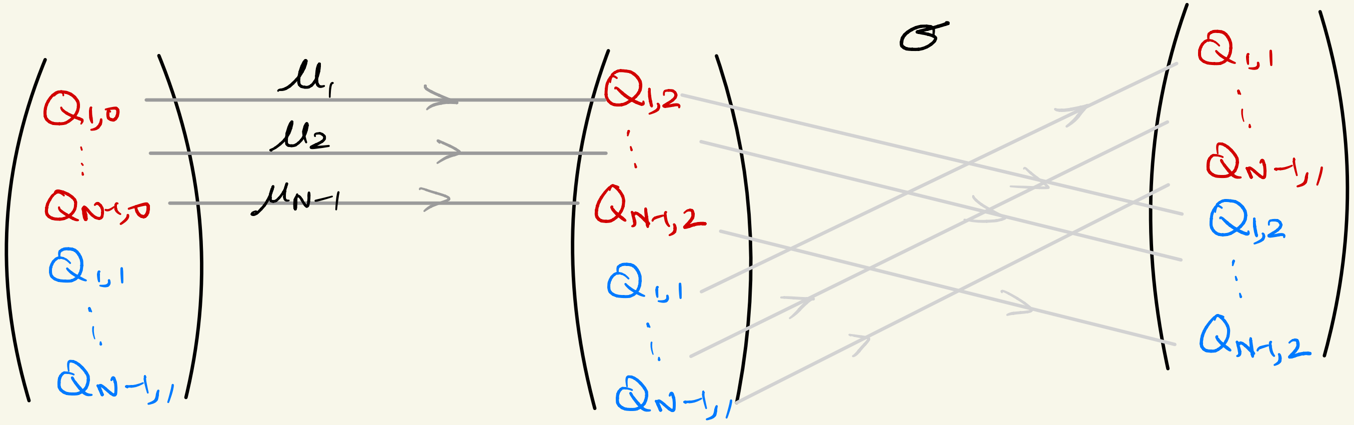
$B_{ij} = \# \, i \to j \; : \; \overline{B} = \begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix}$

C = Cartan

- A-variable mutations* = Q-system equations

- Time translation $\mathcal{J} : k \to k+1$ $\begin{cases} N-1 \text{ mutations at } \circ \\ N-1 \text{ permutations } \circ \overset{\sigma}{\longleftrightarrow} \circ \end{cases}$

$\mathcal{J}$:



$\mathcal{J}(\overline{B}) = \overline{B}$ principal part of exchange matrix $\mathcal{J}$-invariant

[GSV] Poisson structure: $\{A_i, A_j\} = B^{-1}_{ij} A_i A_j$

Quantization: Poisson bracket → Commutation Relations

$$\{A,B\} = \alpha AB \rightsquigarrow AB = q^{\alpha} BA$$

$AB = q^{\alpha} BA$

$:AB: = q^{-\frac{\alpha}{2}} AB$

Weyl-ordering

$\mathcal{X}$-variables (Fock-Goncharov variables)

Define $\quad X_i = :\prod_j A_j^{B_{ji}}:$

q-commute as $X_i X_j = q^{-B_{ij}} X_j X_i \qquad i \circ \xrightarrow{B_{ij}} \circ j$

Definition:

monomial transformation

$$\boxed{\text{q-mutations of } \mathcal{X}\text{-variables: } \mu_j(X_i) = \mathrm{Ad}_{\varphi(X_j)} :X_i X_j^{[B_{ji}]_+ - 2\delta_{ij}}:}$$

q-dilogarithm $\varphi(x) = \prod_{n\geq 0} (1 - xq^n)^{-1}$ (*)

$\boxed{\varphi(qx) = (1-x)\varphi(x)}$ (**)

A-variables $\quad A_i = :\prod_j X_j^{B^{-1}_{ji}}: \qquad$ q-commute as $A_i A_j = q^{B^{-1}_{ij}} A_j A_i$

q-mutations of A-variables follow from $\mathcal{X}$ variables

Quantum Torus:

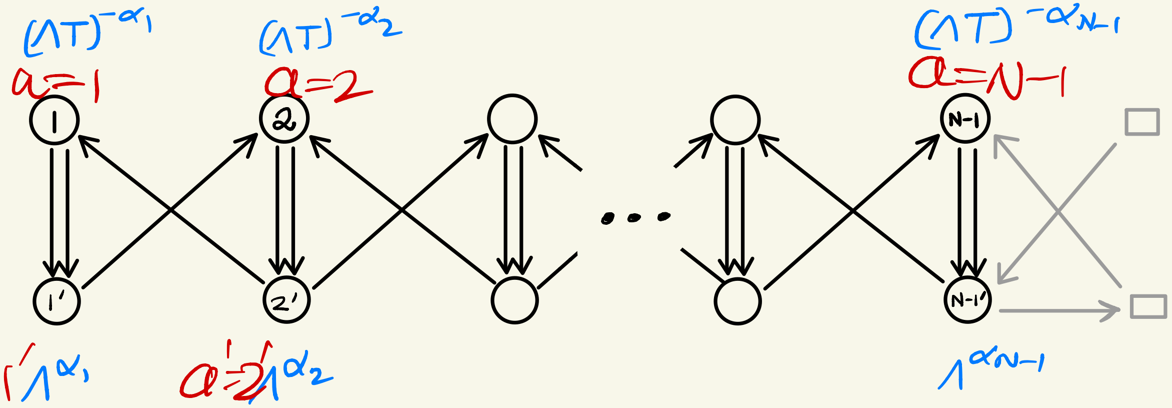
$$\boxed{\langle T_a, \Lambda_a \rangle_{a=1}^{N} \;;\; T_a \Lambda_b = q^{\delta_{ab}} \Lambda_b T_a}$$

$$\bar{B} = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix} \qquad C_{ab} = (\alpha_a^\vee, \alpha_b)$$

simple roots $\alpha_a = e_a - e_{a+1}$

$(\Lambda^{e_a} = \Lambda_a)$

𝒳-variables:



$\Lambda^{\alpha_1} = \Lambda_1/\Lambda_2$

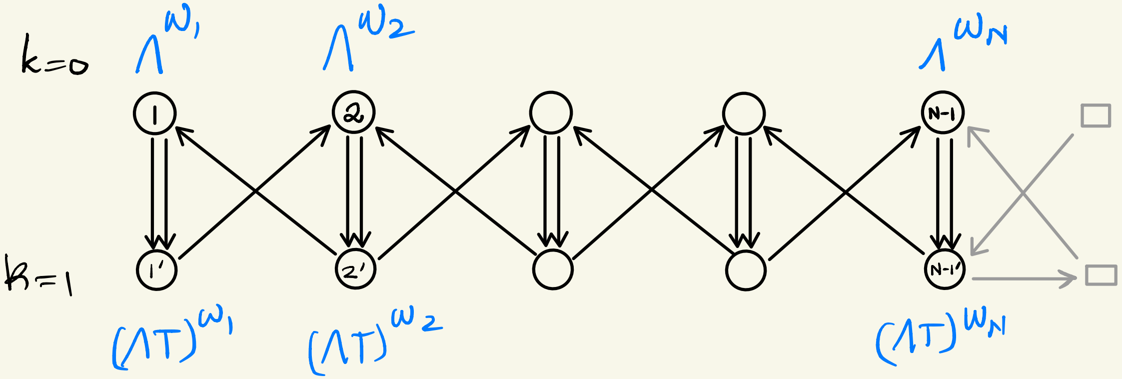
𝒳-variables: $\{X_a \,;\, X_{a'}\}_{a=1,\dots,N-1}$ $\qquad X_a X_{b'} = q^{-C_{ab}} X_{b'} X_a$

Realized by choice: $X_a = :(\Lambda T)^{-\alpha_a}:$, $X_{a'} = \Lambda^{\alpha_a}$ $\qquad$ $(\Lambda^{\alpha_1} = \Lambda_1/\Lambda_2)$.

$\Rightarrow$ A-variables $=$ $Q_{a,0} = A_a = \Lambda^{\omega_a}$ $\qquad$ $\omega_a =$ fundamental $\mathfrak{gl}_N$

$Q_{a,1} = A_{a'} = :(\Lambda T)^{\omega_a}:$ $\qquad$ weights $\omega_a = \sum_{i=1}^{a} e_i$

A-variables:



Time Translation operator $g(\Lambda)$: $\quad \tau(Q_{a,k}) = Q_{a,k+1} = g\, Q_{a,k}\, g^{-1}$

Thm: $\left.\begin{array}{l}\text{mutation at nodes } a = 1,\dots,N-1 \\ \text{Permutation } \sigma \quad a \leftrightarrow a' \quad (a=1,\dots,N-1)\end{array}\right\}$ realized by $\mathrm{Ad}\, g(\Lambda)$

$$\boxed{g(\Lambda) = \gamma(T) \prod_{a=1}^{N-1} \varphi(\Lambda^{-\alpha_a})}$$

monomial transformation
+ permutation
$a \leftrightarrow a'$

- Gaussian

$$\gamma(T) = \prod_{a=1}^{N} \exp \frac{(\log T_a)^2}{2\log q} \qquad \text{Acts as} \quad \mathrm{Ad}_{\gamma(T)}\, \Lambda_a = \,:\Lambda_a T_a:$$

- $\varphi(x) = \prod_{n\geq 0} (1-q^n x)^{-1}$ quantum dilogarithm

$$\Rightarrow \quad \boxed{Q_{a,k+1} = \mathrm{Ad}_{g(\Lambda)}\, Q_{a,k}} \quad \text{initial data} \quad Q_{a,0} = \Lambda^{\omega_a}$$

THM

$$\boxed{g(\Lambda) \text{ commutes with Toda Hamiltonians} \qquad [g(\Lambda), H_a] = 0}$$

Fourier Transform + Bispectral property + Universal functions

c.f. (Sasamoto)

$$D_a(x)\,\Psi(x|\Lambda) = \Lambda_1 \cdots \Lambda_a\,\Psi(x|\Lambda)$$

universal q-Whittaker$\Big|_{\lambda\in P_+}$ = q-Whittaker polynomial

$$t^{\#}e_a(s) \xrightarrow[t\to\infty]{} \Lambda_1\cdots\Lambda_a = \Lambda^{w_a}$$

$$\Big\uparrow \; \text{q-Whittaker} \;\; \lim_{t\to\infty}$$

Macdonald:

$$\boxed{\mathcal{D}_a^{q,t}(x)\,P(x|s) = e_a(s)\,P(x|s)}$$

$$s_i = t^{n-i}q^{\lambda_i}$$

Universal Sol$^{\text{n}}\Big|_{\lambda\in P_+}$ = Macdonald Polynomial

$$\Big\updownarrow \; \text{Bispectral symmetry} \;\; \frac{P(x|s)}{\Delta(x)} = \frac{P(s|x)}{\Delta(s)}$$

Pieri

$$\mathcal{H}_a^{(q,t)}(s)\,P(x|s) = e_a(x)\,P(x|s)$$

$$\Big\downarrow$$

q-Toda

$$H_a(\Lambda)\,\tilde{\Psi}(x|\Lambda) = e_a(x)\,\tilde{\Psi}(x|\Lambda)$$

"q-Whittaker transform" : $(\Lambda_1, \dots, \Lambda_N) \rightsquigarrow (x_1, \dots, x_N)$

There exists a unique* Series in $\{x^{-\alpha_i}\}$ $\qquad \Psi(\underline{x}|\underline{\Lambda}) = x^{\lambda} \sum_{\beta \in Q_+} c_\beta(\Lambda)\, x^{-\beta}$

Solution to eigenvalue equations $D_a(x)\Psi(\underline{x}|\Lambda) = \Lambda^{\omega_a}\Psi(\underline{x}|\underline{\Lambda})$

$$\boxed{D_a = \lim_{t\to\infty} t^{\#} \cdot \mathcal{D}_a(x|q,t) \quad \text{Macdonald operators}}$$

$$D_a = D_{a,0}$$

$$D_{a,k}(x) = \sum_{\substack{I \subset [1,N] \\ |I| = a}} x_I^k \prod_{\substack{i \in I \\ j \notin I}} \frac{1}{1 - x_j/x_i} \Gamma_I = \gamma(x)^{-k} D_a \gamma(x)^k$$

$$\gamma(x) = \prod_{a=1}^{N} e^{\frac{(\log x_a)^2}{2\log q}}$$

$$\Gamma_i x_j = q^{\delta_{ij}} x_j \Gamma_i$$

q-Whittaker transform : $(\Lambda_1, \dots, \Lambda_N) \rightsquigarrow (x_1, \dots, x_N)$

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Theorem:
[DFK]

Satisfy $\boxed{Q_{a,k}(\Lambda, T)\psi(x|\Lambda) = D_{a,k}(x)\psi(x|\Lambda)}$ "Fourier Transform"

Example: $(\Lambda T)^{\omega_a} \psi(x|\Lambda) = \Lambda^{\omega_a} \psi(x|\Lambda q^{\omega_a}) = D_{a,1}(x)\psi(x|\Lambda)$ (Raising operators)

$\Rightarrow$ [
- q-Whittaker transform of $g(\Lambda)$ is $\gamma(x)$.
- $D_{a,k}$ satisfy quantum Q-system

Toda Hamiltonian from Bispectrality

$$\mathcal{D}^{q,t}_a(x)\, P(\{x_i\}|\{t^{N-i}\Lambda_i\}) = e_a(\{t^{N-i}\Lambda_i\})\, P(X|\{t^{N-i}\Lambda_i\})$$

$\uparrow$

Macdonald function $P(X|S)$ series in $\{X^{-\alpha_a}\}$

$\Lambda_i = q^{\lambda_i}$ generic

$s_i = t^{N-i}\Lambda_i$

bispectral symmetry:

$$\frac{P(X|S)}{\Delta(X)} = \frac{P(S|X)}{\Delta(S)}$$

$$\Delta(X) = \prod_{i<j}\prod_{n\geq 0}\frac{(1 - x_i/x_j\, q^n)}{(1 - x_i/x_j\, q^{n+1}/t)}$$

$\Rightarrow$ The q-diff operator $\mathcal{H}_a(S) = \Delta(S)^{-1}\mathcal{H}_a(S)\Delta(S)$ exchange $\underline{x} \leftrightarrow \underline{s}$

satisfy $\mathcal{H}_a(S)\, P(X|S) = e_a(x)\, P(X|S)$ Pieri Operator

q-Whittaker limit $t \to \infty$:

$\Rightarrow$ Toda Hamiltonian $\lim_{t\to\infty}\left(t^{\#}\mathcal{H}_a(S)\right) = H_i(\Lambda)$

The q-Whittaker function $\Psi(X|\Lambda) = \lim_{t\to\infty} t^{\#} P(X|S)$

Remark 1 Cluster algebra structure at finite t : Spherical DAHA

[DFK + Schrader, Shapiro]

$$\mathcal{D}_a(x;q,t) = \sum_{|I|=a} \prod_{\substack{i\in I\\ j\notin I}} \frac{t x_i - x_j}{x_i - x_j}\, \Gamma_I$$ = polynomials in t with Coefficients in upper cluster algebra

Ex sl_2 : $\mathcal{D}_1(x) = \frac{t x_1 - x_2}{x_1 - x_2}\Gamma_1 + \frac{t x_2 - x_1}{x_2 - x_1}\Gamma_2 = t D_{1,0} - e_2(x_1,x_2) D_{1,-2}$

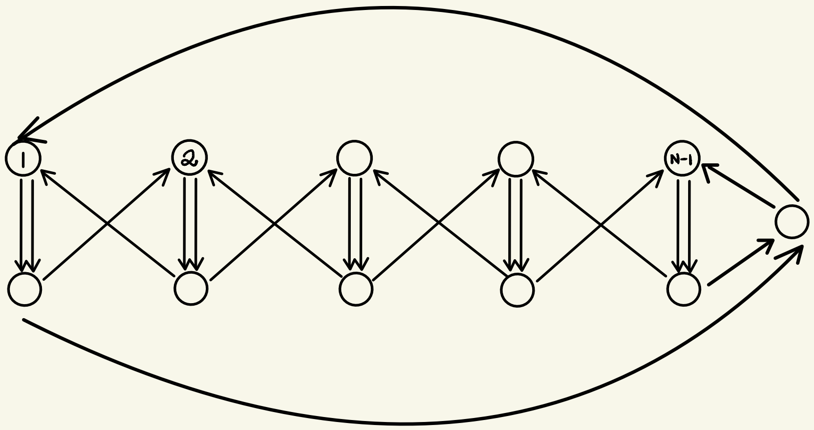
$D_{a,k}$ = cluster $\mathcal{A}$-variables

$e_a(x)$ = Toda Hamiltonians

sDAHA $= \langle e_a(x), \mathcal{D}_a(x) \rangle \subset$ Upper CA

Has $SL_2(\mathbb{Z})$-symmetry $\tau_+ = \begin{pmatrix}1&1\\0&1\end{pmatrix} = \mathrm{Ad}\,\gamma(x)$, τ_--action?

Thm / conjecture: $SL_2(\mathbb{Z})$ = MCG acts by sequence of mutations on extended quiver



Reduction of FG higher Teichmüller quiver on punctured torus

Remark 2: Other root systems: Q-system for type $\mathfrak{g}$

simple roots $\{\alpha_1,\dots,\alpha_N\}$, $\mathcal{X}$-variables $\{X_a = (\Lambda T)^{-\alpha_a},\ X_{a'} = \Lambda^{\alpha_a}\}_{a=1}^{N}$

$$B = \begin{array}{c|c} \left[\begin{array}{c|c} C - C^T & C^T \\ \hline -C & 0 \end{array}\right] & \begin{array}{l} \}a \\ \}a' \end{array} \end{array} \qquad C_{ab} = (\alpha_a^\vee, \alpha_b), \qquad t_a = \frac{(\alpha_{long}, \alpha_{long})}{(\alpha_a, \alpha_a)} = 1, 2, 3$$

Time translation τ: [DFK08]

- $(\text{mutate at short labels } a,\ \text{permute } a \leftrightarrow a')^{t_a}$
- mutate at long labels a, Permute $a \leftrightarrow a'$

$$\Rightarrow \quad g(\Lambda) = \Big(\gamma(T)^{1/t} \prod_{\alpha_a \text{ short}} \varphi(\Lambda^{-\alpha_a})\Big)^{t} \prod_{\alpha_a \text{ long}} \varphi(\Lambda^{-\alpha_a})$$

[DFK24 for BCD]

Initial $\mathcal{A} = \{Q_{a,0} = \Lambda^{\omega_a^\vee},\ Q_{a,1} = \Lambda^{\omega_a^\vee} T^{\omega_a}\}$

Thm: $\boxed{Q_{a,k+t_a} = \mathrm{Ad}_{g(\Lambda)} Q_{a,k}}$ Evolution faster for α short

Q-system in affine $\widehat{sl}_N$

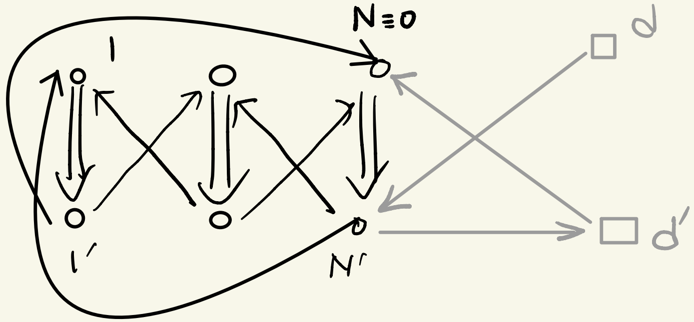
[DFK, w/ M. Bershteen, J-E Bourgine, J. Shiraishi]

$\hat{C}$ = affine Cartan matrix (singular): Extend via coefficients $(\Lambda T)^d$, Λ^d

$(\alpha_i,\alpha_j)=\hat{C}_{ij}$, $(d,\alpha_0)=1$, $(d,\alpha_i)=0$ $(i\neq 0)$ $(d,d)=0$

Construct Cluster Algebra: exchange matrix

$$\overline{B}=\left[\begin{array}{c|c} 0 & \hat{C} \\ \hline -\hat{C} & 0 \end{array}\right]$$



quantum torus:

$$\left.\begin{array}{l} \Lambda_1,\dots\Lambda_{N-1},\Lambda_N,\Lambda_d \\ T_1,\dots T_{N-1},T_N,T_d \end{array}\right\} \quad T_i\Lambda_j=q^{\delta_{ij}}\Lambda_jT_i$$

2(N+1)-dimensional

$\mathcal{X}$-variables: $X_a=(\Lambda T)^{-\alpha_a}$, $a\in\mathbb{Z}/n\mathbb{Z}$

$X_{a'}=\Lambda^{\alpha_a}$

Coefficients: $X_d=(\Lambda T)^d$, $X_{d'}=\Lambda^d$

$\Longrightarrow$ $\mathcal{A}$-variables satisfy affine Q-system (Definition)...

$\widehat{sl_N}$ Q-system:

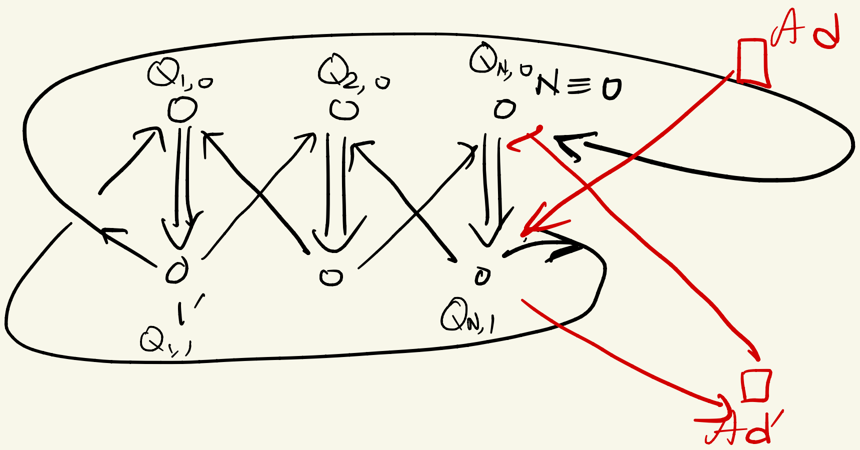
$$\boxed{Q_{a,k+1}Q_{a,k-1} = Q_{a,k}^2 - P_k^{\delta_{a,0}} Q_{a-1,k} Q_{a+1,k}} \qquad a \in \mathbb{Z} \bmod N$$

A-variables $Q_{a,k}$

$$P_k = \underbrace{A_d^{k-1} A_{d'}^{k}}_{\text{coefficients}} = \Lambda^{-\delta} T^{-k\delta} \qquad \delta = \alpha_0 + \cdots + \alpha_{N-1}$$

$$\boxed{\begin{aligned} A_a &= Q_{a,0} = \Lambda^{w_a} \\ A_{a'} &= Q_{a,1} =: (\Lambda T)^{w_a} \end{aligned}} \qquad a \in \mathbb{Z}/N\mathbb{Z}$$

$w_0, \ldots, \omega_{N-1}$: affine weights



Time-Translation: $\mathcal{J}$:

$$Q_{a,k} \longmapsto Q_{a,k+1}$$

(i) mutate at $a = 1, \ldots, N$
(ii) permute $a \leftrightarrow a'$

$$T_d \Lambda_d = q \Lambda_d' T_d$$

g-operator :

$$\mathrm{Ad}_g(Q_{a,k}) = Q_{a,k+1}$$

$$\hat{g} = \hat{\gamma}(T) \prod_{a=1}^{N} \varphi(\Lambda^{-\alpha_a}) \quad \text{where } \hat{\gamma}(T) = \gamma(T)\, e^{\frac{(\log T_d)^2}{2\log q}}$$

$T_d \in$ extended torus

For affine system, we look for :

- q-Whittaker transform: Universal function "$\hat{g}(\Lambda)\hat{\Psi} = \gamma(x)\hat{\Psi}$"
- t-deformation to affine Macdonald theory
- Bispectral property
- Affine qToda Hamiltonians
- Solutions of affine Q-system
- Character interpretation
- Algebra structure

$\vdots$

Finite t: Affine Macdonald Theory

Universal function [Shiraishi 2019]:

$\varphi_{q,t}(x,p|s,\kappa)$ "non-stationary Ruijsenaars function" $\underset{t\to\infty}{\rightsquigarrow}$ $\Psi = \Psi_q(x,p|\Lambda,\kappa)$

↑ affine q-Whittaker

series in $\{x^{-\alpha_i}\}_{i=0}^{N-1}$ coefficients ratios of Nekrasov factors

("matrix elements of vertex ops for quantum toroidal $\mathfrak{gl}_N$")

Shiraishi Conjectures:

- Bispectral Symmetry $\varphi(x,p|s,\kappa) = \varphi(s,\kappa|x,p)$
- Specialization to "Affine Macdonald polynomials" [Etingof-Kirillov]

Ruijsenaars operator: $\mathcal{D}_1(x;q,p) = \sum_{i=1}^{N} \prod_{j\neq i} \frac{\Theta_p(t\, x_i/x_j)}{\Theta_p(x_i/x_j)} T_i$ for $\kappa = 1$

$\Theta_p(x) = (x;p)_\infty (p/x;p)_\infty (p;p)_\infty$

Eigenvalue equation at $\kappa = 1$

$$\mathcal{D}_1^{(p)}(x)\,\tilde{\varphi}\,\Big|_{\kappa=1} = \varepsilon(s)\,\tilde{\varphi}, \quad \varepsilon(s) = e_1(s) + \mathcal{O}(p) \quad \text{(stationary state)}$$

Affine q-Whittaker limit $t \to \infty$:

$\delta =$ null root of $\hat{A}_{N-1}$: $(\delta, \alpha_i) = 0 = (\delta, \delta)$, $(\delta, d) = 1$.

$$\boxed{p = \Lambda^{-\delta}, \qquad \kappa = T^{-\delta}}$$

$\delta = \alpha_0 + \dots + \alpha_{N-1}$

$\alpha_i = e_i - e_{i+1}$, $i = 1, \dots, N-1 \Rightarrow \Lambda^{-\alpha_i} = \frac{\Lambda_{i+1}}{\Lambda_i}$

$\alpha_0 = e_N - e_1 + \delta \Rightarrow \Lambda^{-\alpha_0} = p \frac{\Lambda_N}{\Lambda_1}$

Closed qToda Hamiltonians

$$\boxed{H_p = \sum_{a=1}^{N} (1 - \Lambda^{-\alpha_{a-1}}) T_a} = \left(1 - p \frac{\Lambda_N}{\Lambda_1}\right) T_1 + \left(1 - \frac{\Lambda_2}{\Lambda_1}\right) T_2 + \cdots$$

Closed Toda Hamiltonian is not a conserved quantity!

$$\hat{\mathcal{T}}(H_p) = \hat{g} H_p \hat{g}^{-1} = H_{\kappa p}$$ Dynamical symmetry

when $\kappa = 1$: $H_p \psi = (e_1(x) + \mathcal{O}(p)) \psi$ Eigenfunction closed Toda

when $\kappa \neq 1$ $\hat{g}(\Lambda)$: $p \to \kappa p$ Non-stationary state

Combinatorics of affine Q-system solutions

Classical: $q=p=1$

$$\boxed{Q_{a,k+1}Q_{a,k-1} = Q_{a,k}^2 - p_k^{\delta_{a,0}} Q_{a-1,k}Q_{a+1,k}}$$

$$p_\kappa = p\kappa^k \Big|_{p=1} = \kappa^k$$

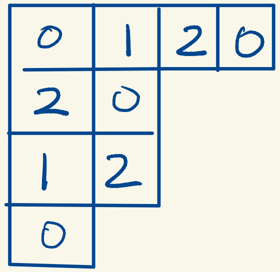
$$\boxed{\tilde{Q}_{a,k} = \sum_{\substack{\pi \text{ plane} \\ \text{partitions of} \\ \text{height} \le k}} \prod_{i,j} z_{a+j-i}^{\pi_{ij}}}$$

Periodicity

$z_{a+N} = z_N$

$z_0 z_1 \cdots z_{N-1} = \kappa$

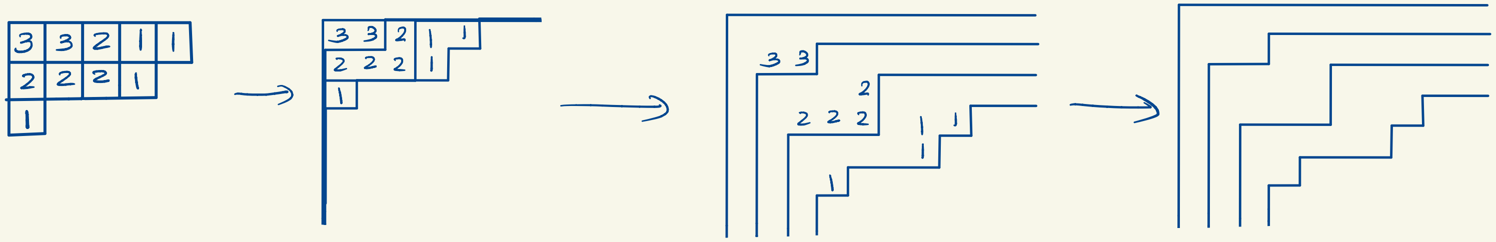
$N=3$
$a=0$:



(content mod N)

$$\tilde{Q}_{a,0} = 1 \,, \quad \tilde{Q}_{a,1} = \sum_{\lambda:\text{Young diagram}} \prod_{i,j\in\lambda} z_{a+i-j}$$

plane partitions of height $\le k \xrightarrow{\sim} k$ paths



Lindström Gessel Viennot $\Rightarrow$ $\tilde{Q}_{a,k+1}\tilde{Q}_{a,k-1} = \tilde{Q}_{a,k}^2 - z_a^k \tilde{Q}_{a+1,k}\tilde{Q}_{a-1,k}$, $a\in \mathbb{Z}/N$

(Renormalized) closed Q-system with $p=1$

Affine characters

$$\boxed{Q_{a,k} = \sum_{\lambda}{}' \prod_{i,j\in\lambda} z_{a+j-i}}$$

Thm: At $q=p=1$, $Q_{a,1}$ = Characters of integrable, level-1 $\widehat{gl}_N$-modules

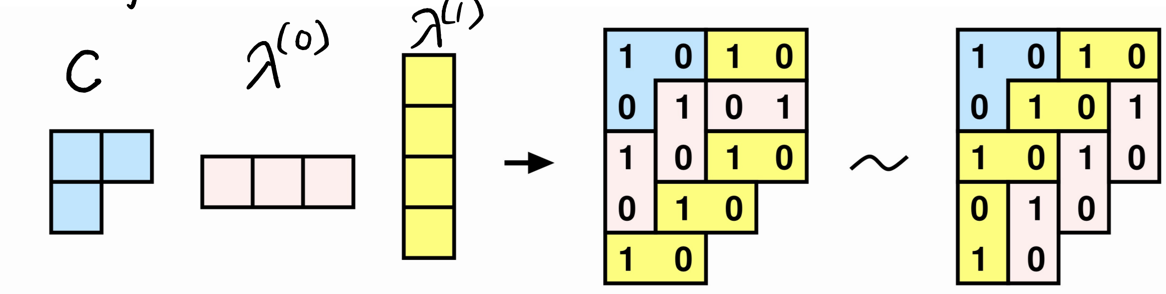
Example N=2: $Q_{1,1} = \frac{1}{(\kappa;\kappa)^2_\infty}\sum_{n\in\mathbb{Z}}{}' \kappa^{n^2} z_0^{n} = \frac{1}{(\kappa;\kappa)^2_\infty}\sum_{n\geq 0}{}' z_0^{n(n+1)}\left(z_1^{n^2} + z_1^{(n+1)^2}\right)$

$(\kappa = z_0 z_1)$

Proof: Plane partition π of height ≤ 1 is a partition λ

$$Q_{1,1} = \sum_{\lambda}{}' \prod_{\square\in\lambda} z_{\mathrm{cont}(\square)}$$

use core-quotient combinatorics:



$$\lambda \longleftrightarrow (\underset{\substack{\uparrow\\ \text{2-core}}}{C}, \lambda^{(0)}, \lambda^{(1)})$$

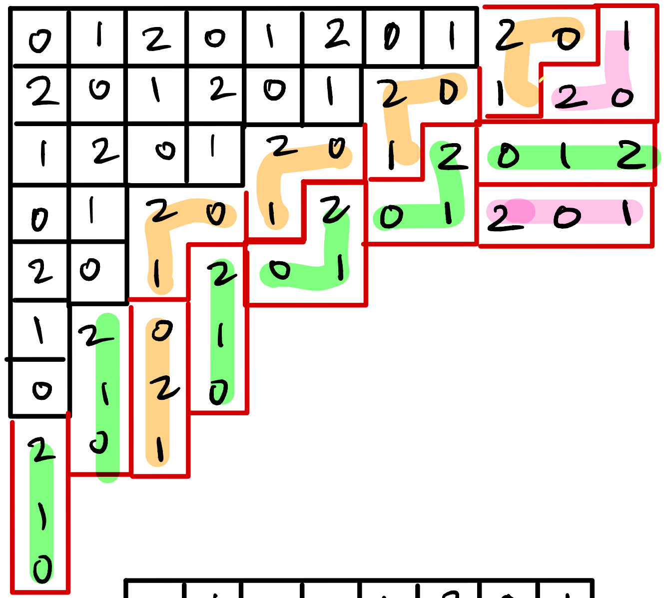
2-quotient

- $C = (j, j-1, \ldots, 1)$ has weight $z_0^{n(n+1)} \times \begin{cases} z_1^{n^2}, & (j=2n) \\ z_1^{(n+1)^2}, & (j=2n+1) \end{cases}$

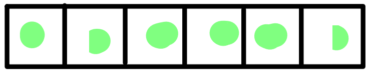
$$\lambda^{(0)} = \left\{ \boxed{0\,|\,1}\,,\ \begin{array}{|c|}\hline 1\\ \hline 0\\ \hline\end{array} \right\} \quad wt = (z_0 z_1)^{|\lambda^{(0)}|} = \kappa^{|\lambda^{(0)}|}$$

$$\lambda^{(1)} = \left\{ \boxed{1\,|\,0}\,,\ \begin{array}{|c|}\hline 0\\ \hline 1\\ \hline\end{array} \right\} \quad wt = (z_0 z_1)^{|\lambda^{(0)}|} = \kappa^{|\lambda^{(1)}|}$$

N=3 :

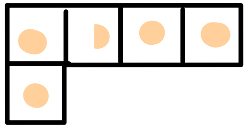


$\lambda^{(0)} =$



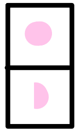
κ^6

$\lambda^{(1)} =$



κ^5

$\lambda^{(2)} =$



κ^2

$$C = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0&1&2&0&1&2&0&1 \\ \hline 2&0&1&2&0&1 \\ \cline{1-6} 1&2&0&1 \\ \cline{1-4} 0&1 \\ \cline{1-2} 2&0 \\ \cline{1-2} 1 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array}$$

$$Q_{0,1} = \frac{1}{(\kappa;\kappa)^3_\infty} \sum_{k_1,k_2\in\mathbb{Z}} \kappa^{h_2(k_1,k_2)} z_1^{k_1+k_2} z_2^{k_2}$$

level-1 $\hat{gl}_3$-char

$\uparrow$

affine counterpart of $e_a = S_{w_a}$ for open Toda

Summary:

- For any root system, Cartan $\rightsquigarrow$ Quantum "Q-system"

  Cluster mutation sequence $\Rightarrow$ time translation operator $g(\Lambda)$

  (Λ, T) - space

- q-Whittaker transform: Macdonald-type difference operators

  (x, Γ) -space

Finite root systems:

$$\mathcal{D}_a(x)\,\Psi(x|\Lambda) = \Lambda^{\omega_a}\,\Psi(x|\Lambda)$$

$$\uparrow t \to \infty$$

$$\mathcal{D}_a(x)\,P(x|s) = e_a(s)\,P(x|s)$$

$\Downarrow$ Bispectrality, $t \to \infty$

$$H_a(\Lambda)\,\Psi(x|\Lambda) = e_a(x)\,\Psi(x|\Lambda)$$

Open q-Toda Hamiltonians

Commute with g : $g H_a g^{-1} = H_a$

conserved.

Algebra: spherical DAHA

Affine

Non-stationary states when $\kappa \neq 1$:

$\varphi(x|s;\kappa,p,q,t)$ series in $\{x^{-\alpha_i}\}$

$\kappa = 1$: Ruisenaars operators

Bispectrality

$\Downarrow$

$$H(\Lambda) = H_p(\Lambda), \quad p = \Lambda^{-\delta}$$

"Dynamical" $g H_p = H_{\kappa p}\, g$

Algebra ...