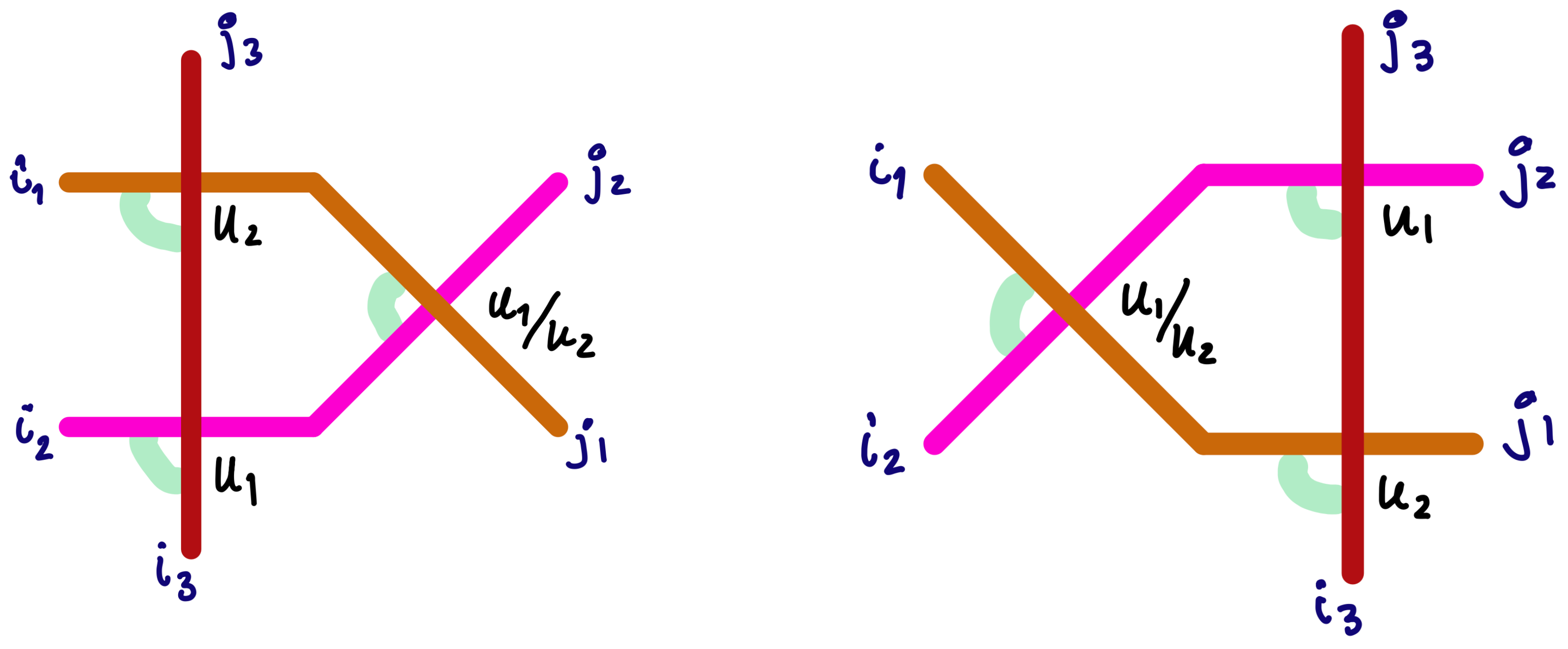
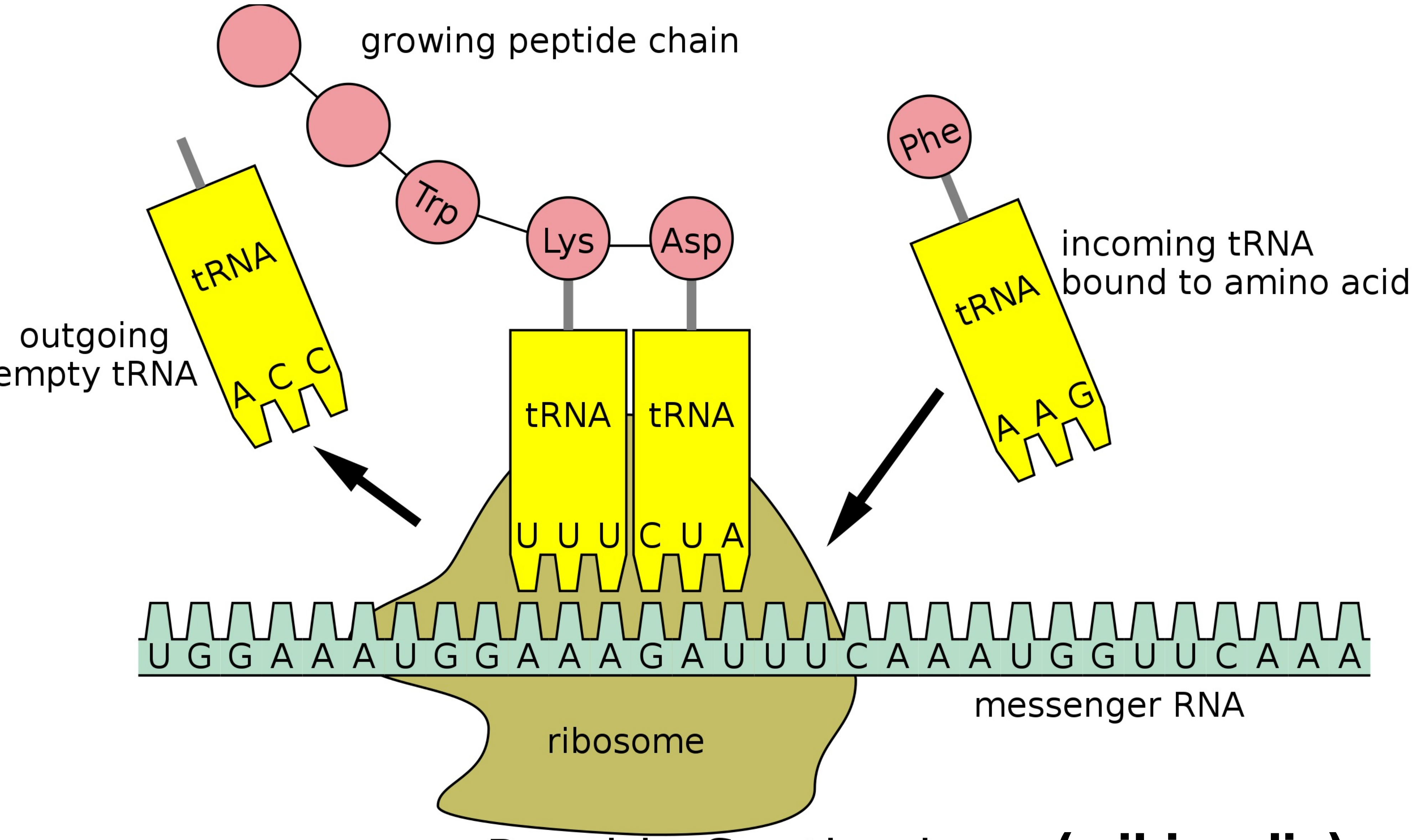
Vertex model integrability for stochastic particle systems





Peptide Synthesis **(wikipedia)**

Leonid Petrov
(University of Virginia)

May 20, 2024
IPAM Workshop IV: Vertex Models: Algebraic and Probabilistic Aspects of Universality

- P.-Saenz **arXiv:2212.01643**
- Aggarwal-Nicoletti-P. **arXiv:2309.11865**

TL;DR

Goal:

Apply the Yang-Baxter equation to stochastic particle systems. **Graphical manipulations** lead to interesting **probabilistic consequences**

Corollaries:

- Lax-type equations for (q-)TASEP
- Description of stationary measures for ASEP

I. TASEP with inhomogeneous speeds and symmetry on the line

- [C. MacDonald, J. Gibbs, A. Pipkin 1969]
- [Spitzer 1970]
- [Rost 1981]
- [Johansson 2000]
- [Matetski, Quastel, Remenik 2017]
- [Dauvergne, Ortmann, Virag 2018]
- [Busani, Seppäläinen, Sorensen 2022+]
- ...

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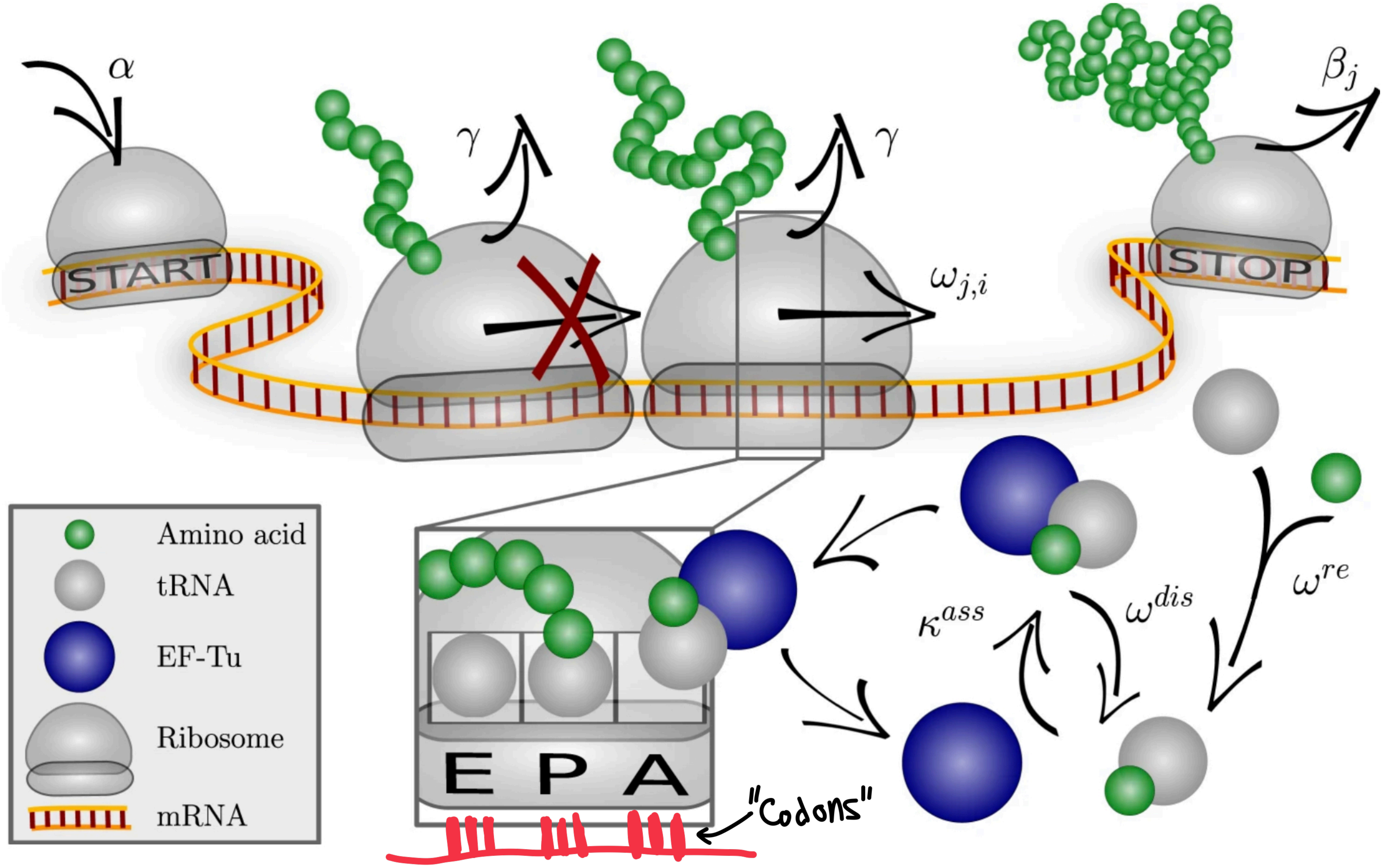

Sketch of the codon-specific elongation model (COSEM). Ribosomes attach to a codon sequence labeled by *j* with initiation rate *α*, which is determined by the ribosome concentration if the initiation site is not occupied, and move to the next position with the elongation rate $\omega_{j,i}$ specific to the codon at position *i* in the sequence *j* as well as to the organism under consideration.

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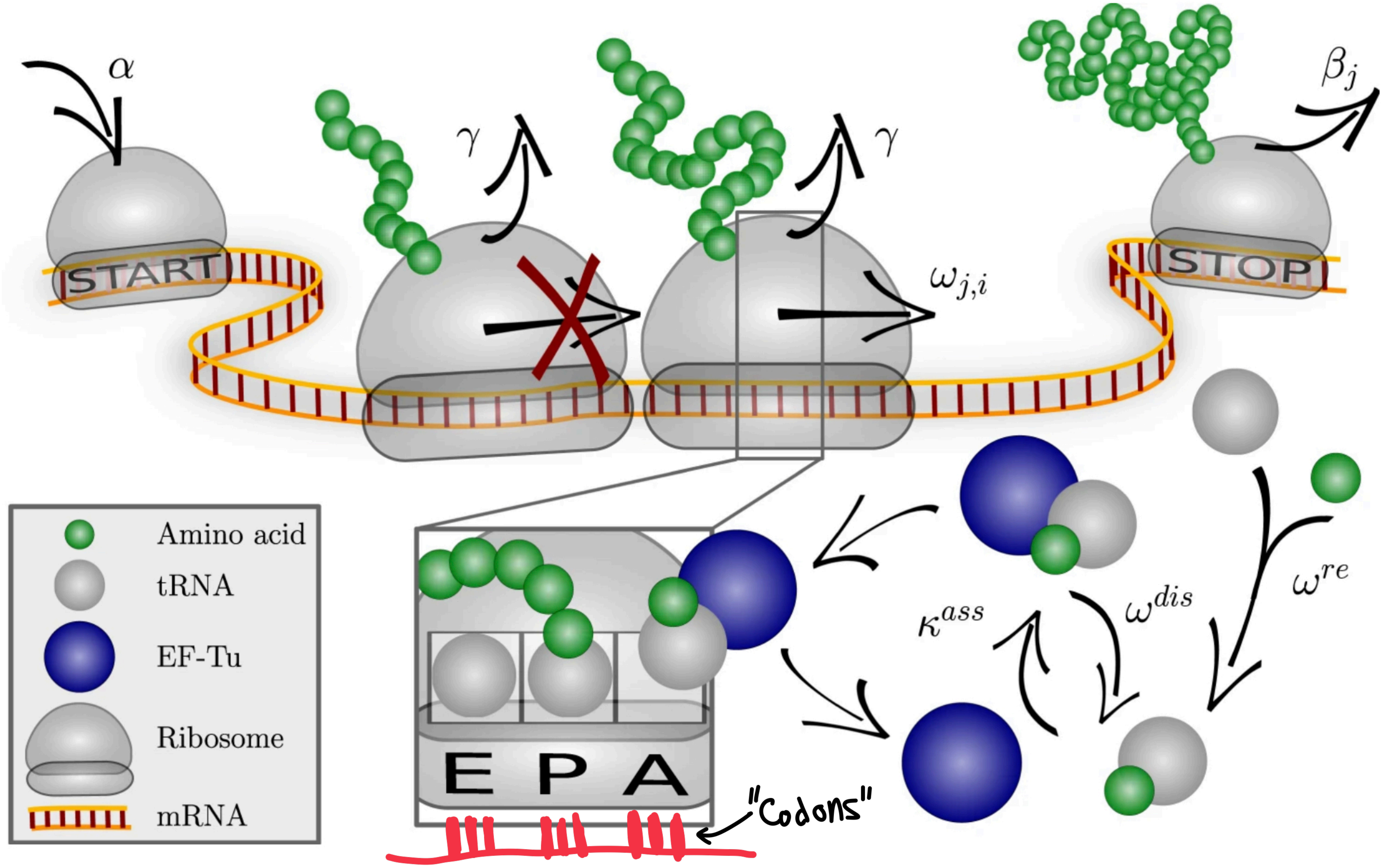

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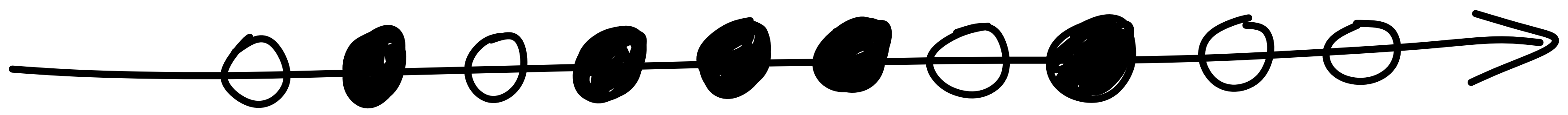

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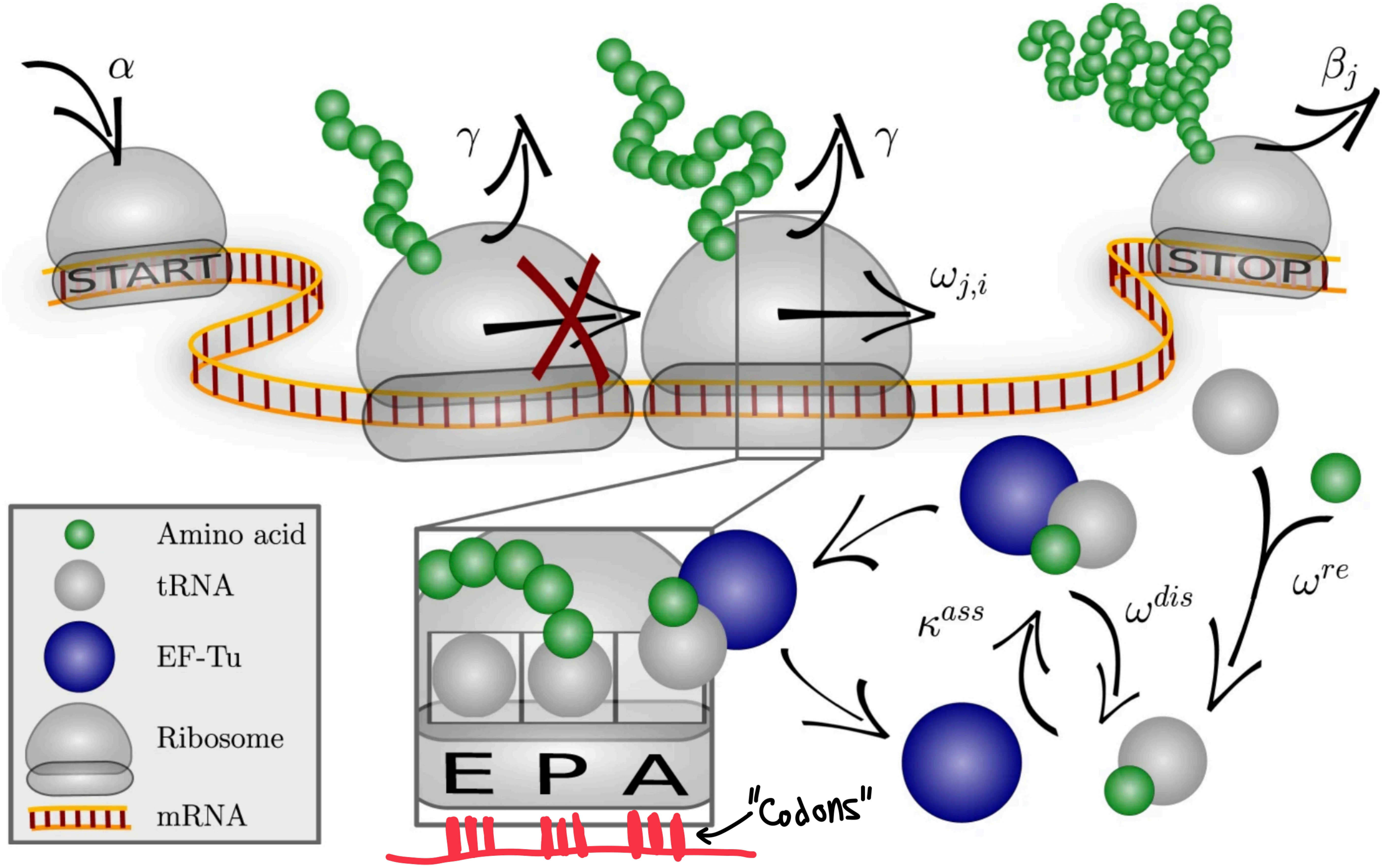

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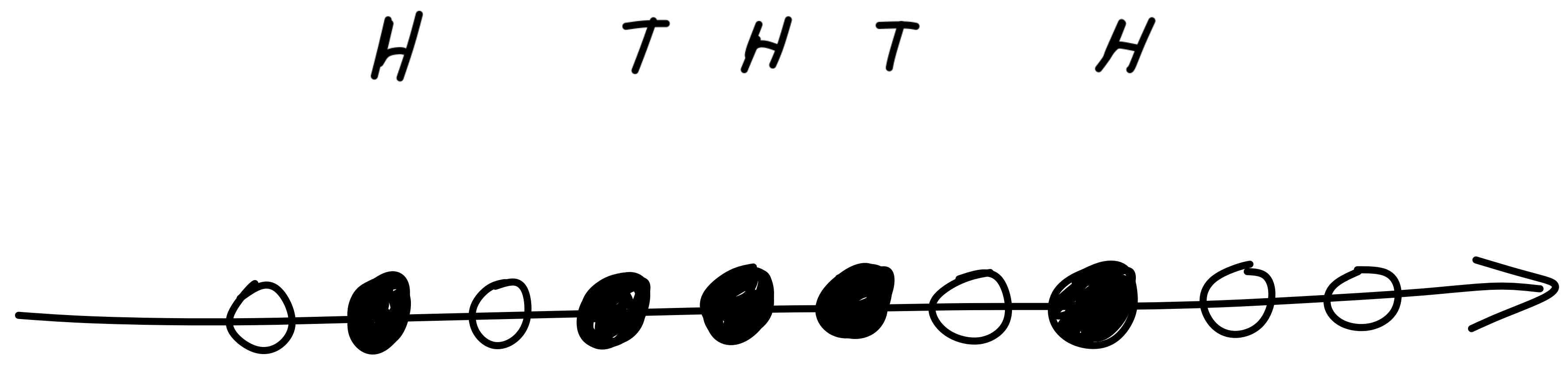

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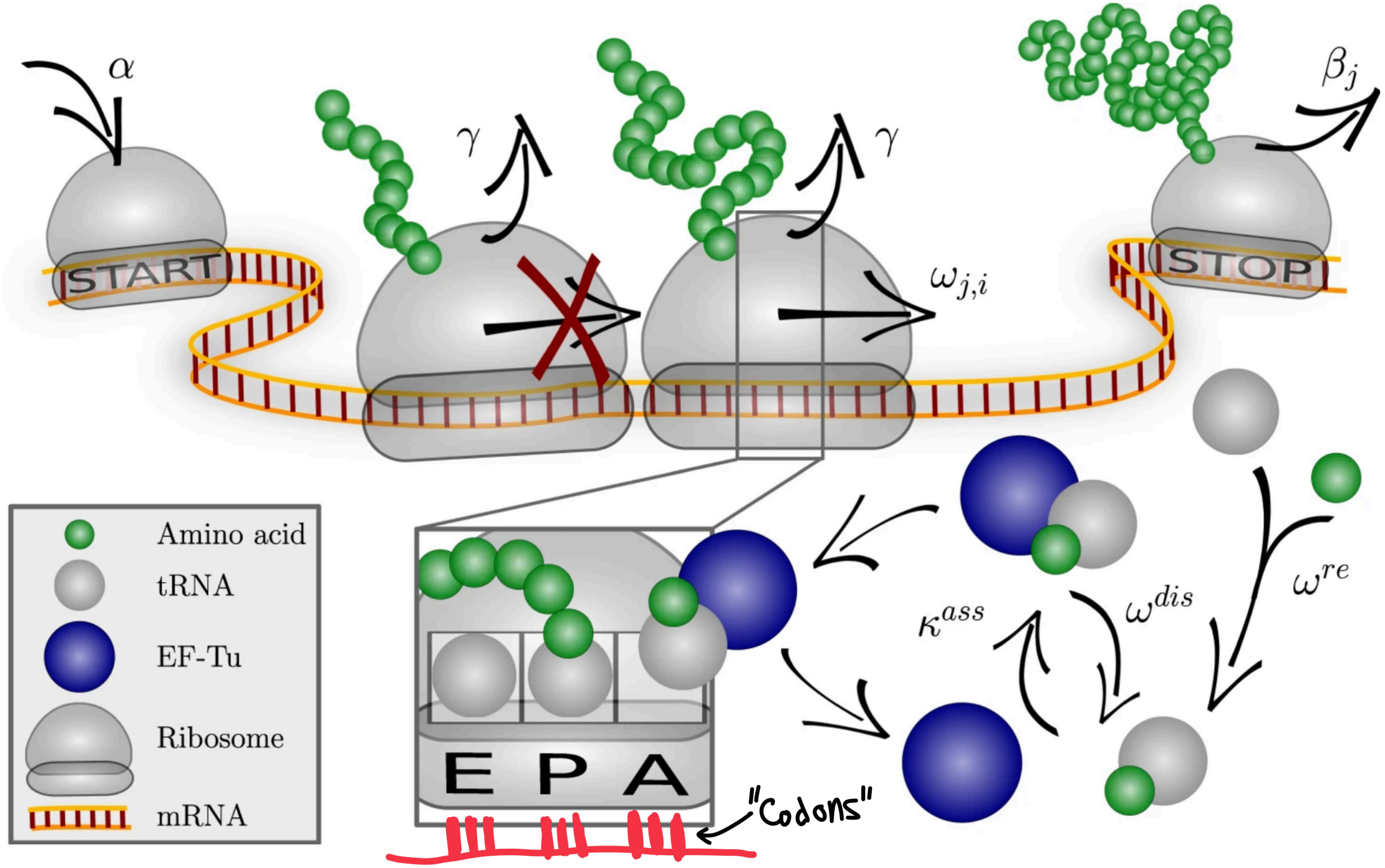

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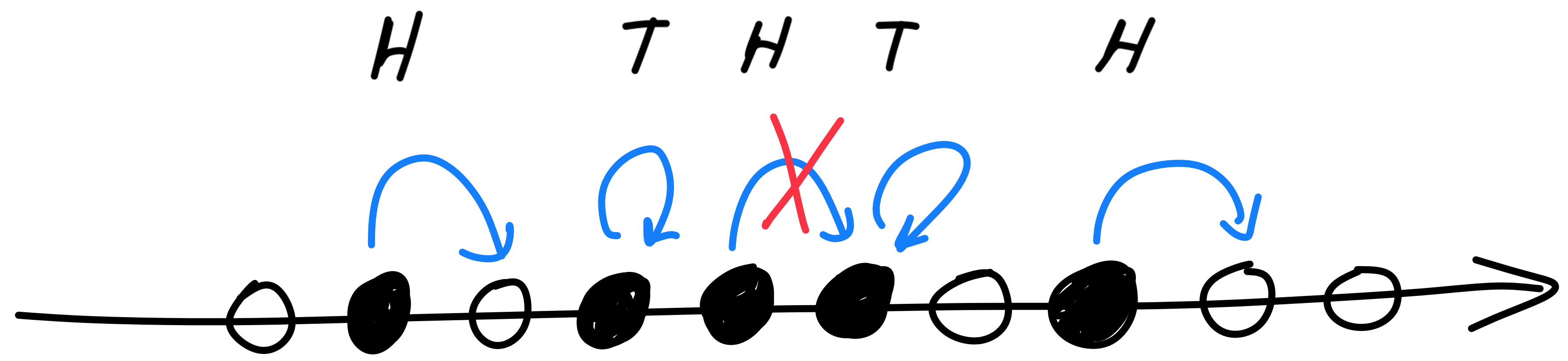

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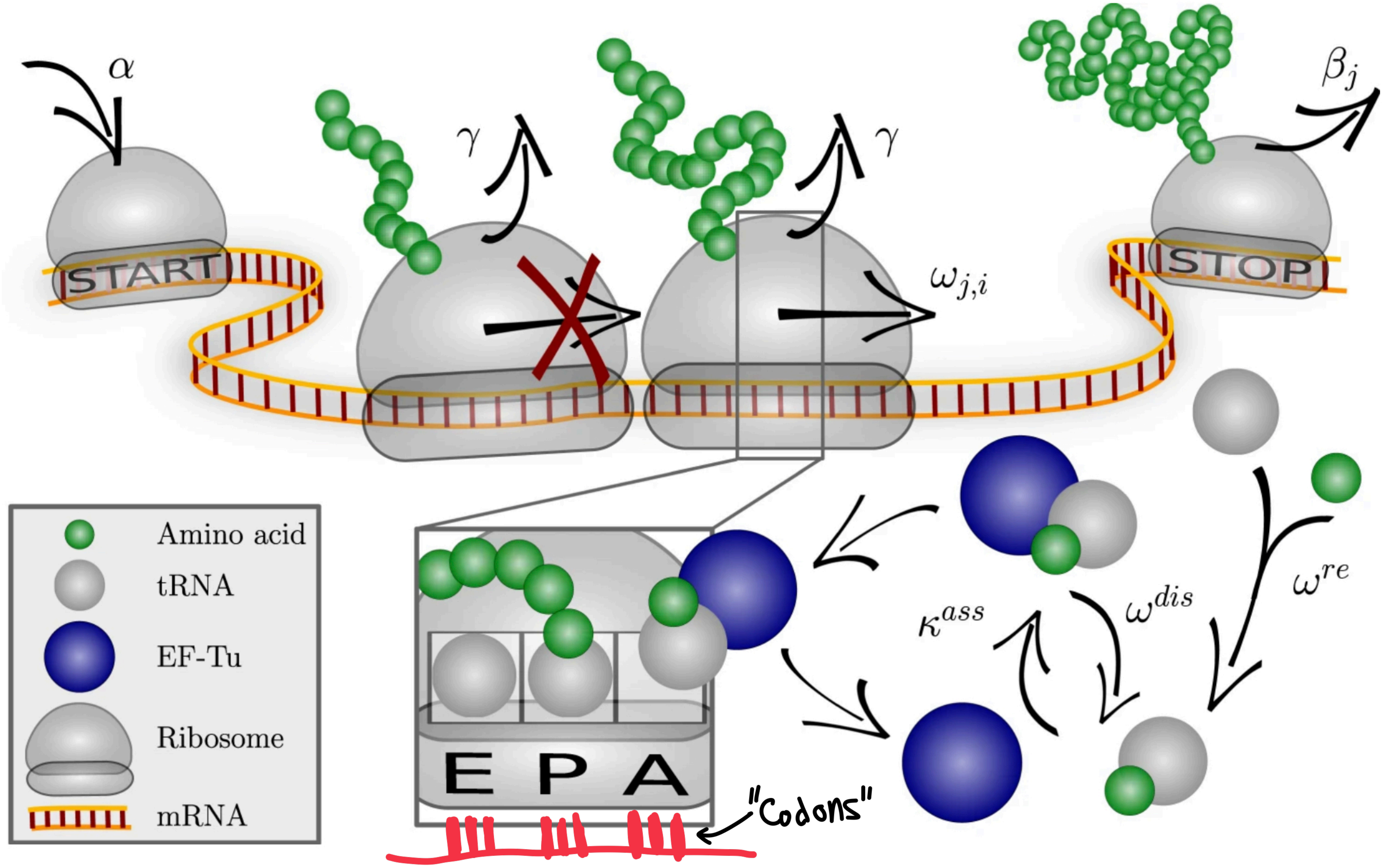

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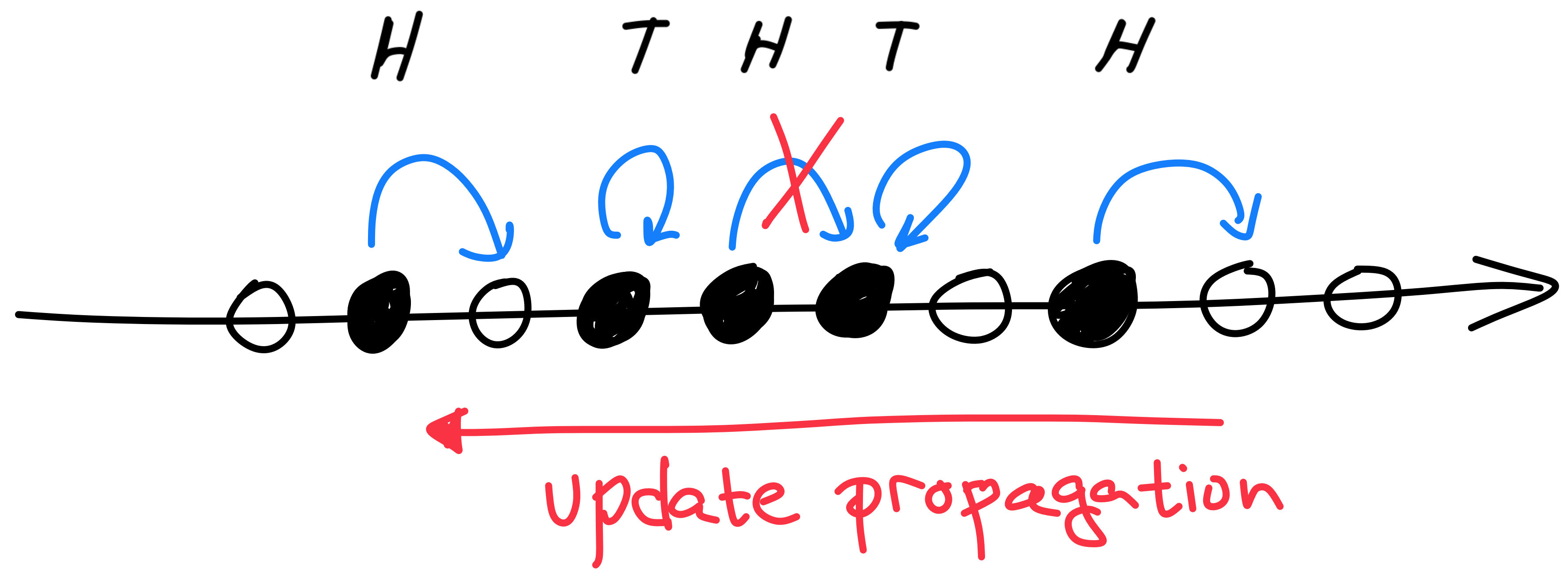

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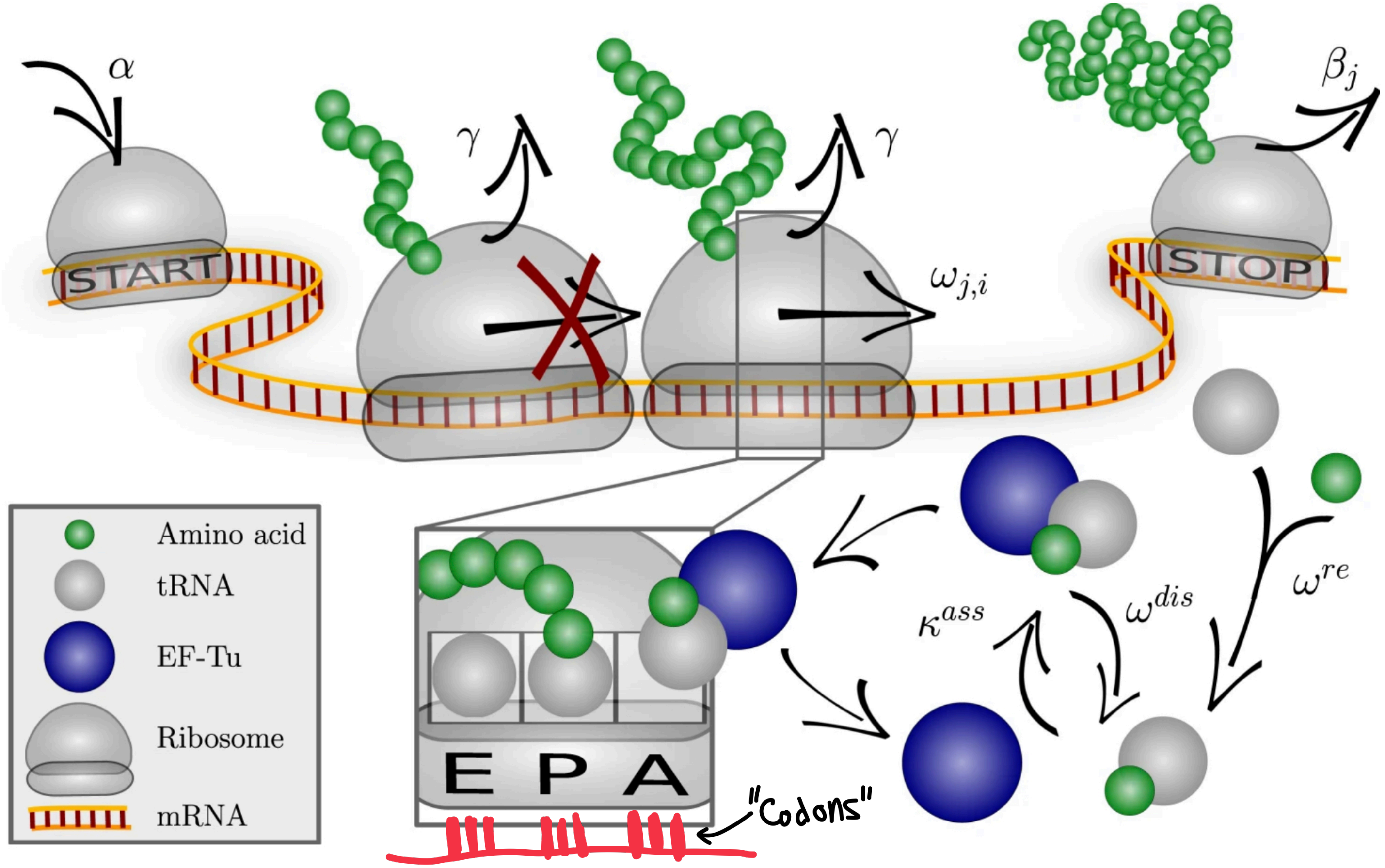

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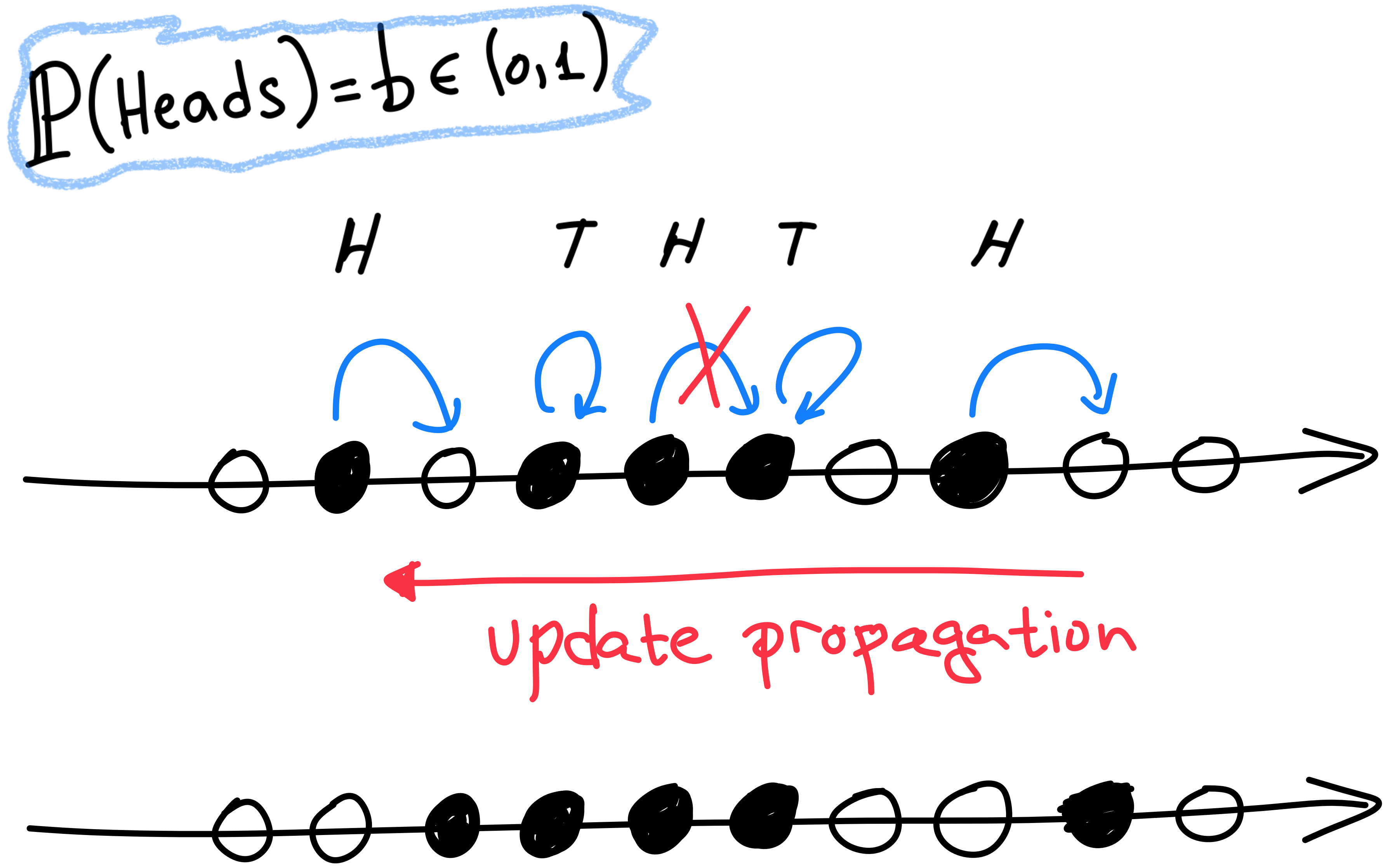

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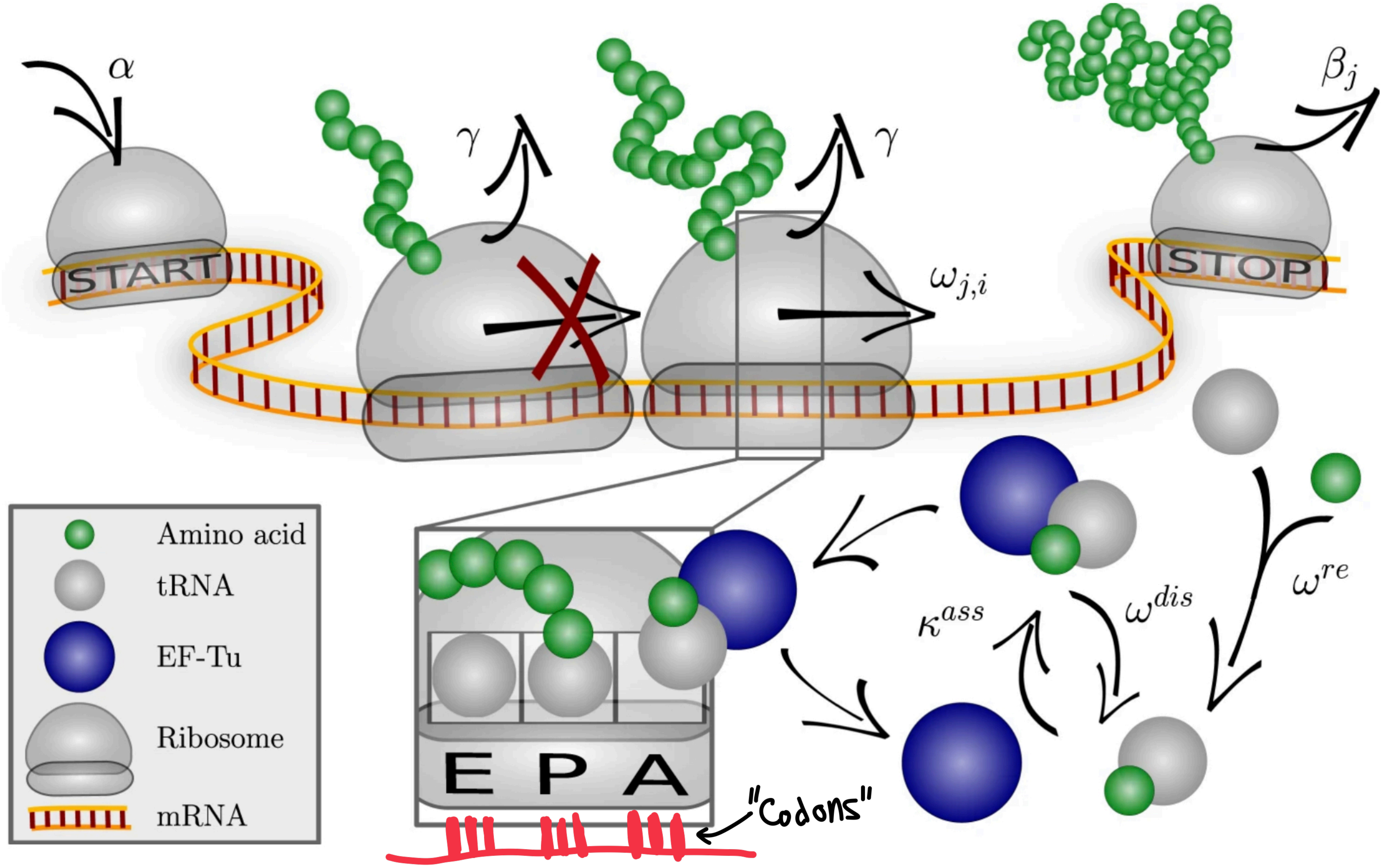

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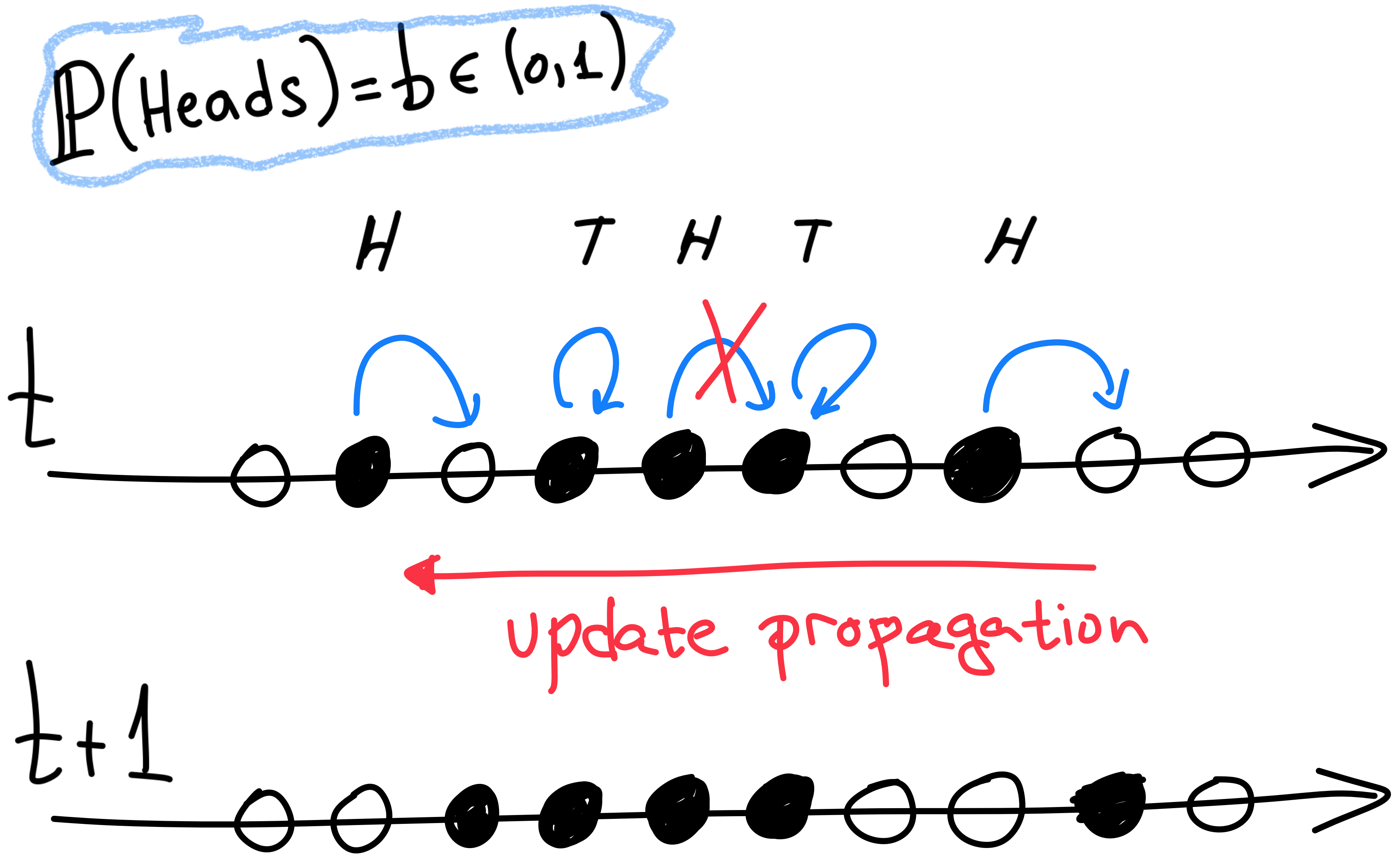

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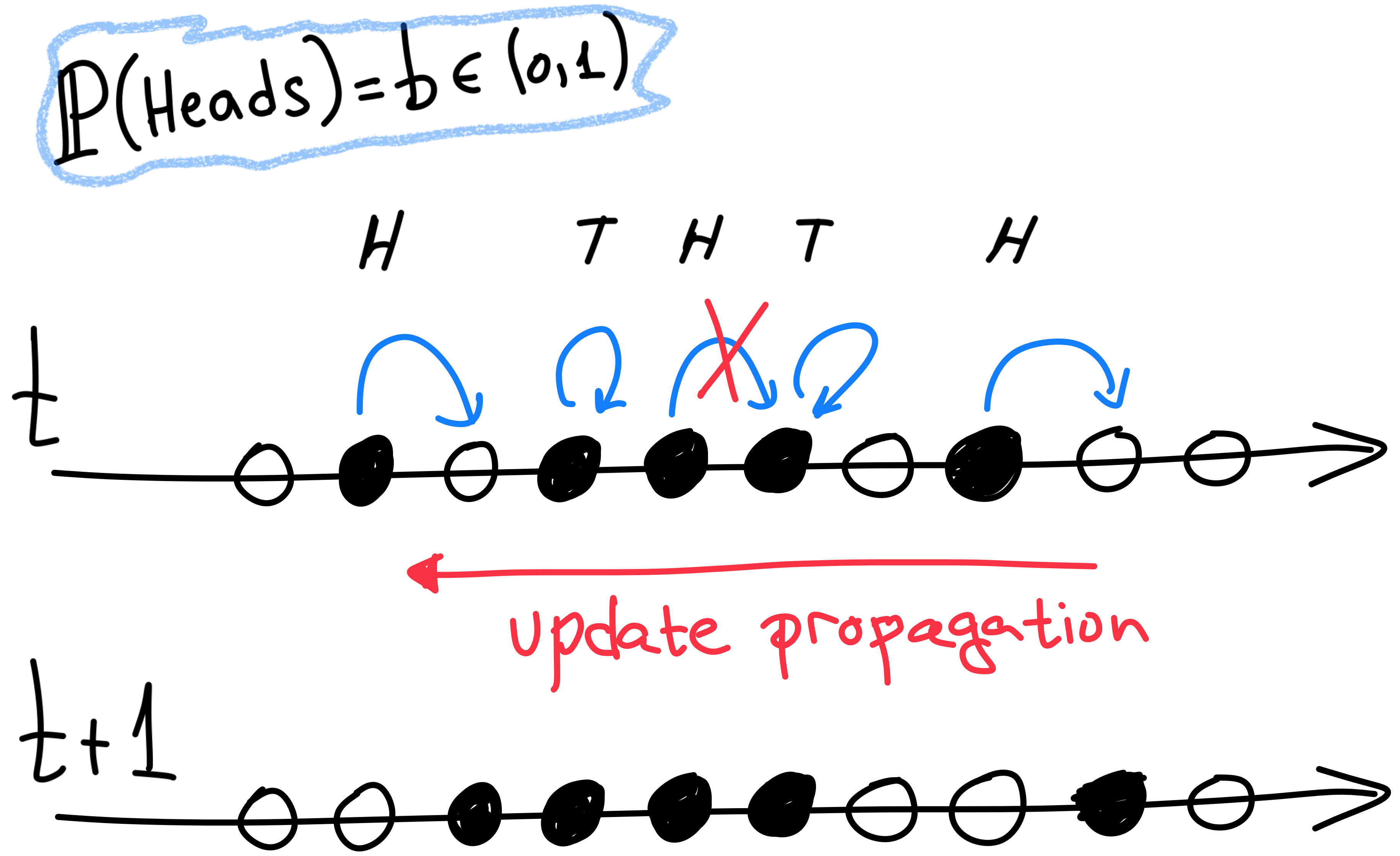

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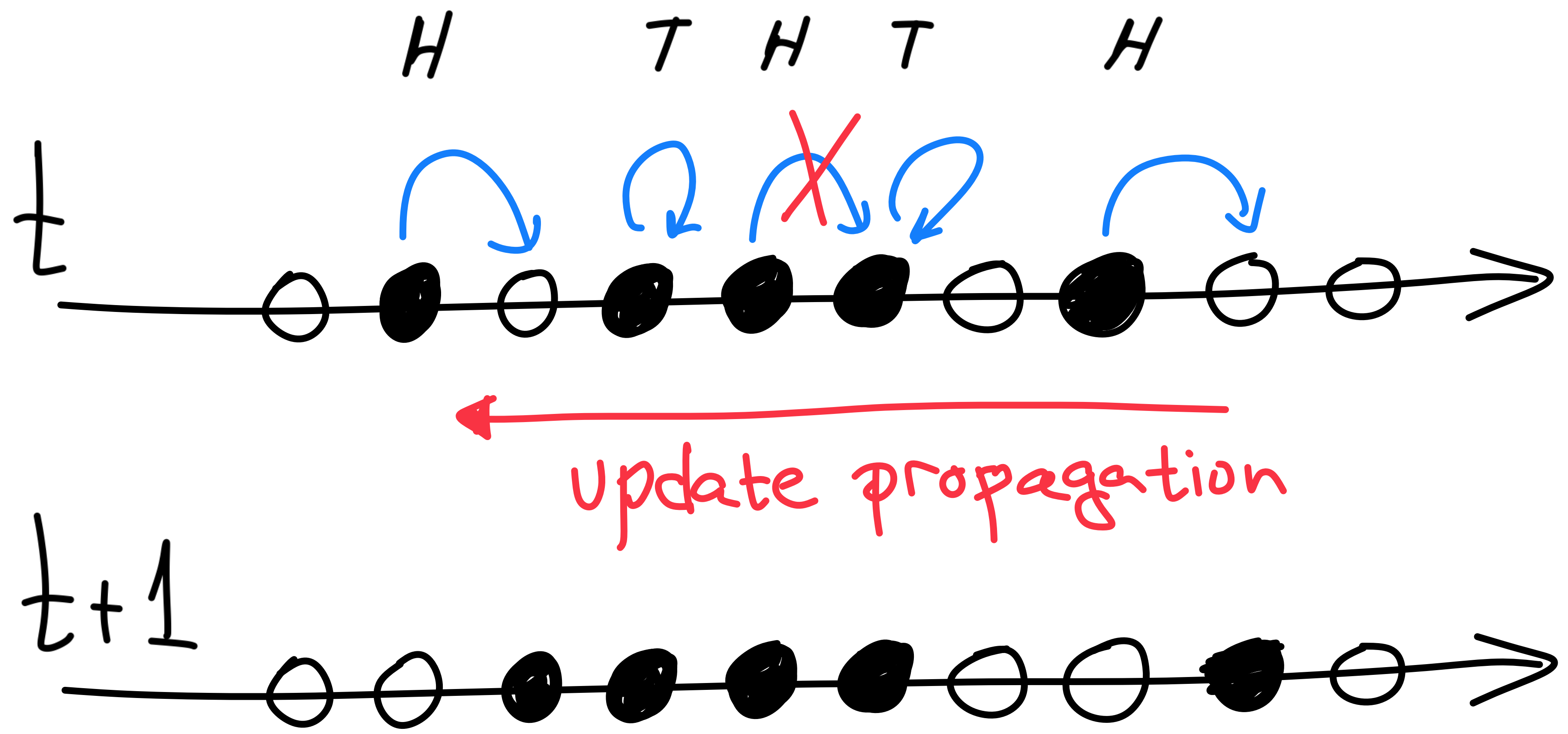

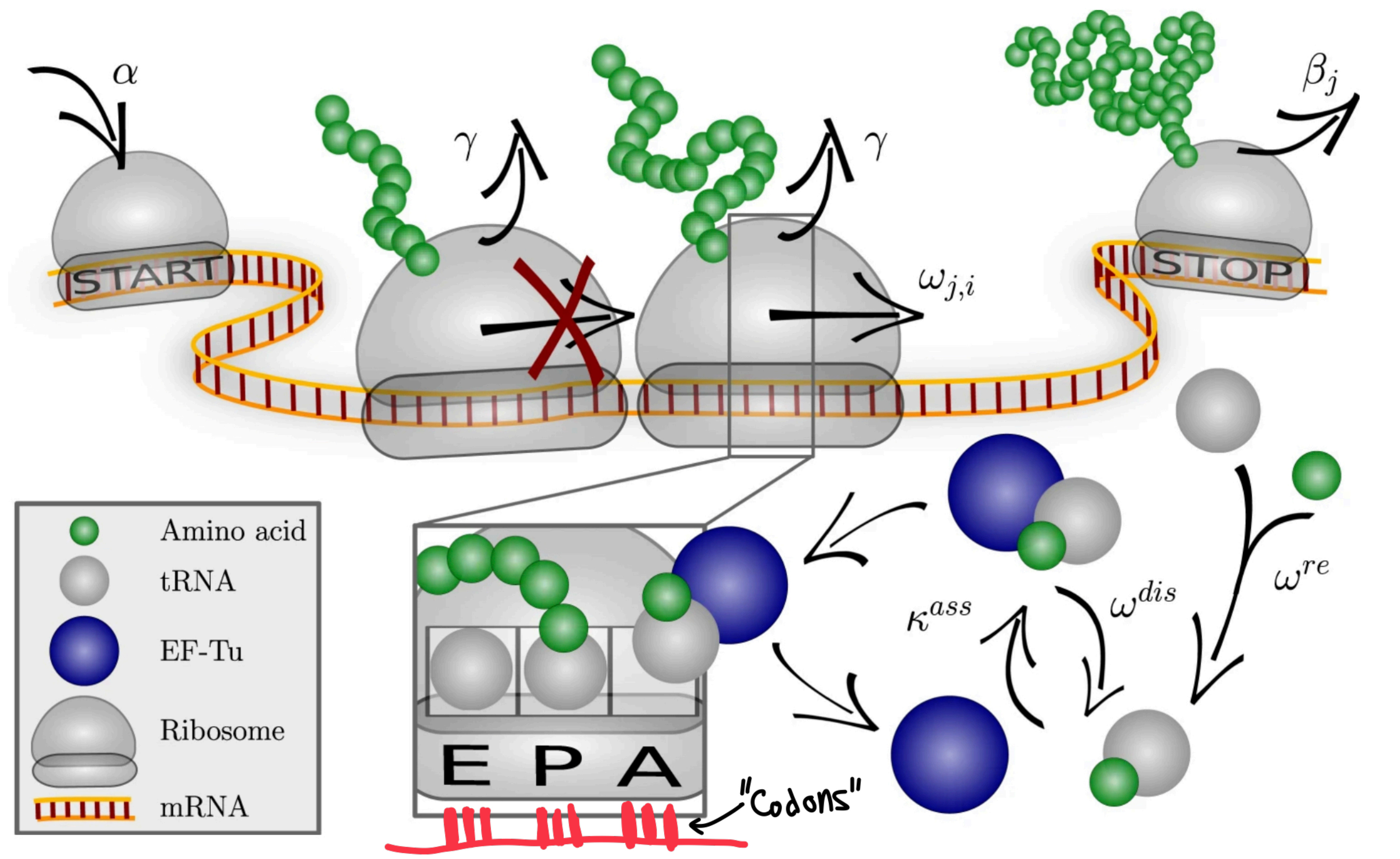

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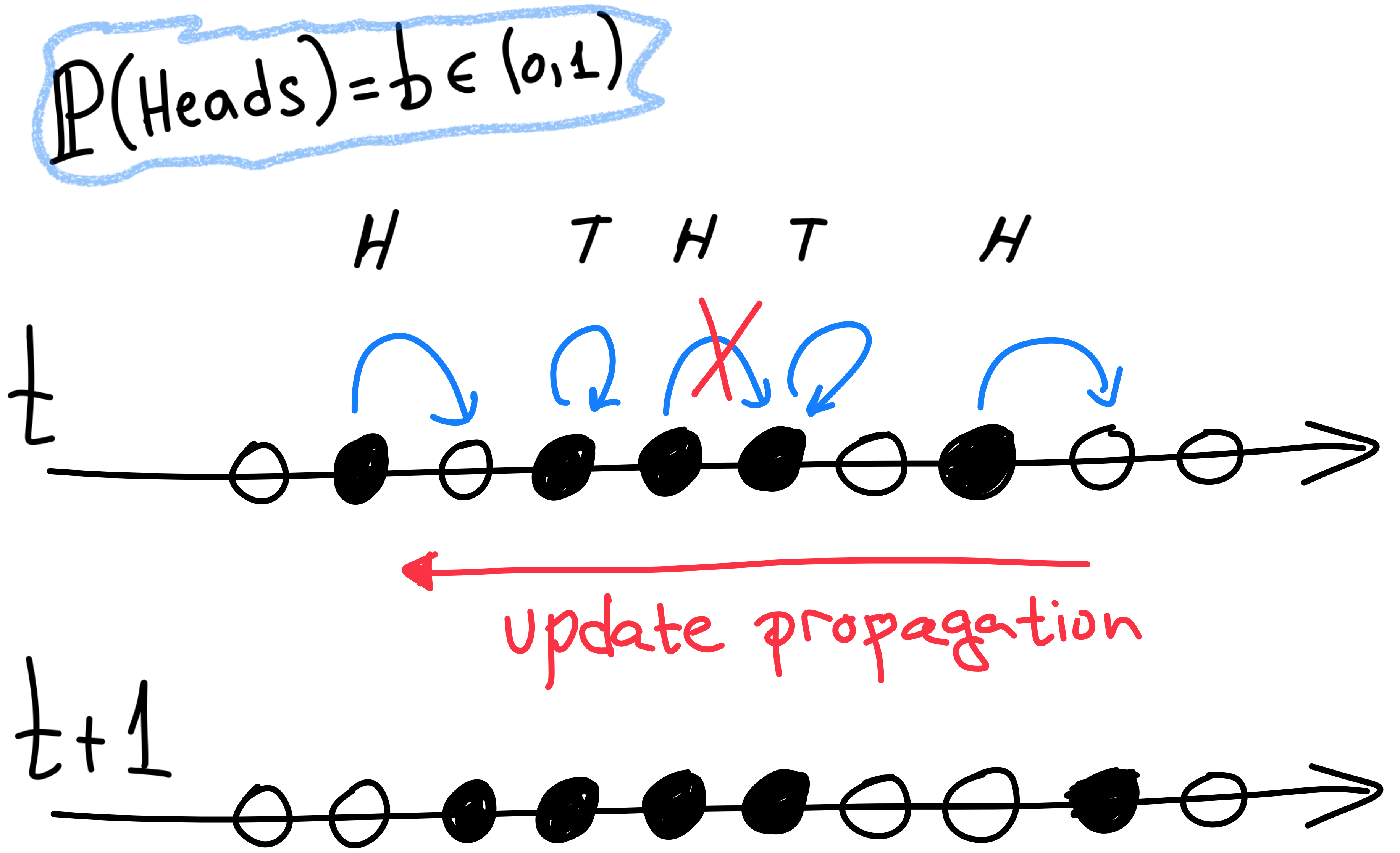
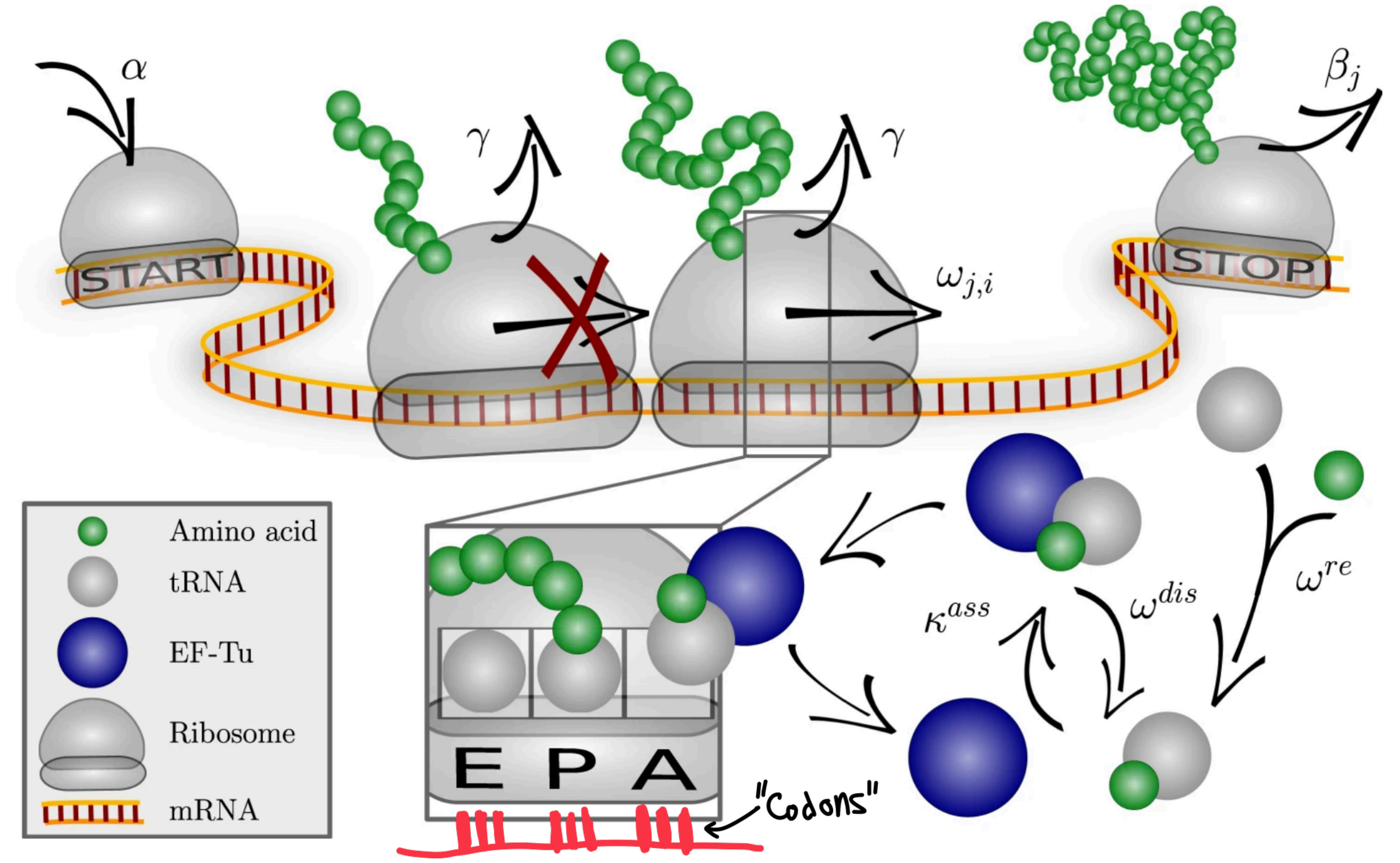
TASEP (Totally Asymmetric Simple Exclusion Process), with discrete time, Bernoulli jumps, and sequential update



- mRNA tells us that speed b depends on the *particle’s location* (and particles randomly change type)

- Mathematically, this is ***challenging***

- More tractable **(integrable)** if the speeds are the same, or may depend on the *particles* (leads to **symmetric** dependence on particle’s speeds)



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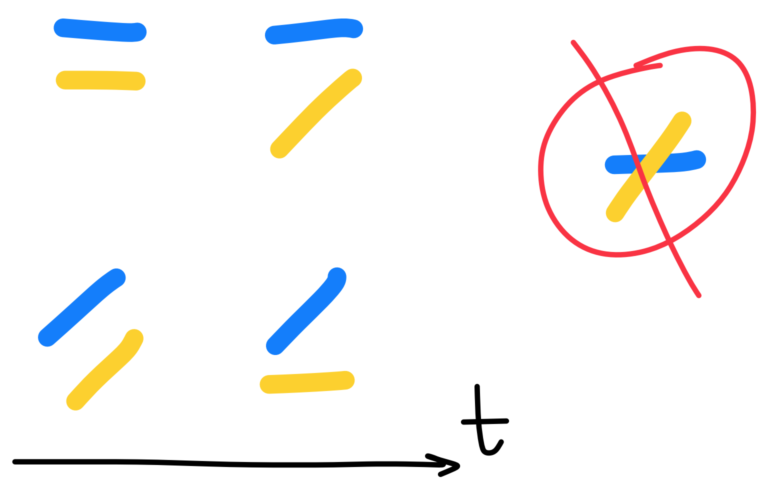
Two cars (two-particle discrete time TASEP)

Time $t \in \mathbb{Z}_{\geq 0}$, 2 cars with speeds $a_1 > a_2 > 0$, probabilities of jumps $b_i = \dfrac{a_i}{1 + a_i}$



One-lane highway: no passing. Consider two systems: FS and SF

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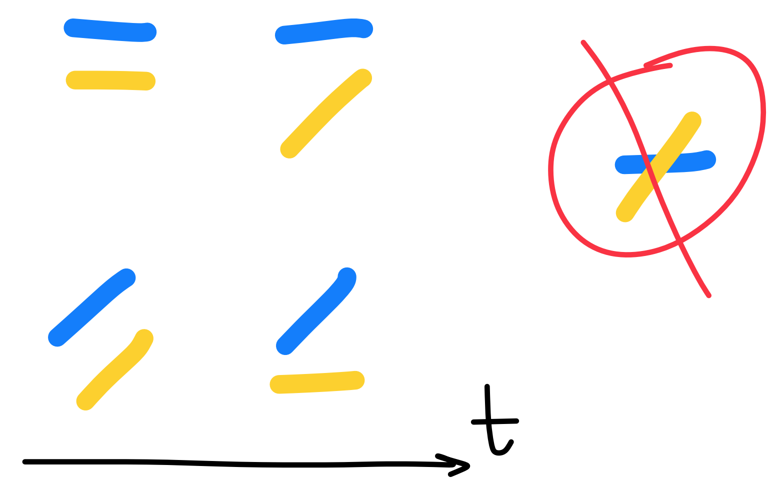

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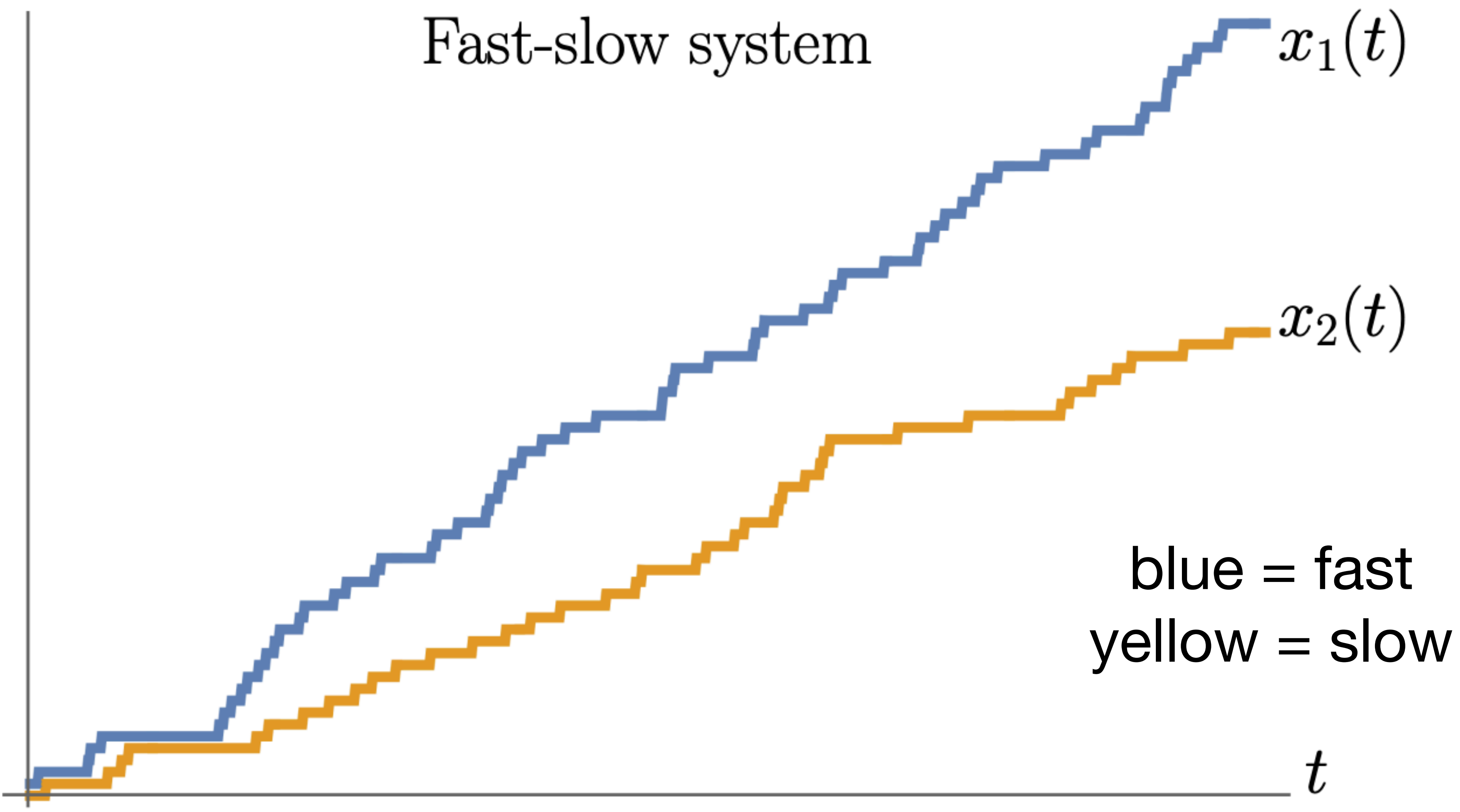
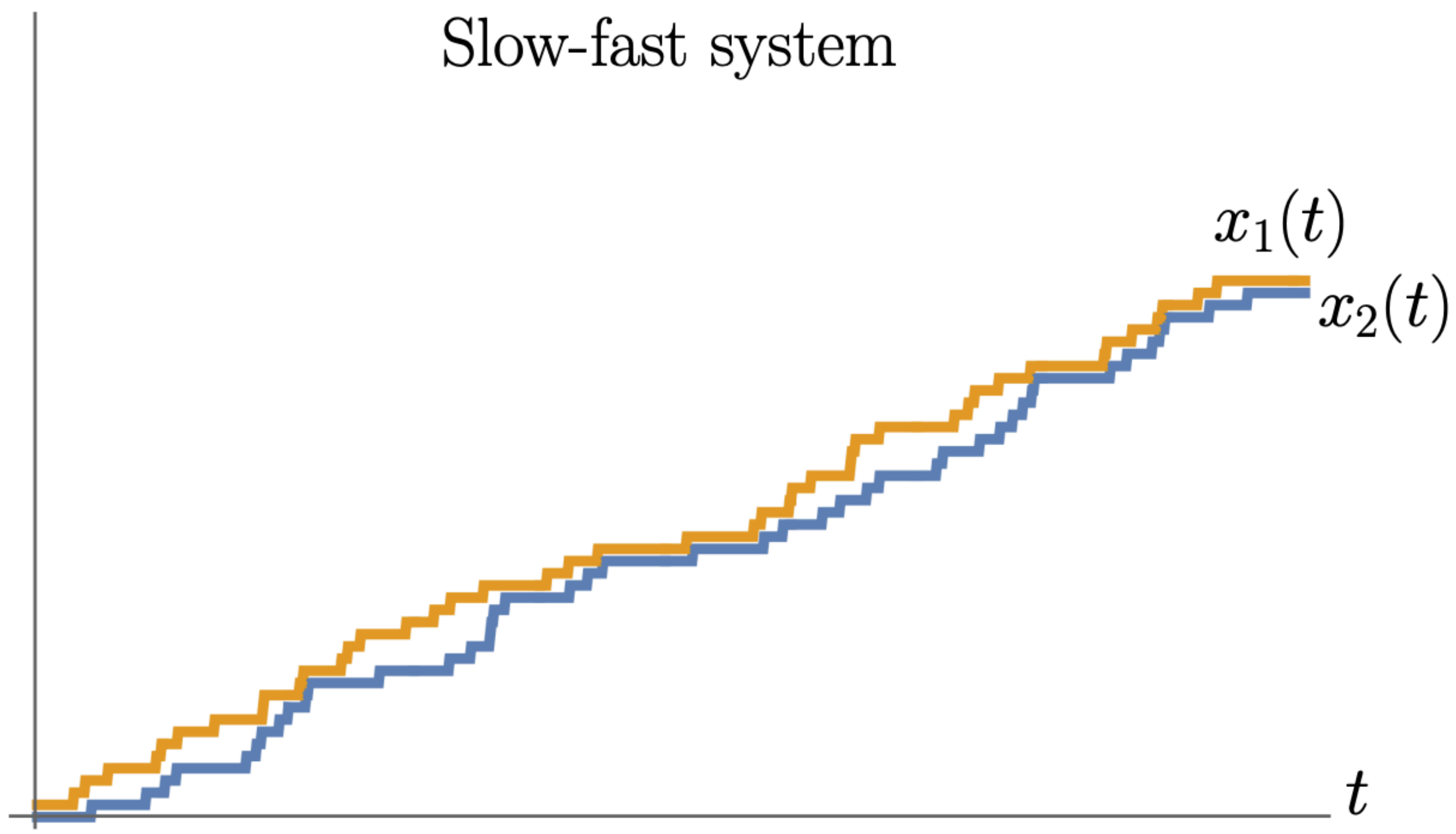


blue = fast
yellow = slow



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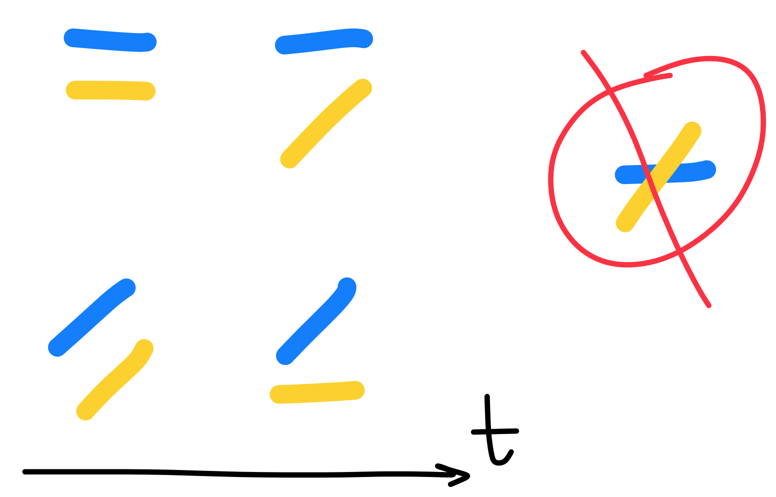

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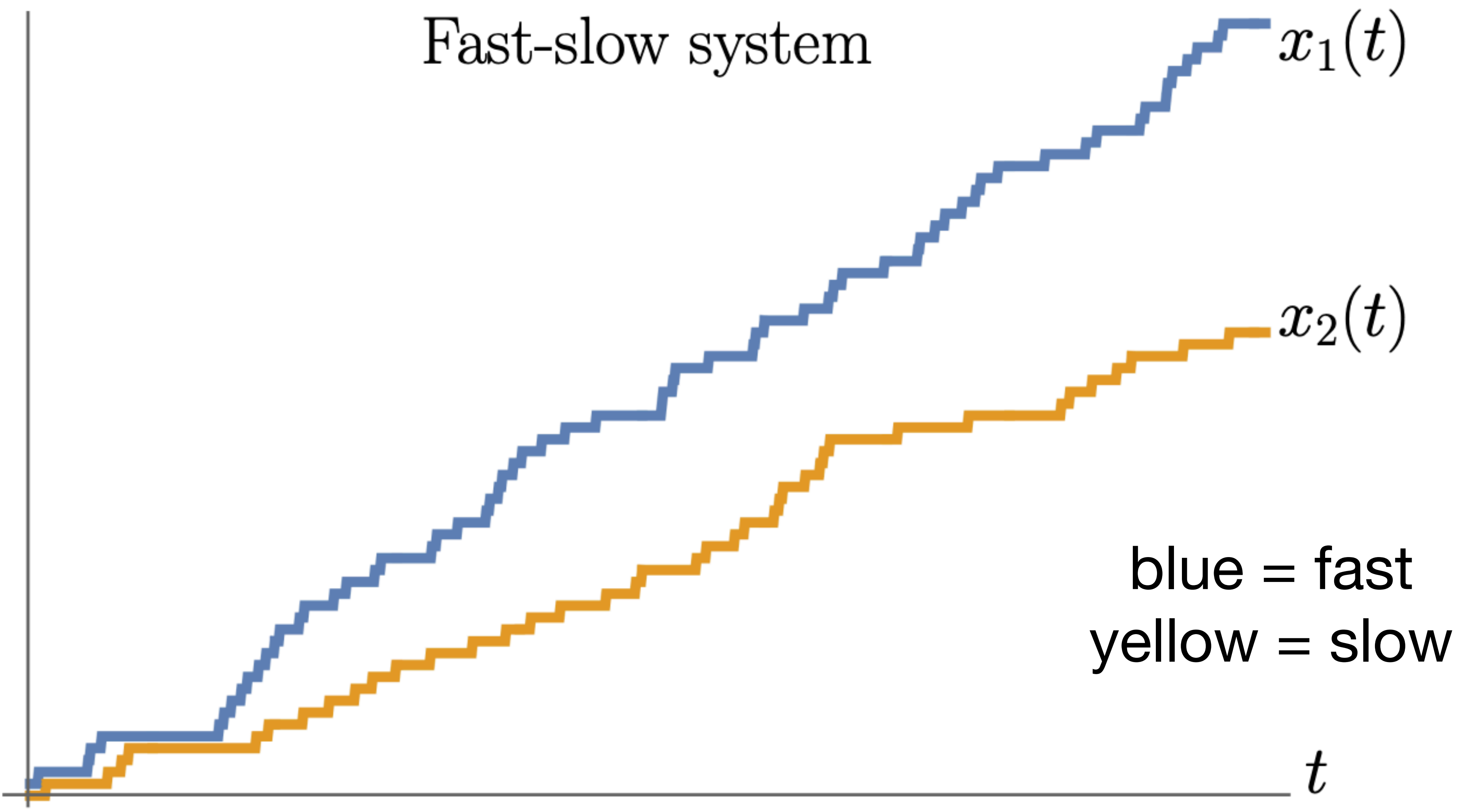

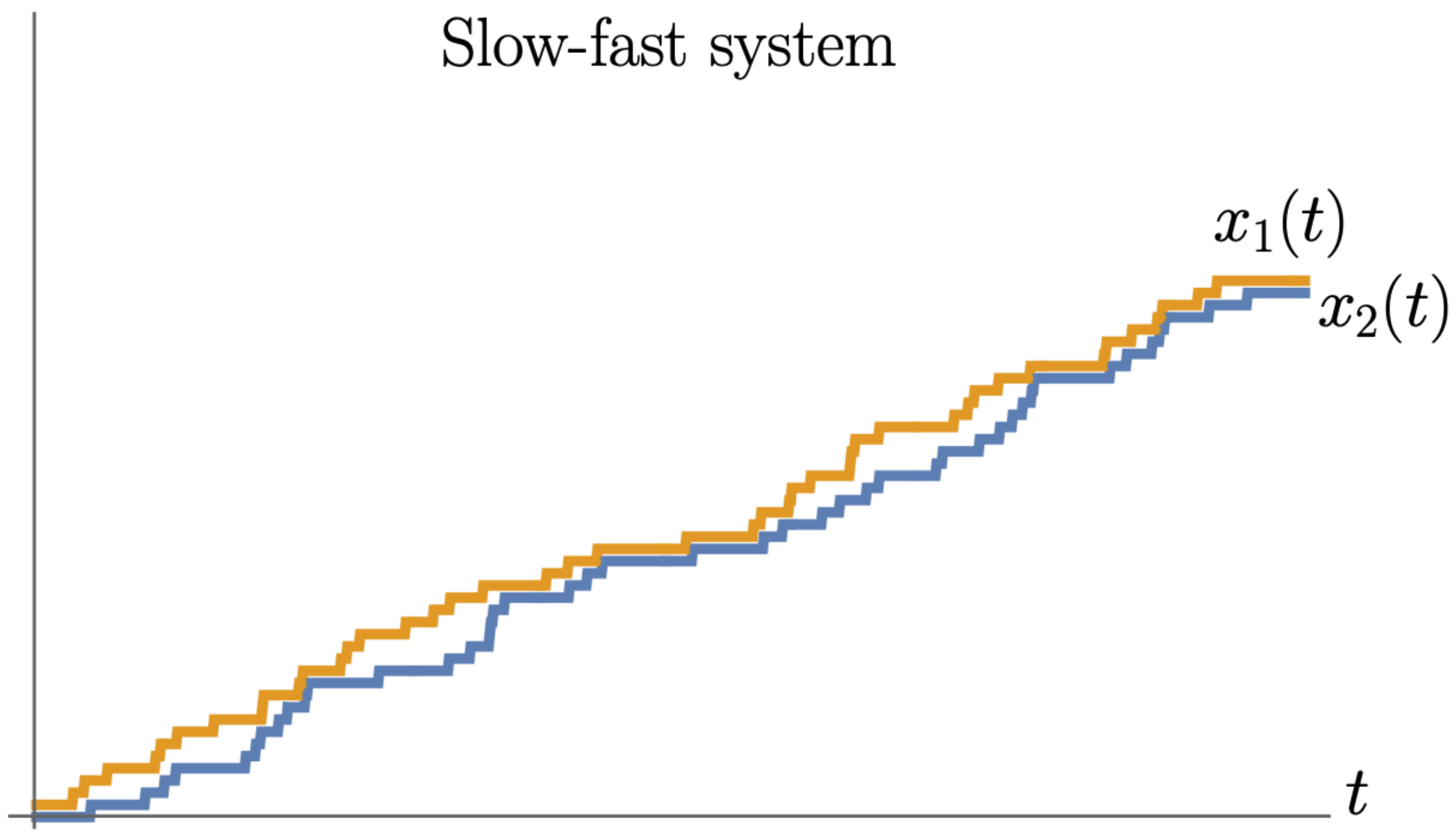

The long-time speed of the car ahead depends on which car is first; for the car behind it **does not** depend on the order

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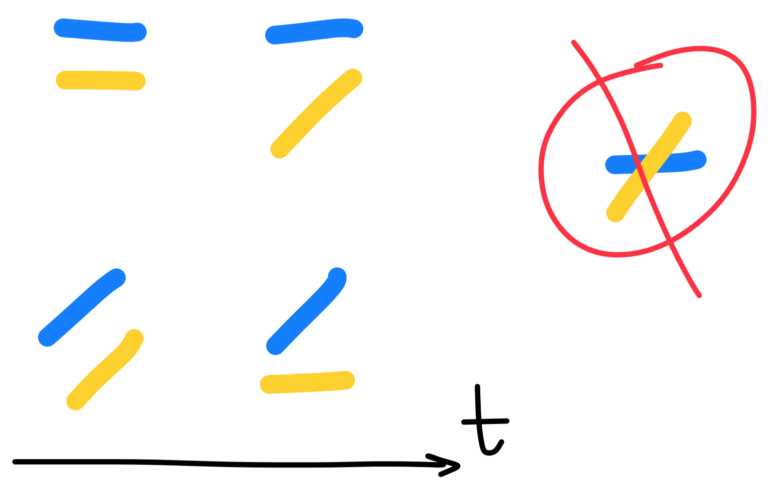

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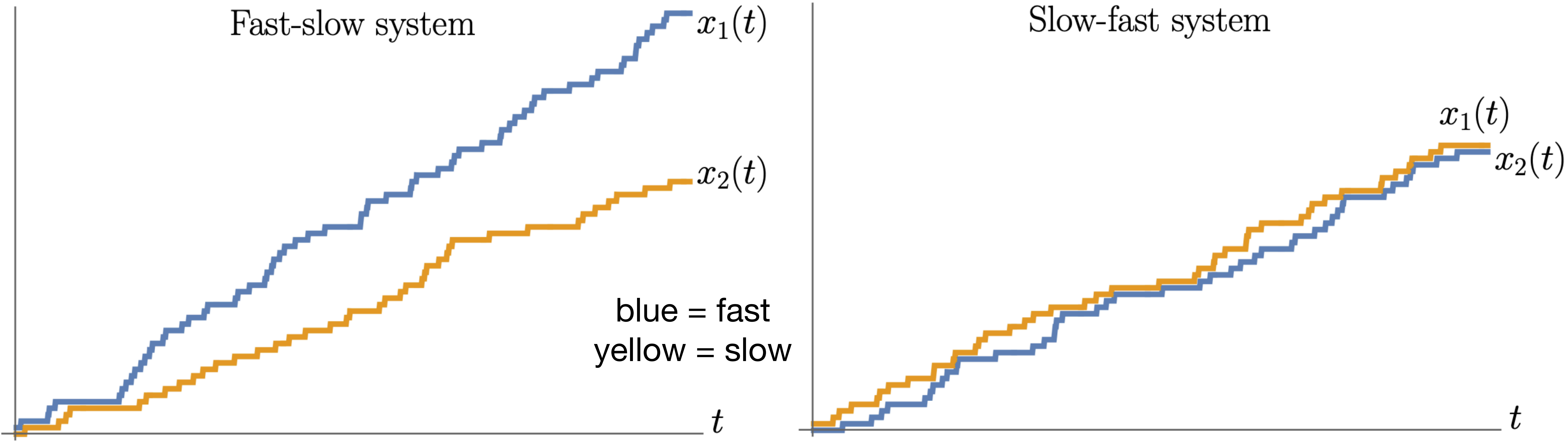
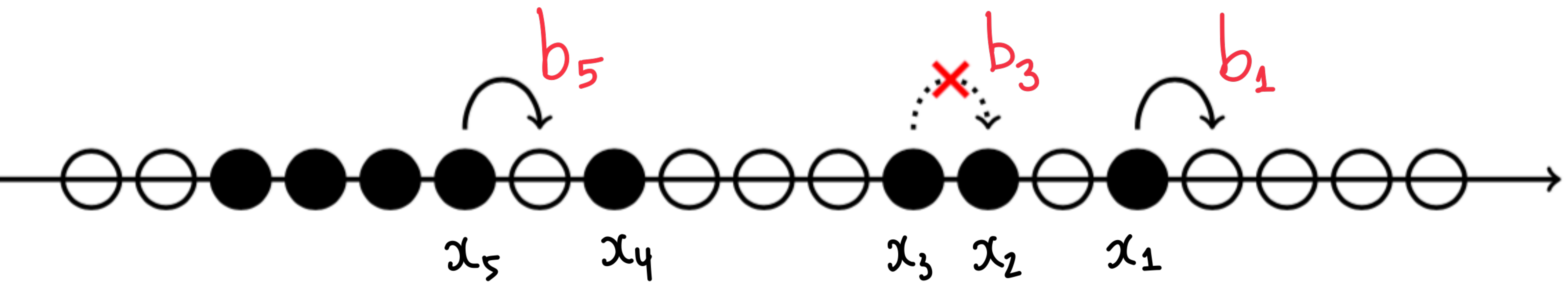


The long-time speed of the car ahead depends on which car is first; for the car behind it **does not** depend on the order

Fact 0 - symmetry of trajectory. (Vershik-Kerov ~1981; O'Connell 2003)

Cars start at 0,1 (*step initial configuration*)
$\Rightarrow$ the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Why symmetry is nice?



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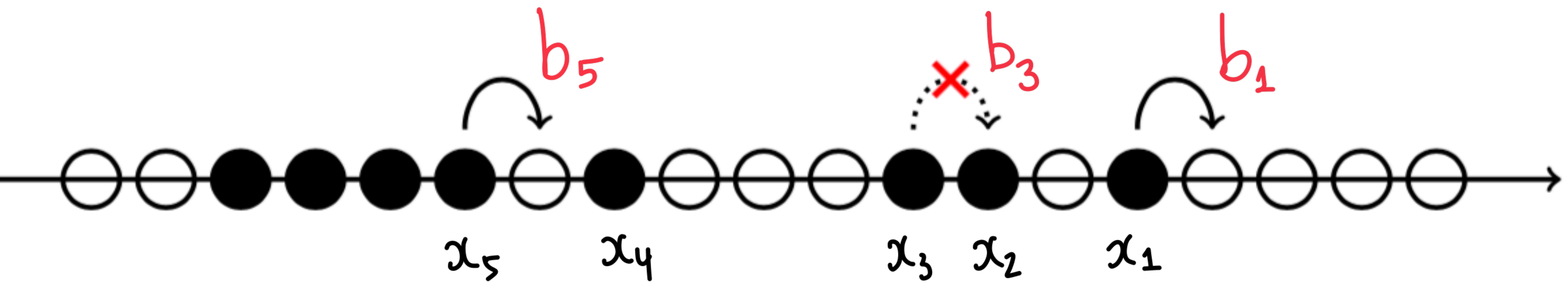

- Add parameters $b_i > 0$, where b_i is the jumping probability / rate of particle x_i

- If $x_i(0) = -i$ (**step**), then the trajectory of the particle $x_n(t)$ depends on parameters $b_1, b_2, \ldots, b_n$ **symmetrically**

- We can use **symmetric polynomials** to study this trajectory. Expanding probabilities in **symmetric function bases** yields exact formulas suitable for asymptotic analysis

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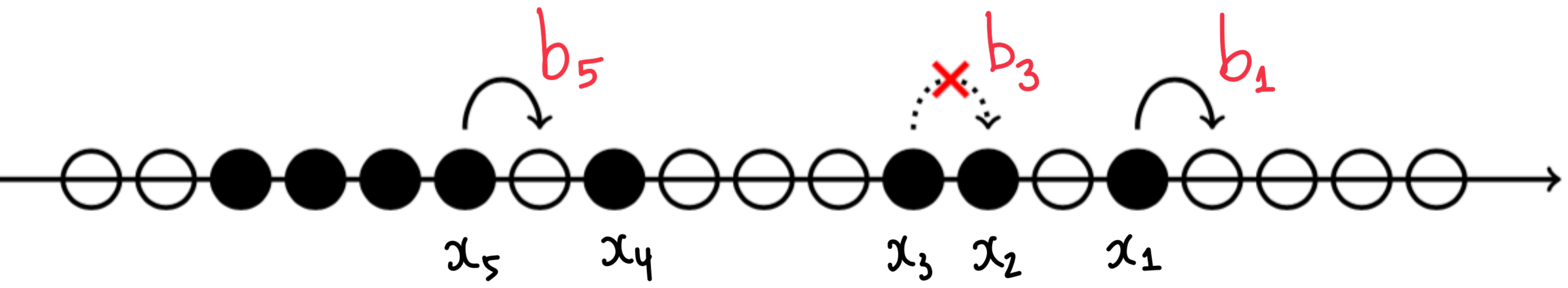

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- The linear space of **symmetric polynomials** in n variables has numerous interesting bases, like Schur polynomials $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$:

$$s_\lambda(u_1, \dots, u_n) = \frac{\det[u_i^{\lambda_j + n - j}]_{i,j=1}^n}{\prod_{i<j}(u_i - u_j)}$$

- Enumerative / algebraic combinatorics
- Representation theory / $\mathbb{C}$ (of S_n, GL_N)
- Geometry (Hilbert schemes, …)
- Random matrices, particle systems, statistical mechanics, …

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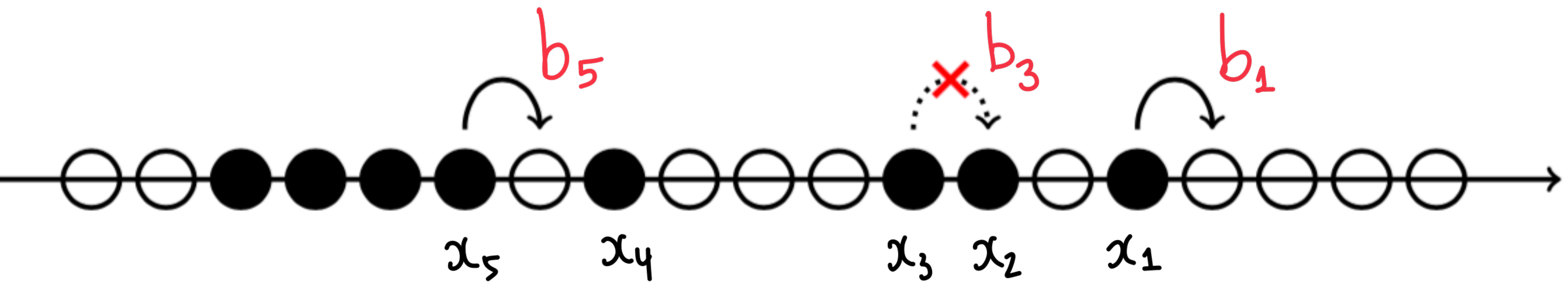

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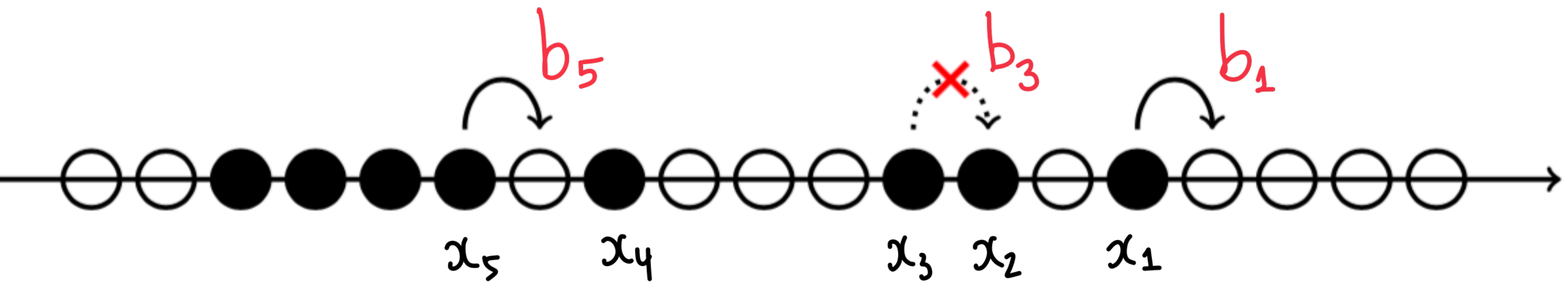

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- What if we swap $b_{n-1} \leftrightarrow b_n$?
- Then the trajectories of $x_n, x_{n-2}, \ldots, x_1$ do not change in distribution, but the **trajectory of x_{n-1} changes**
- We constructed a **coupling** between two particle systems that moves only x_{n-1} [P.-Saenz 2019, 2022]
- Also extended this to **arbitrary initial conditions** [P.-Saenz 2022]
- Next, I discuss the latter setup for two particles, and show how it follows from the **Yang-Baxter equation**

Randomized initial conditions

Fact 0. Cars start at 0,1 (*step initial configuration*) $\Rightarrow$ the distribution of the trajectory of the car behind is **independent** of the order of the speeds

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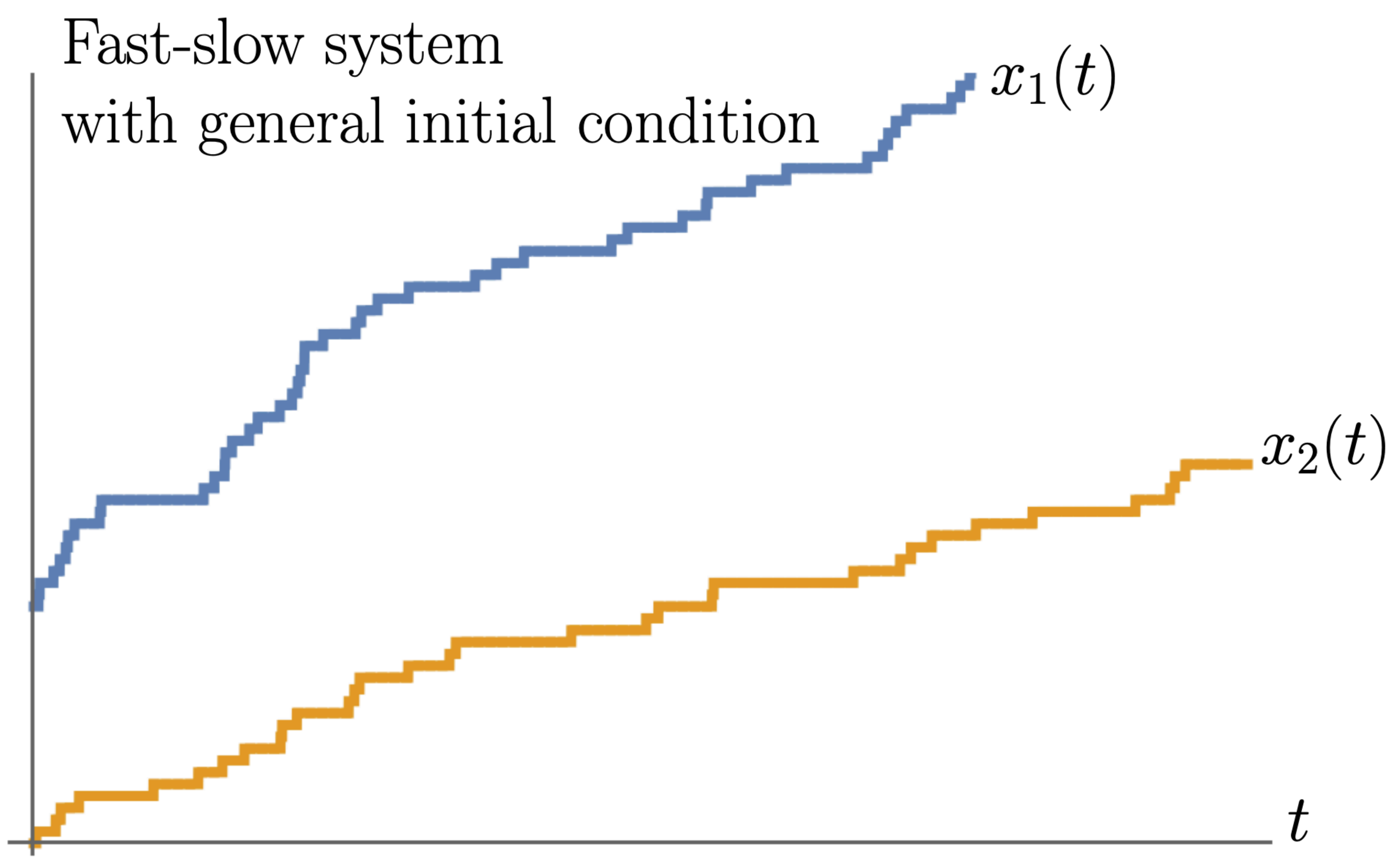

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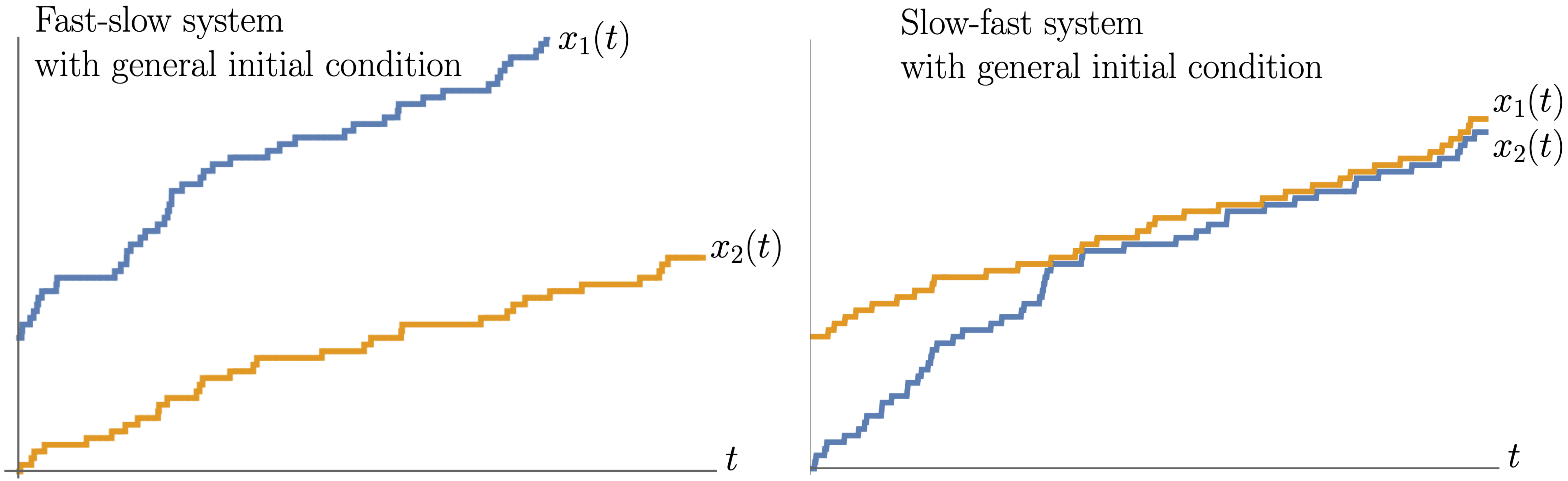

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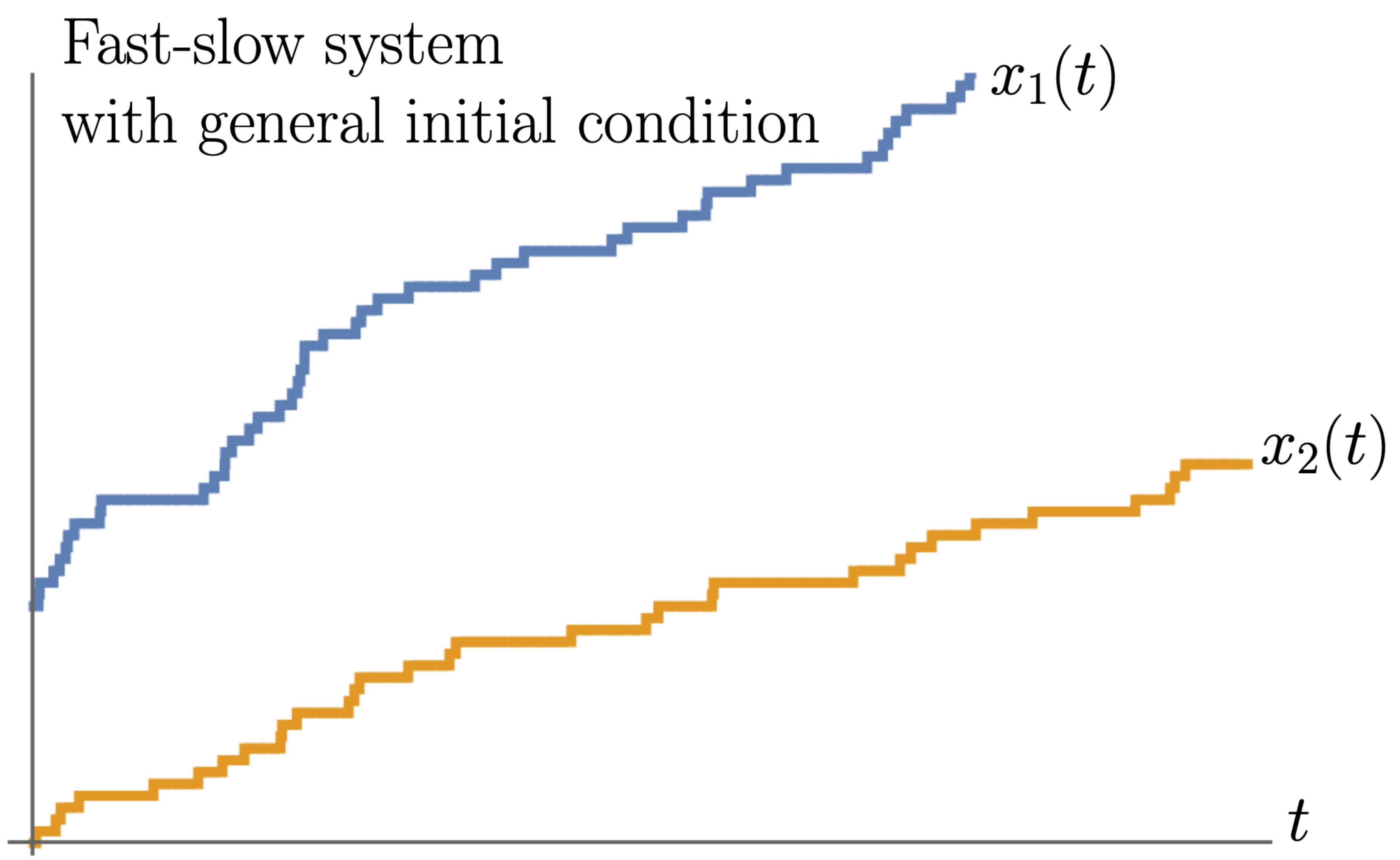

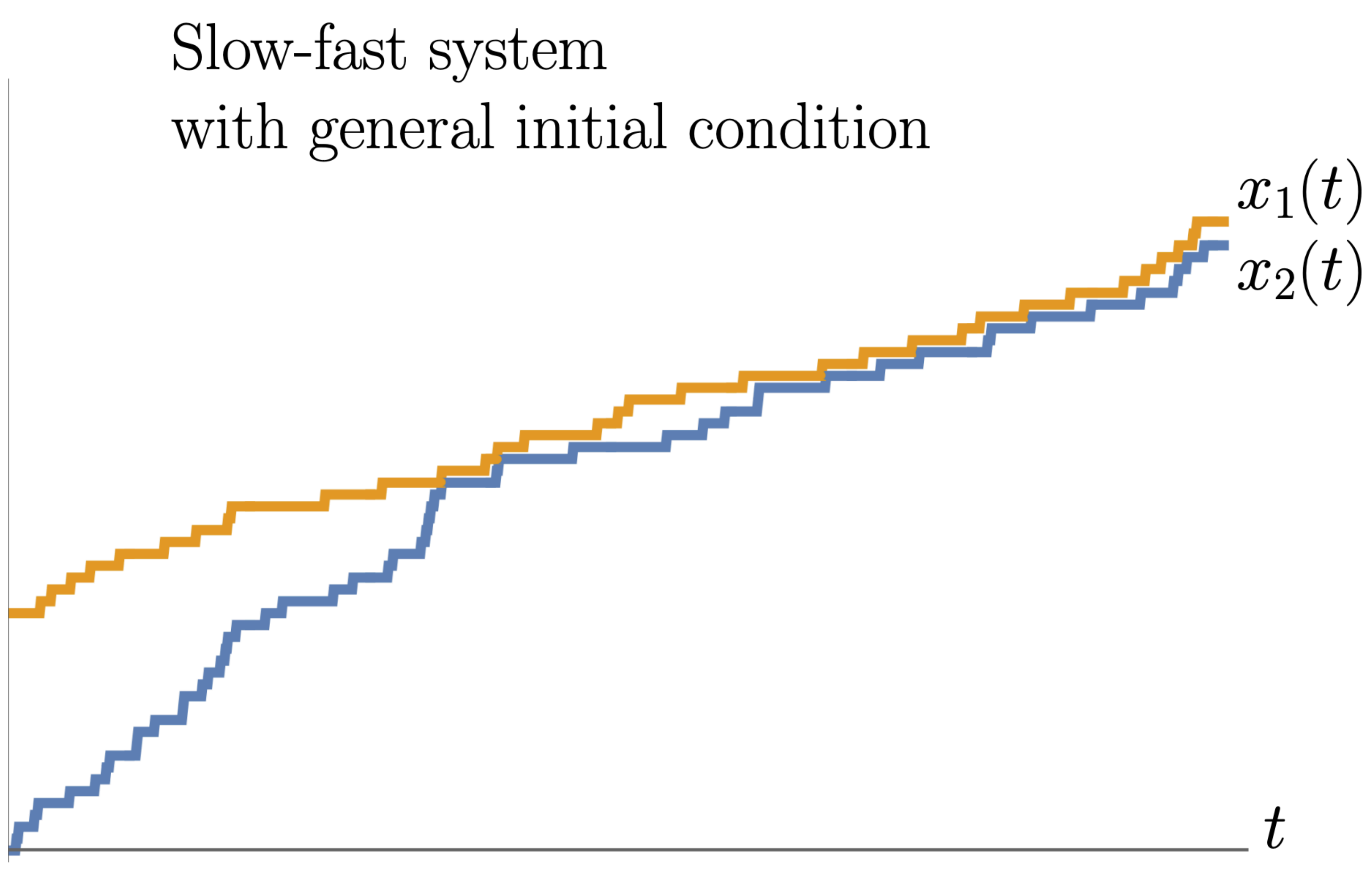

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Let $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$. Start SF from $(y_1(0), x_2(0))$.

Then the trajectories of the second particle become **the same in distribution**.

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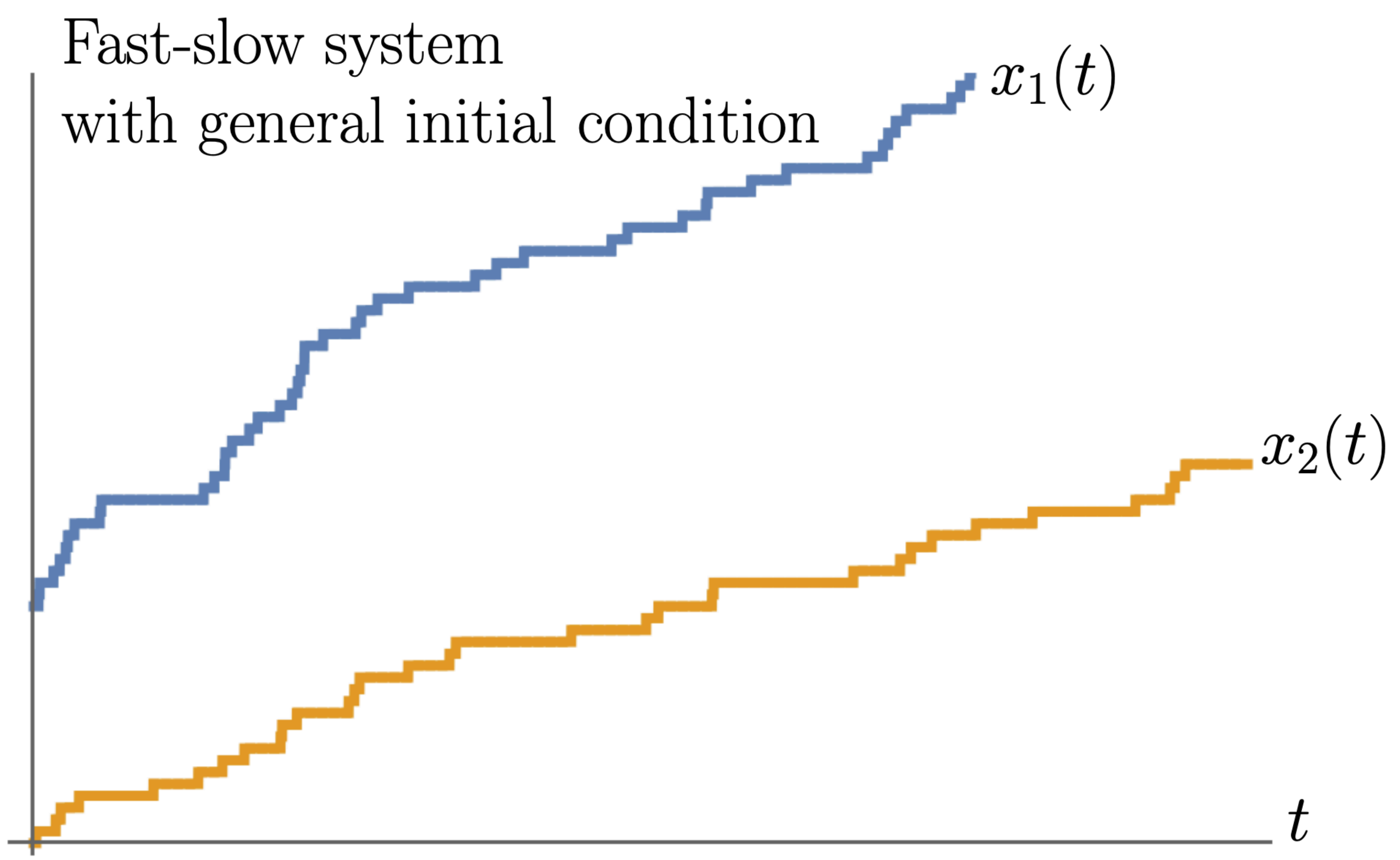

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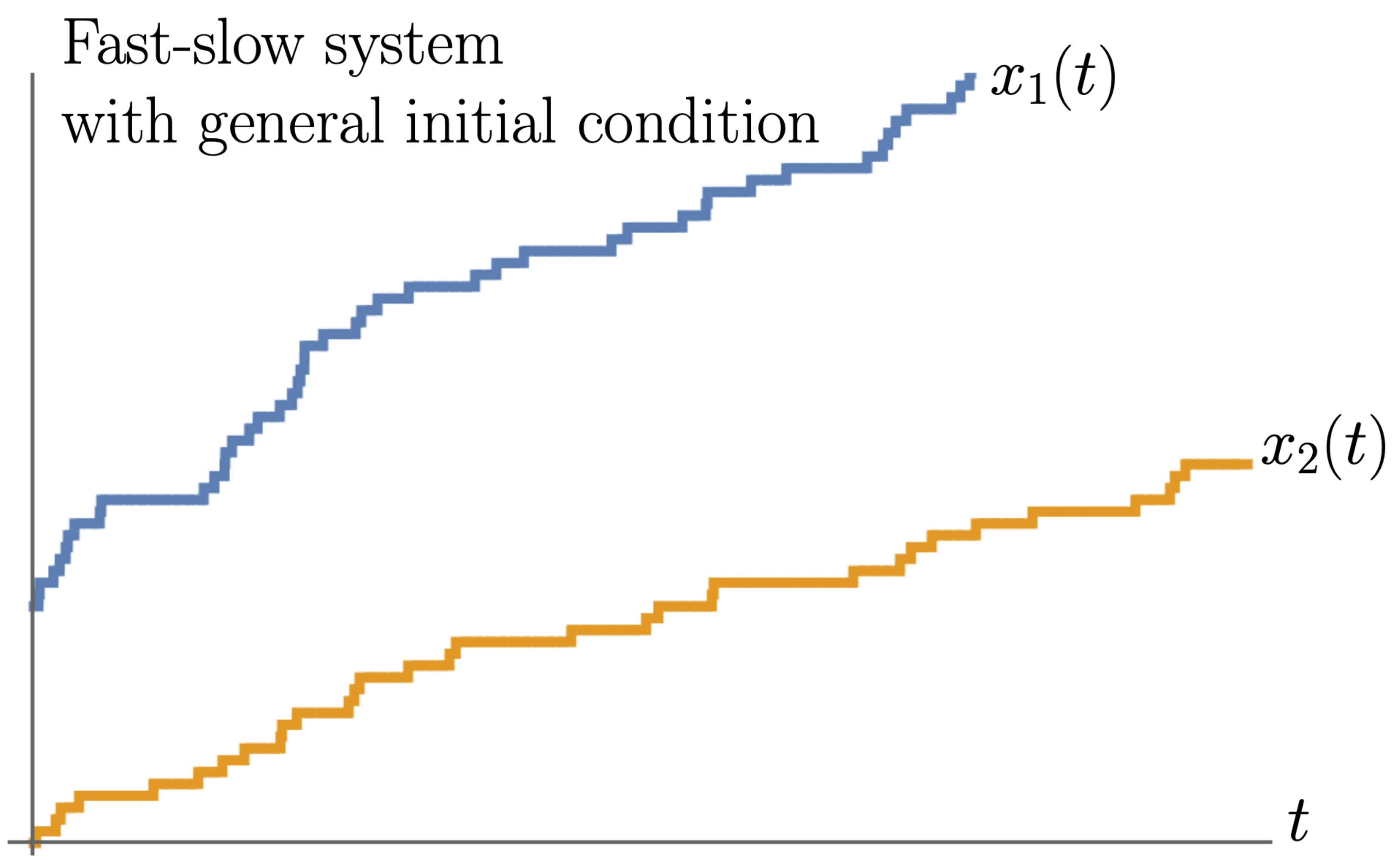

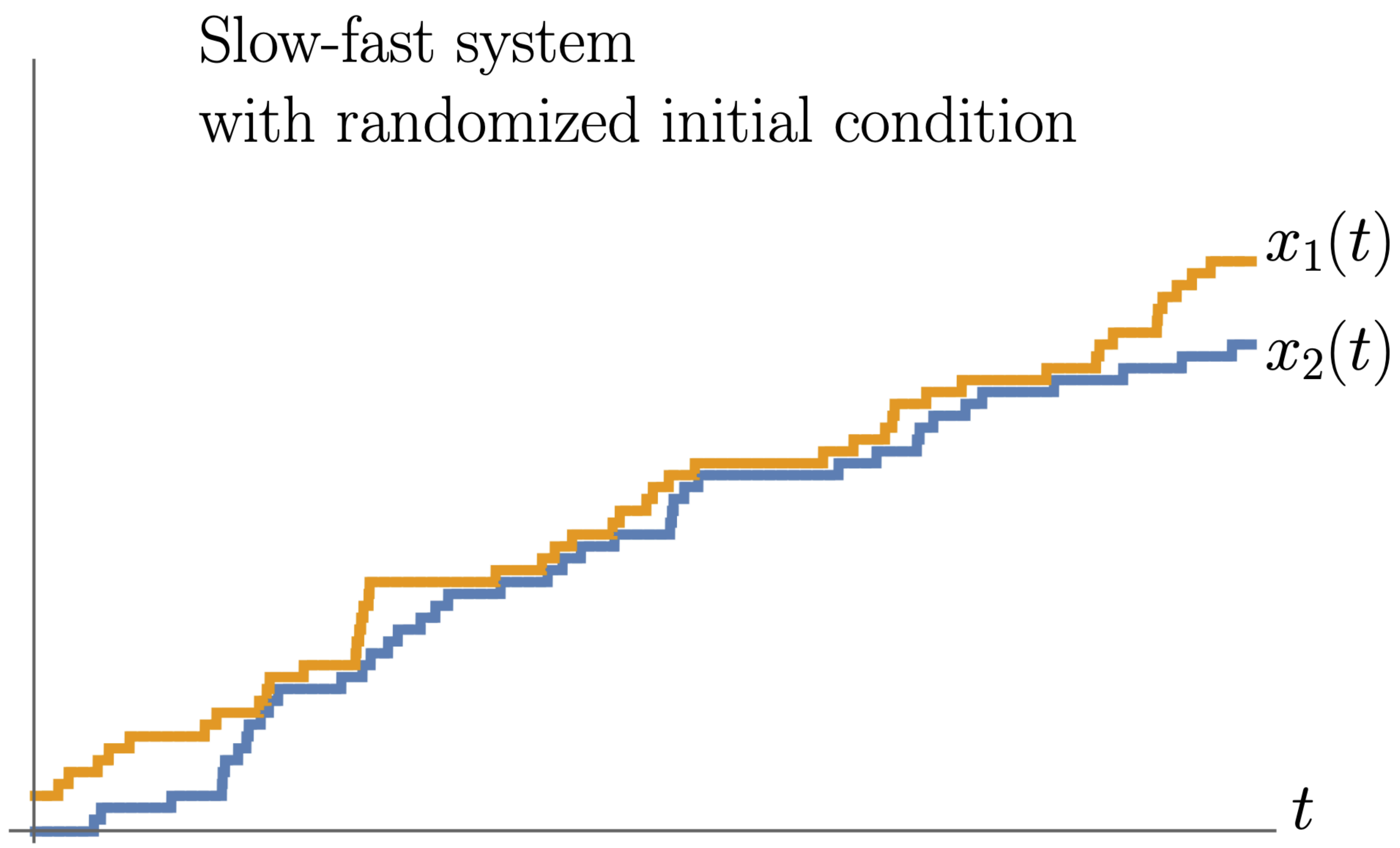

Theorem 1 [P.-Saenz 2022]. "*Be wise - randomize*". Recall $a_1 > a_2 > 0$, $b_i = a_i/(a_i+1)$. Let $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$. Start SF from $(y_1(0), x_2(0))$.

Then the trajectories of the second particle become **the same in distribution**.

Randomized initial conditions

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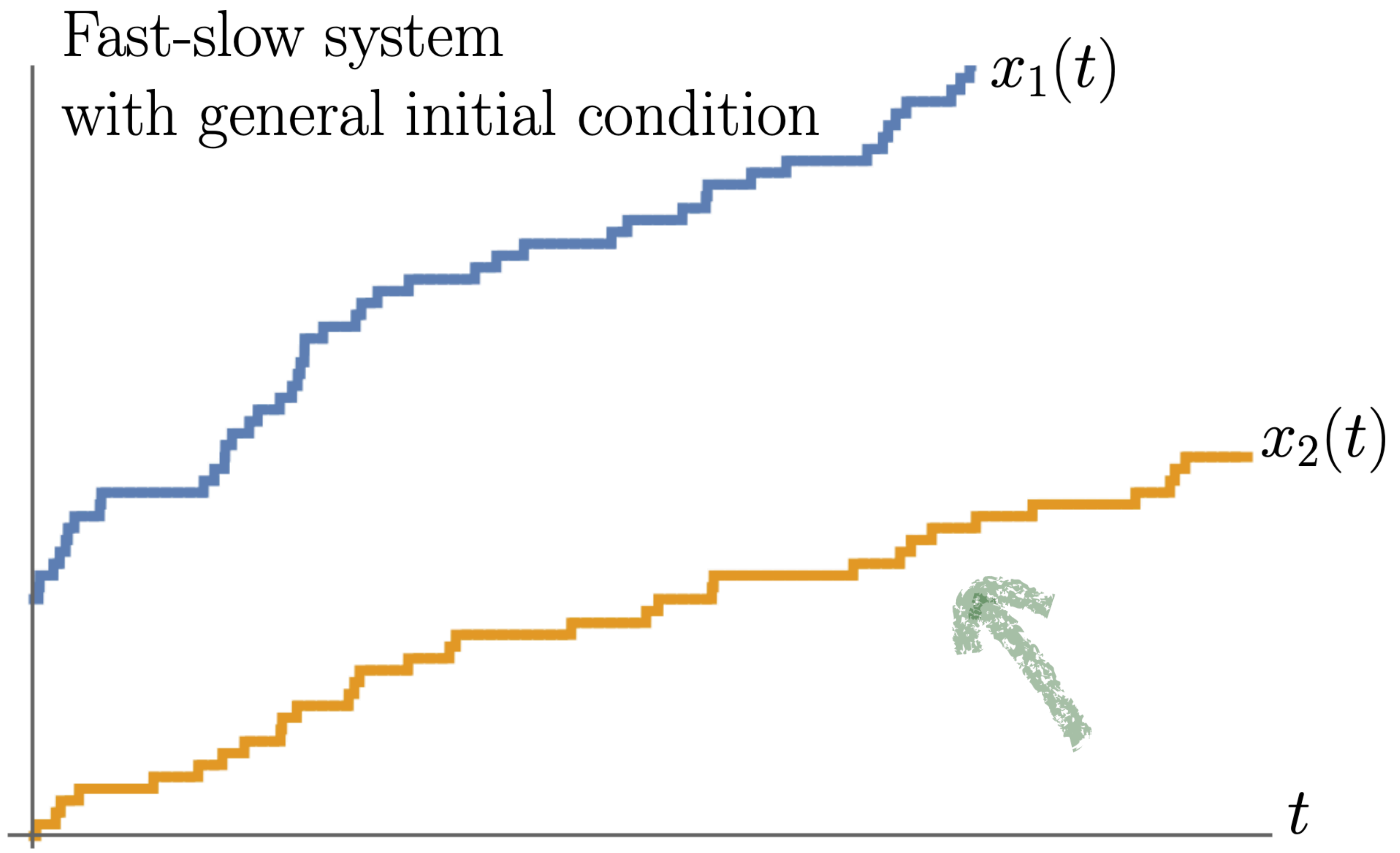

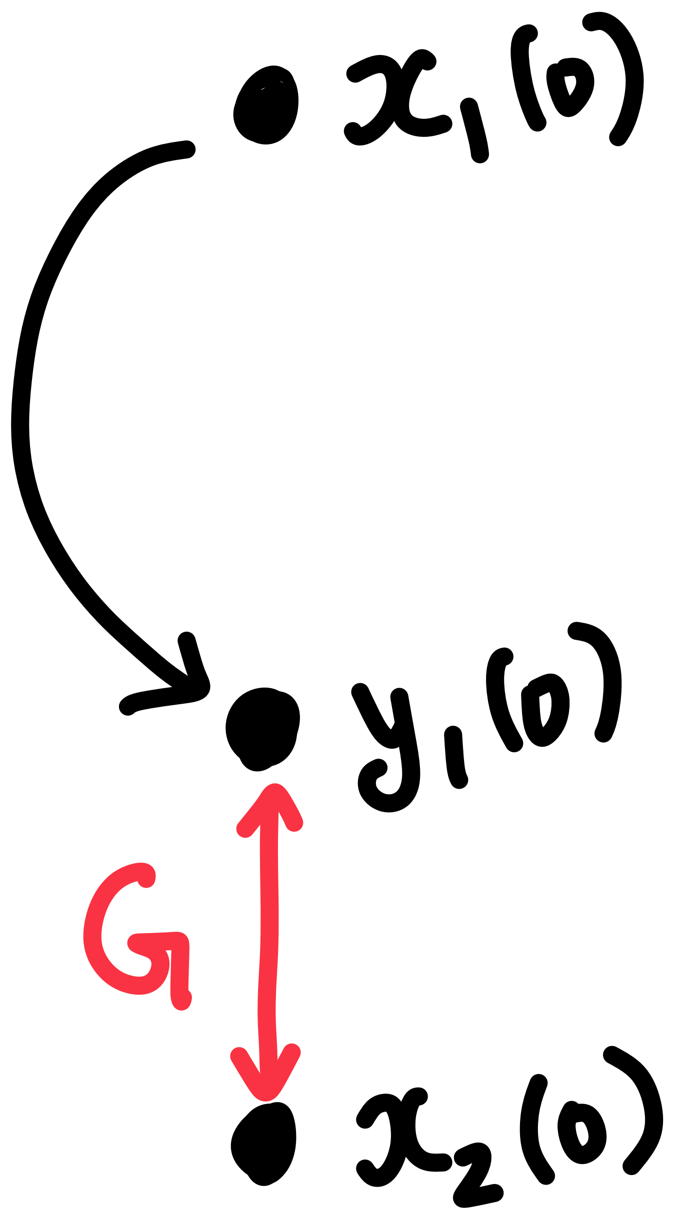

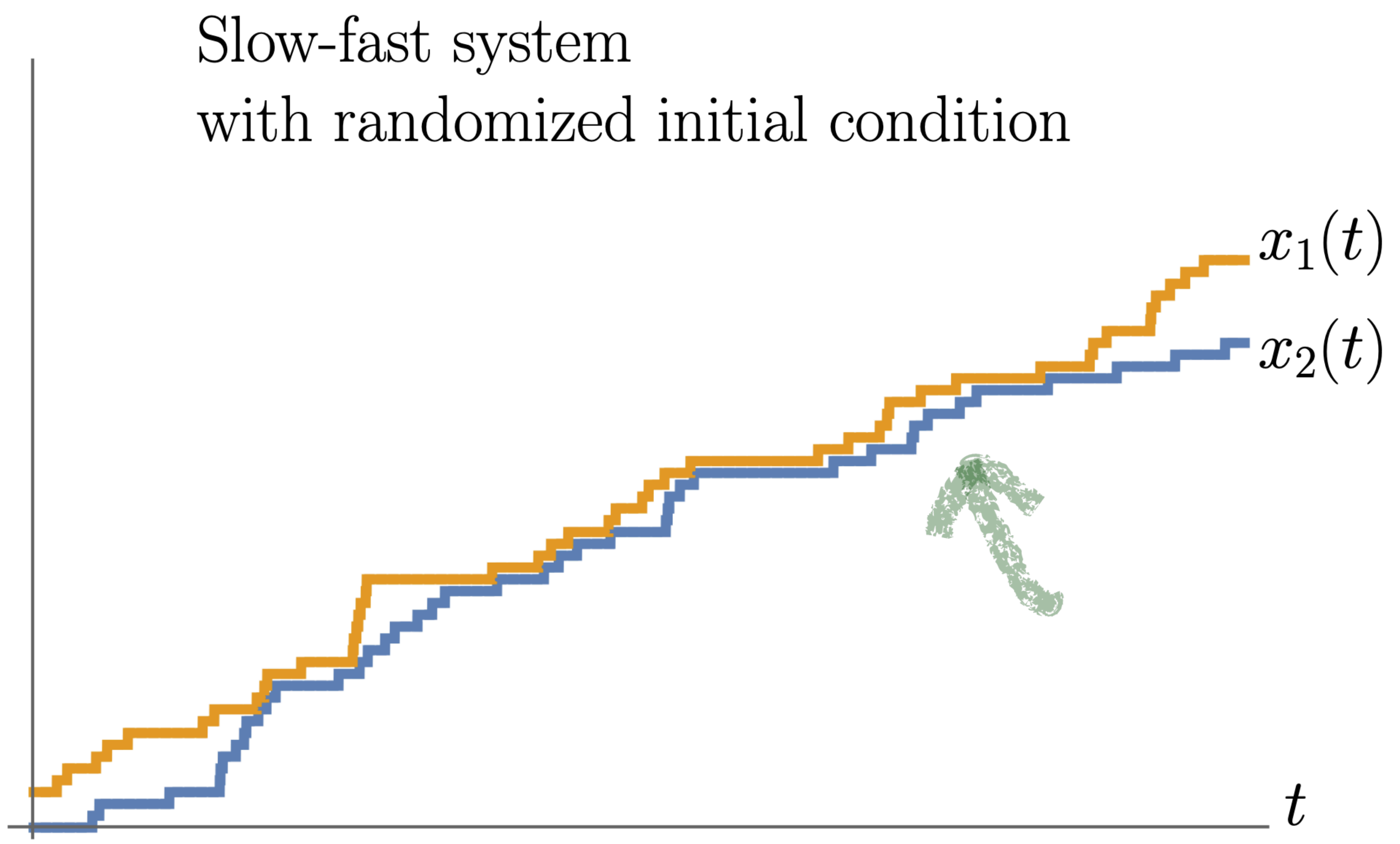

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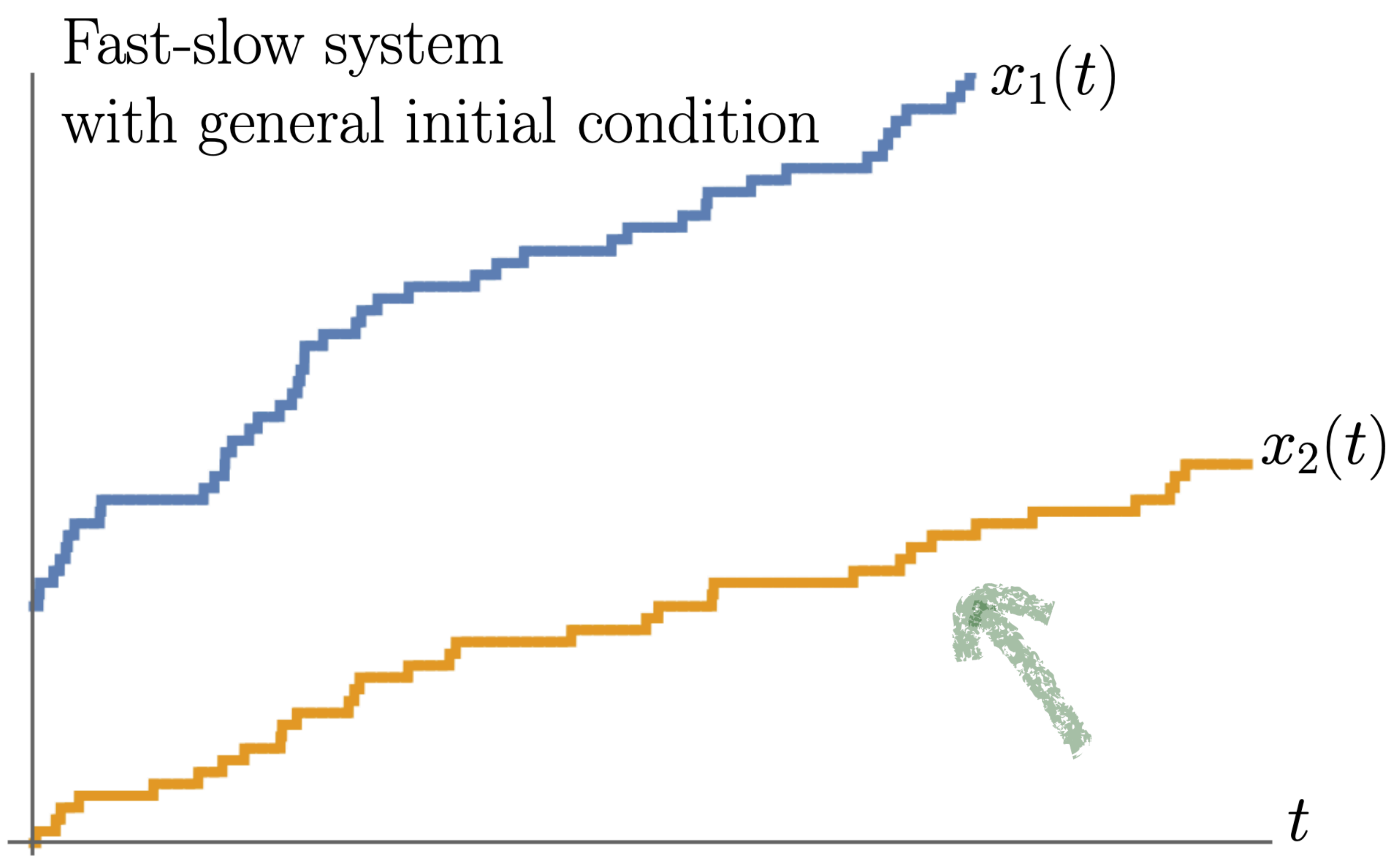

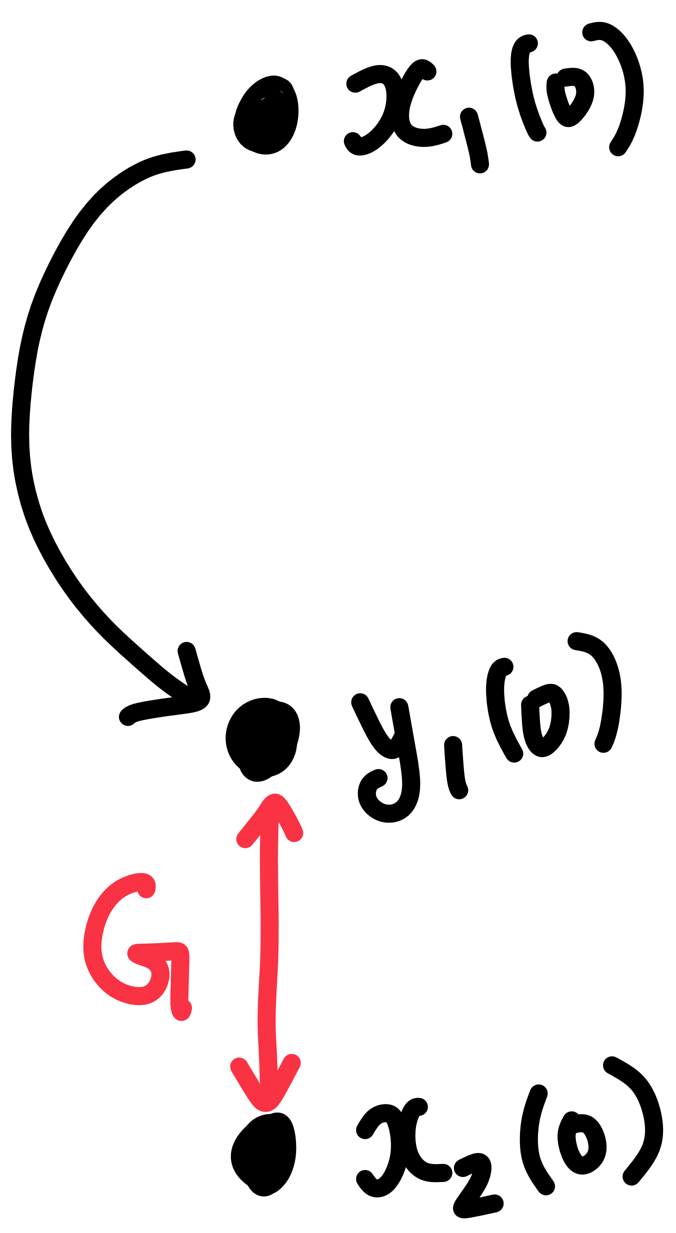

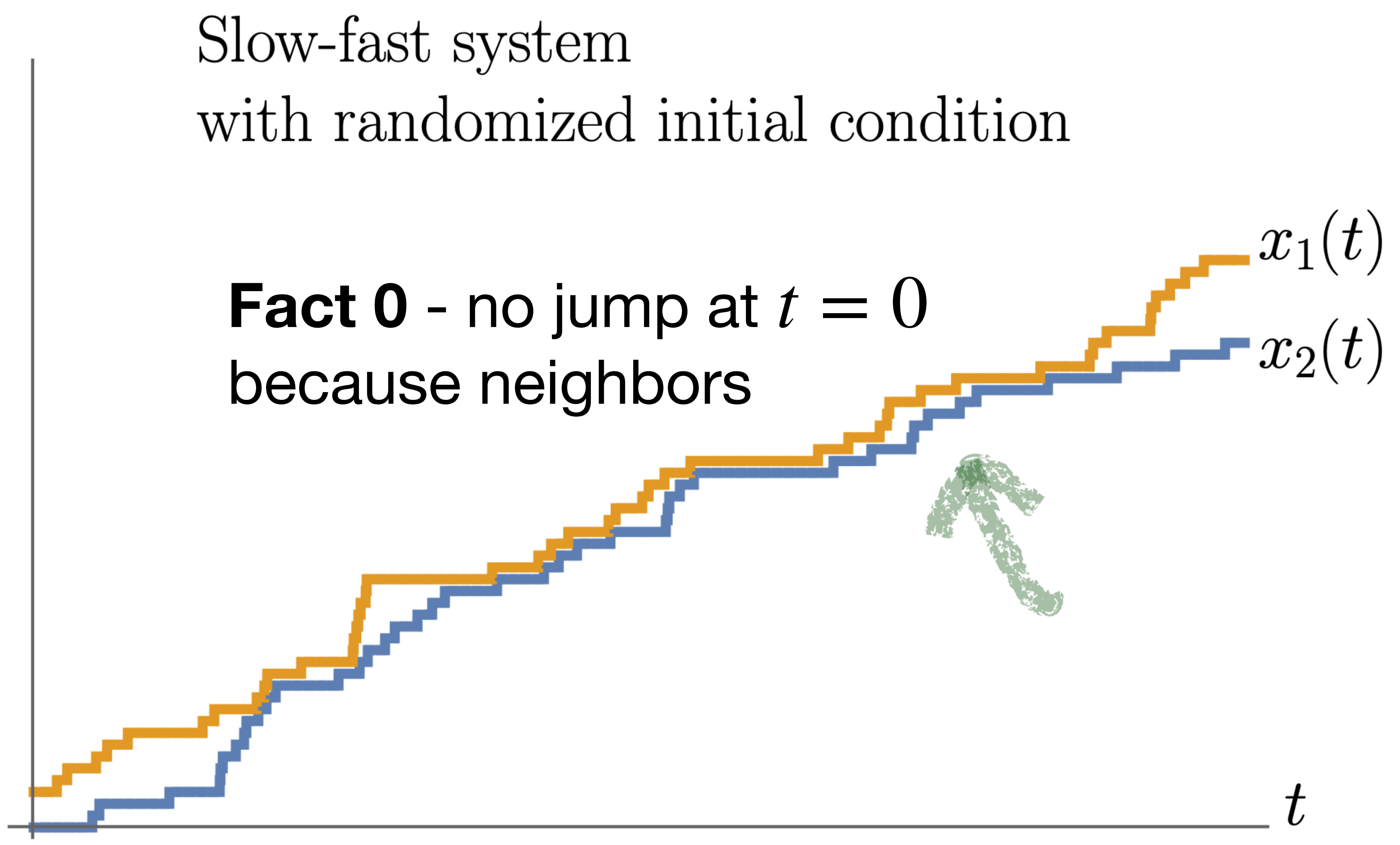

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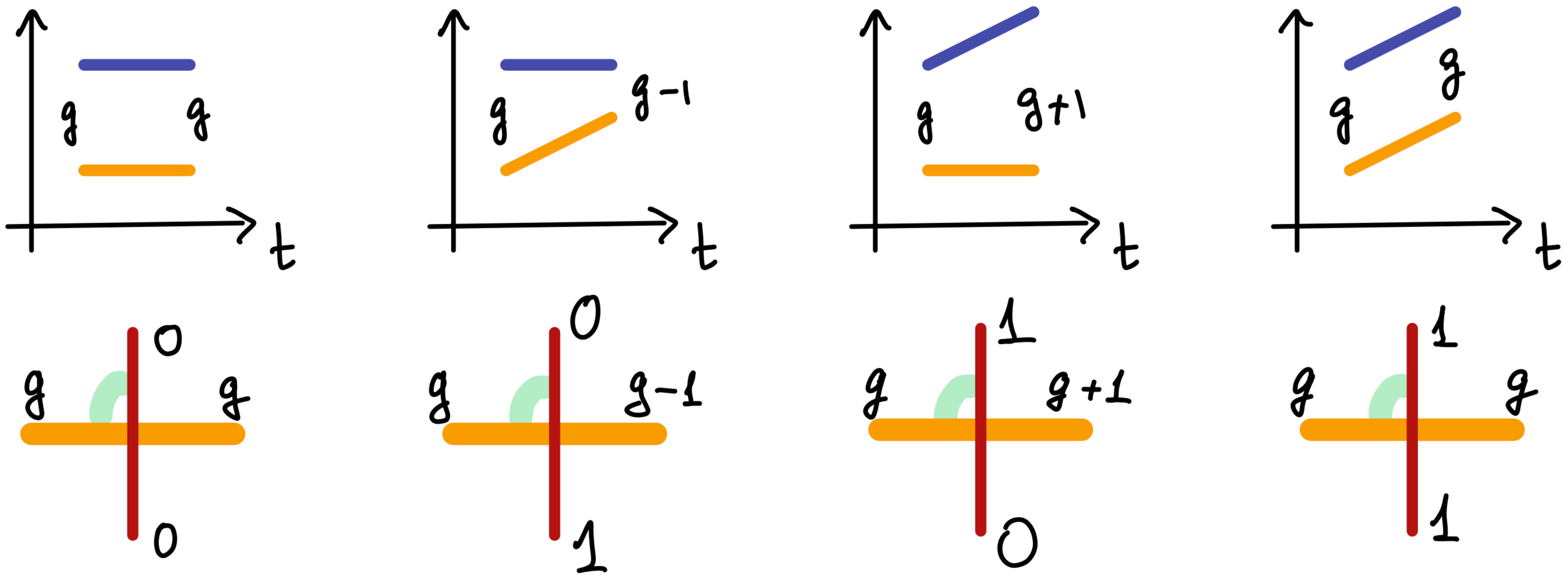
Proof via Yang-Baxter equation



$$\frac{1+a_2\mathbf{1}_{g=0}}{1+a_2} \qquad \frac{a_2\mathbf{1}_{g>0}}{1+a_2} \qquad \frac{1}{1+a_2} \qquad \frac{a_2}{1+a_2}$$

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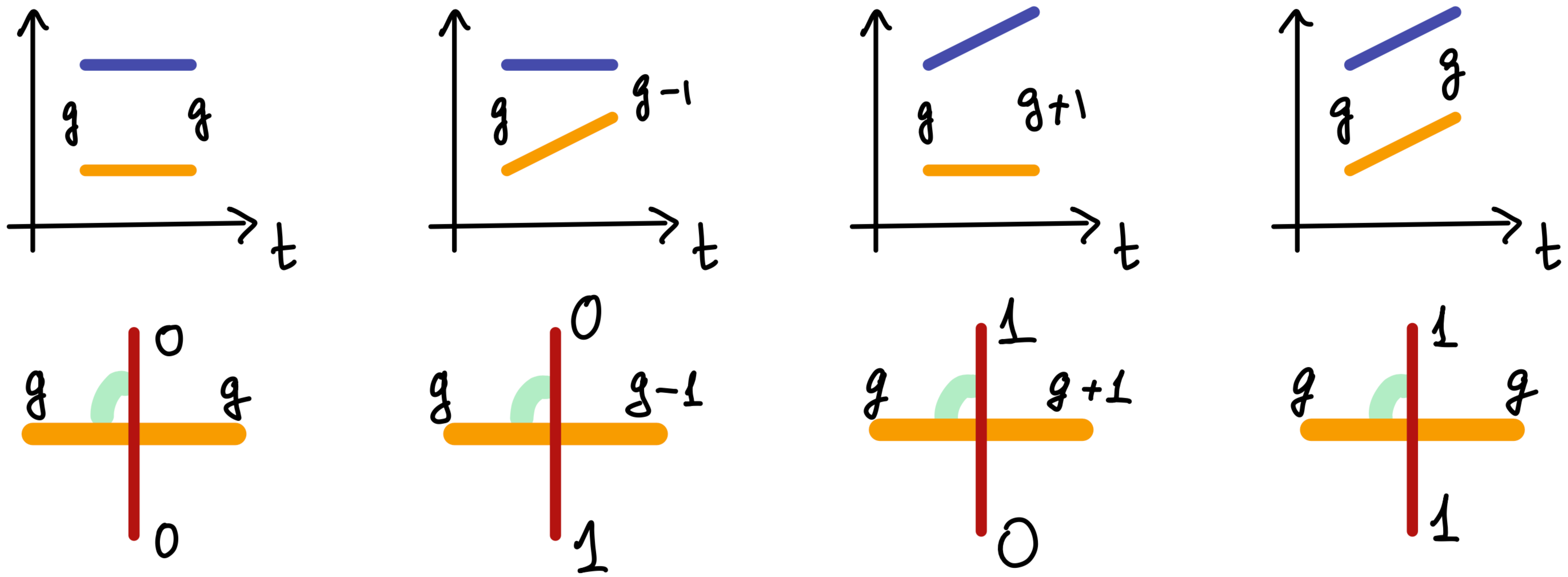

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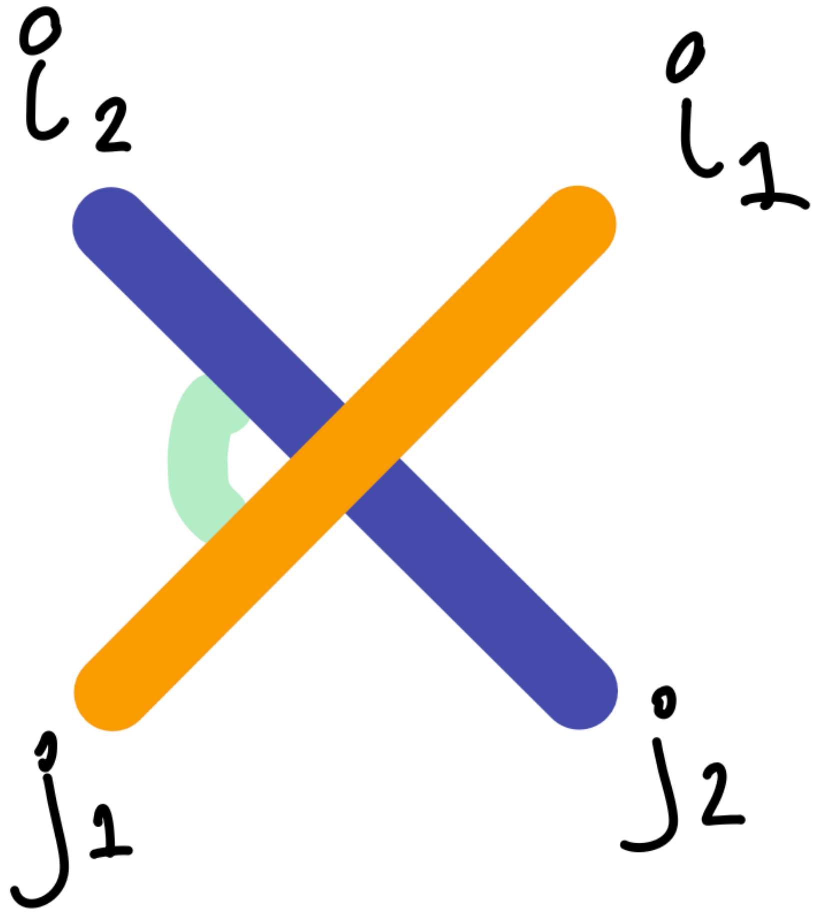
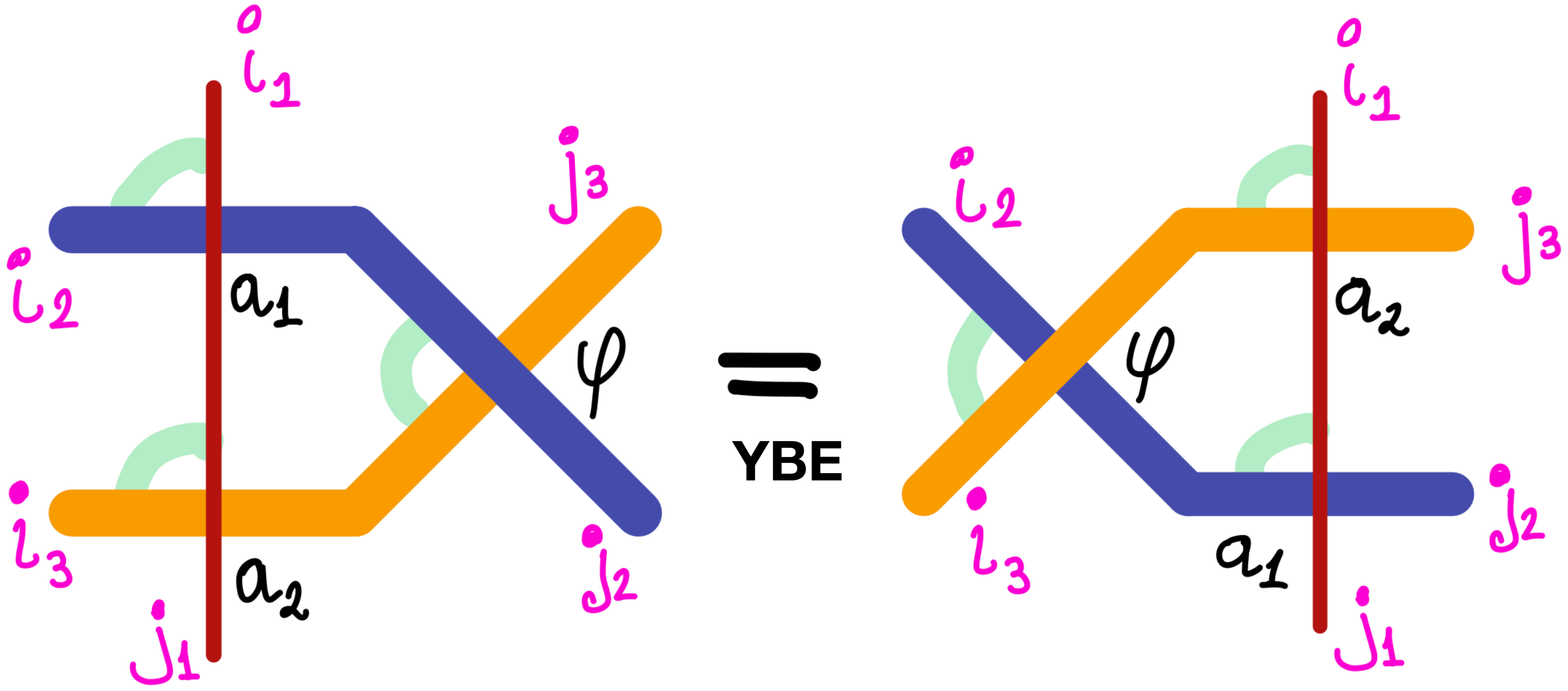


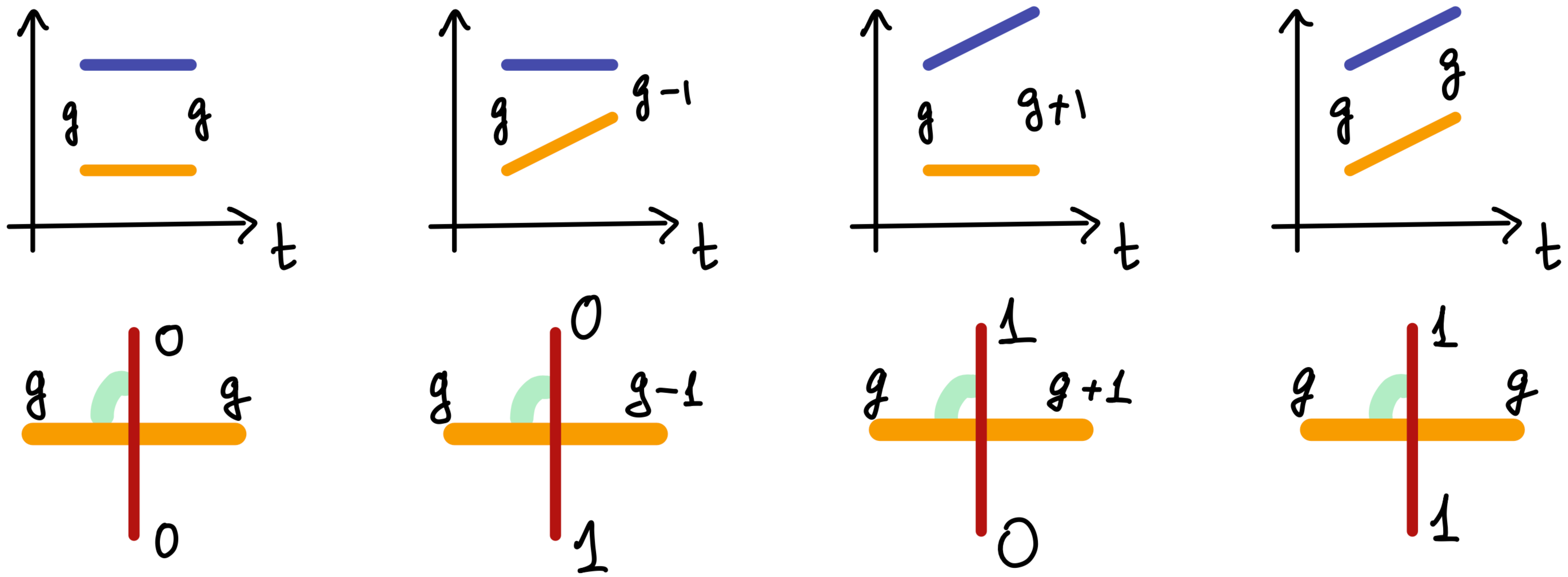
φ_{a_2/a_1} - geometric jumps

$$(a_2/a_1)^{j_2}(1-\mathbf{1}_{j_2<j_1}a_2/a_1)\,\mathbf{1}_{j_2\le j_1}$$



Proof via Yang-Baxter equation

- B_{a_2/a_1} - geometric jump operator on configurations of two particles
- T_{a_1,a_2}, T_{a_2,a_1} - FS and SF transition operators



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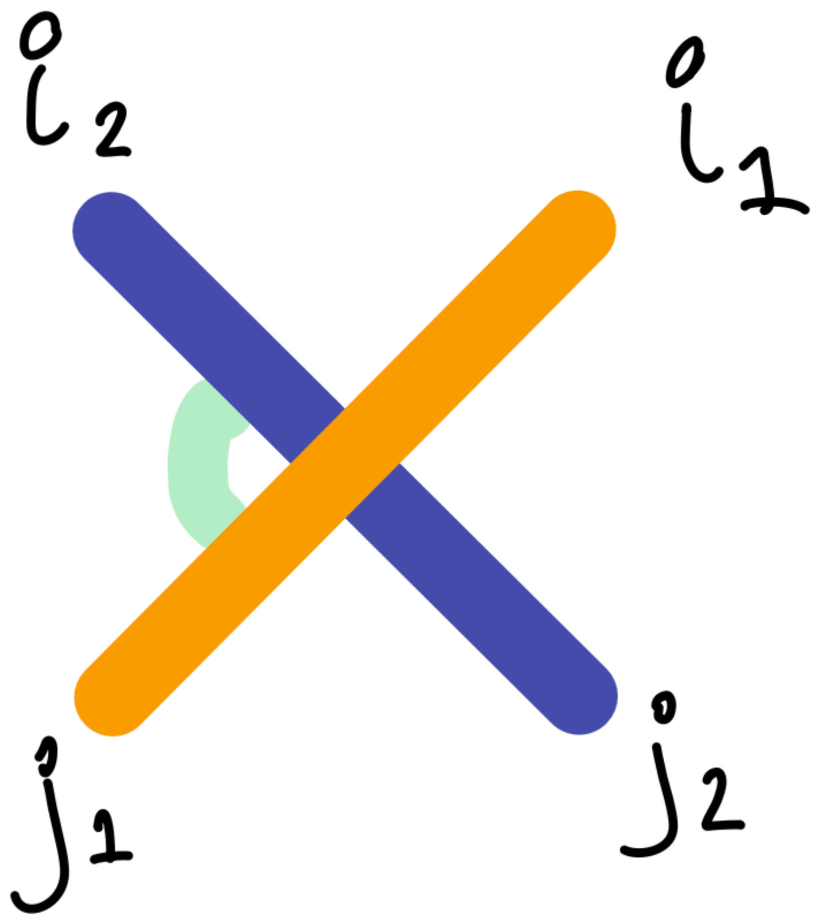

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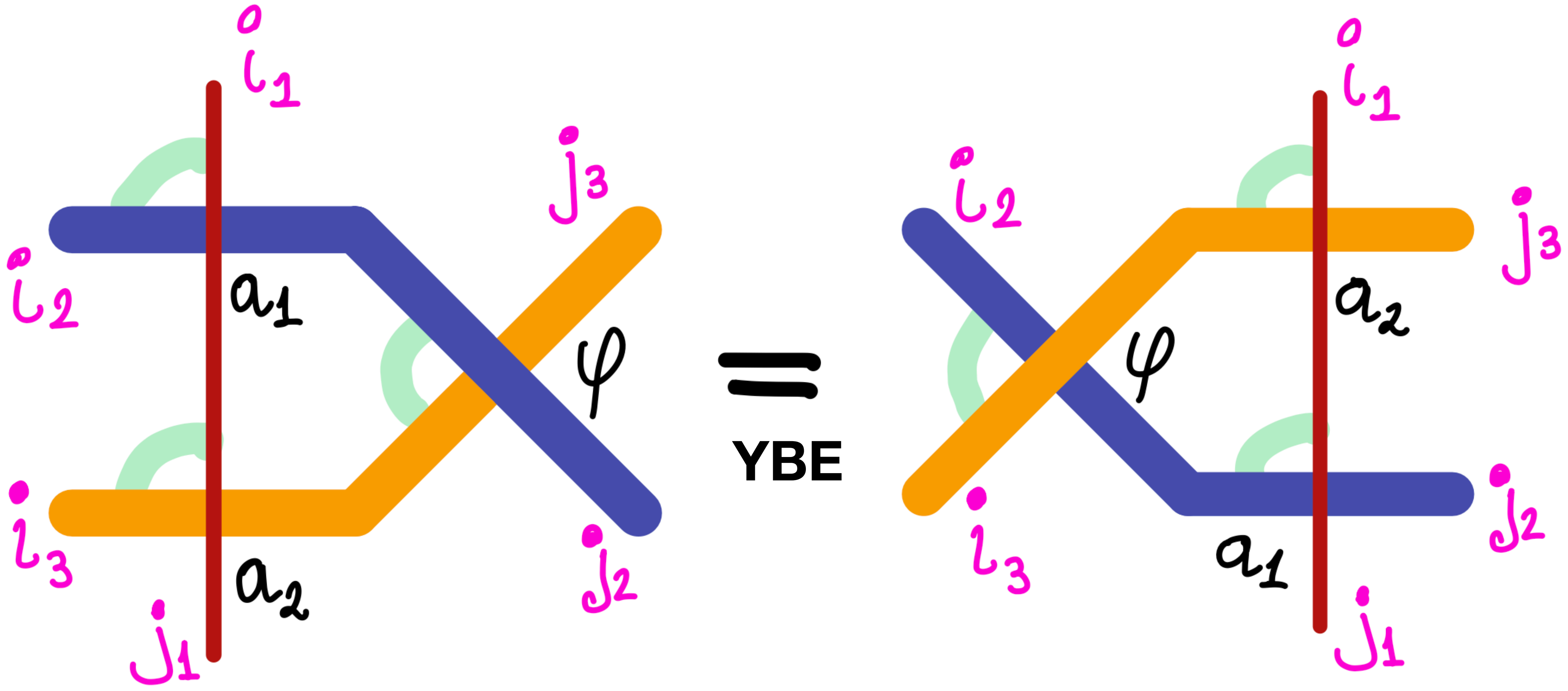

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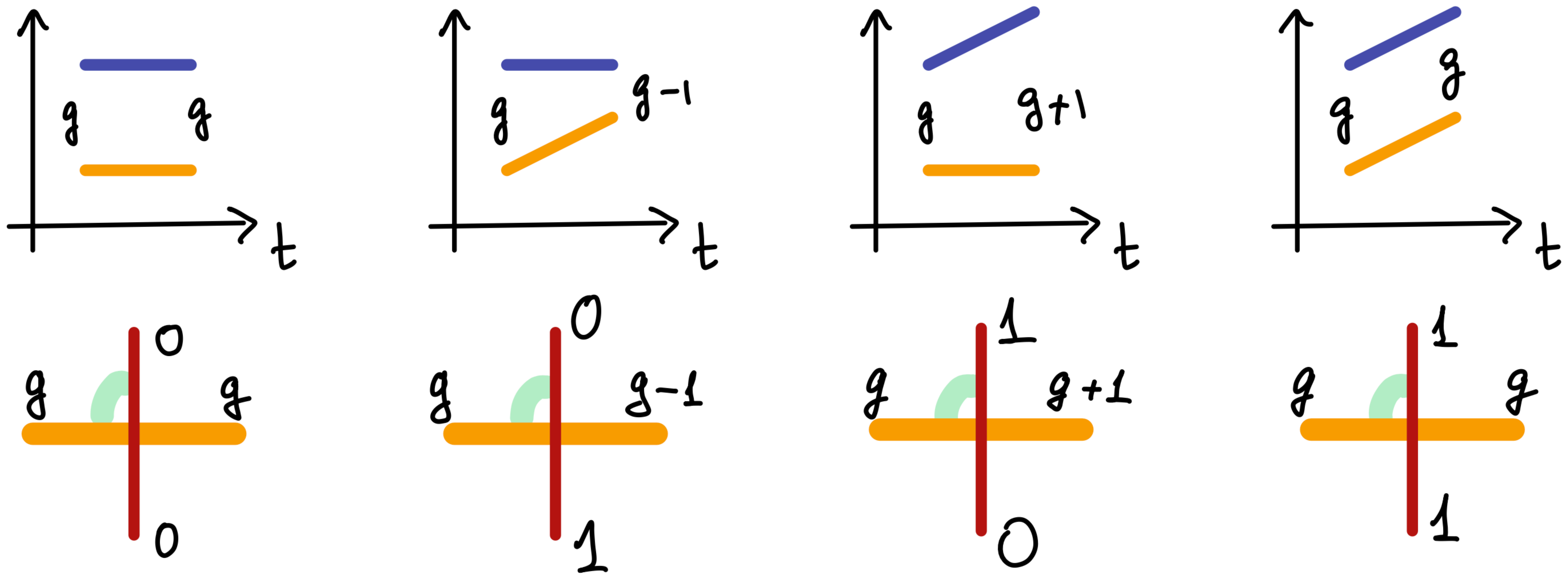

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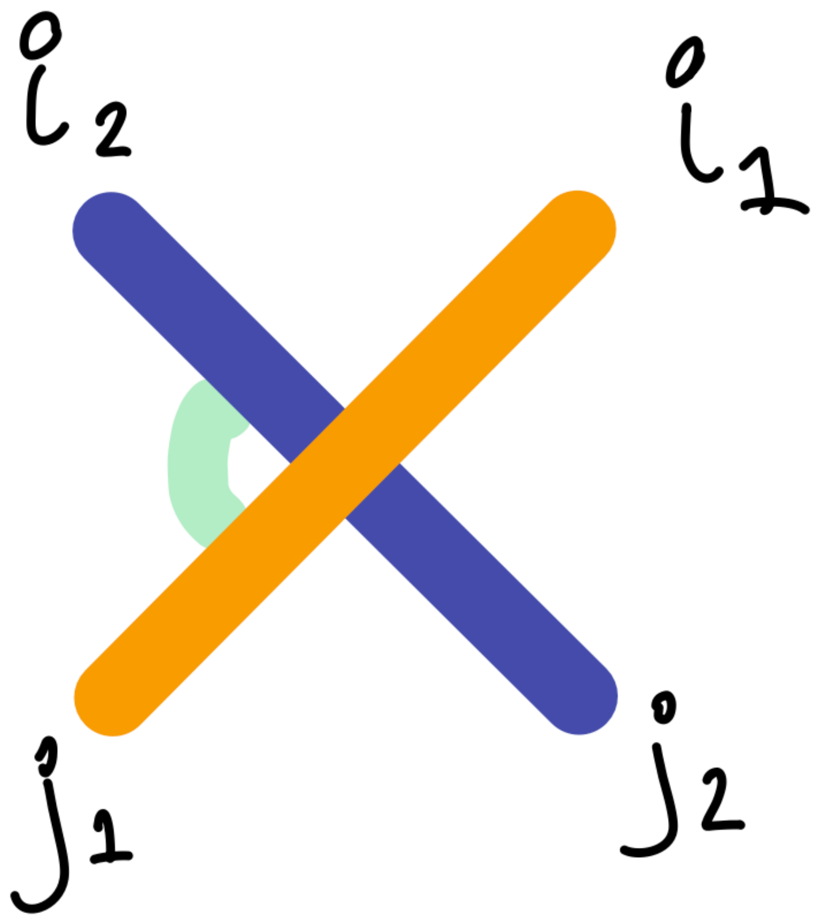
- B_{a_2/a_1} - geometric jump operator on configurations of two particles
- T_{a_1,a_2}, T_{a_2,a_1} - FS and SF transition operators

Theorem 2 (P.-Saenz 2022).
Intertwining $T_{a_1,a_2} B_{a_2/a_1} = B_{a_2/a_1} T_{a_2,a_1}$



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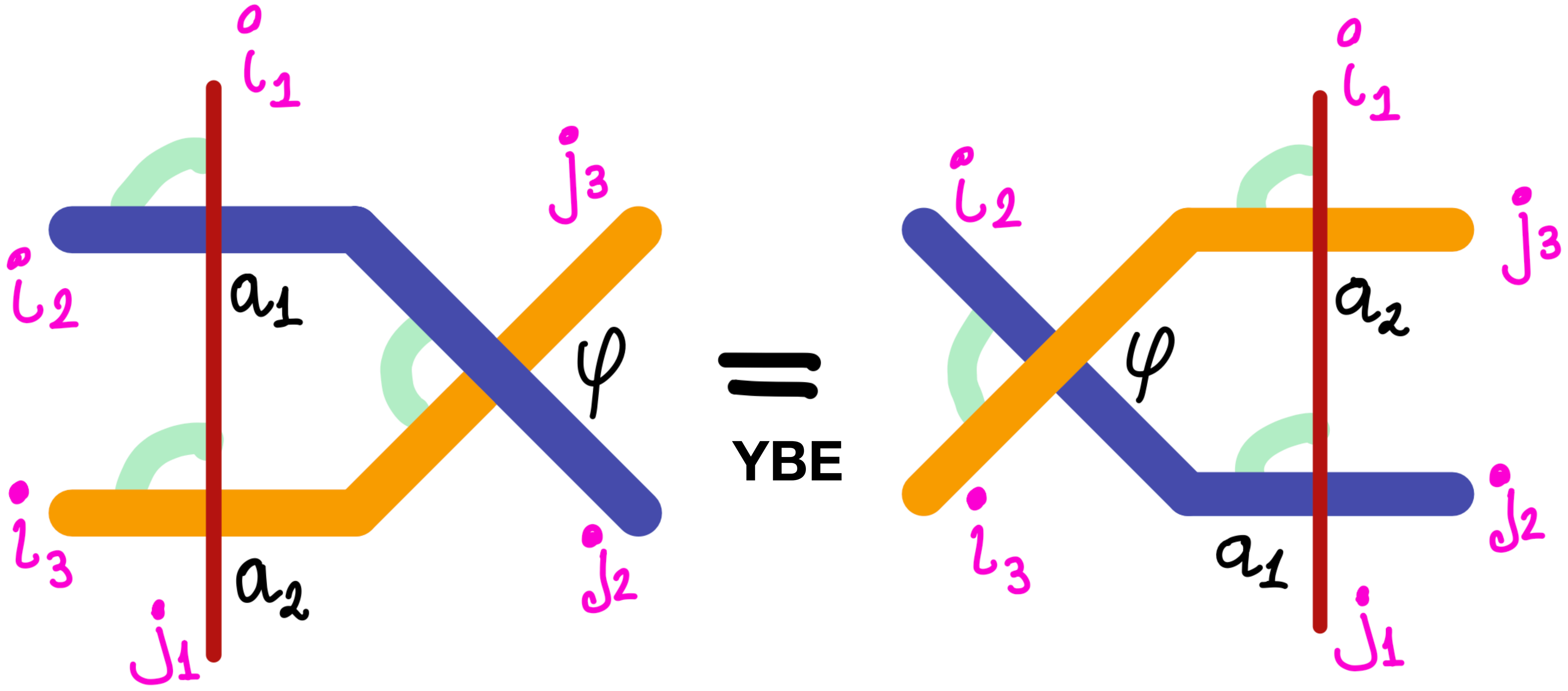
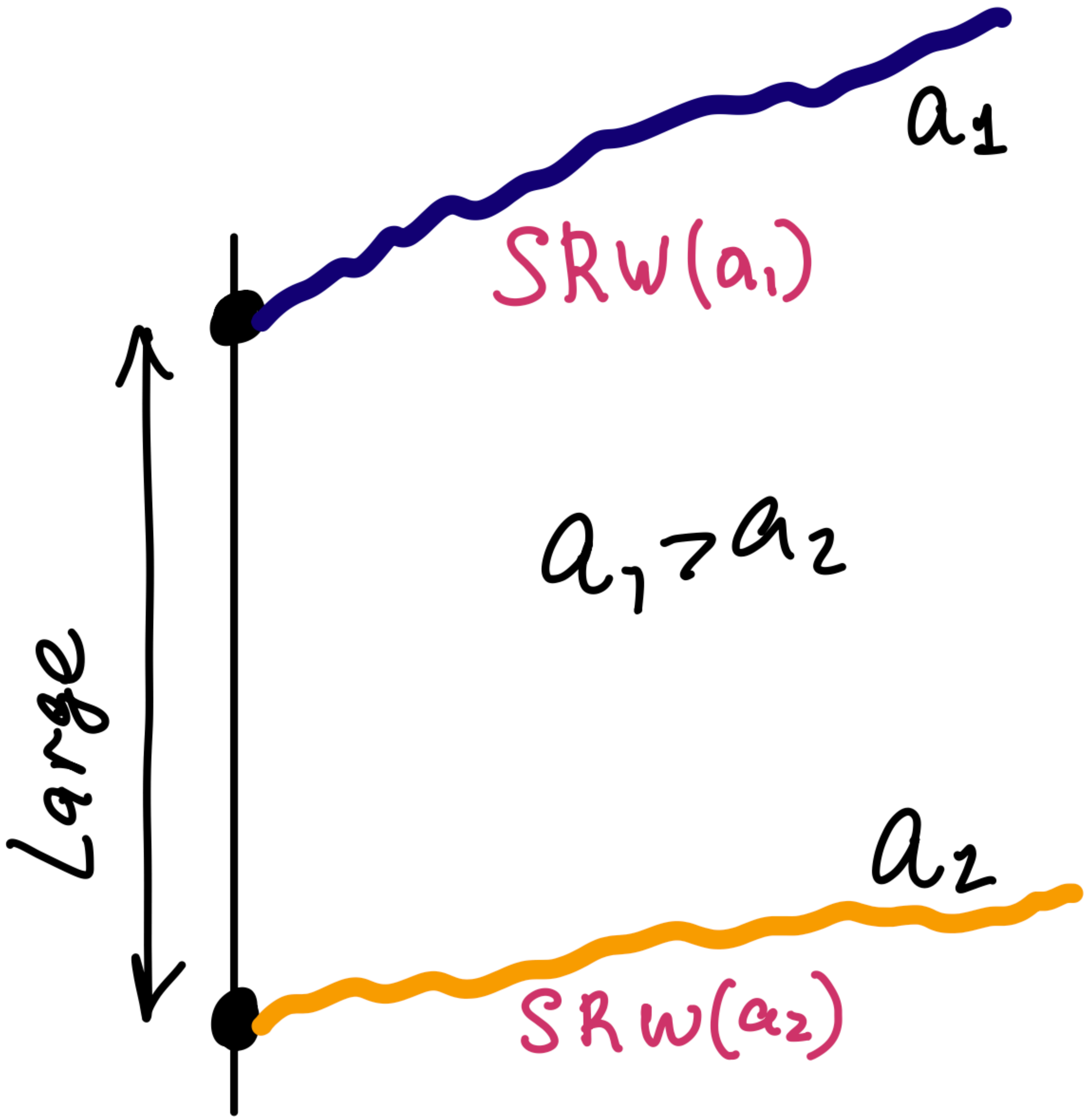


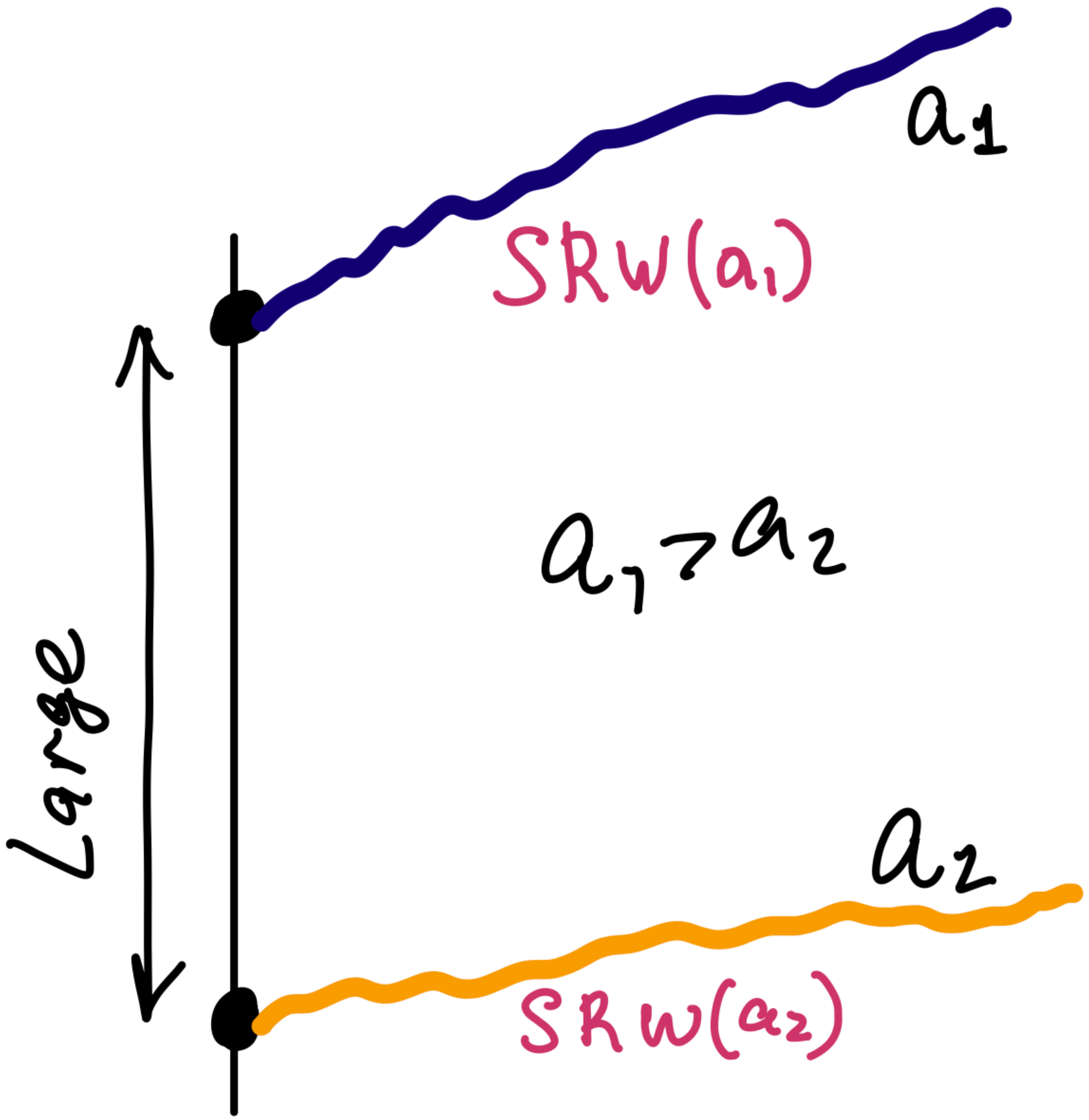
II. Three applications

Application 1: Stationarity in TASEP



Geometric jump of the top particle down

Application 1: Stationarity in TASEP



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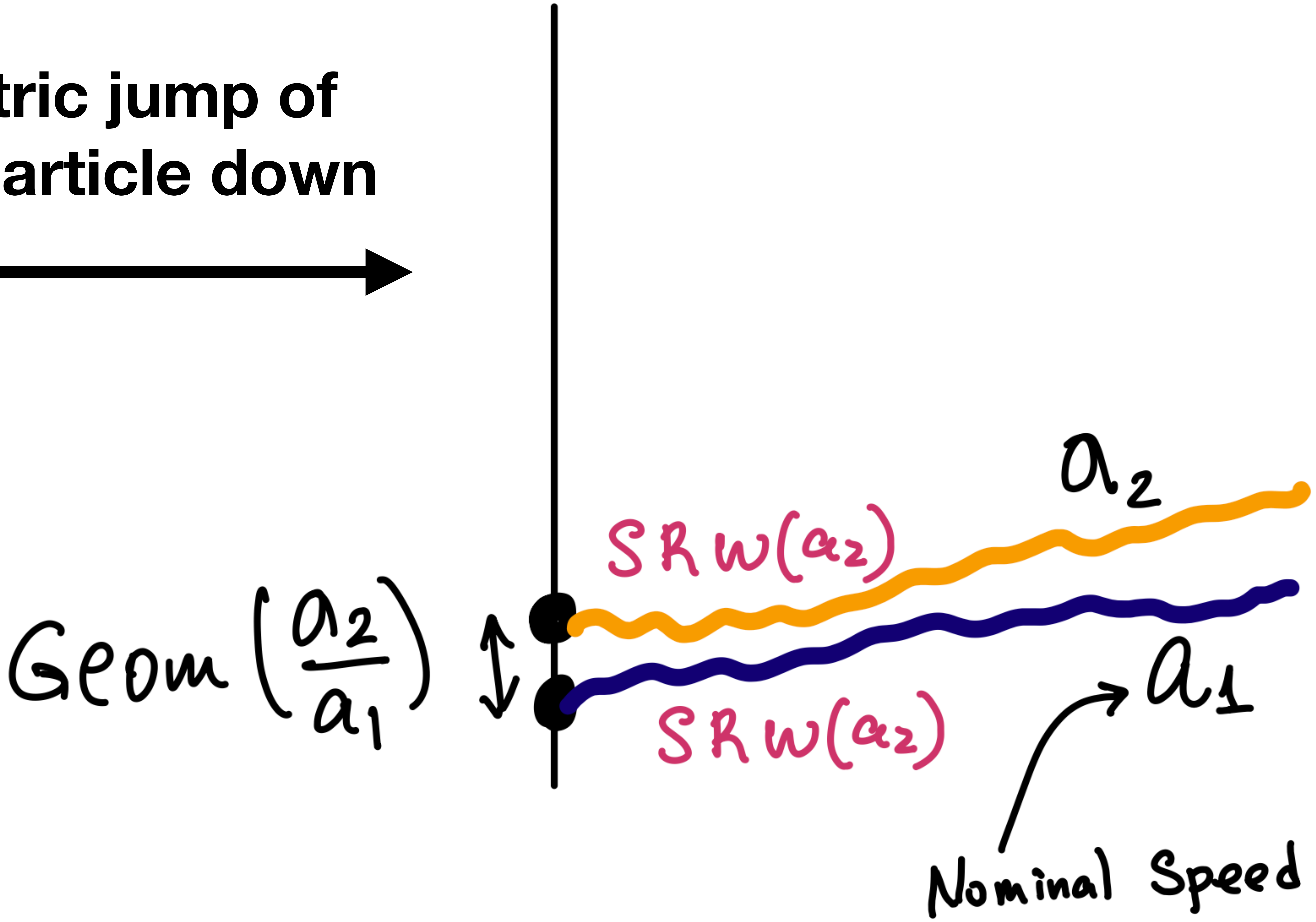

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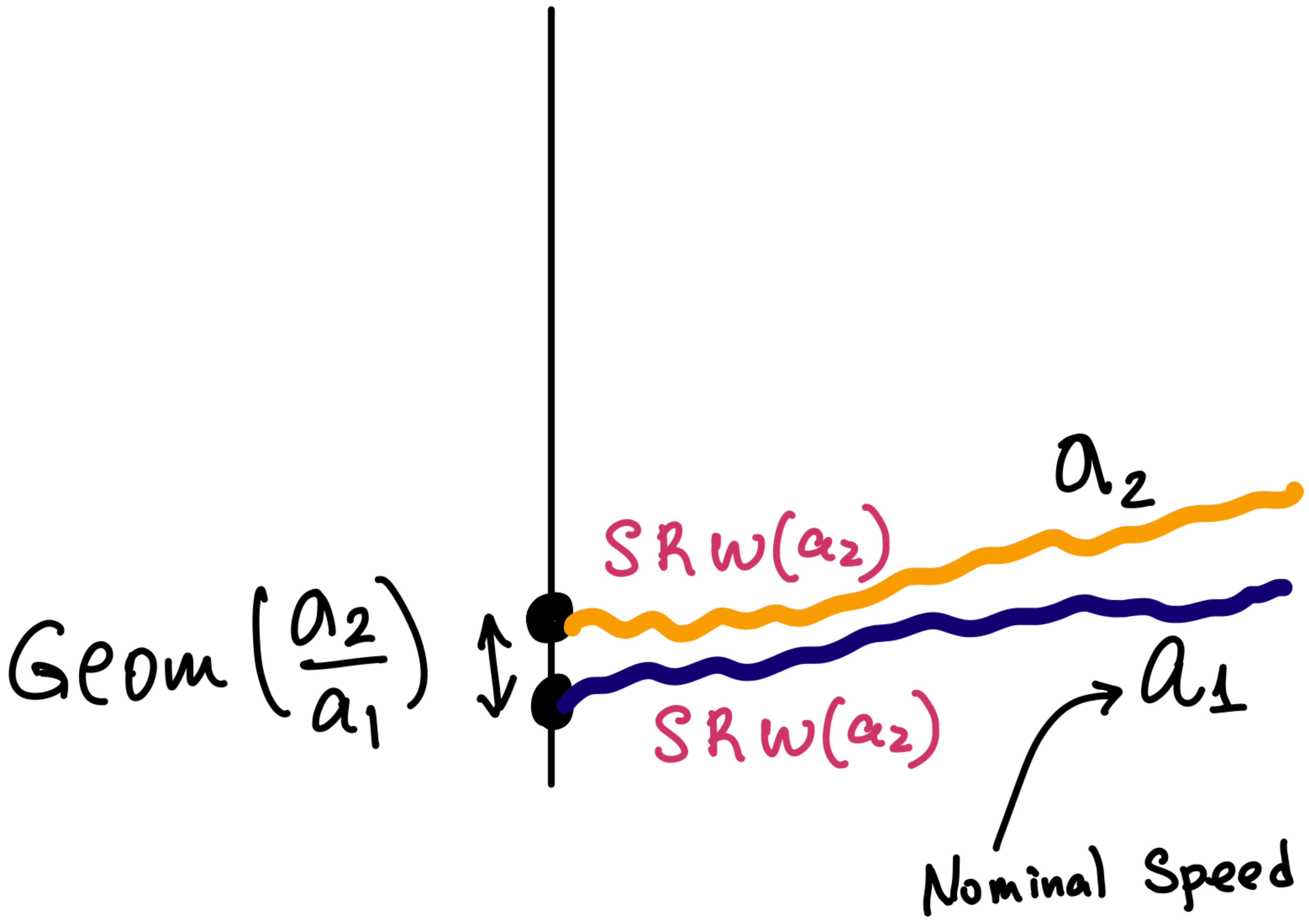
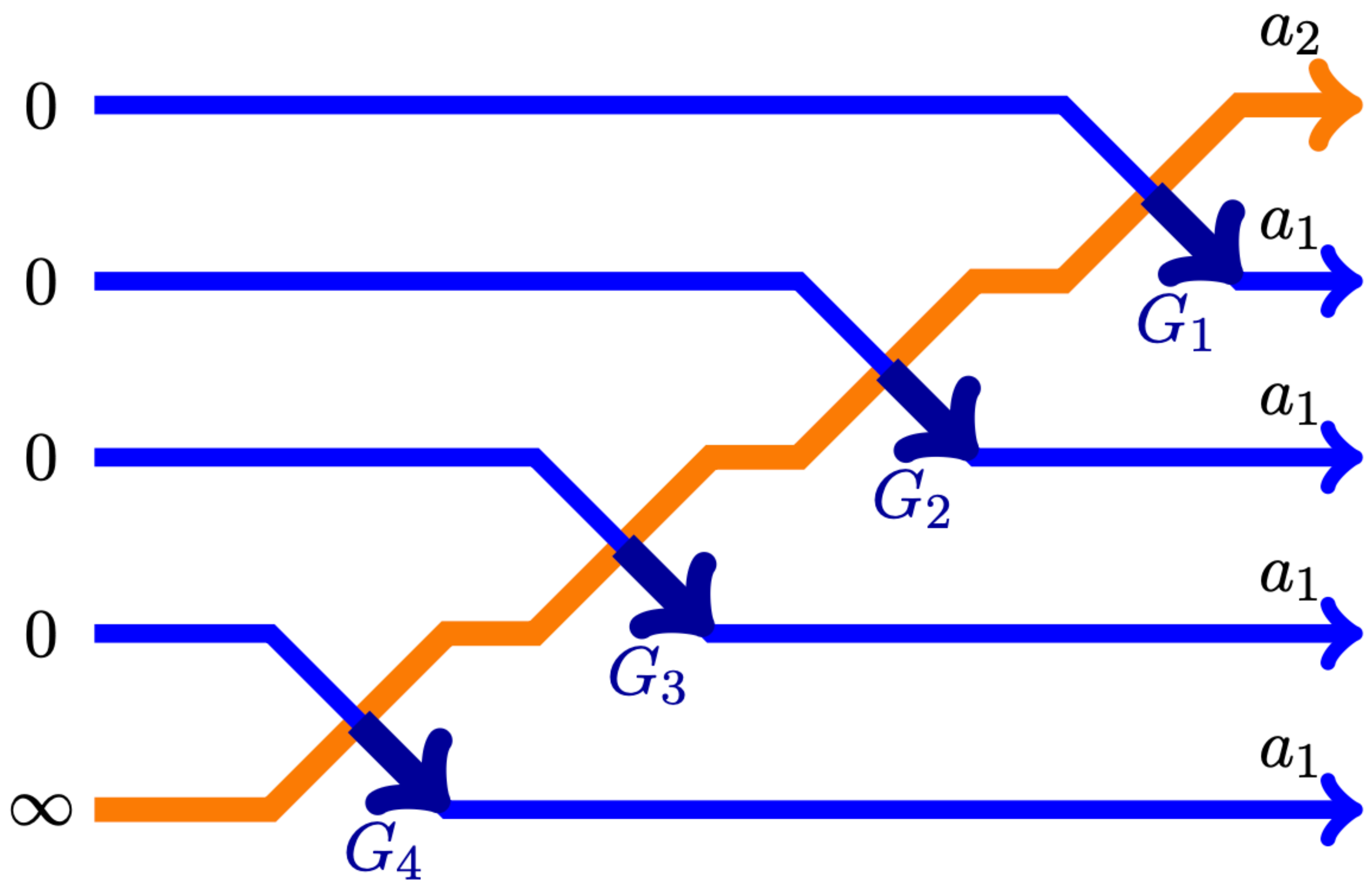


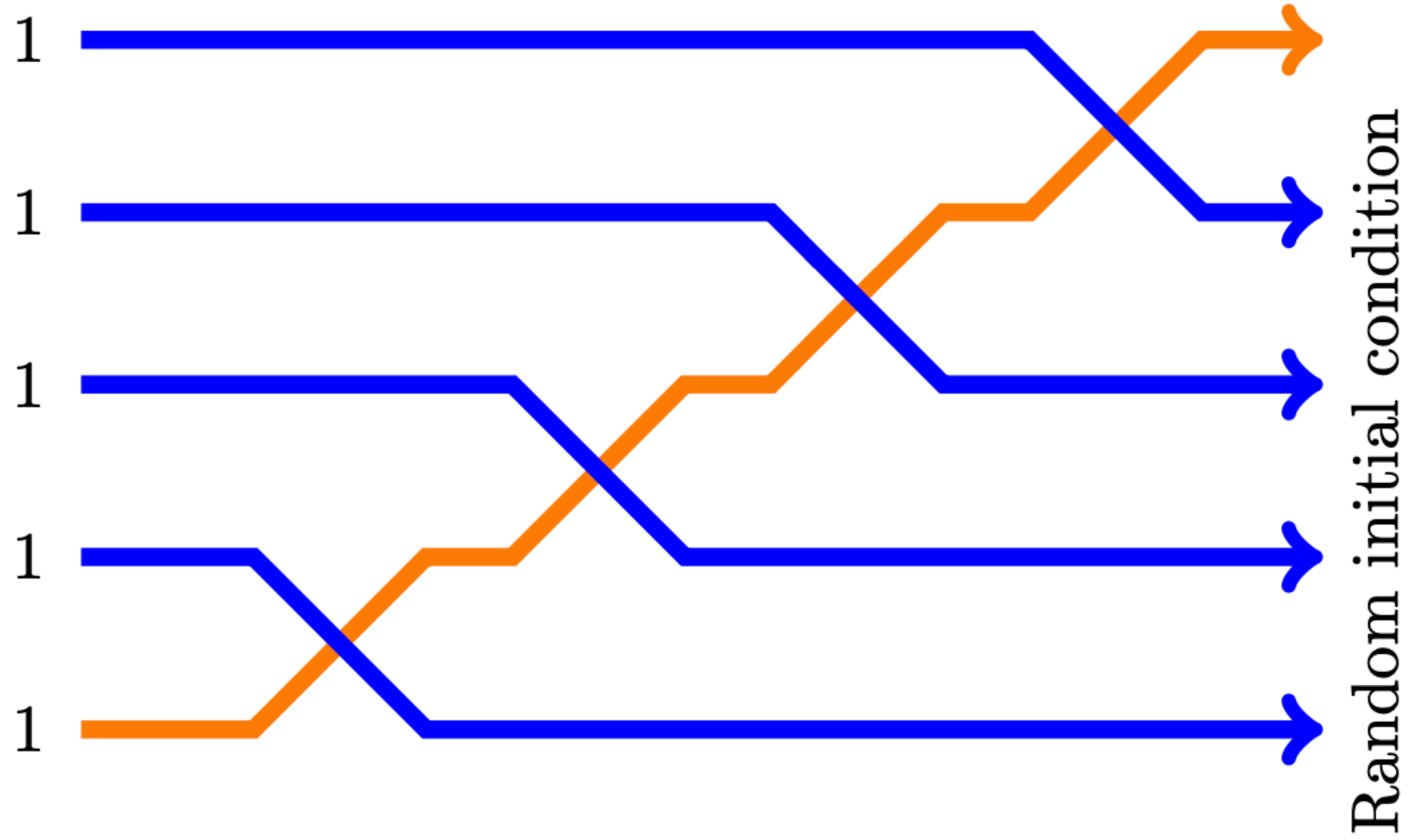
Application 1: Stationarity in TASEP

Same, but in terms of Yang-Baxter crosses ($a_1 > a_2$)



The TASEP stationary distribution ($G_i \sim$ iid Geometric(a_2/a_1)) is *deposited* by a stochastic vertex with infinitely many arrows along one direction.

Application 2: Flat initial condition ($a_1 = 1$)



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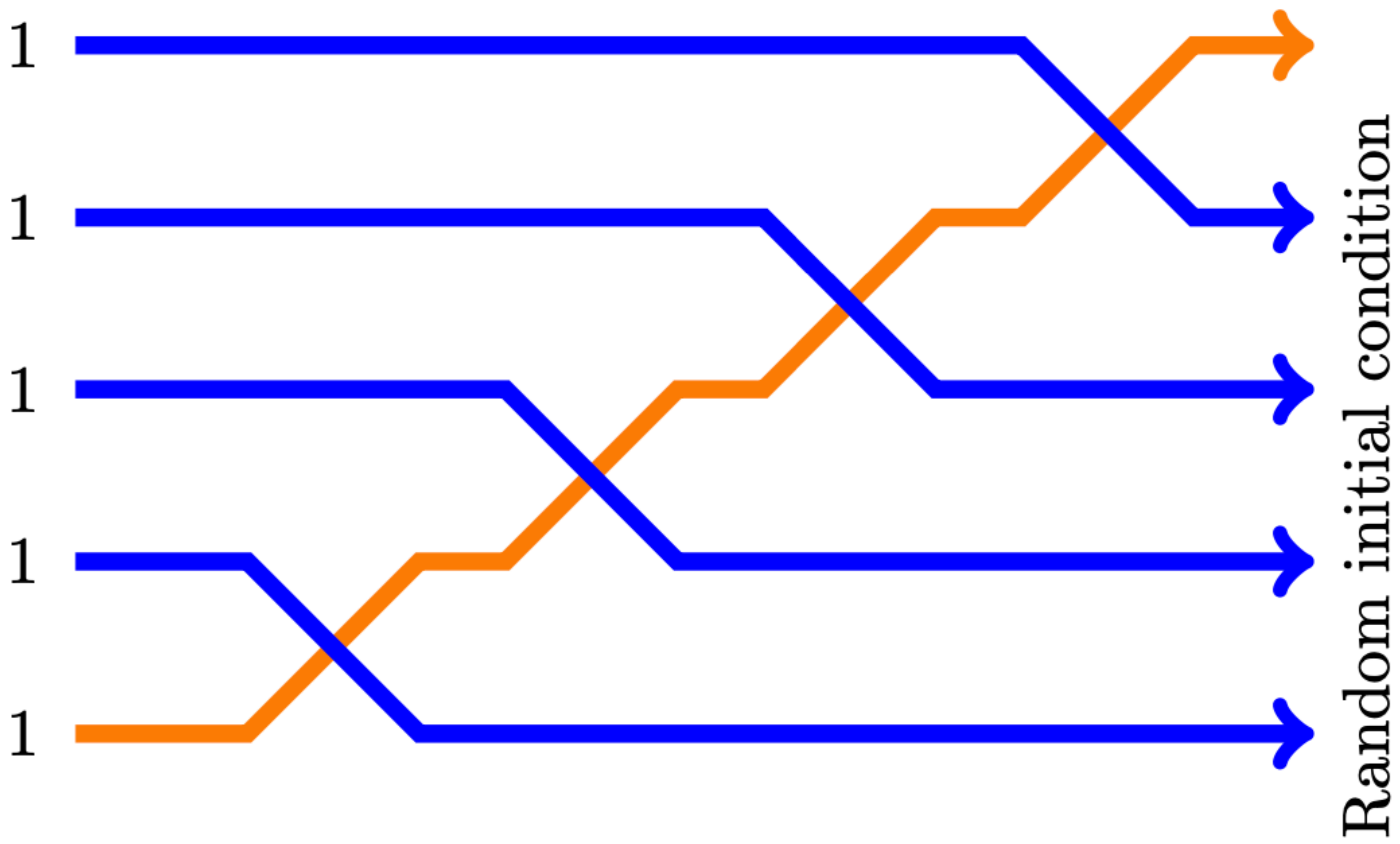
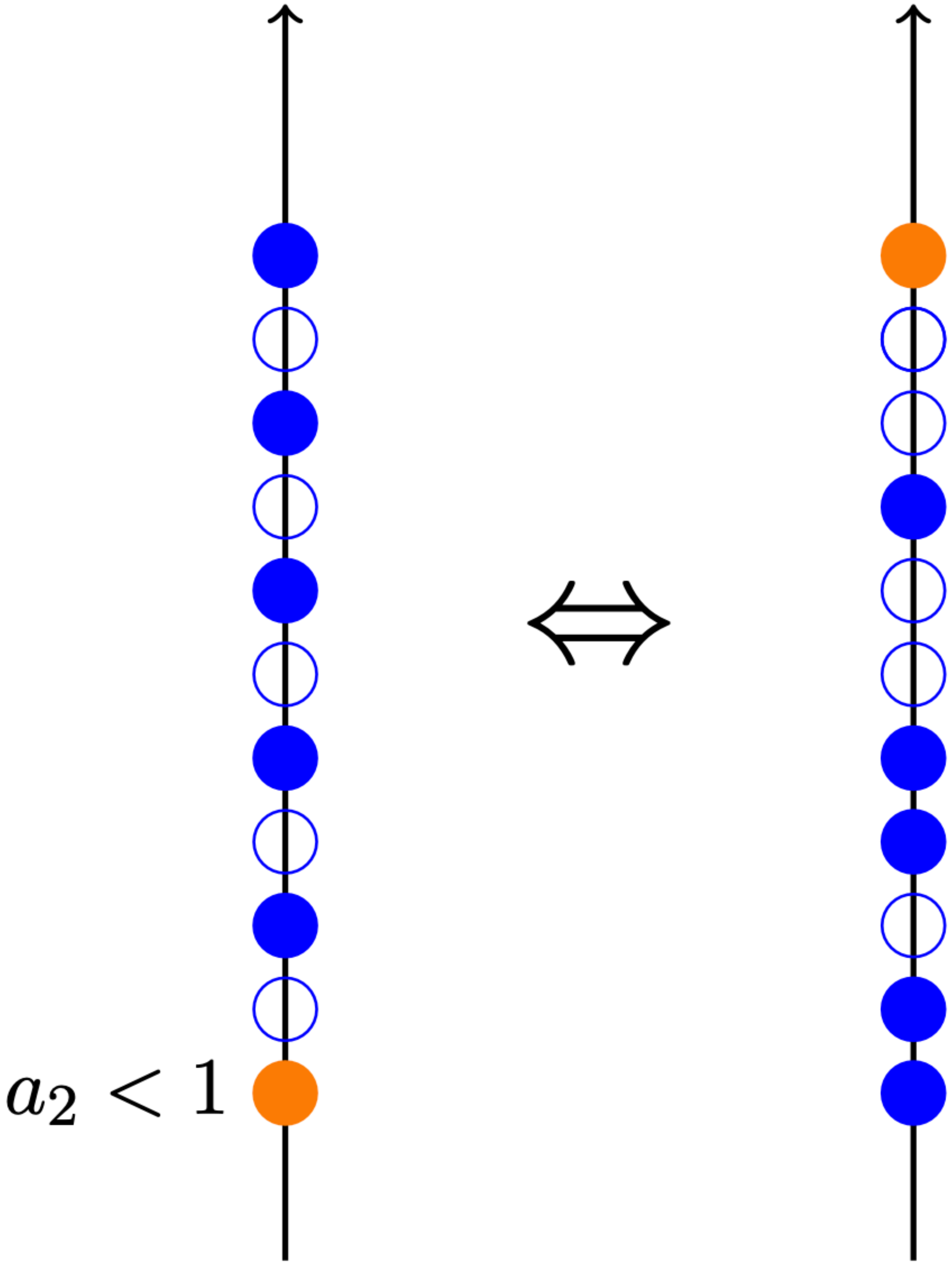
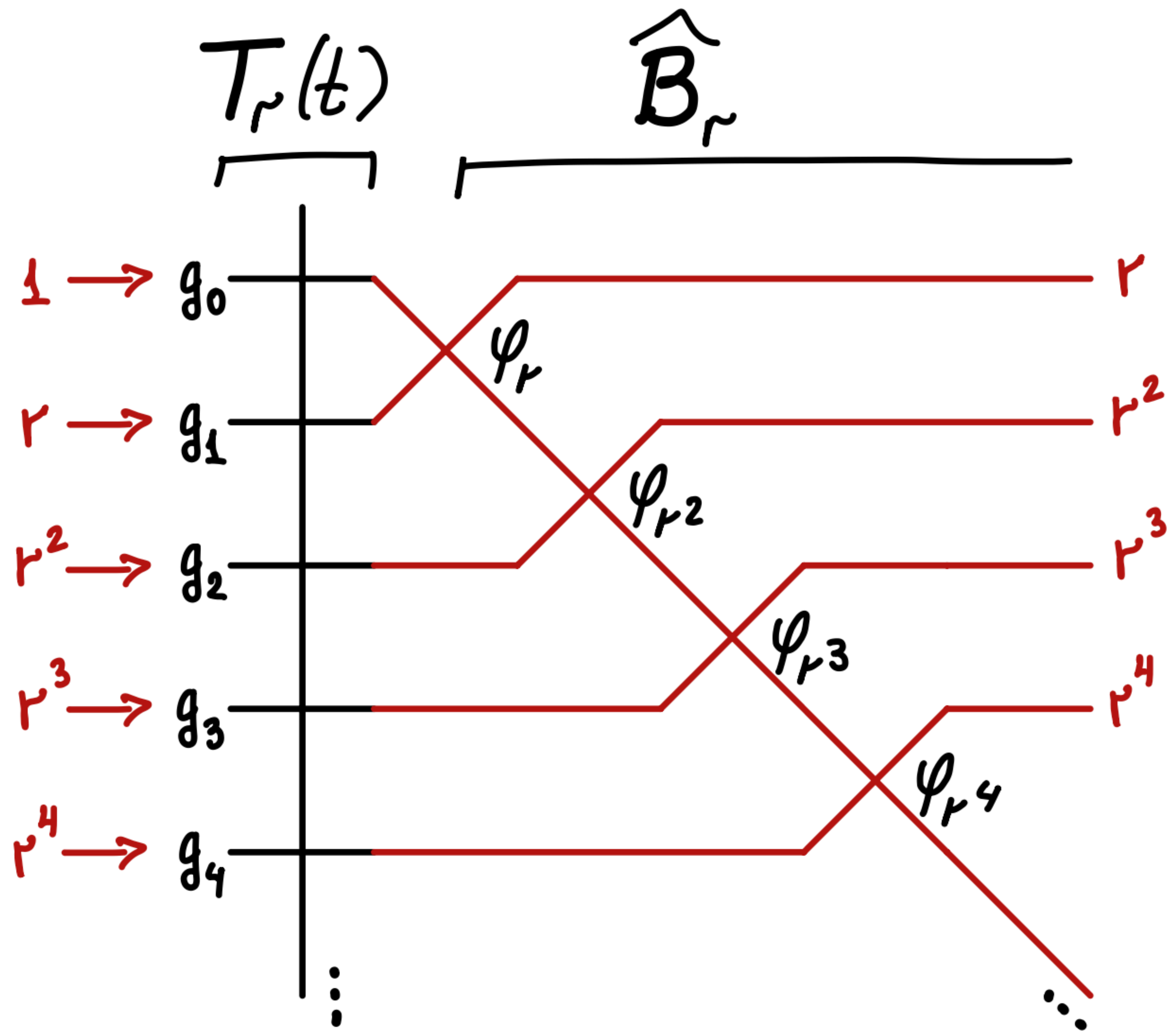




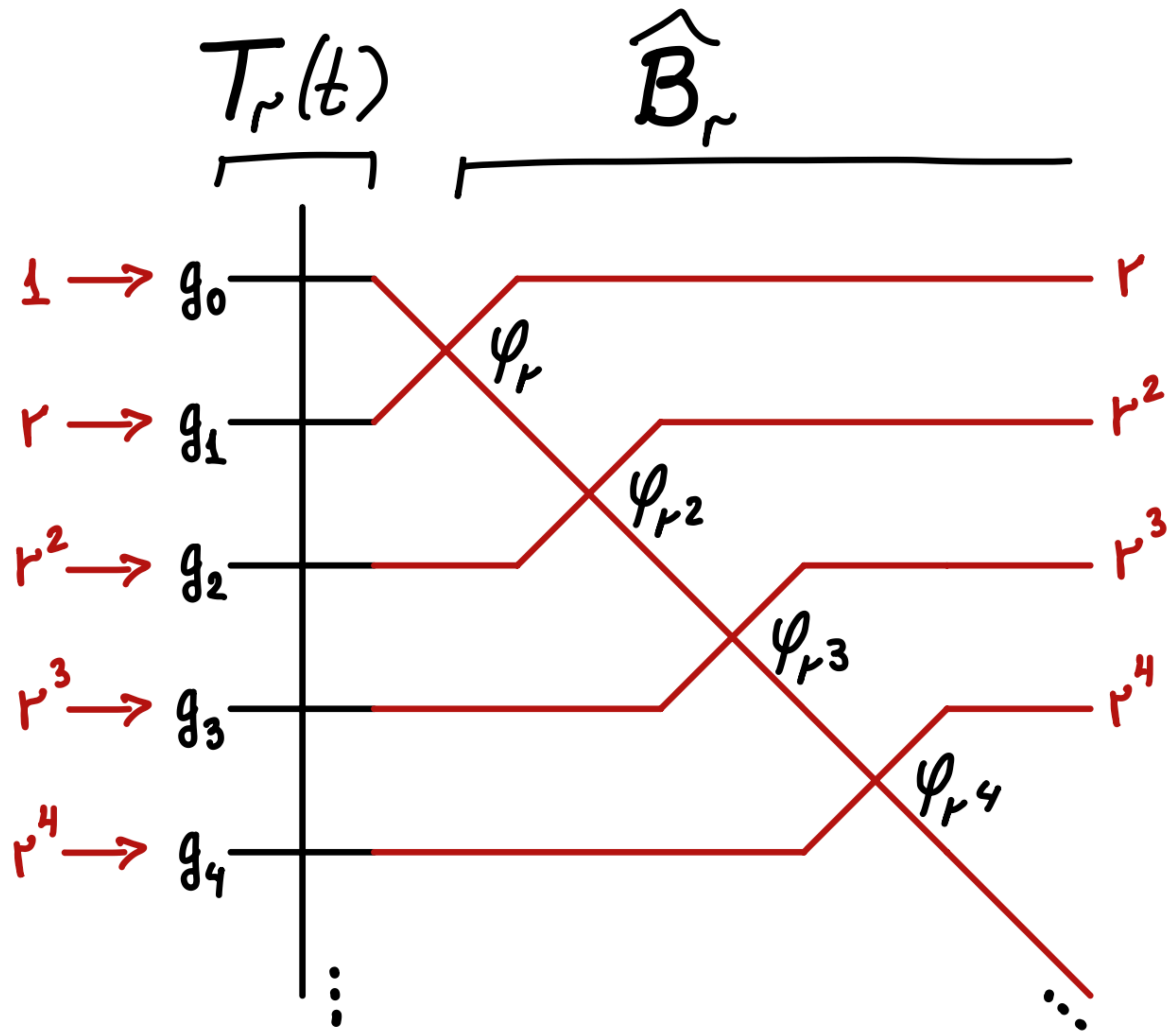
The trajectories of the back particle are the same in both situations

Application 3: Lax equation for TASEP semigroup



Application 3: Lax equation for TASEP semigroup

Take continuous time TASEP with $a_n = r^n$. Then $T_r(t)\hat{B}_r = \hat{B}_r T_r(rt)$, which is a **time change.**



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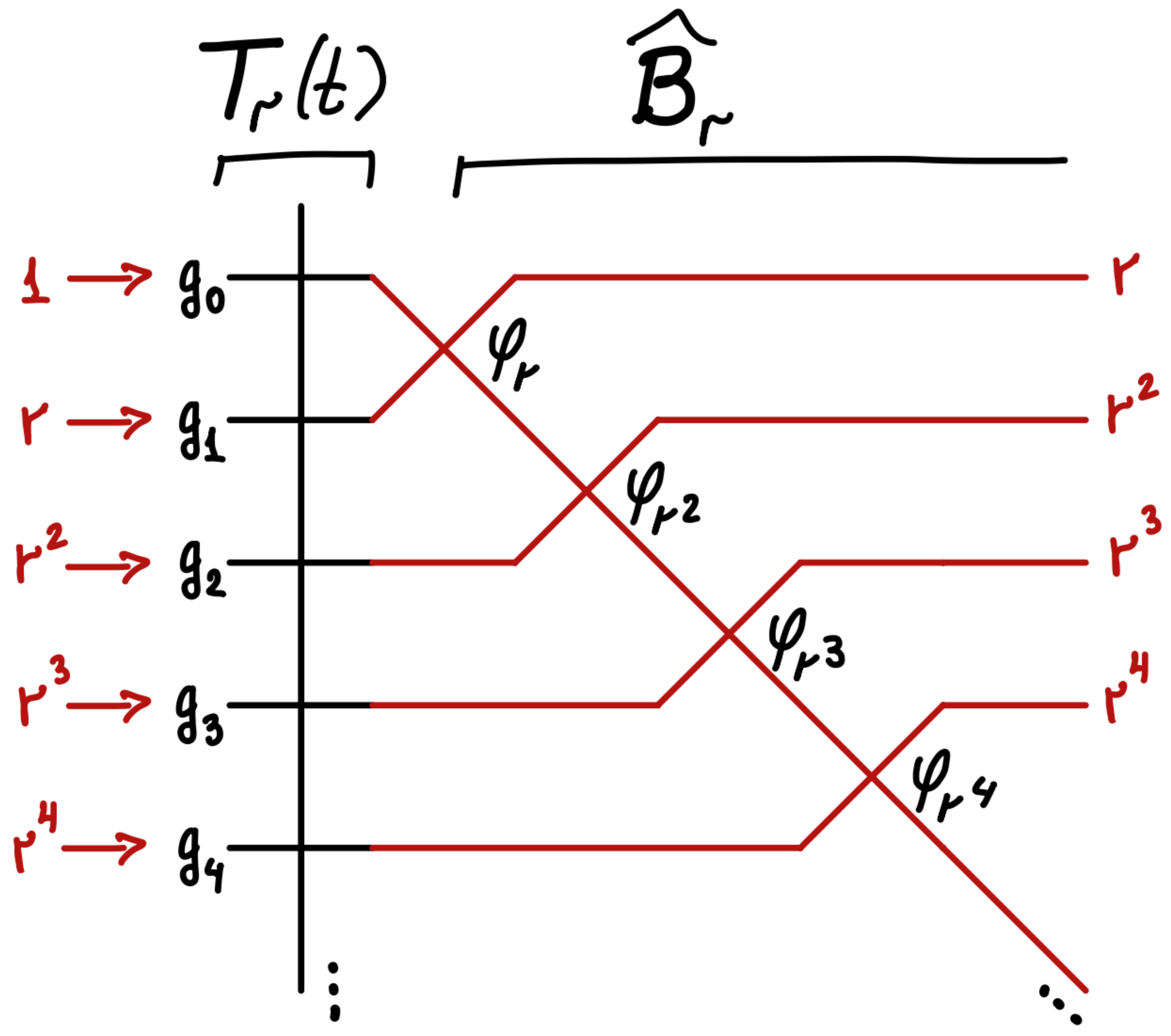

Send $r \to 1$, to **homogeneous** TASEP

In continuous time Poisson limit of $(\hat{B}_r)^{\tau/(1-r)}$, we get a Markov semigroup $B(\tau)$ - **backwards Hammersley process**.

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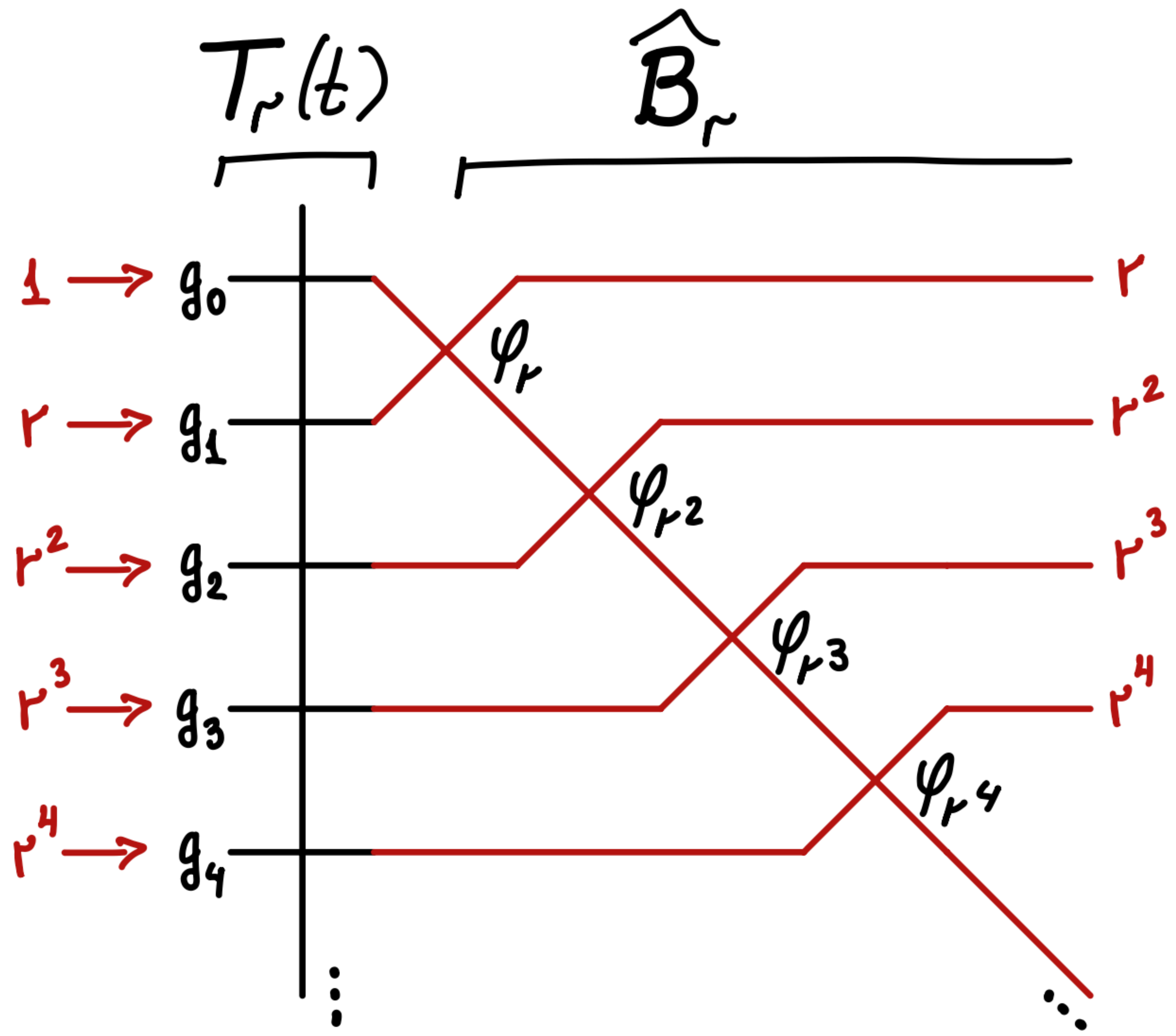

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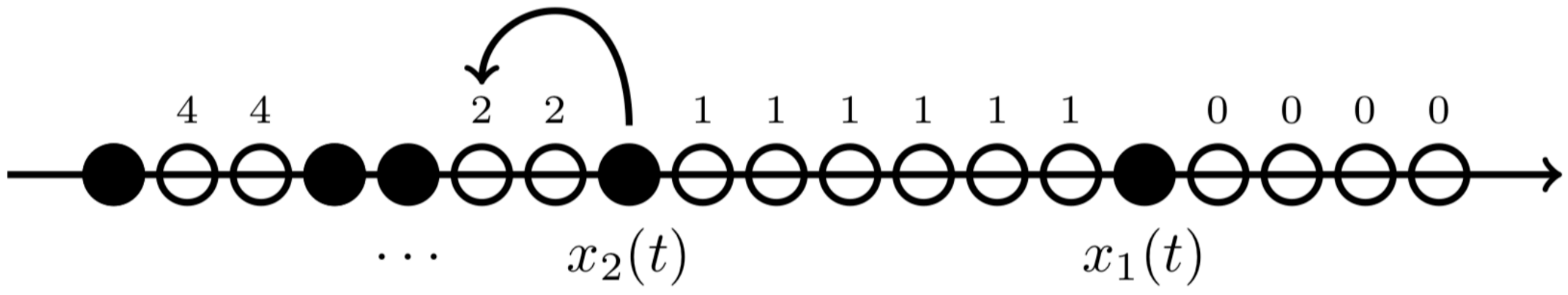

When the clock rings, the nearest particle to the right instantaneously jumps into this hole.

Application 3: Lax equation for TASEP semigroup

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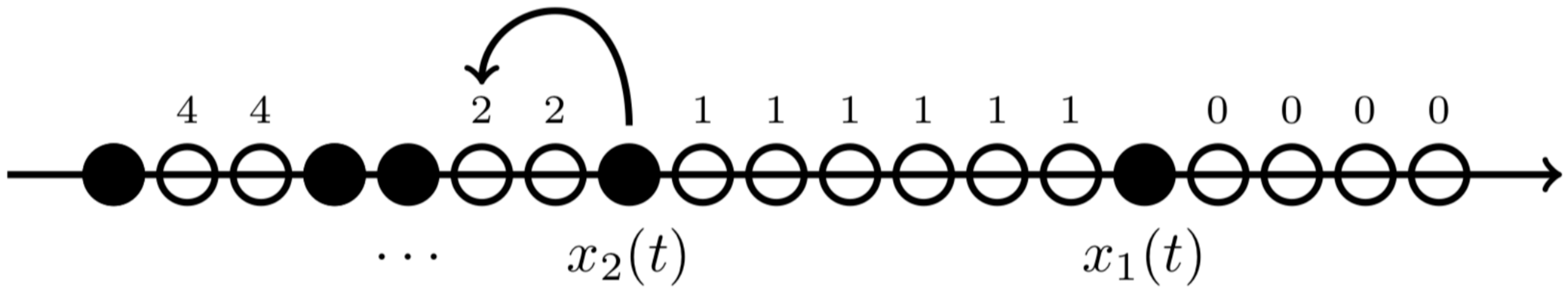

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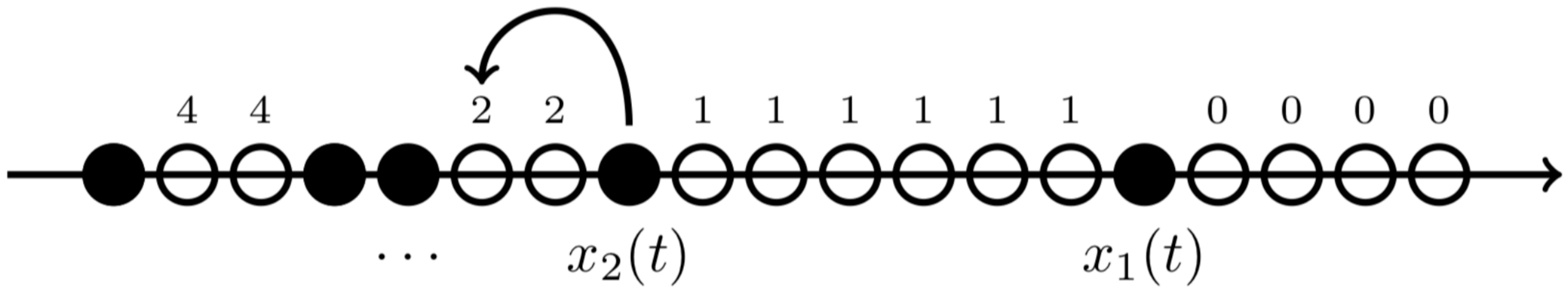
Send $r \to 1$, to **homogeneous** TASEP

Intertwining $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$

In continuous time Poisson limit of $(\hat{B}_r)^{\tau/(1-r)}$, we get a Markov semigroup $B(\tau)$ - **backwards Hammersley process**.

Lax equation (differential form)

$$\frac{d}{dt}T(t) = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$$



When the clock rings, the nearest particle to the right instantaneously jumps into this hole.

Lax equation (for expectations)

$$t\,\frac{\partial}{\partial t}\mathbb{E}_{\vec{y}}\left[F(\vec{x}(t))\right] = \mathsf{B}\,\mathbb{E}_{\vec{y}}\left[F(\vec{x}(t))\right] - \mathbb{E}_{\vec{y}}\left[(\mathsf{B}F)(\vec{x}(t))\right]$$

Application 3: Lax equation for TASEP semigroup

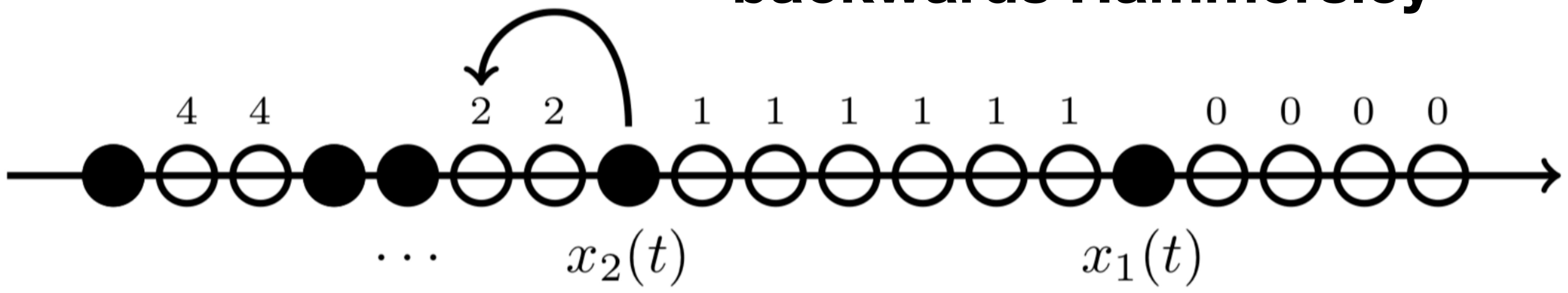
- Let $T(t)$ be the TASEP semigroup, and $B(\tau)$ be another process which we call **backwards Hammersley**

- **Theorem [P-Saenz '22].** Lax equation

$$\frac{d}{dt}T(t) = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$$

- $\Rightarrow$ Lax equation for expectations

backwards Hammersley



($q = 0$) Each hole has exponential clock of rate $h(t, x)$

When the clock rings, the nearest particle to the right instantaneously jumps into this hole.

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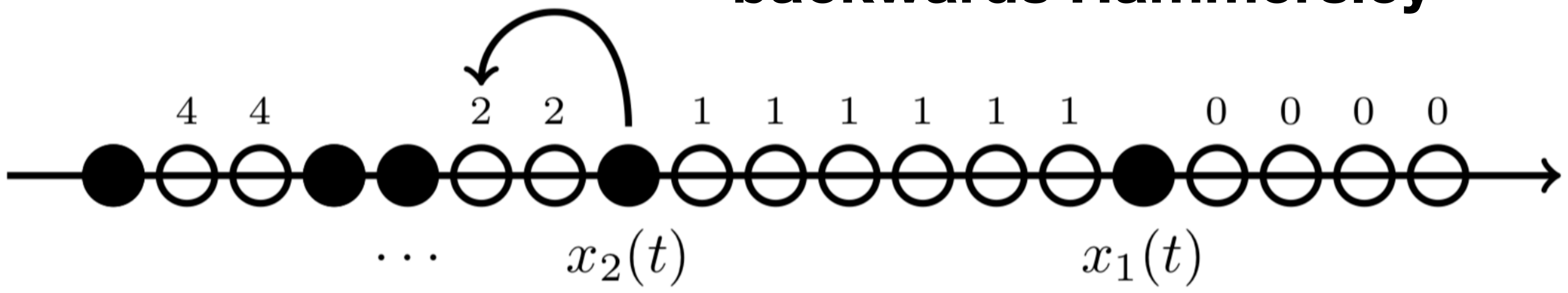

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- **Theorem** **[P-Saenz '22]**. Lax equation

$$\frac{d}{dt}T(t) = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$$

- $\Rightarrow$ Lax equation for expectations

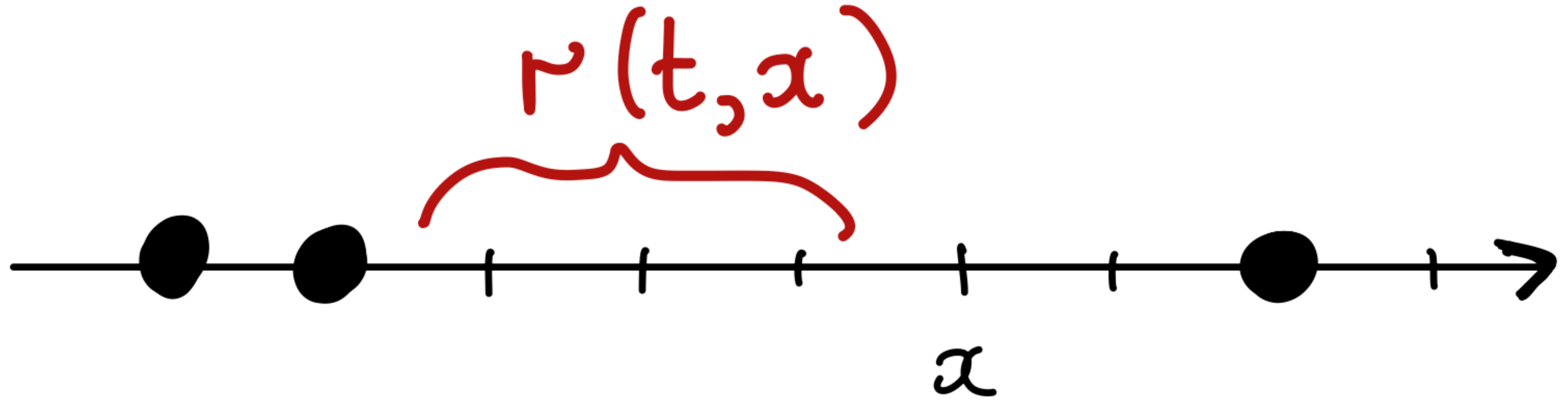
$$t\,\frac{\partial}{\partial t}\,\mathbb{E}_{\vec{y}}\left[F(\vec{x}(t))\right] = \mathsf{B}\,\mathbb{E}_{\vec{y}}\left[F(\vec{x}(t))\right] - \mathbb{E}_{\vec{y}}\left[(\mathsf{B}F)(\vec{x}(t))\right]$$

$\oplus$ for **any** function F, so characterizes the dynamics

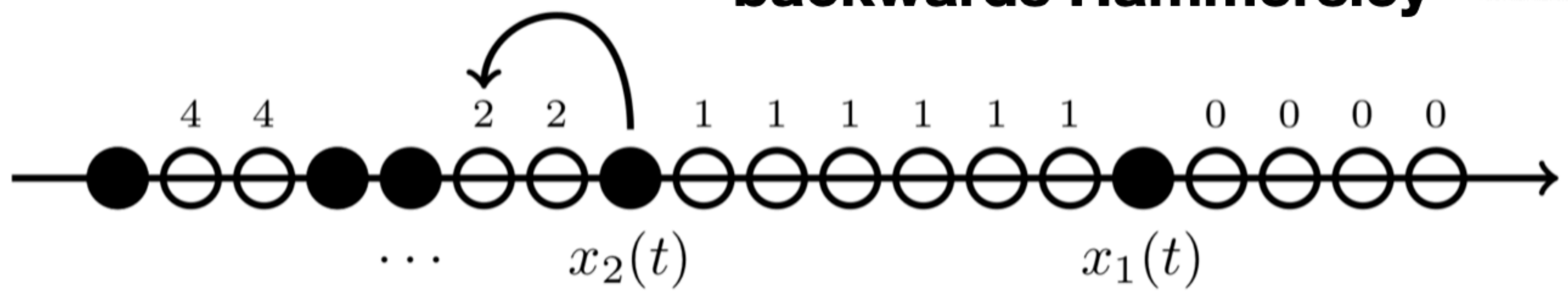
$\oplus$ same form for TASEP and q-TASEP (differ in B)

$\ominus$ hard to extract asymptotic information from

Example



backwards Hammersley



($q = 0$) Each hole has exponential clock of rate $h(t, x)$

When the clock rings, the nearest particle to the right instantaneously jumps into this hole.

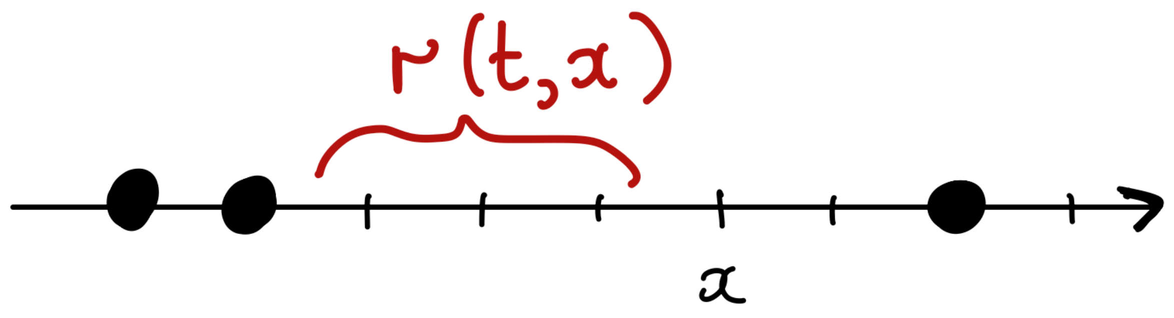
$$t \frac{\partial}{\partial t} \mathbb{E}_{\vec{y}} \left[F(\vec{x}(t)) \right] = \mathsf{B}\, \mathbb{E}_{\vec{y}} \left[F(\vec{x}(t)) \right] - \mathbb{E}_{\vec{y}} \left[(\mathsf{B}F)(\vec{x}(t)) \right]$$

If $y = \textbf{step}$ and $F(x(t)) = g(h(t, x))$, then

$$\frac{\partial}{\partial t} \mathbb{E}_{\textbf{step}} \left[g(h) \right] = -\frac{1}{t} \mathbb{E}_{\textbf{step}} \left[h \cdot \overbrace{(g(h-1) - g(h))}^{=-1 \text{ for } g(h)=h} \cdot r(x, t) \right]$$

where $r(t, x)$ is the number of holes to the left of x at time t for $q = 0$

Example



Let $g(h) = h(h-1)\ldots(h-k+1)$, then

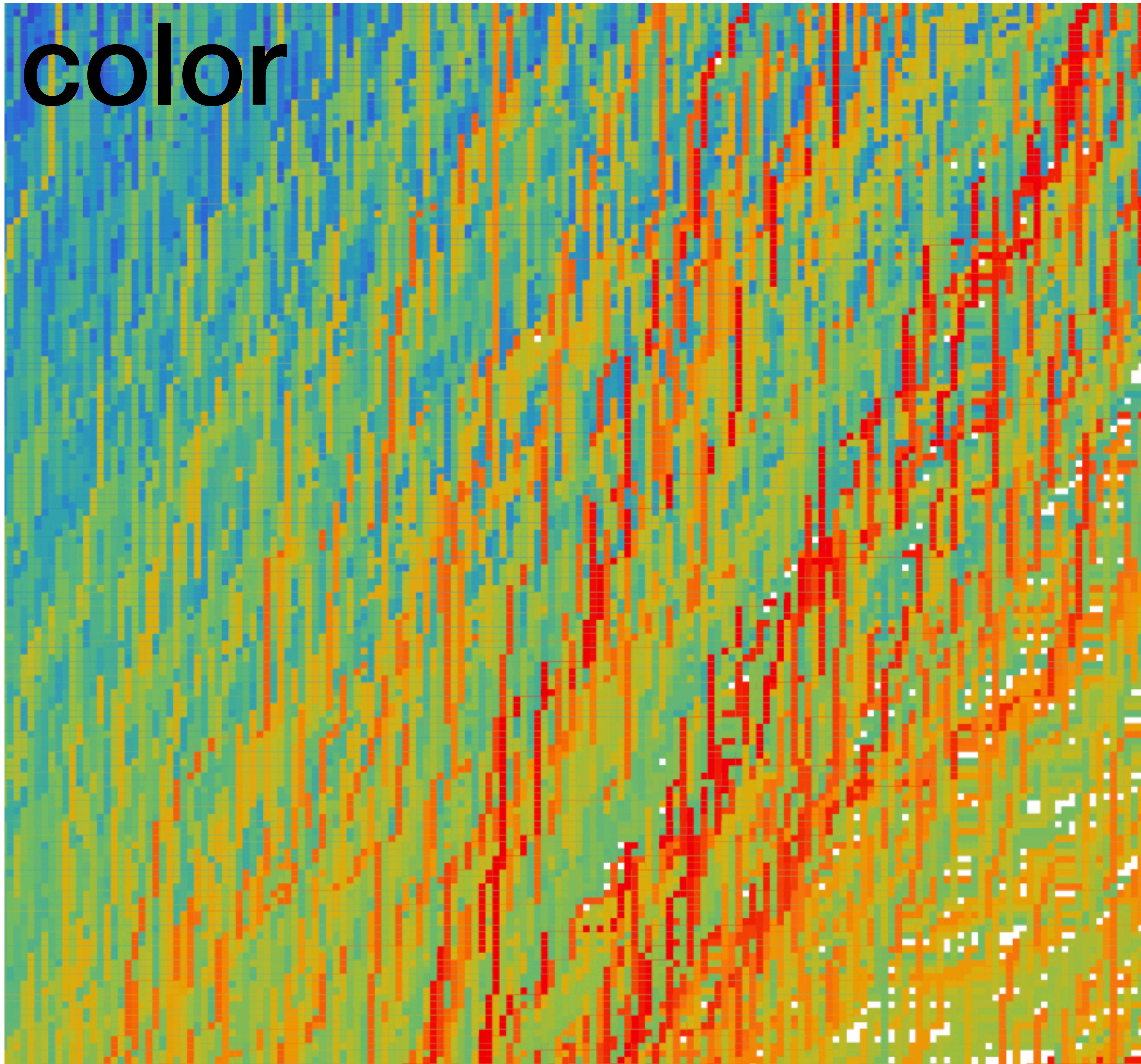
(under local equilibrium assumptions)

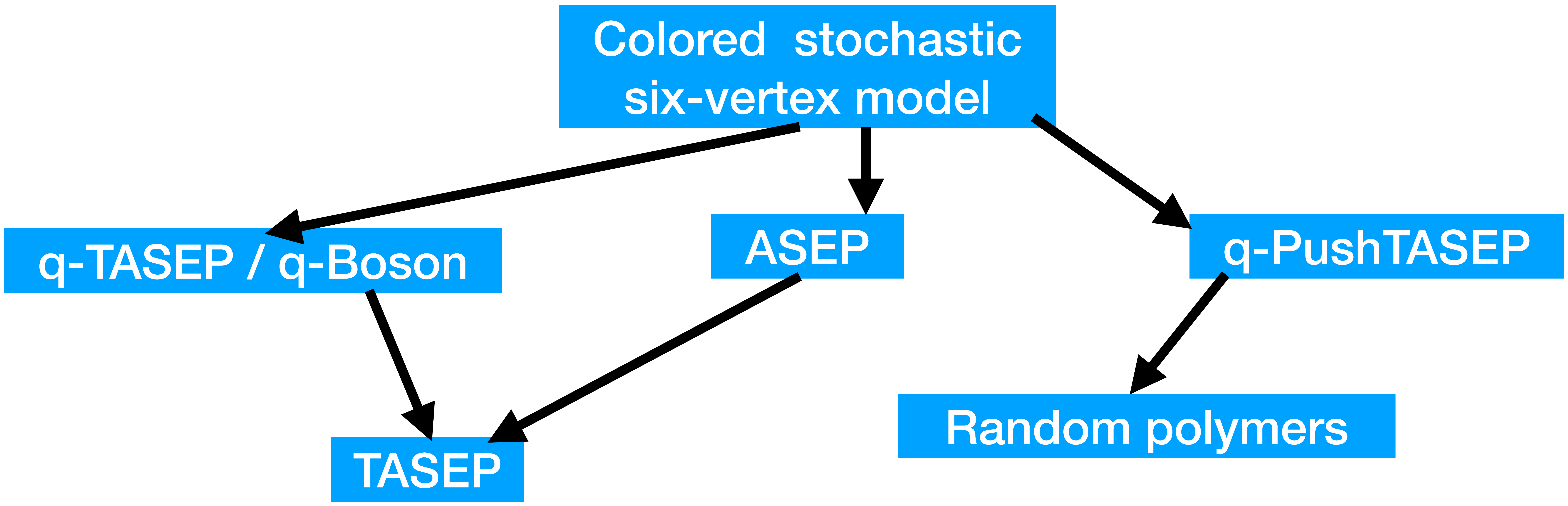
$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{step}}\left[g(h)\right] \sim \frac{k}{t}\mathbb{E}_{\mathbf{step}}\left[g(h)\right]\cdot\mathbb{E}_{\mathbf{step}}[r(t,x)]$$

We have $\mathbb{E}_{\mathbf{step}}[r(t,x)] = \dfrac{1-\rho(t,x)}{\rho(t,x)}$, where $\rho(t,x)$ is the local density of particles

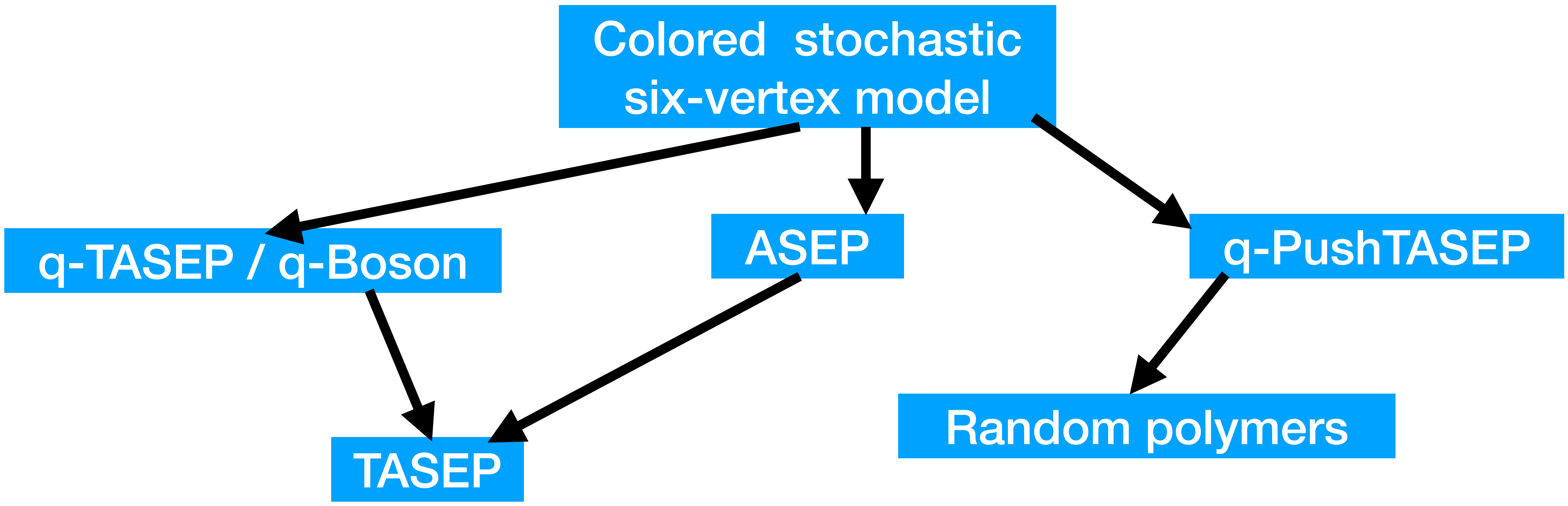
Same computations can be done for $q > 0$, where $\mathbb{E}_{\mathbf{step}}[r(t,x)]$ is a certain explicit q-sum, and it gives a different coefficient of growth.

III. Stationarity in color and on the circle

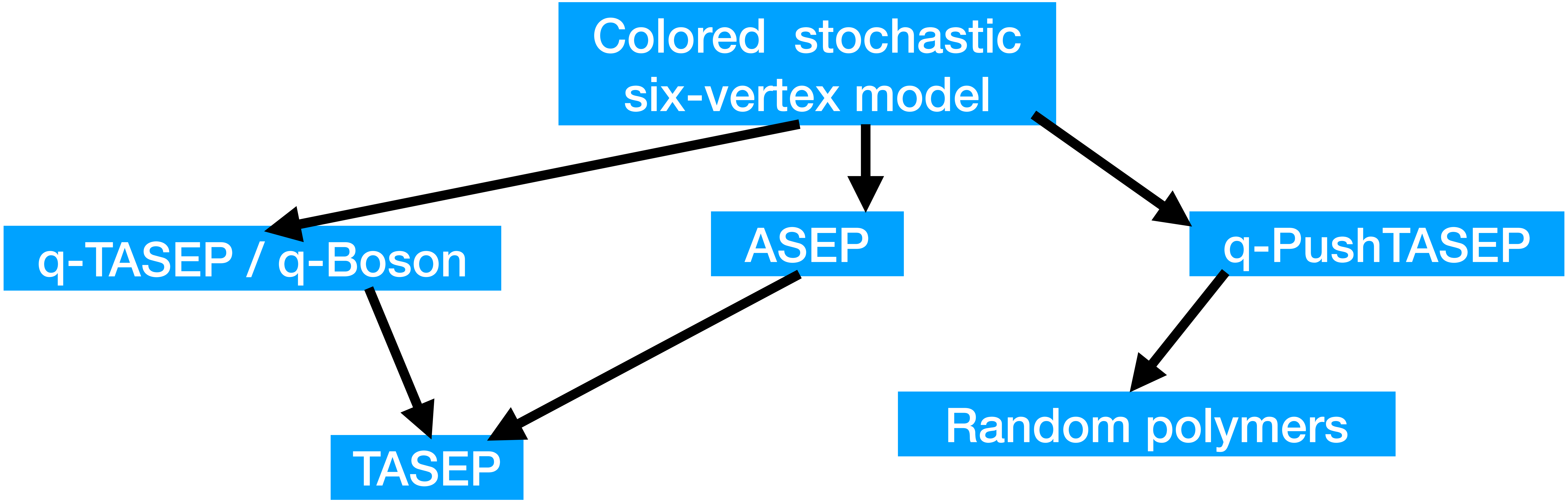




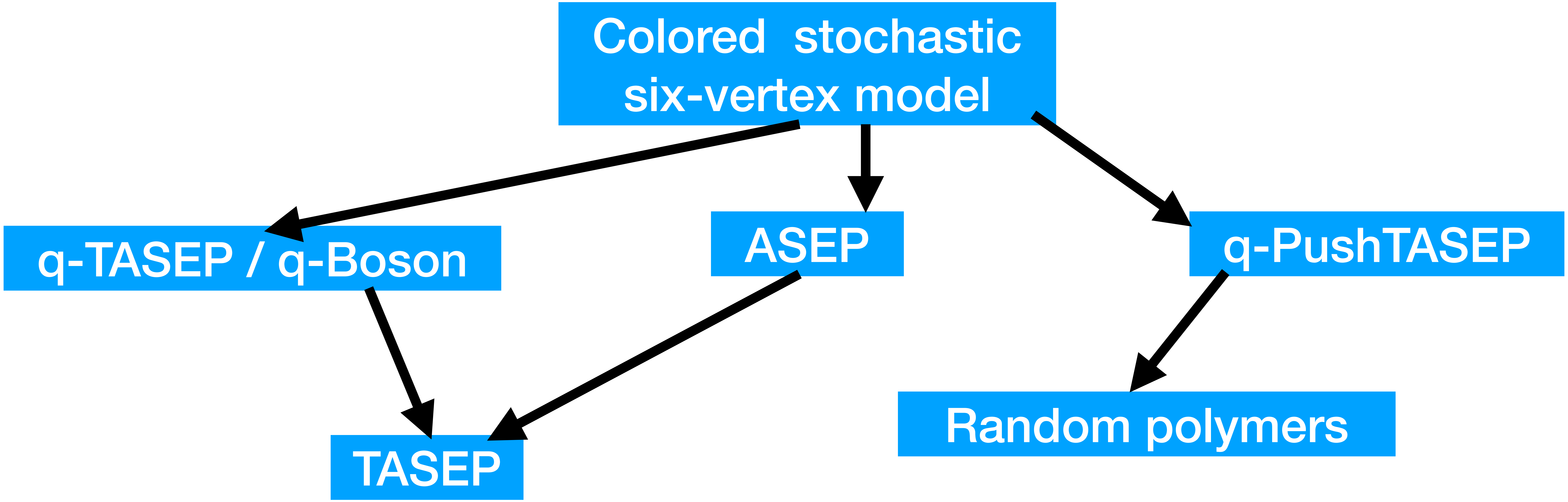
Colored stochastic six-vertex model
q-TASEP / q-Boson
ASEP
q-PushTASEP
TASEP
Random polymers



- All previous symmetry results work for TASEP and q-TASEP

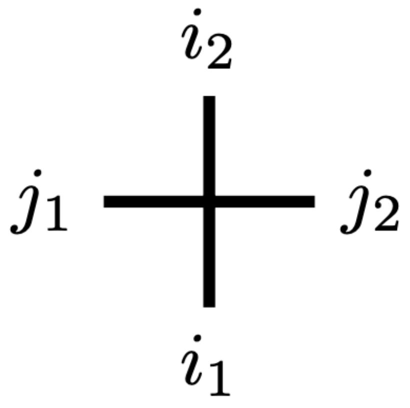


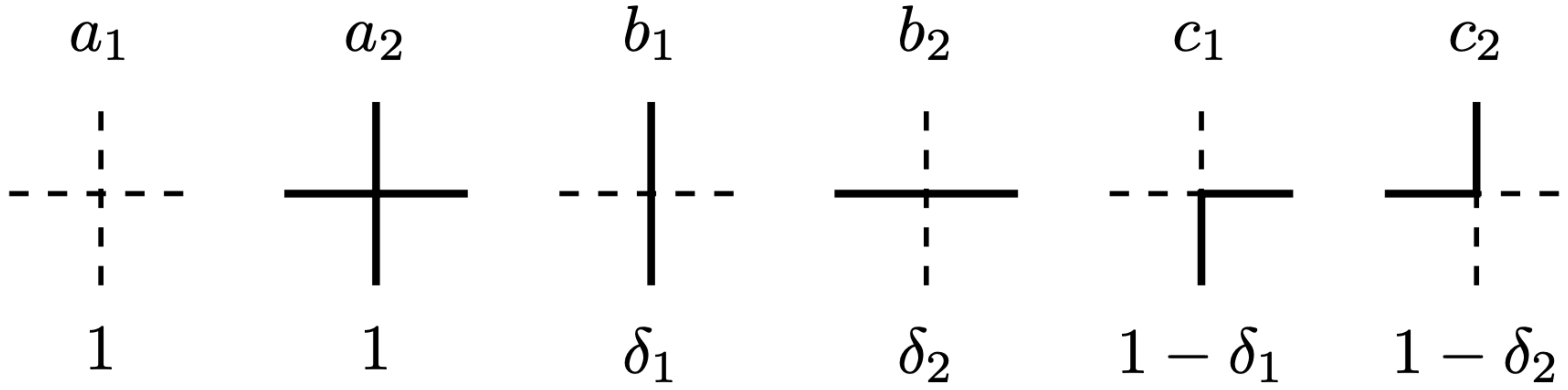
- All previous symmetry results work for TASEP and q-TASEP
- Not for ASEP



- All previous symmetry results work for TASEP and q-TASEP
- Not for ASEP
- Back to the drawing board, start at the level of stochastic six-vertex model

Stationarity for the single-color stochastic six-vertex model





[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

Stationarity for the single-color stochastic six-vertex model

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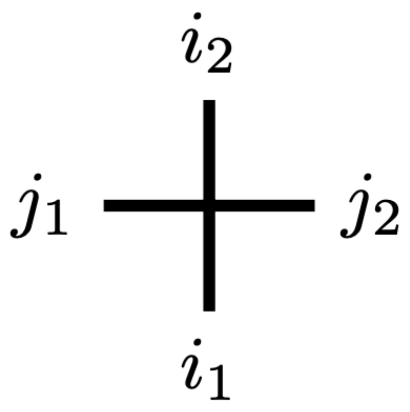
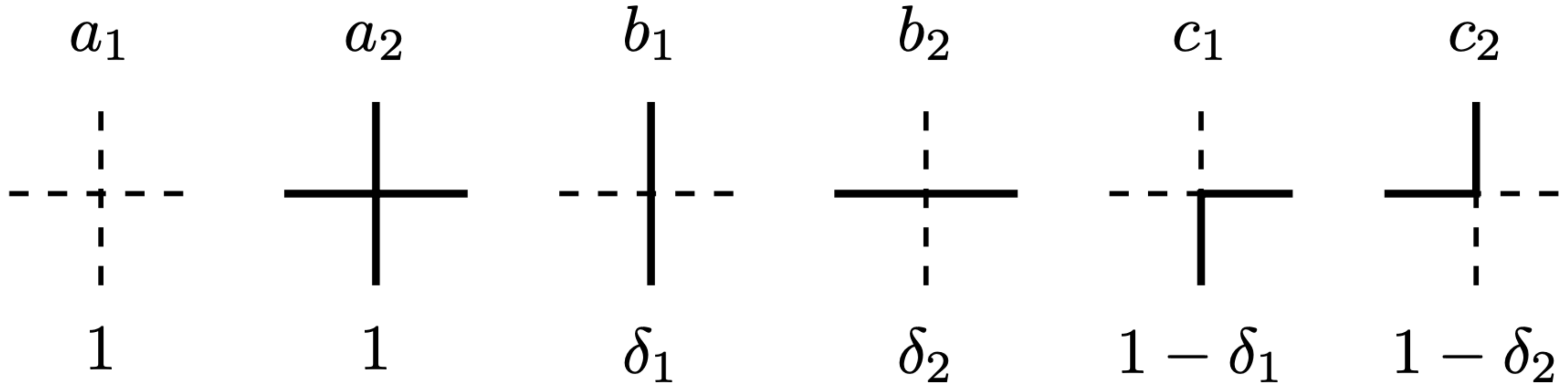
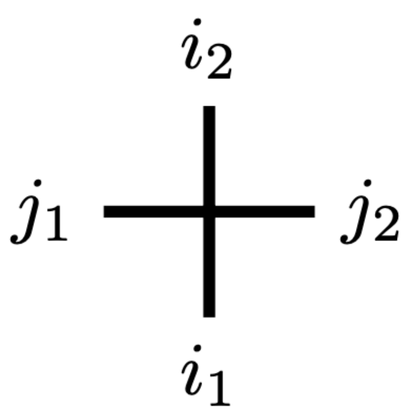
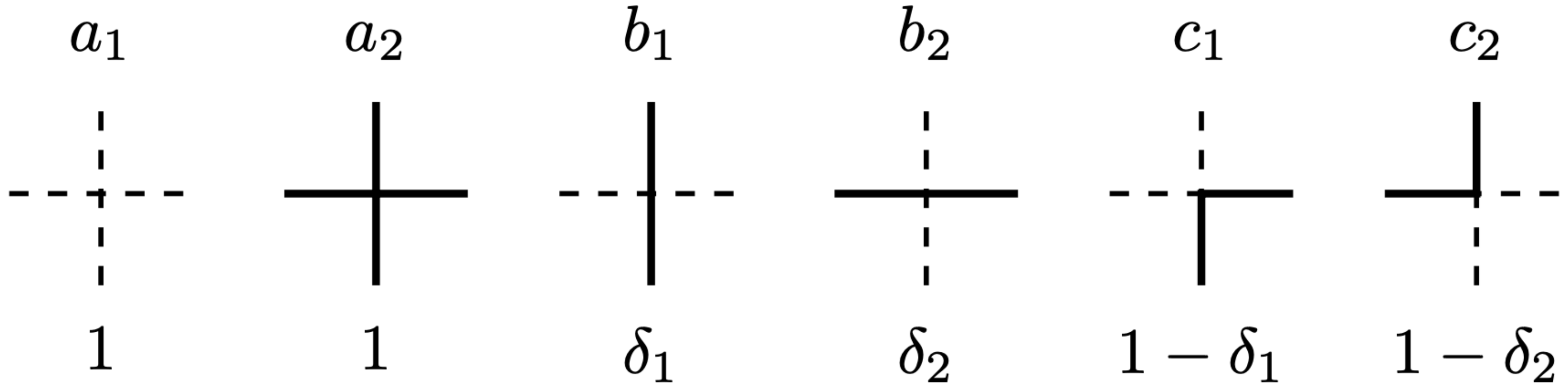




The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1 - \delta_1$, $b_2 = \delta_2$, $c_2 = 1 - \delta_2$ are *stochastic*: $\sum_{i_2, j_2} w(i_1, j_1; i_2, j_2) = 1$.

Stationarity for the single-color stochastic six-vertex model





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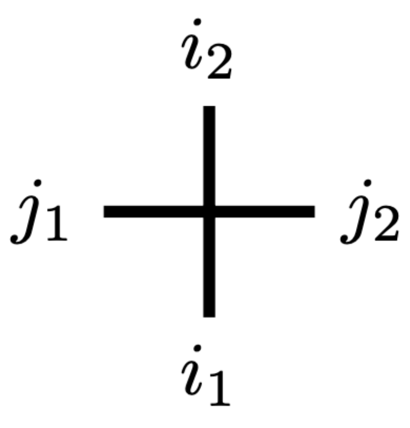

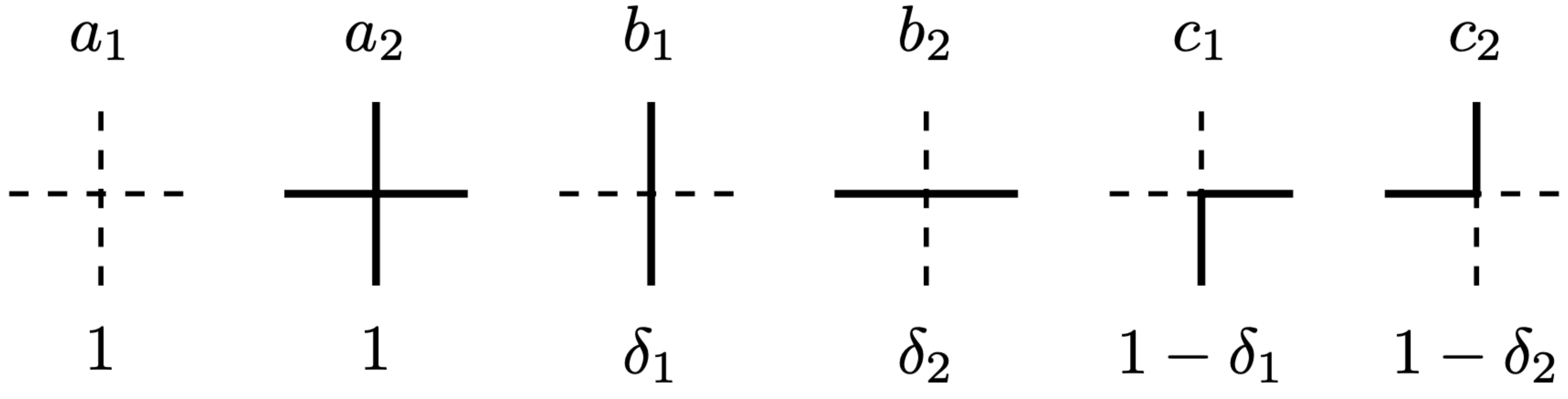

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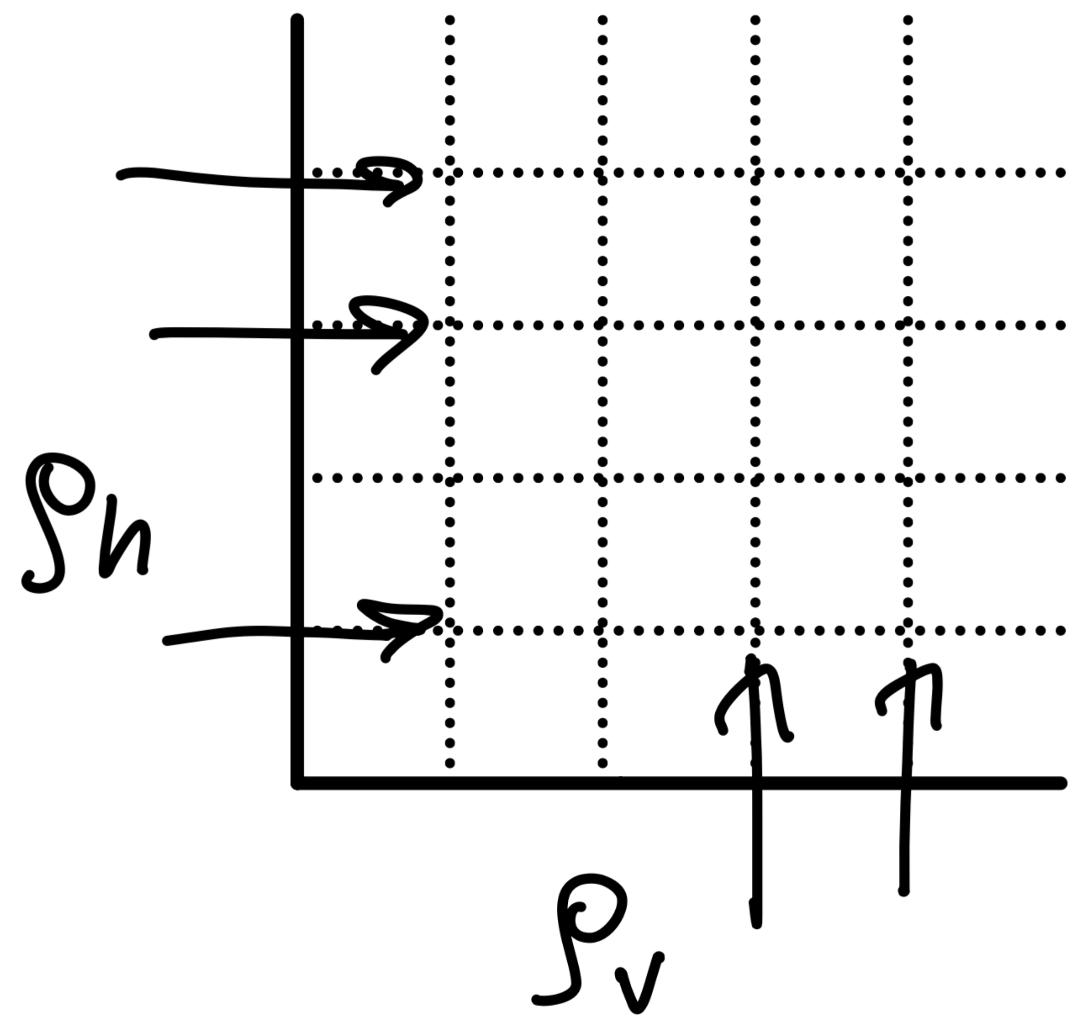

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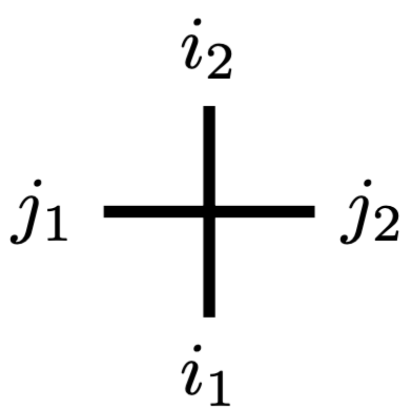

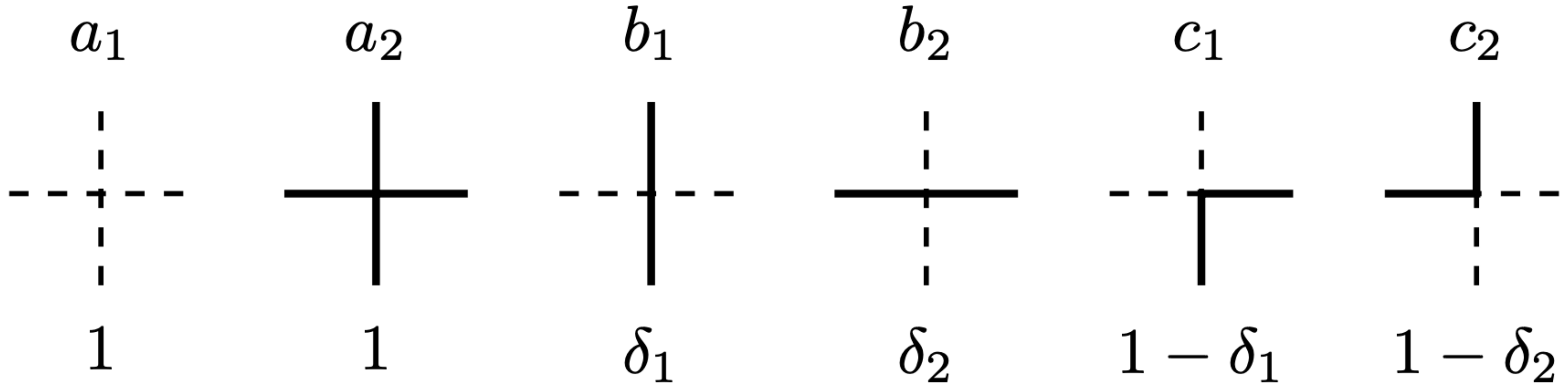

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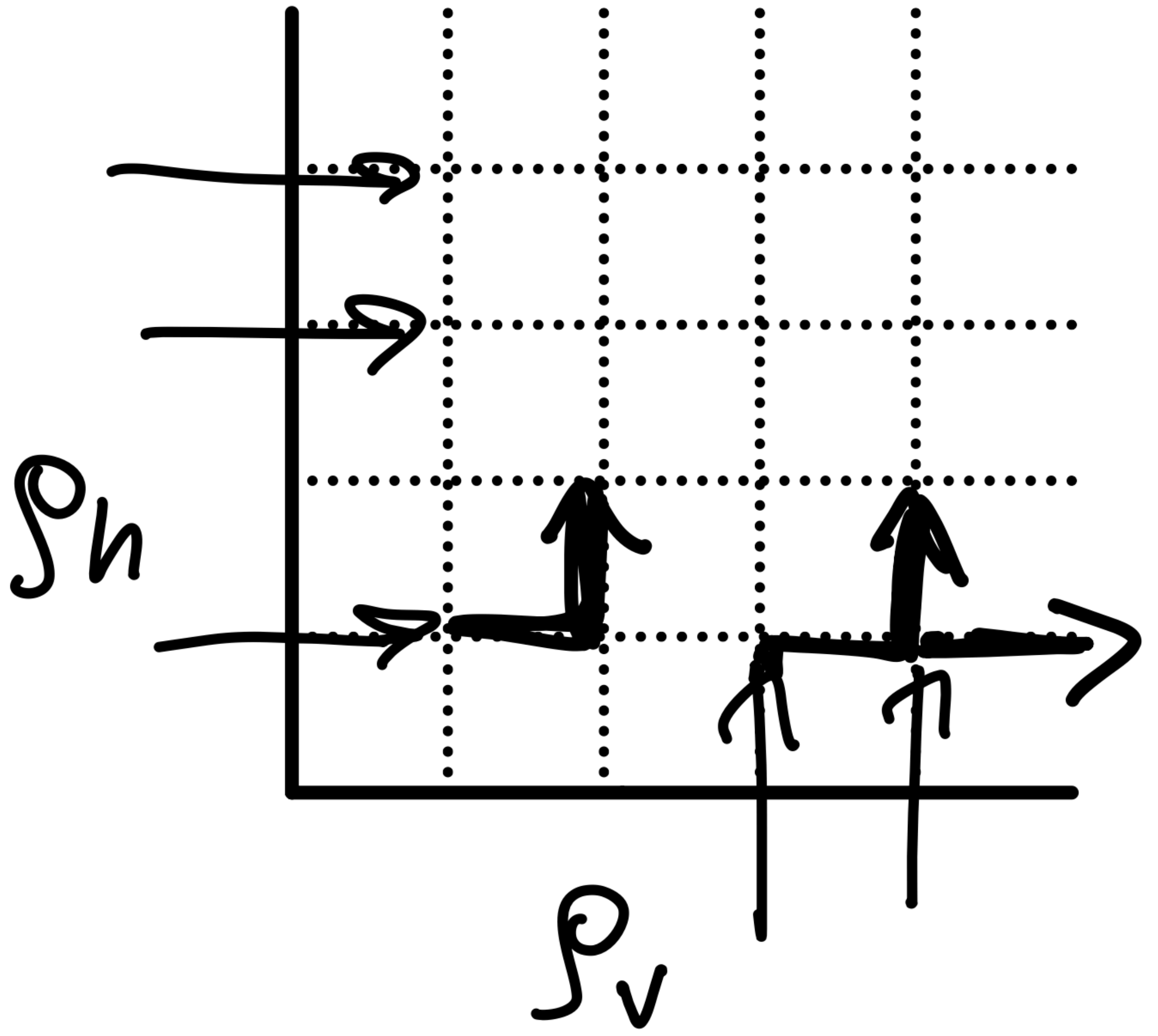
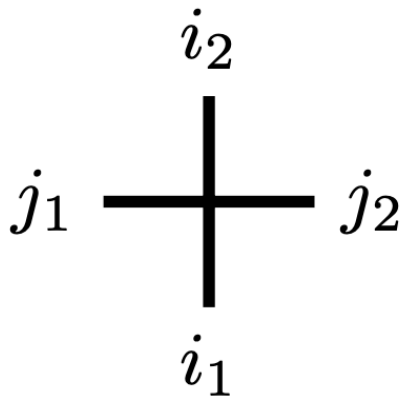
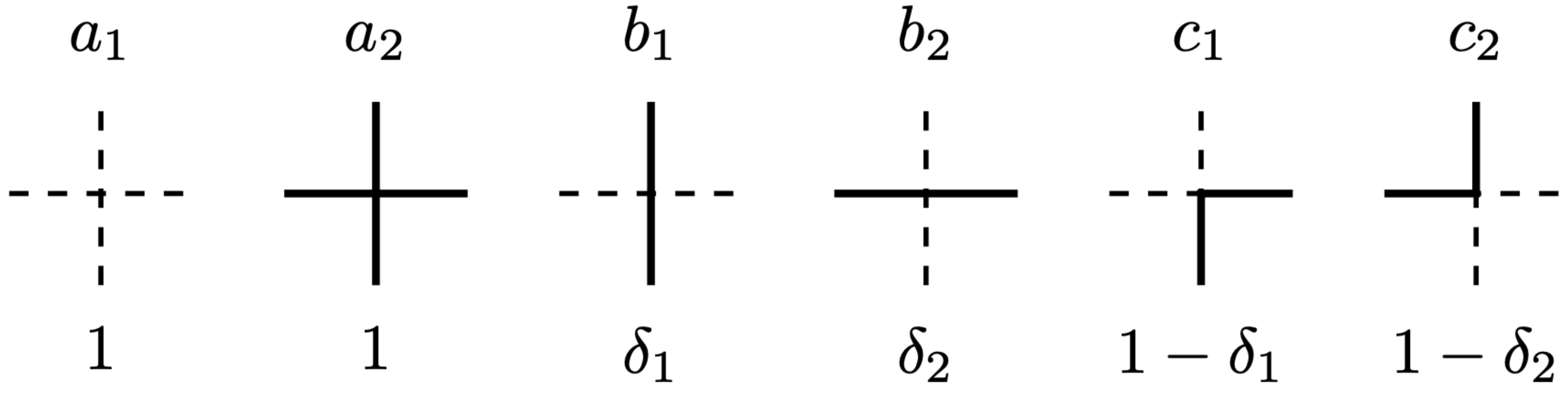


Stationarity for the single-color stochastic six-vertex model





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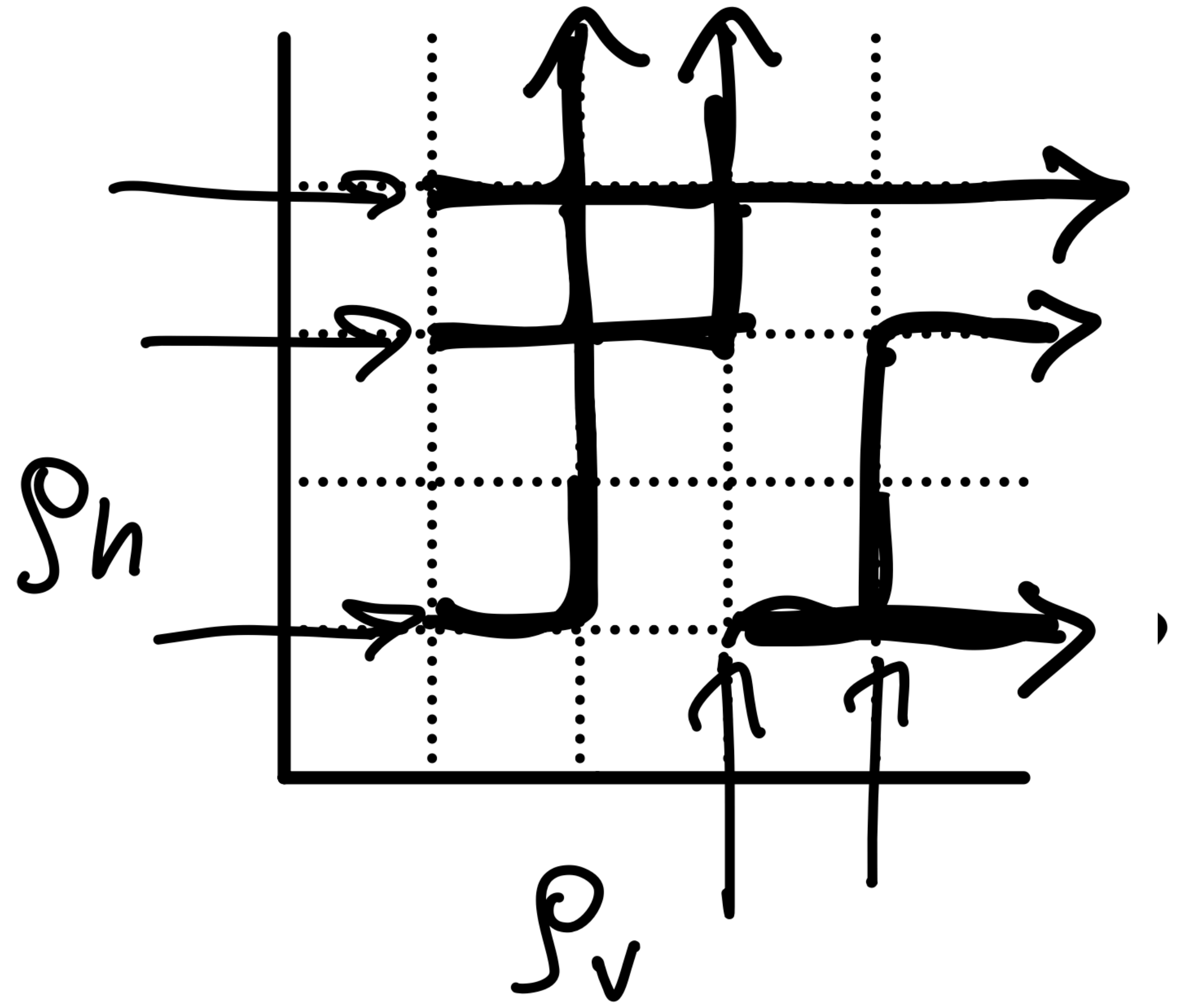
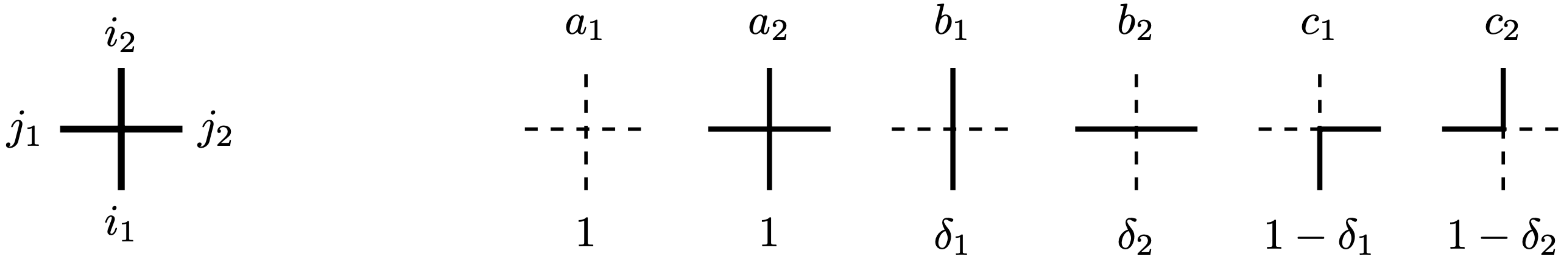


Stationarity for the single-color stochastic six-vertex model



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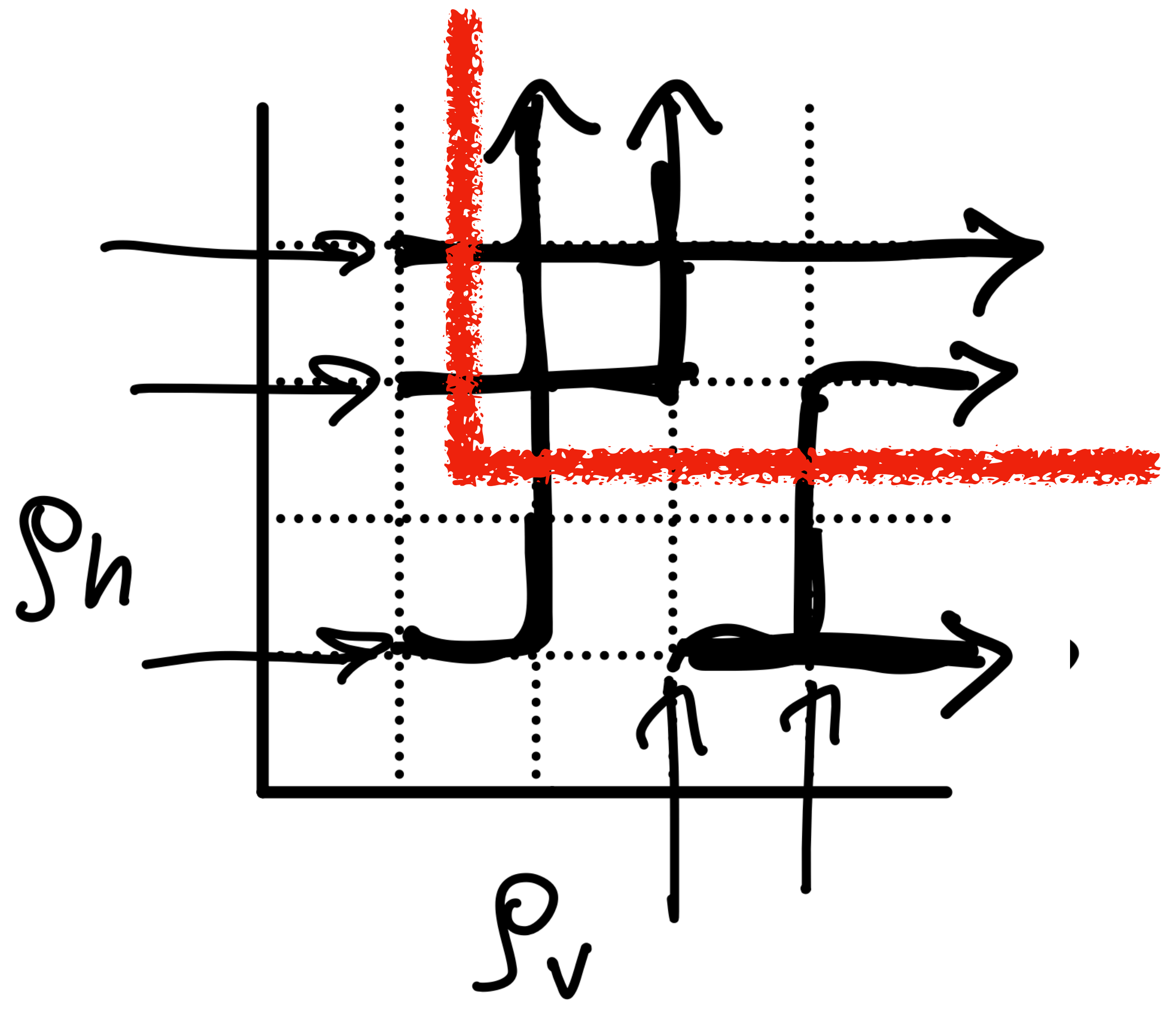

Stationarity for the single-color stochastic six-vertex model

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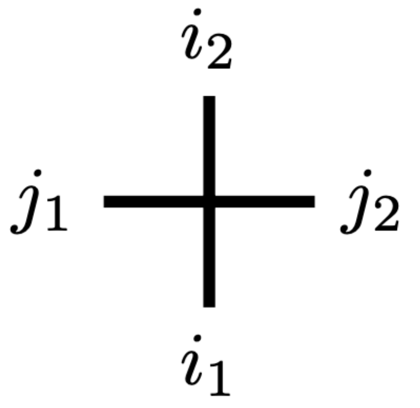

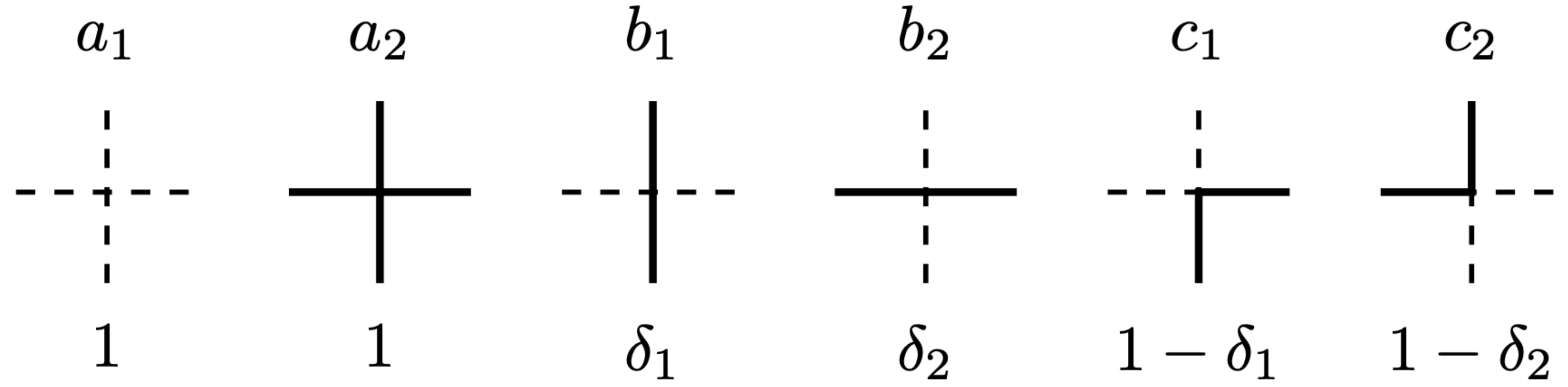

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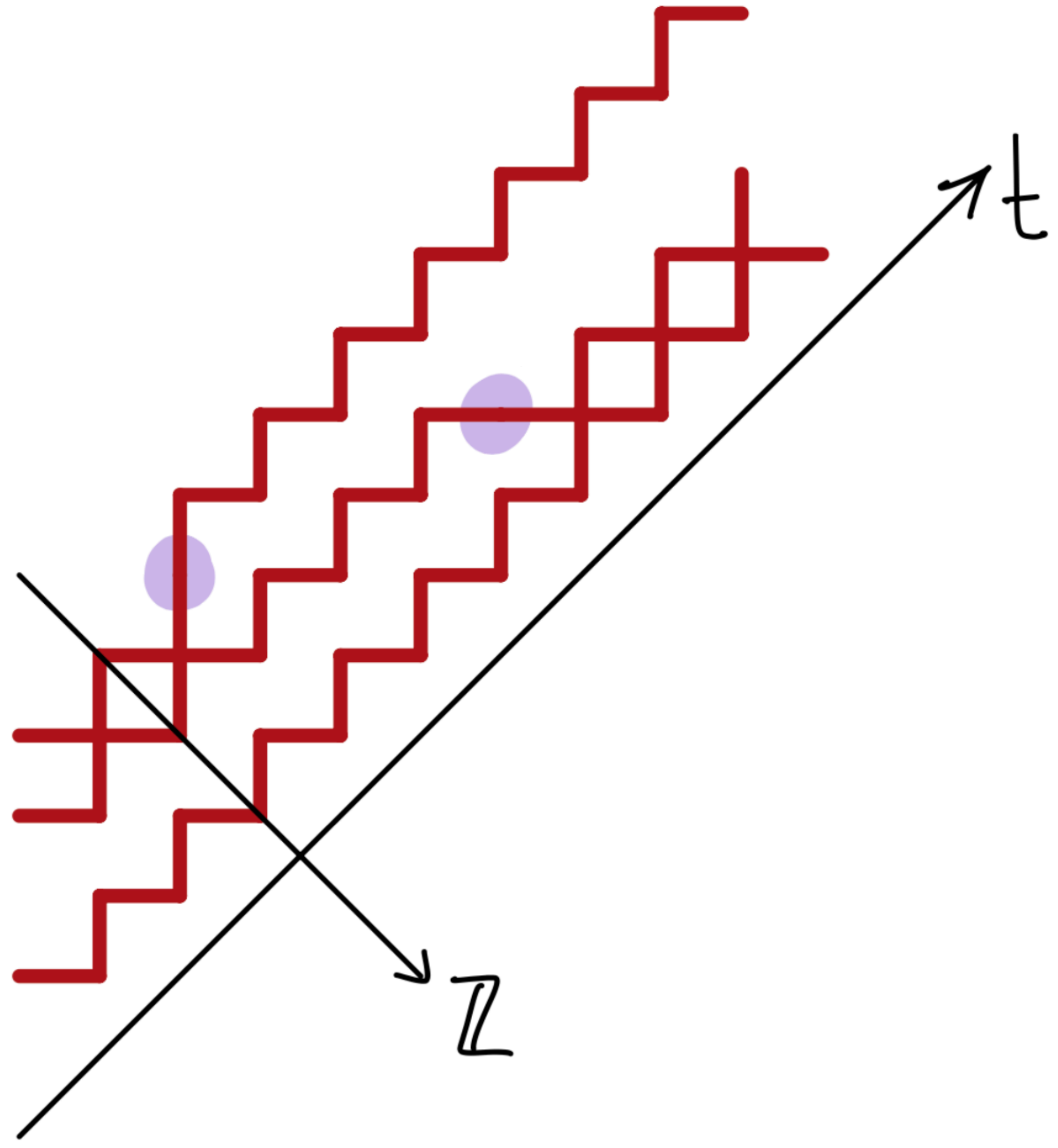
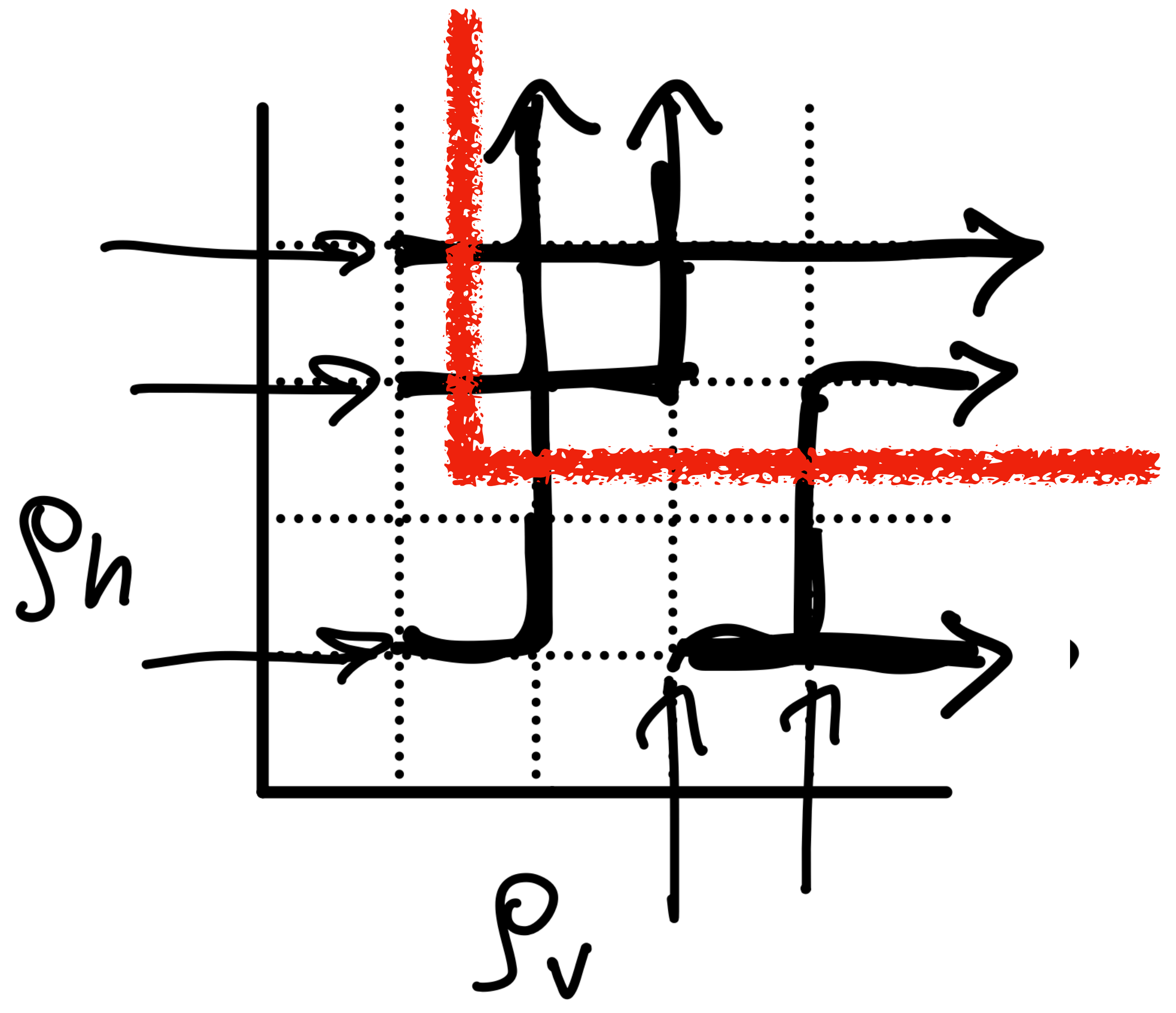
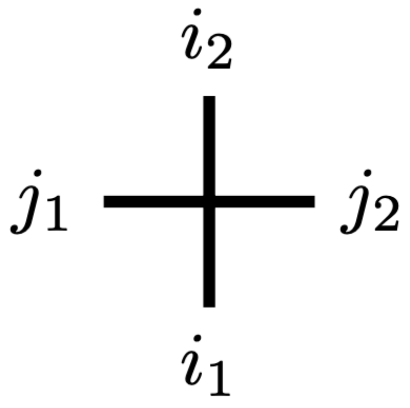
Converges to ASEP along the diagonal as

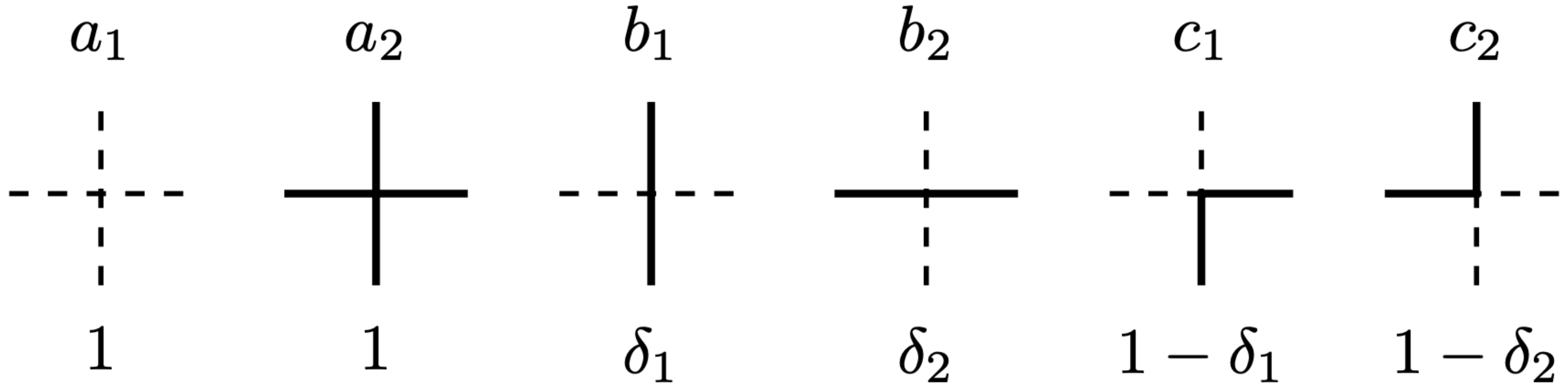
$\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)





Stationarity for the single-color stochastic six-vertex model





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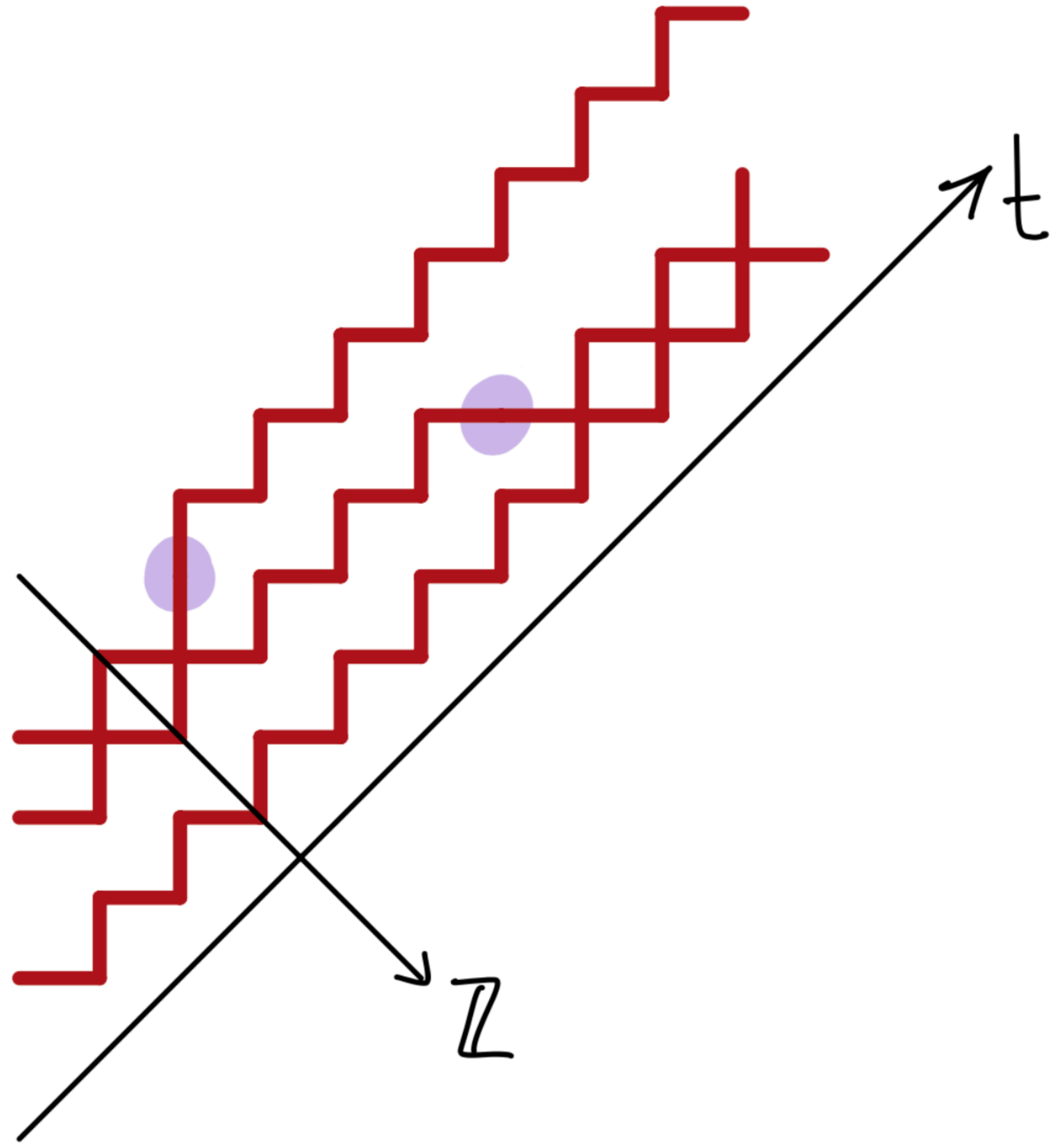
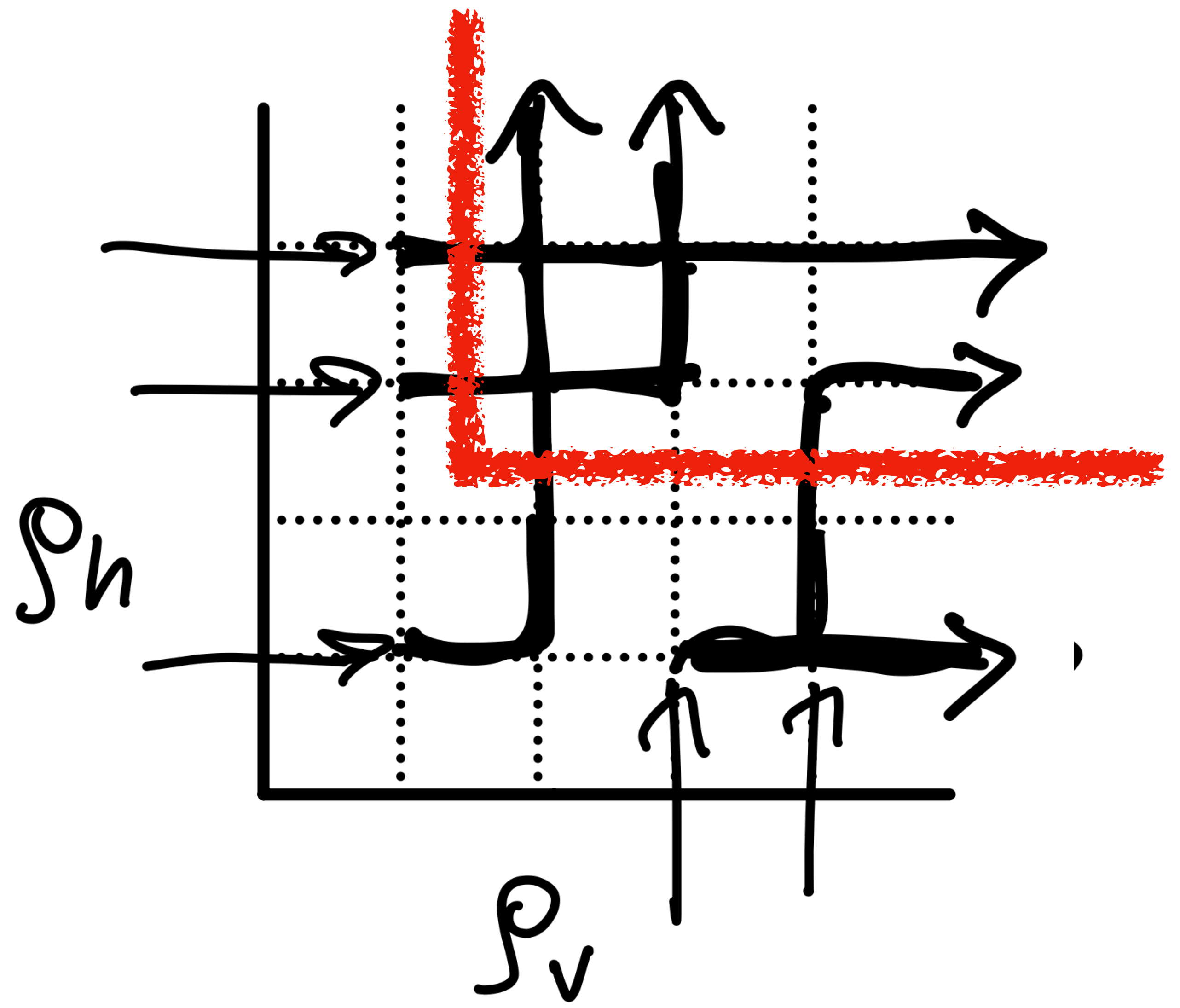
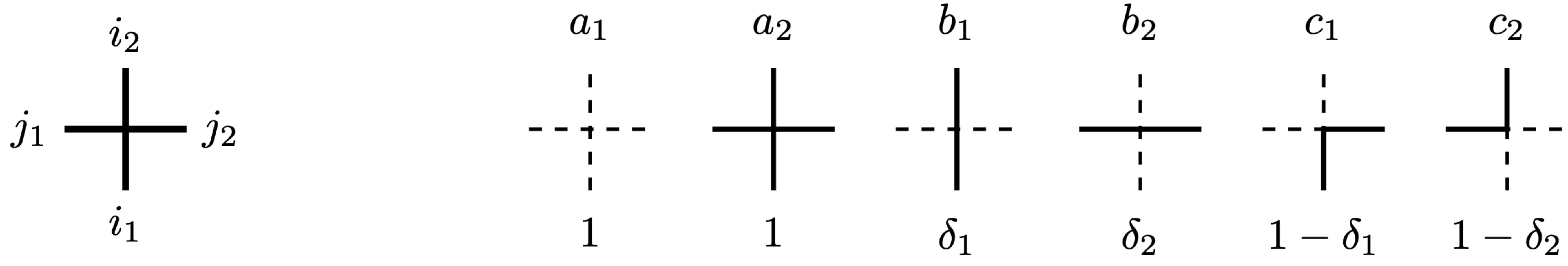


- **Stationarity.** Assume that the boundary conditions are Bernoulli with densities ρ_h, ρ_v. Then for $\rho_h = \dfrac{u\rho_v}{1-\rho_v+u\rho_v}$, the distribution is stationary in the quadrant.



Stationarity for the single-color stochastic six-vertex model



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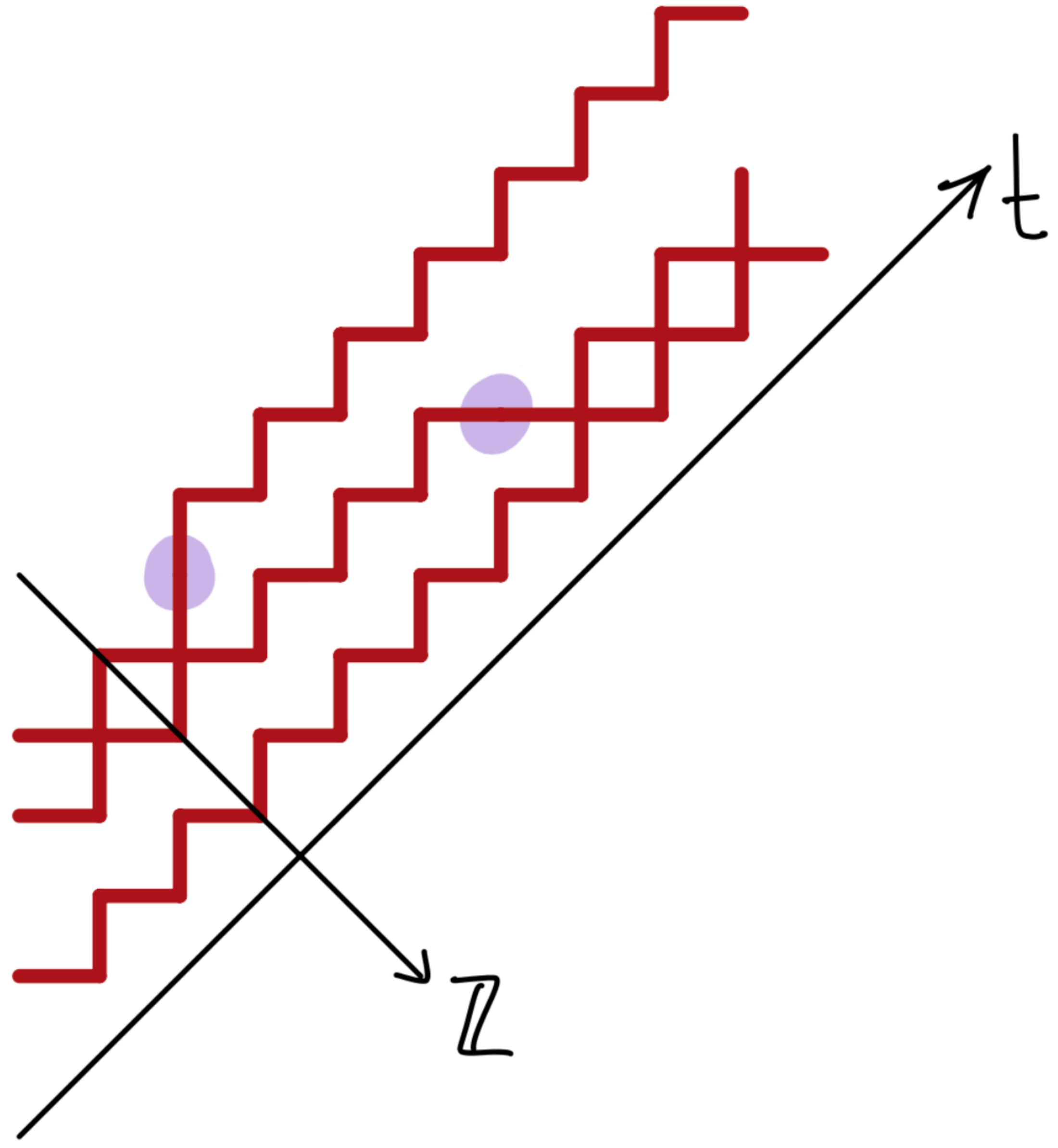
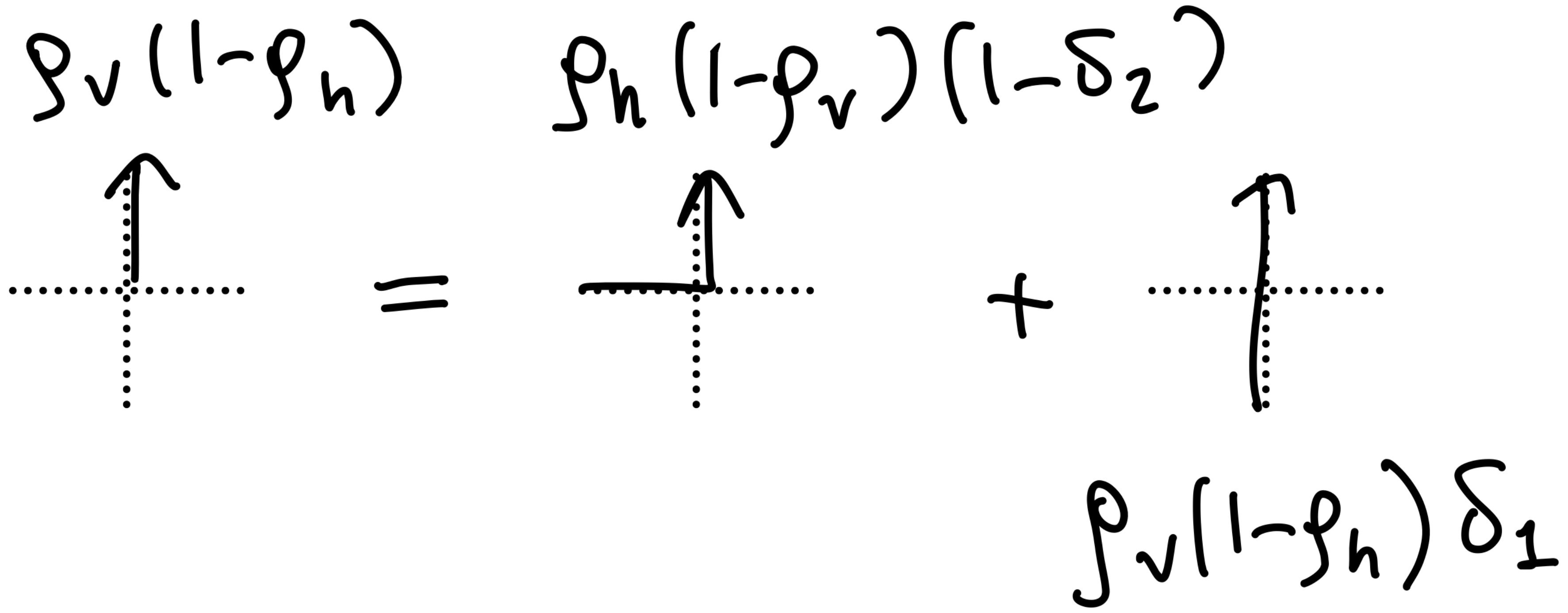
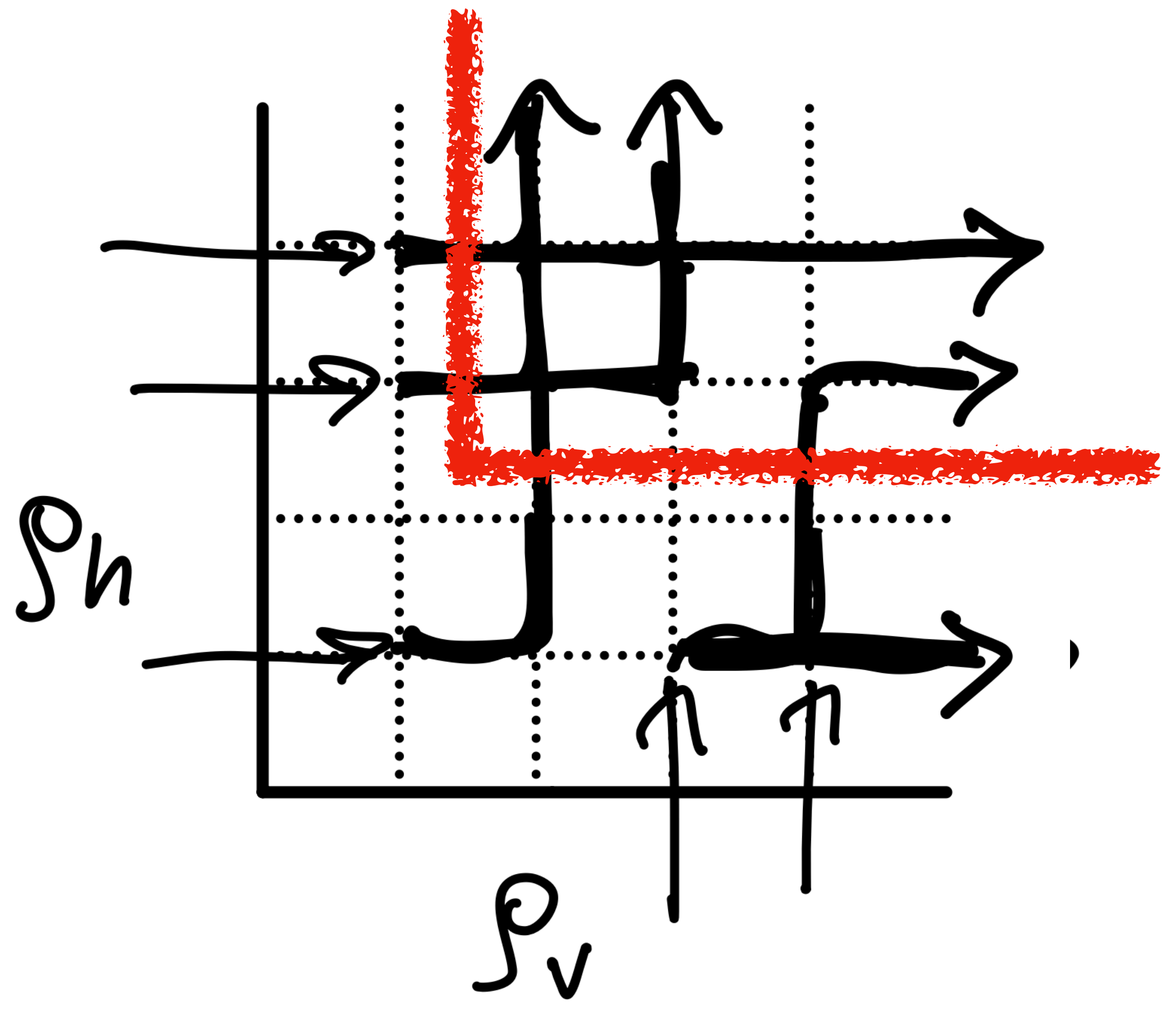
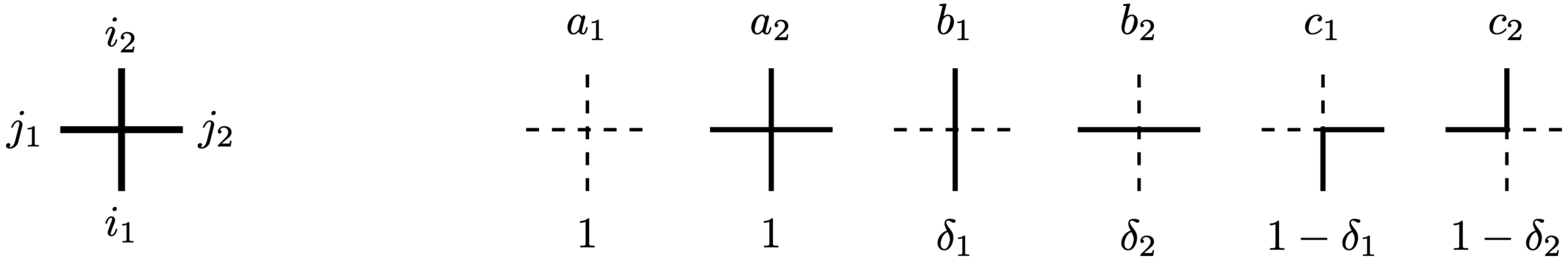






Stationarity for the single-color stochastic six-vertex model



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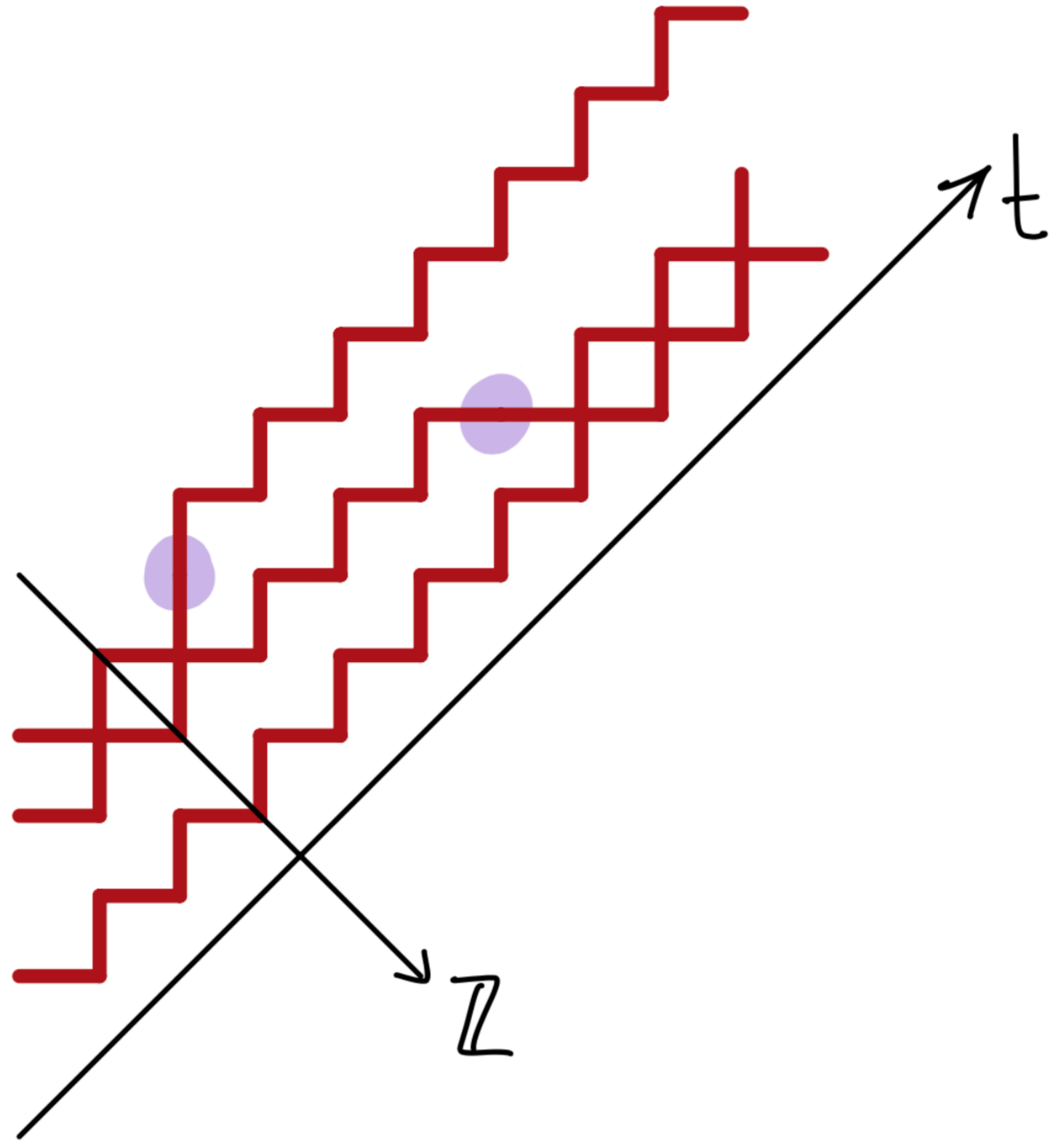
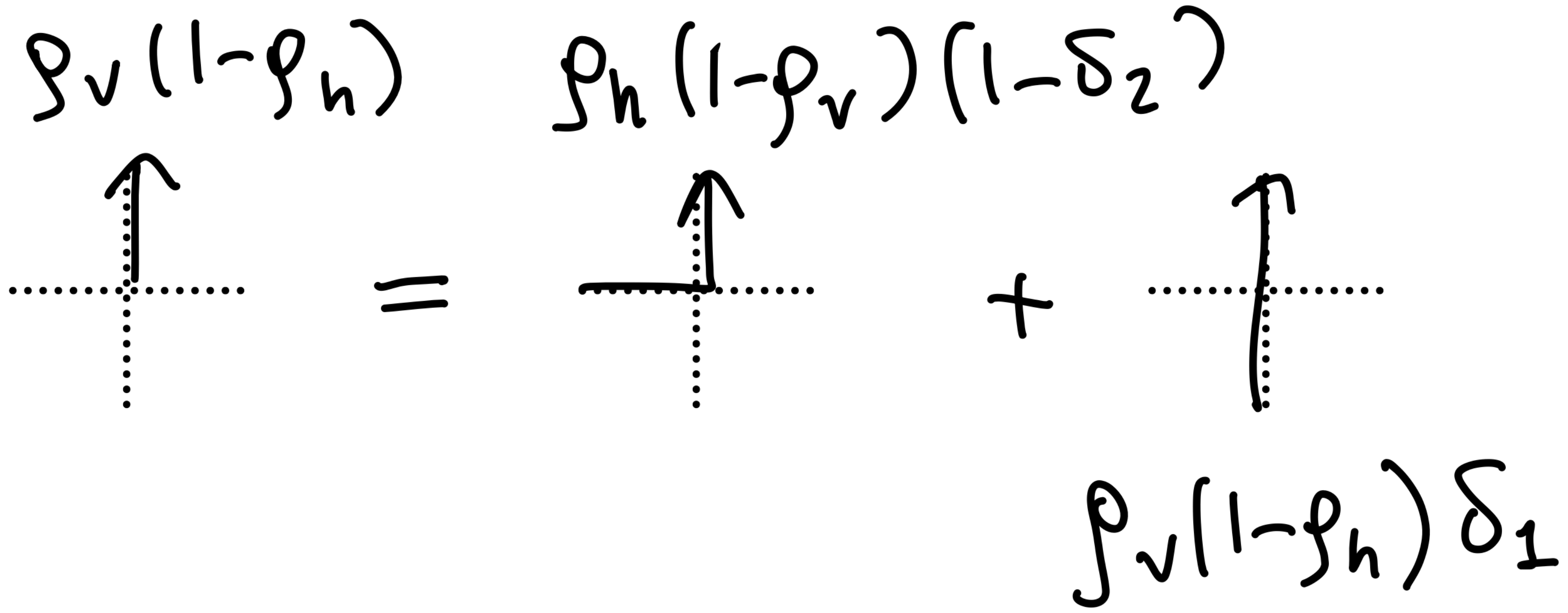
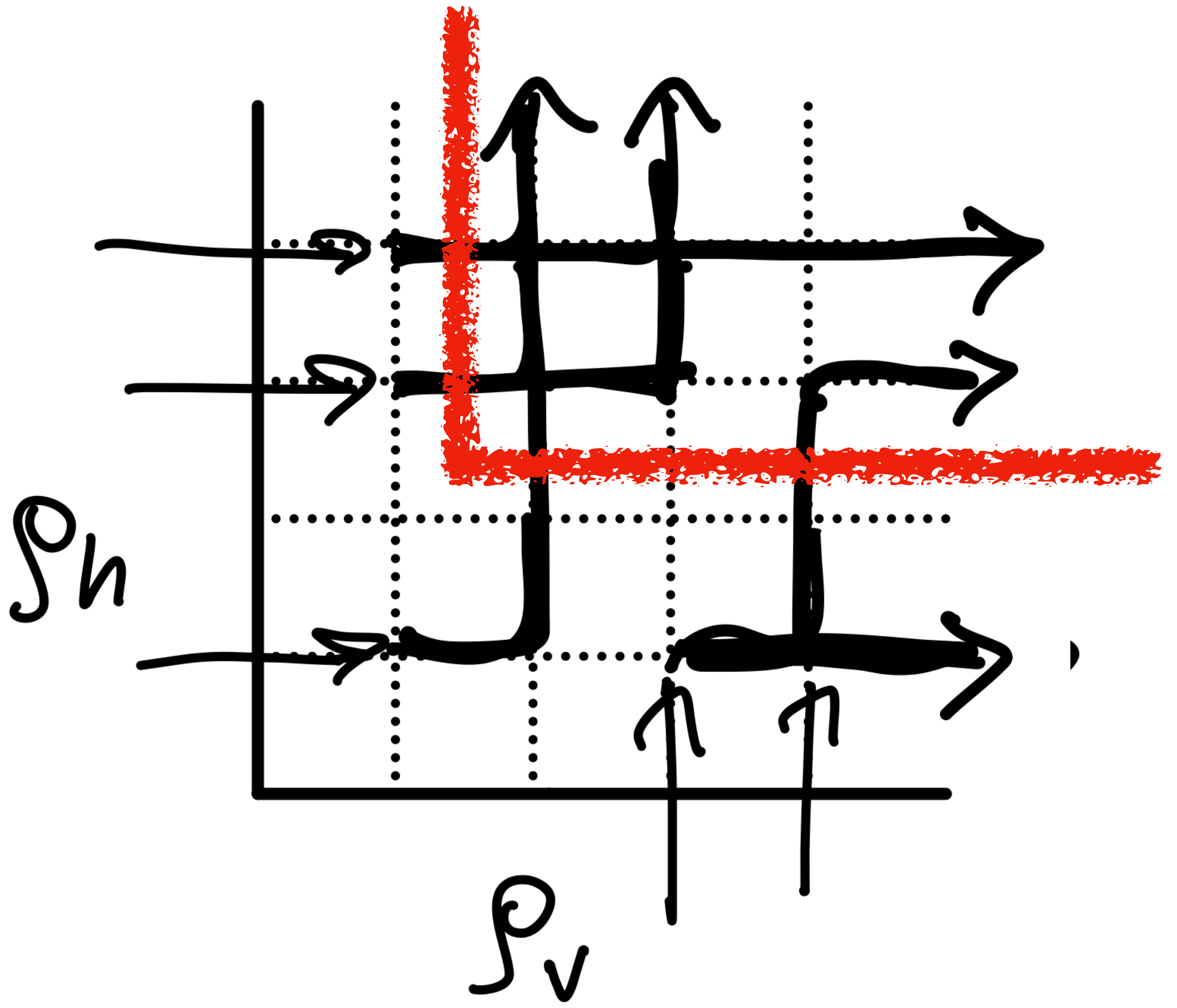




“Burke property”



Stationarity for the single-color stochastic six-vertex model

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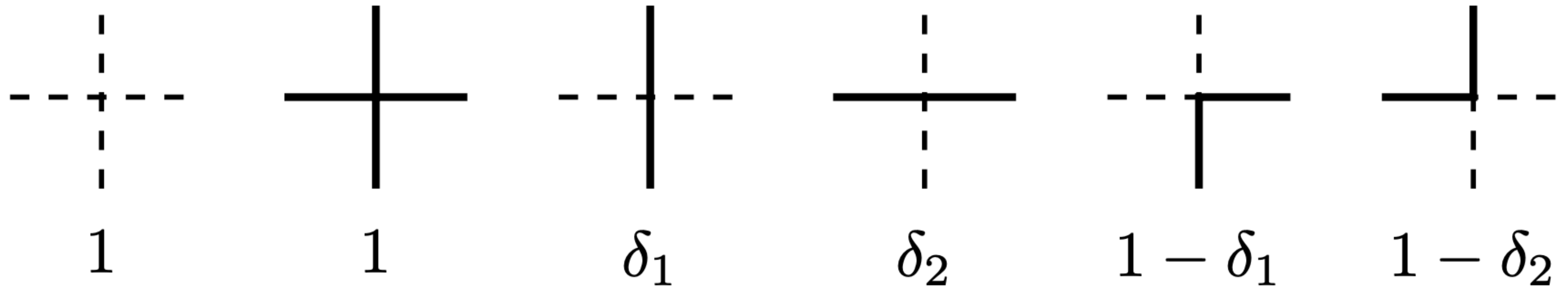
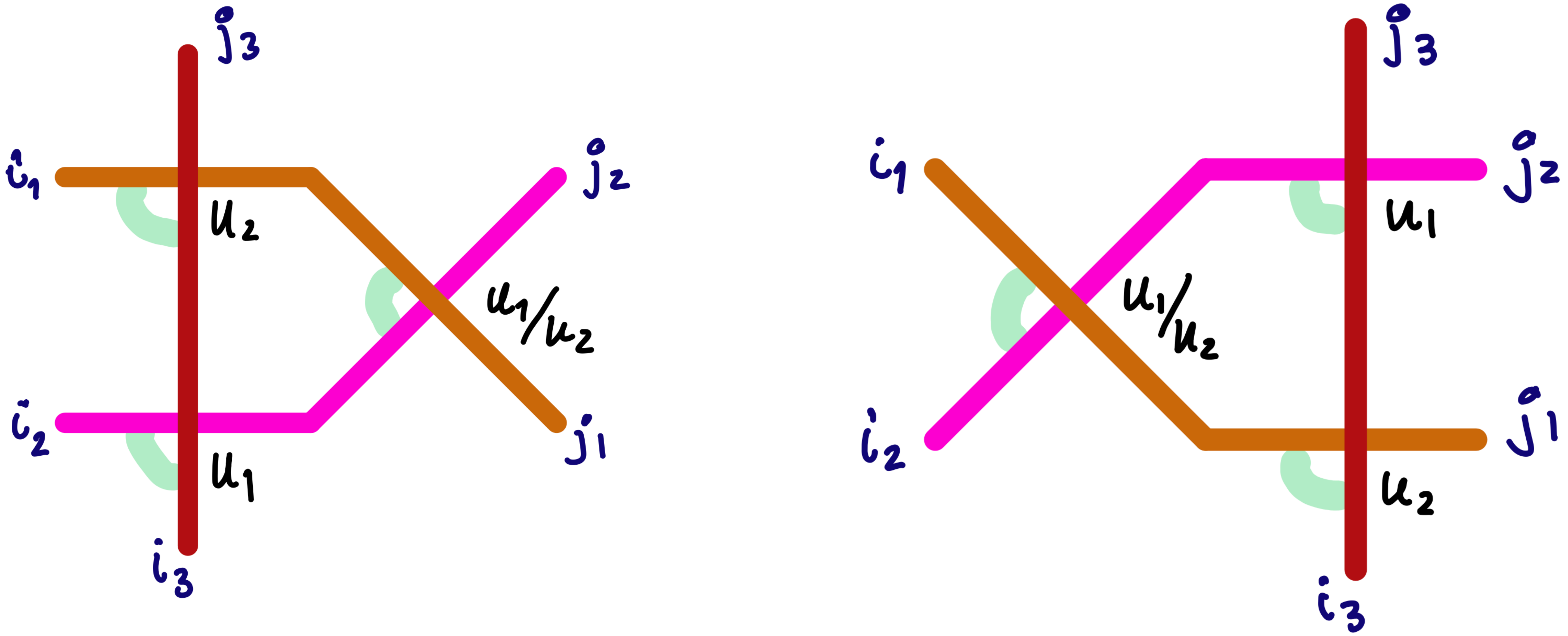


$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

- **Yang-Baxter equation.** For fixed q, and fixed $i_1, i_2, i_3 \in \{0,1\}$, the joint distribution of j_1, j_2, j_3 in two pictures is the same:



Stationarity for the single-color stochastic six-vertex model

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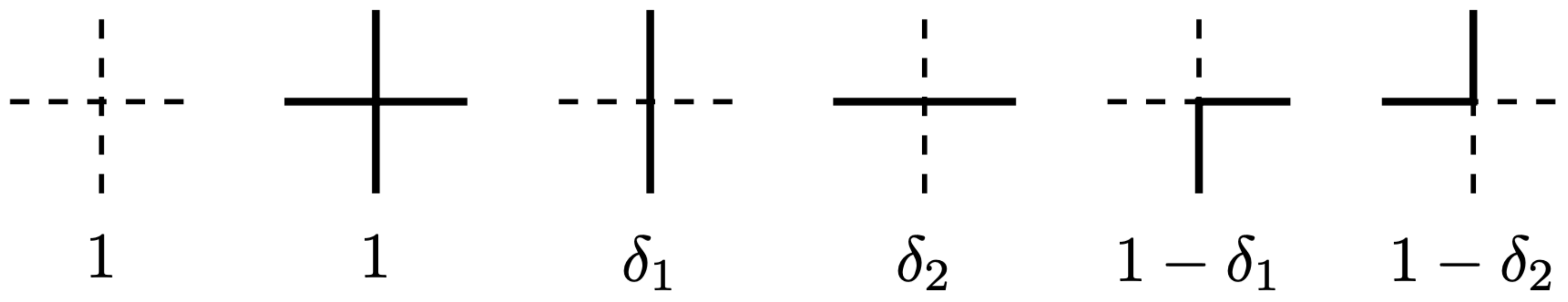

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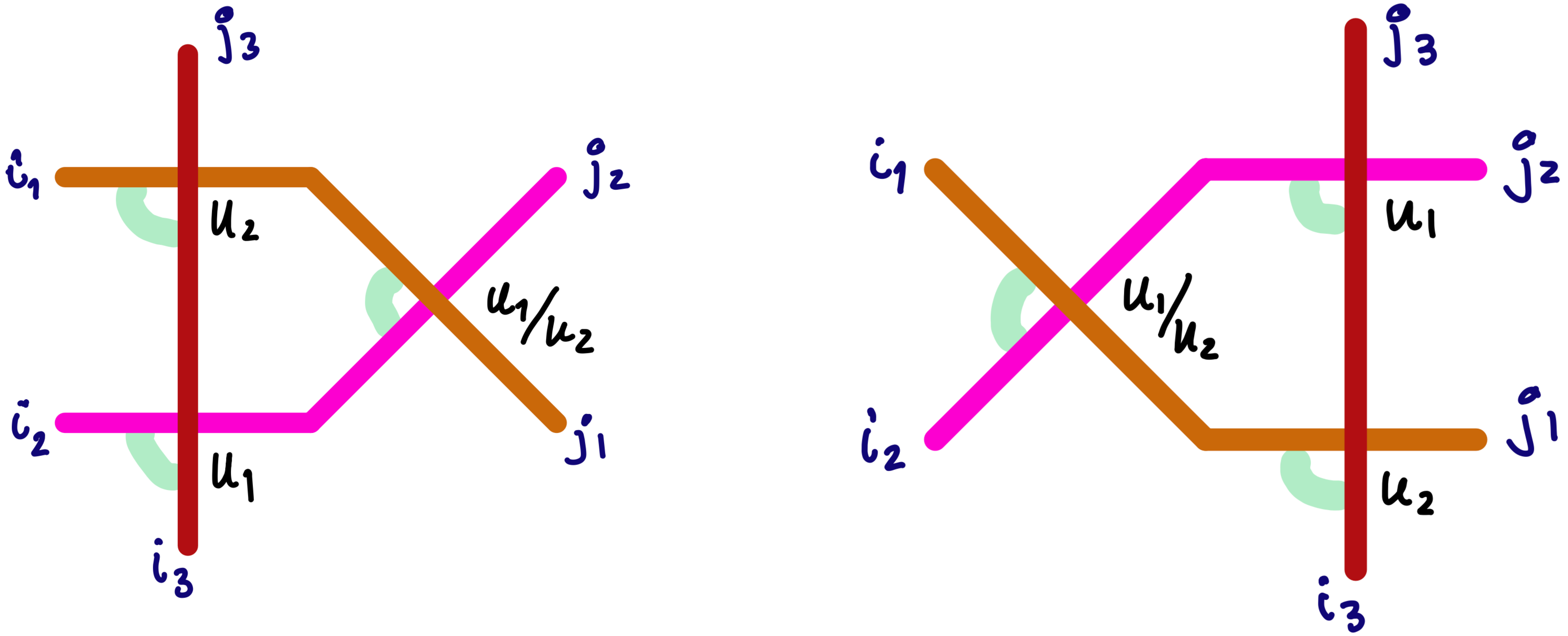

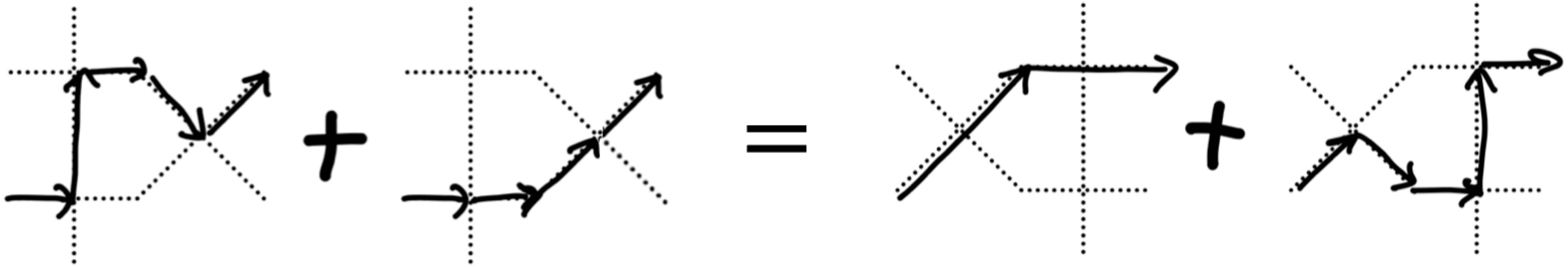

Stationarity for the single-color stochastic six-vertex model

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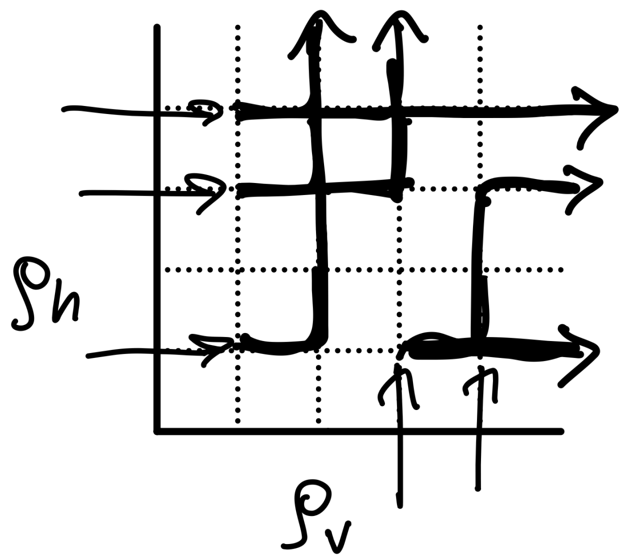

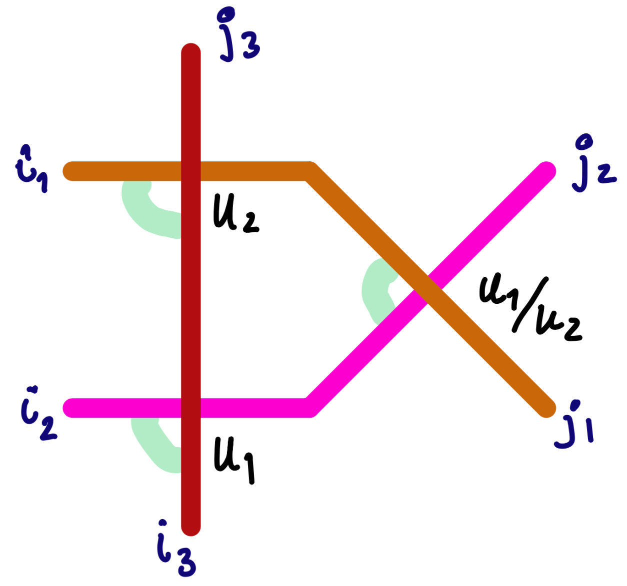

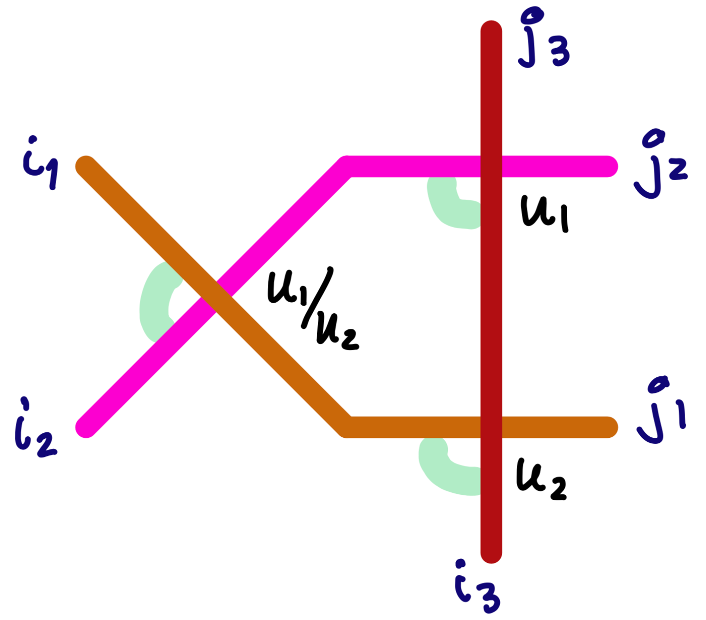

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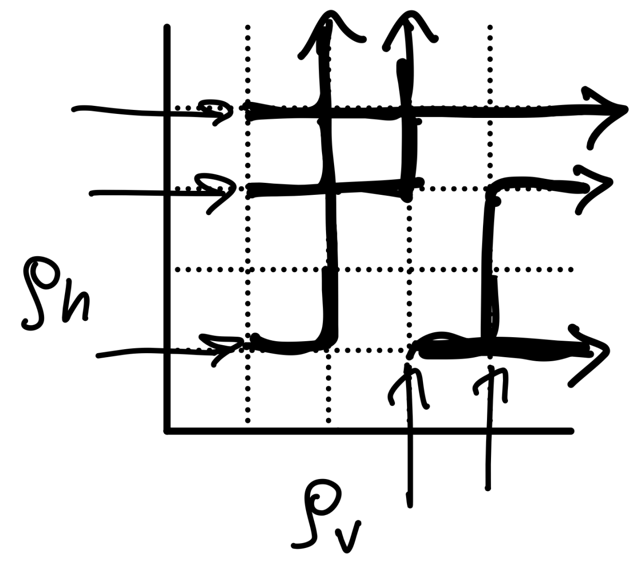

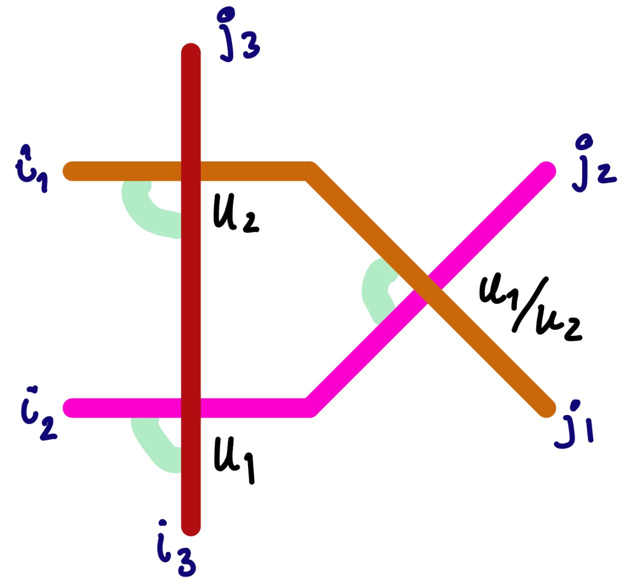

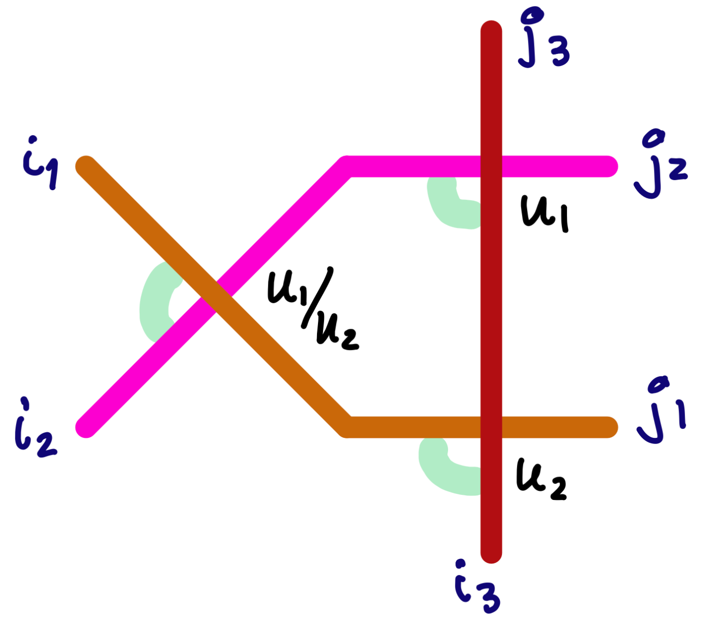

- **Fusion [Kulish-Reshetikhin-Sklyanin 1983], [Corwin-P. 2015]** - a way to construct new YBE solutions from existing ones. **Inserts ∞ arrows.**

Stationarity for the single-color stochastic six-vertex model

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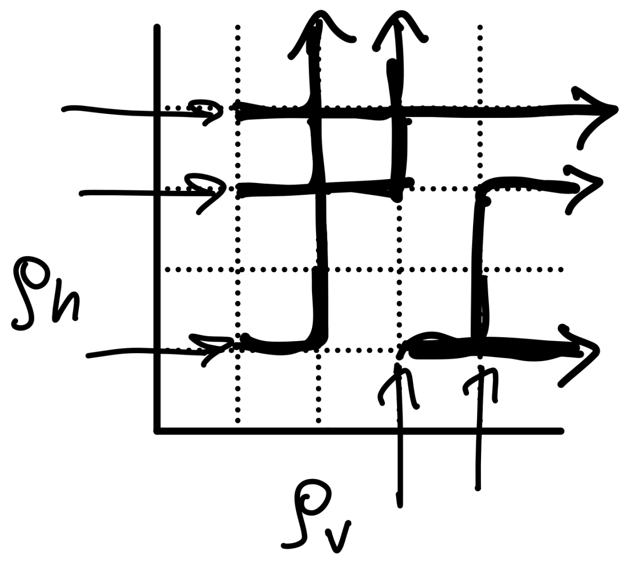

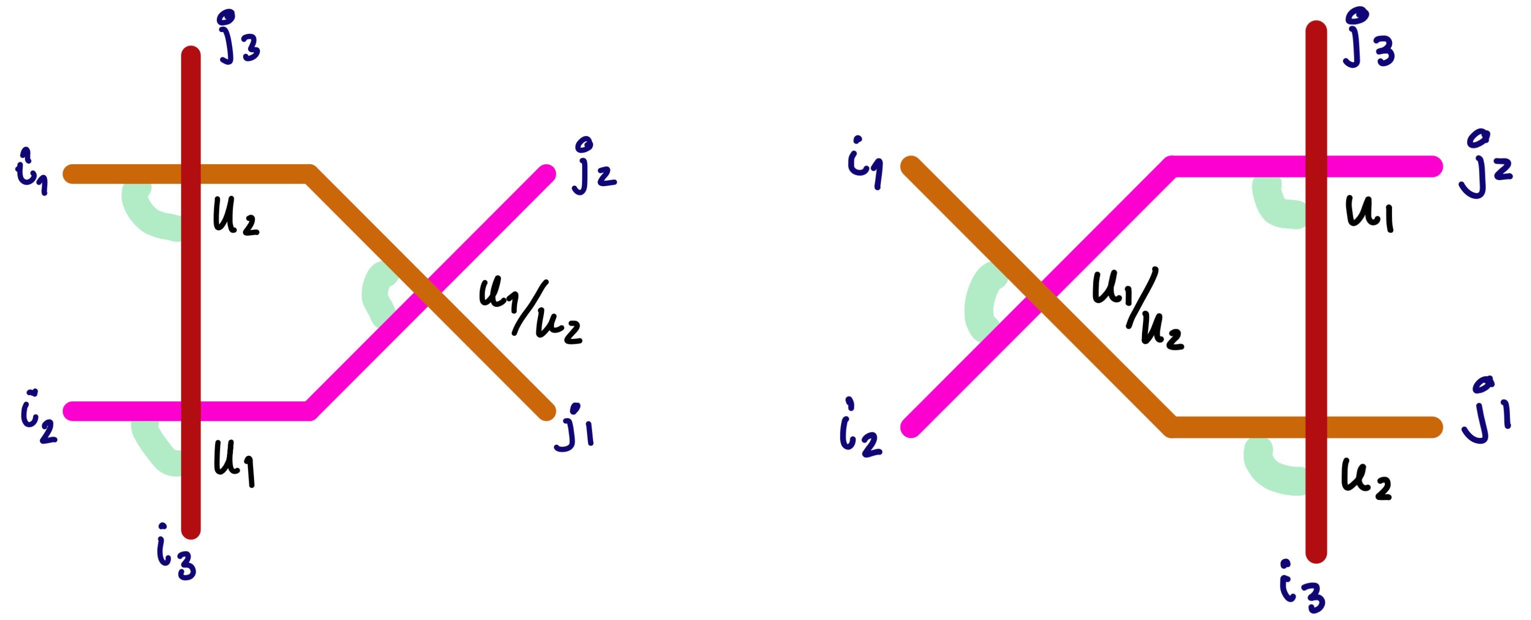

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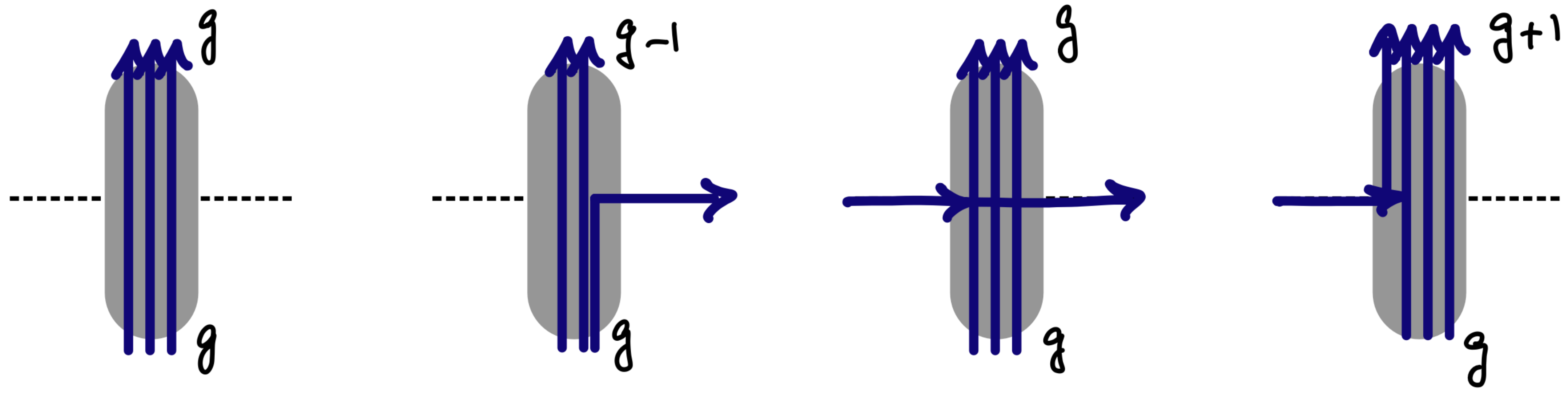

$$\frac{1-xq^g}{1+x} \qquad \frac{x(1-q^g)}{1+x} \qquad \frac{x}{1+x} \qquad \frac{1}{1+x}$$

Stationarity for the single-color stochastic six-vertex model

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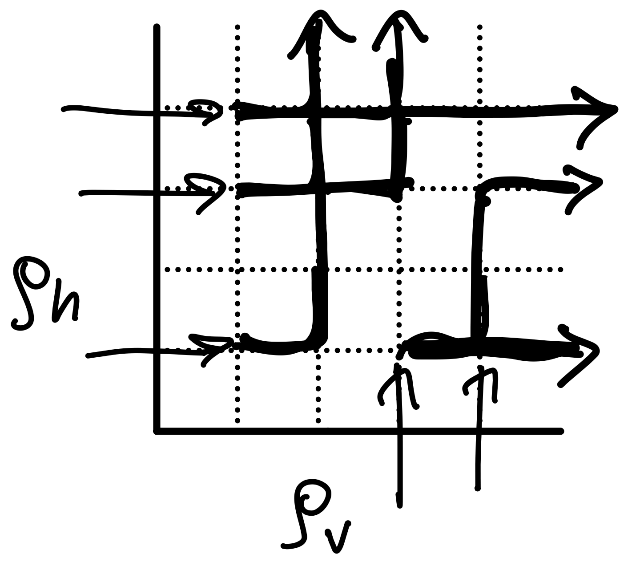

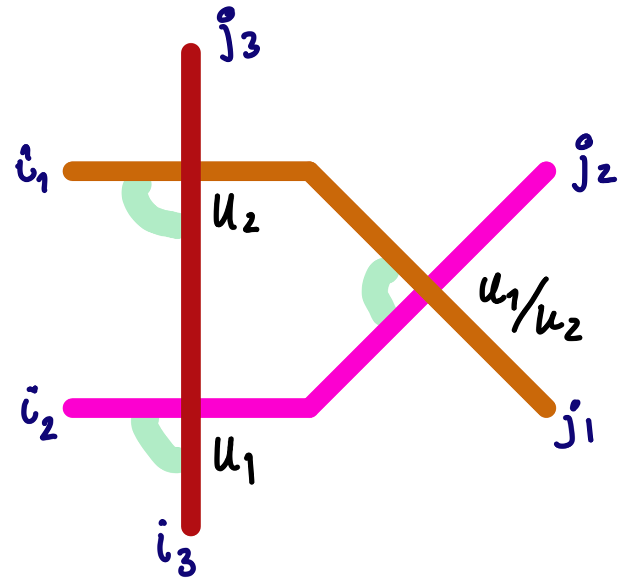

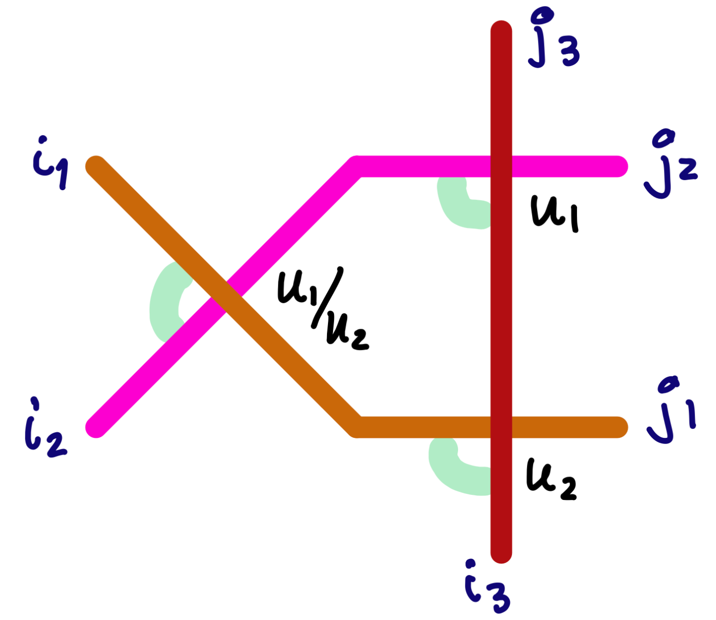

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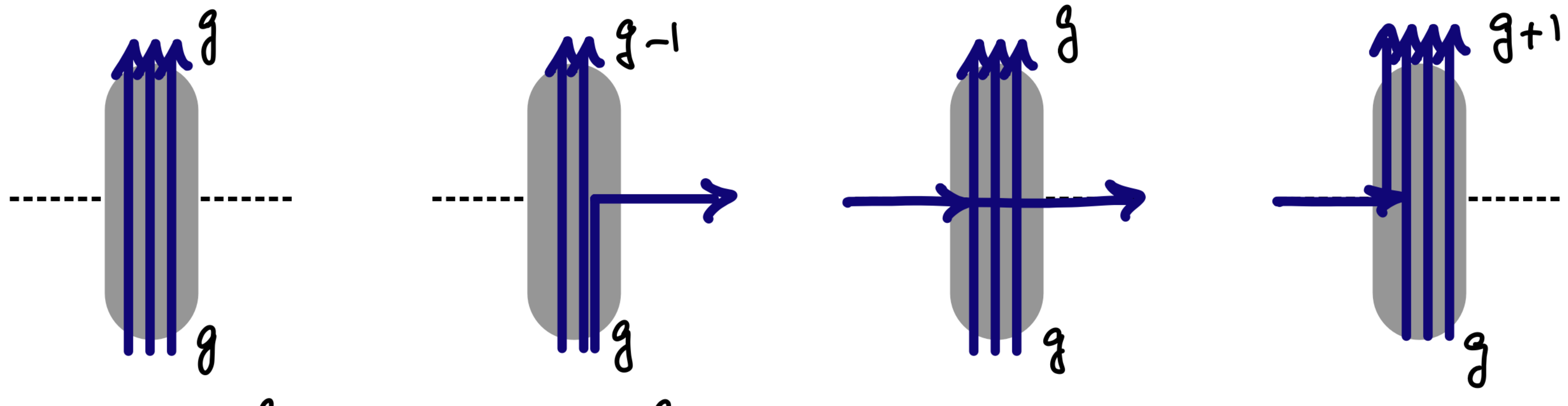
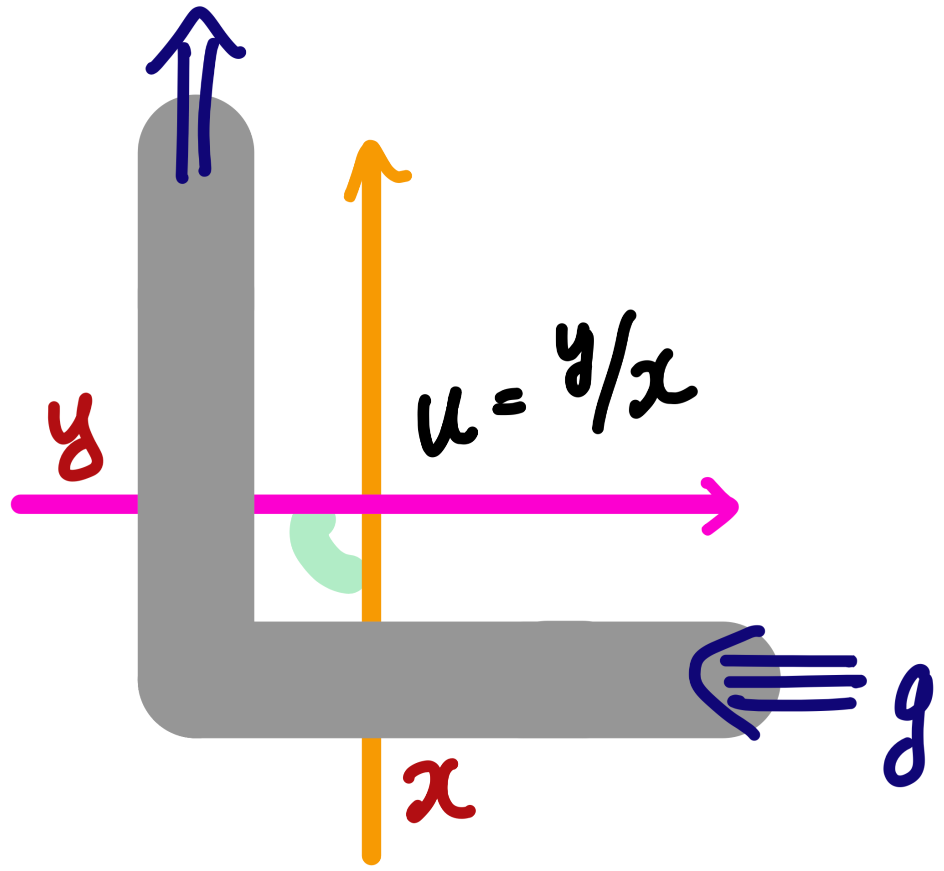
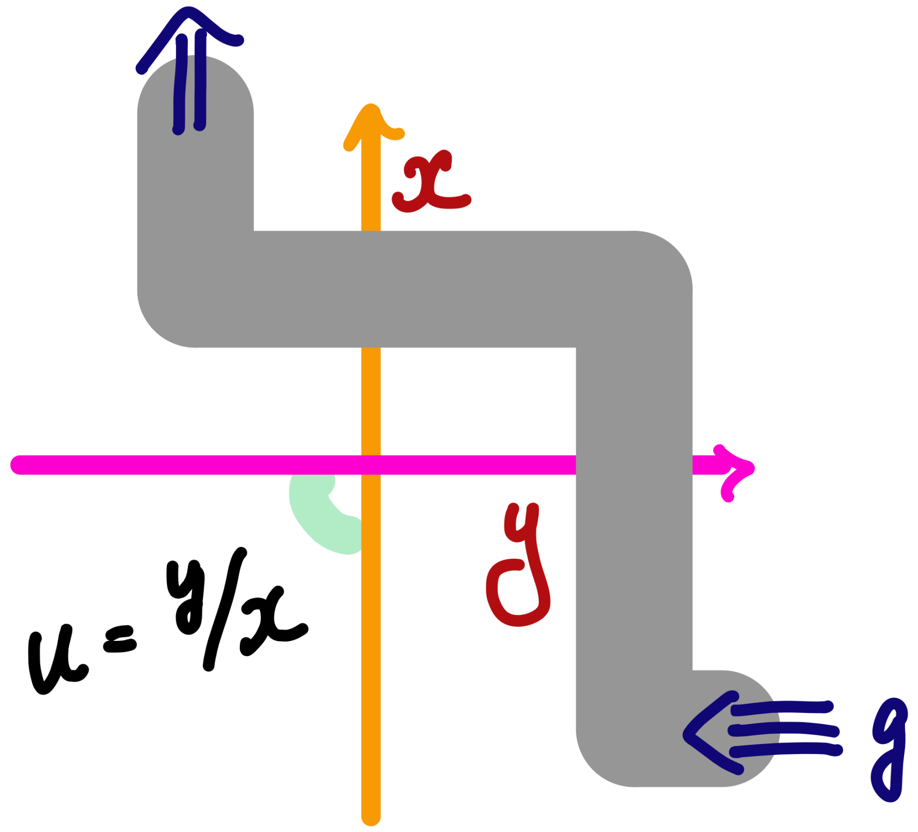


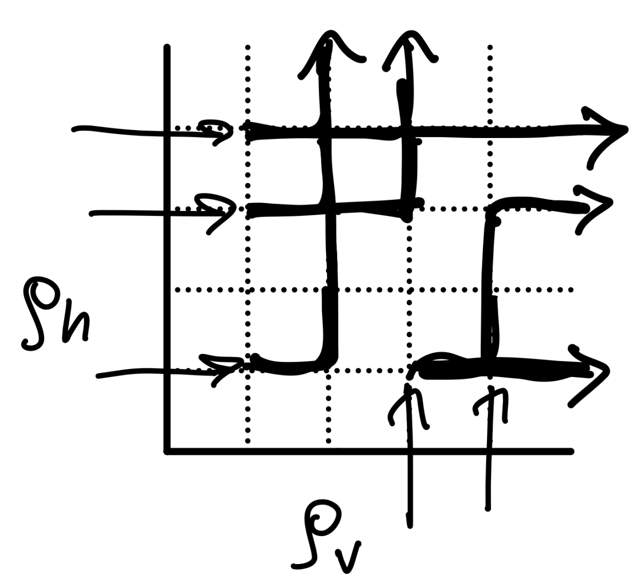
$$\frac{1-xq^g}{1+x} \qquad \frac{x(1-q^g)}{1+x} \qquad \frac{x}{1+x} \qquad \frac{1}{1+x}$$

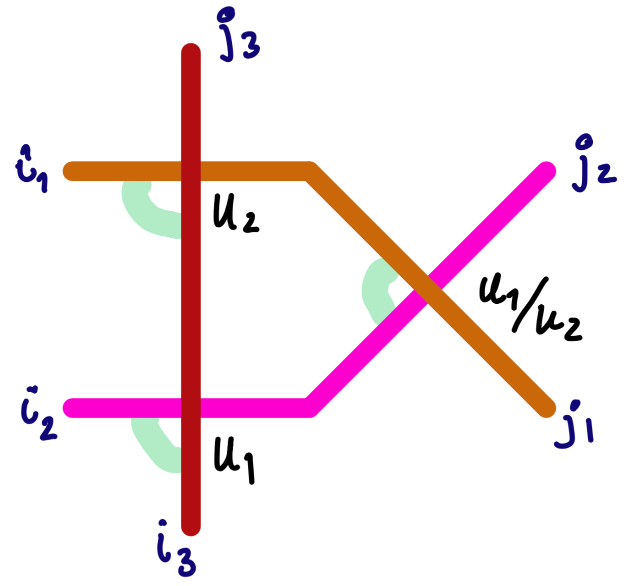


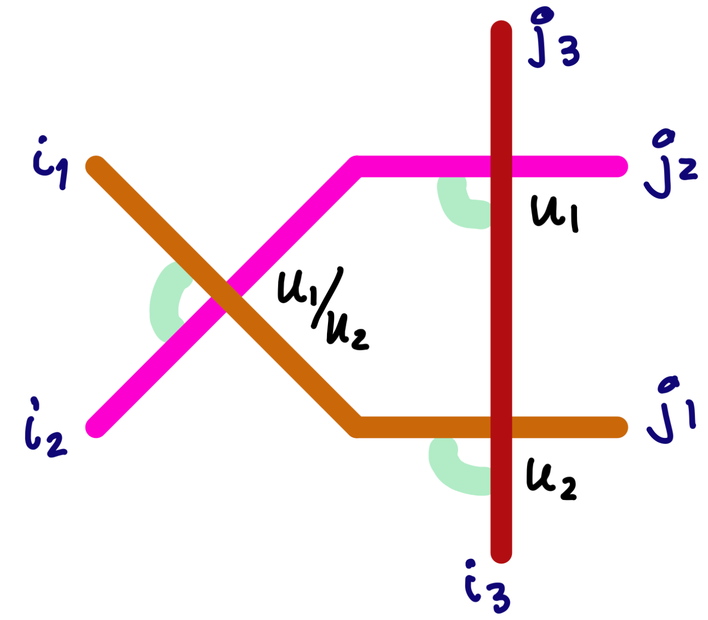


Stationarity for the single-color stochastic six-vertex model

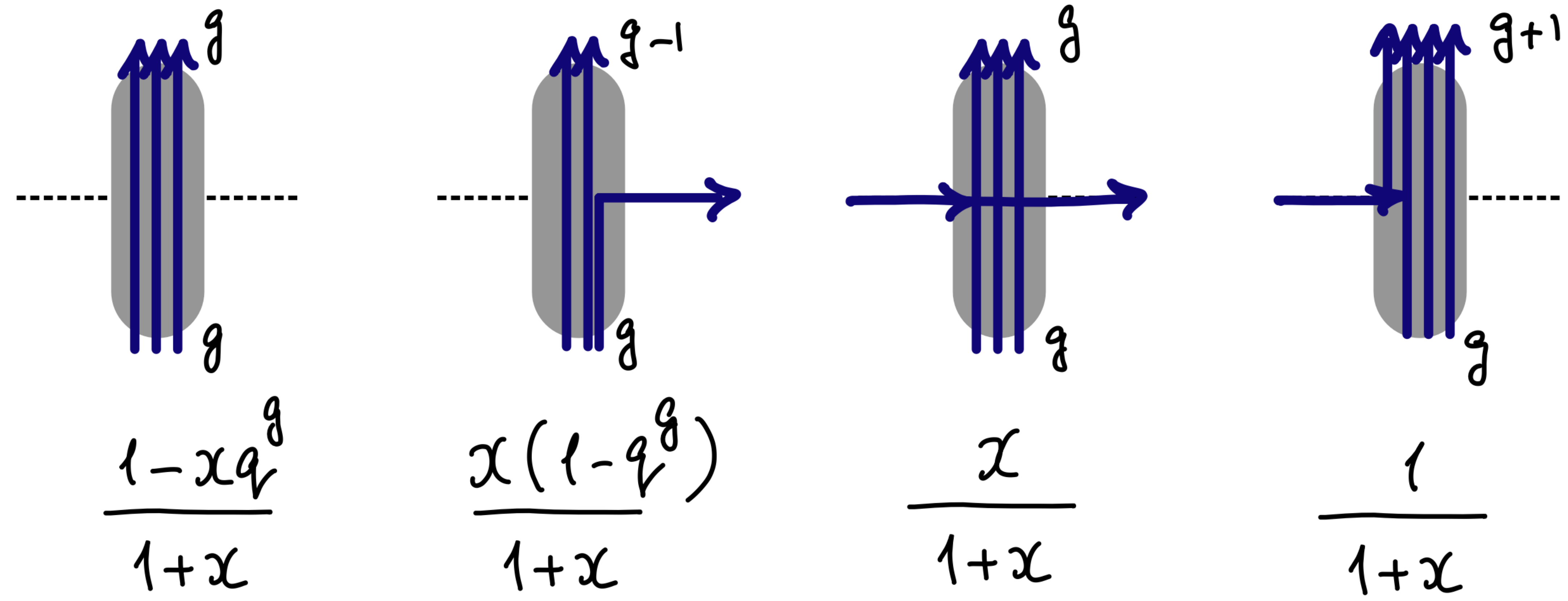
$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

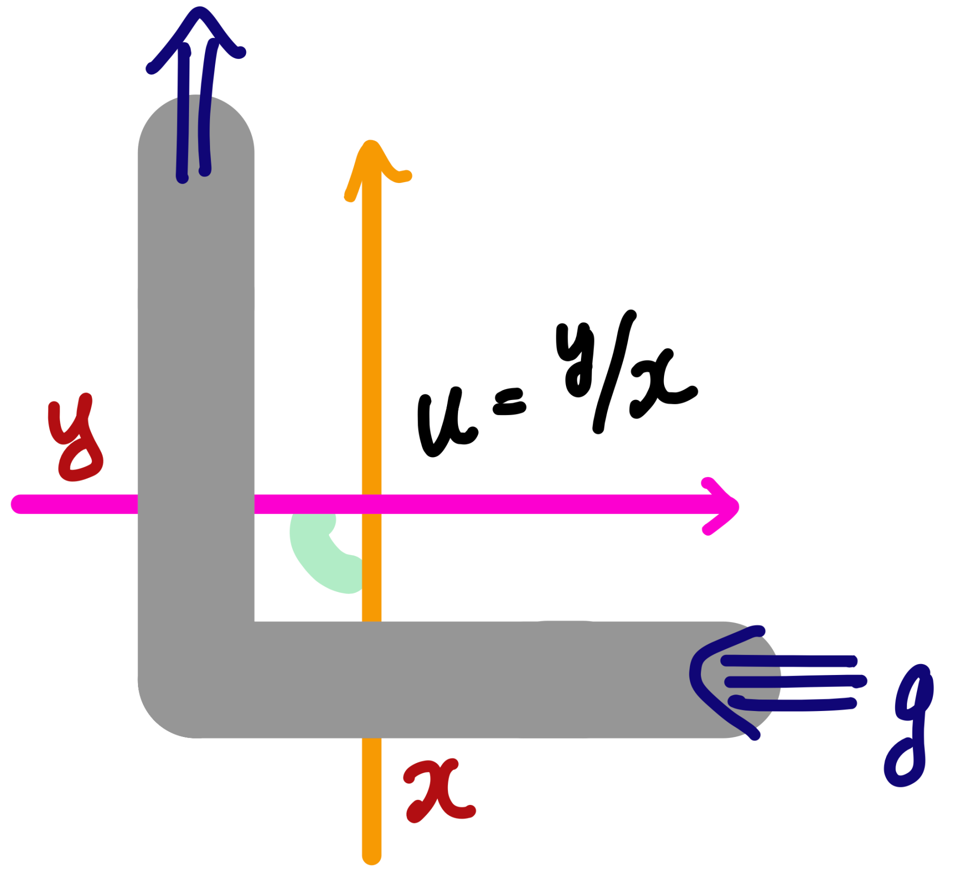


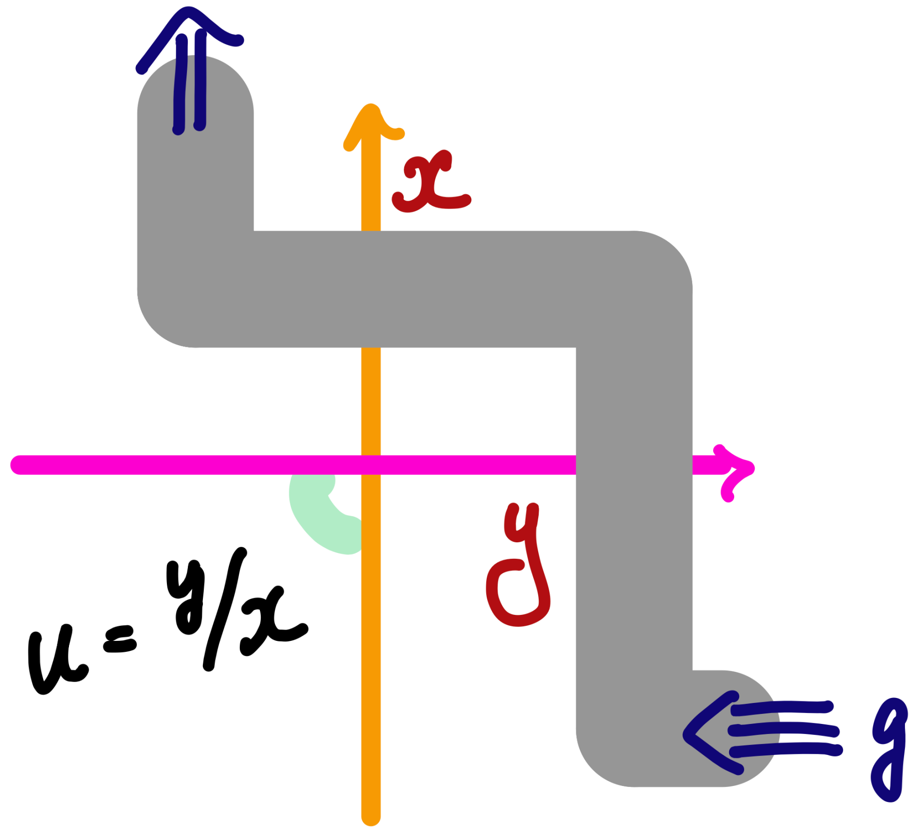




- **Fusion [Kulish-Reshetikhin-Sklyanin 1983], [Corwin-P. 2015]** - a way to construct new YBE solutions from existing ones. **Inserts ∞ arrows.**





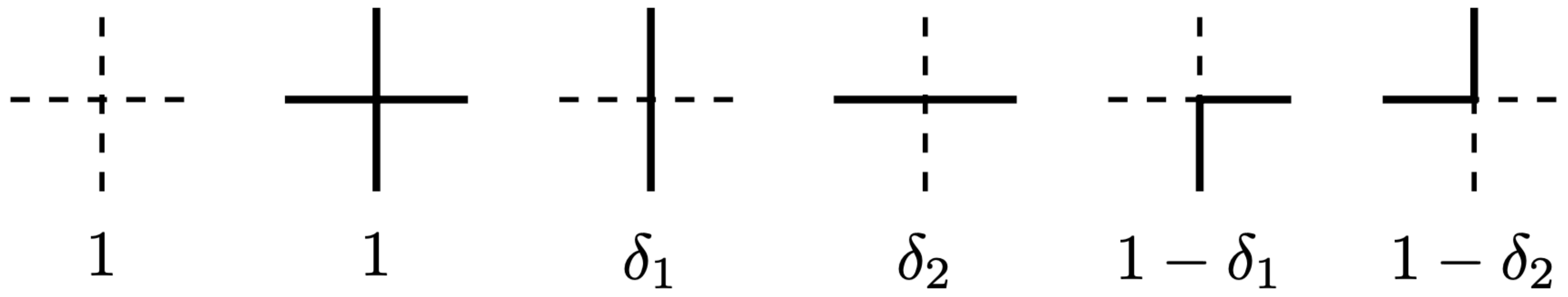


Stationarity via Yang-Baxter

- For $g = +\infty$, the right output of the fat vertex is $Bernoulli(\frac{x}{x+1})$, independent of the bottom and the left inputs.
- The Yang-Baxter equation is equivalent to the previous "Burke" computation: $\rho_v = \dfrac{x}{x+1}$,

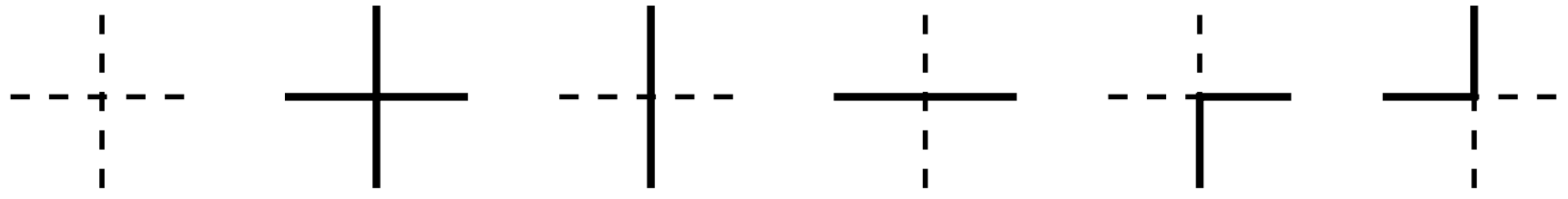
$$\rho_h = \frac{ux}{ux+1} \Rightarrow \rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v}.$$

Stationarity for the single-color stochastic six-vertex model

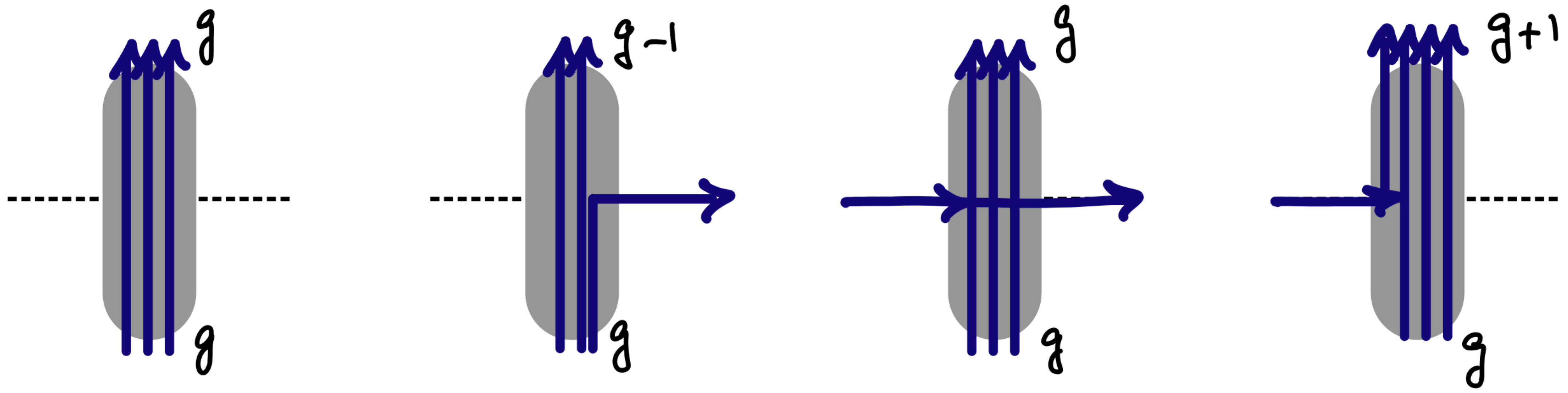


$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

Stationarity for the single-color stochastic six-vertex model

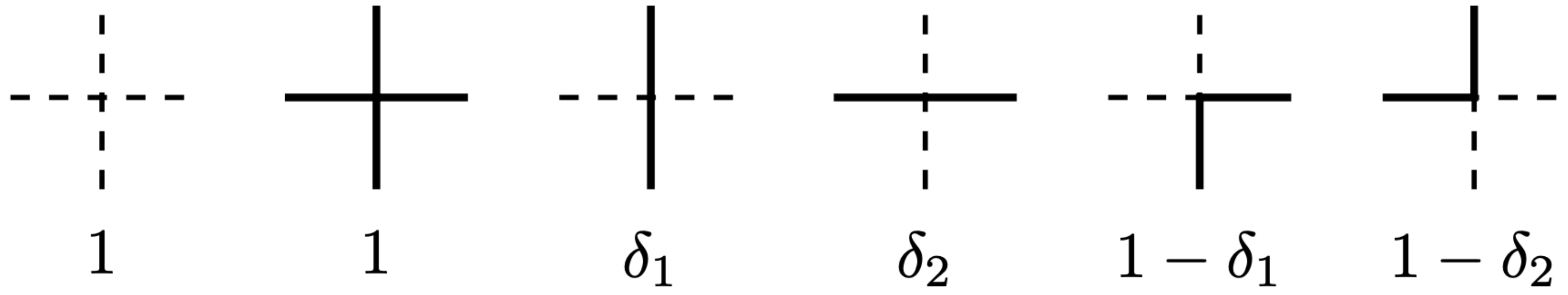


$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

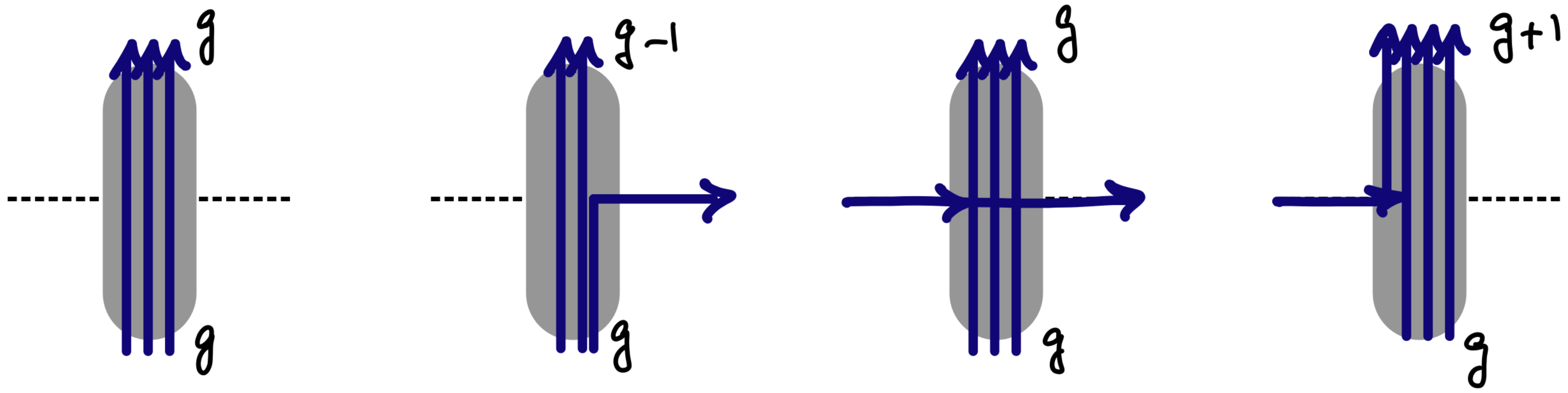


$$\frac{1-xq^g}{1+x} \qquad \frac{x(1-q^g)}{1+x} \qquad \frac{x}{1+x} \qquad \frac{1}{1+x}$$

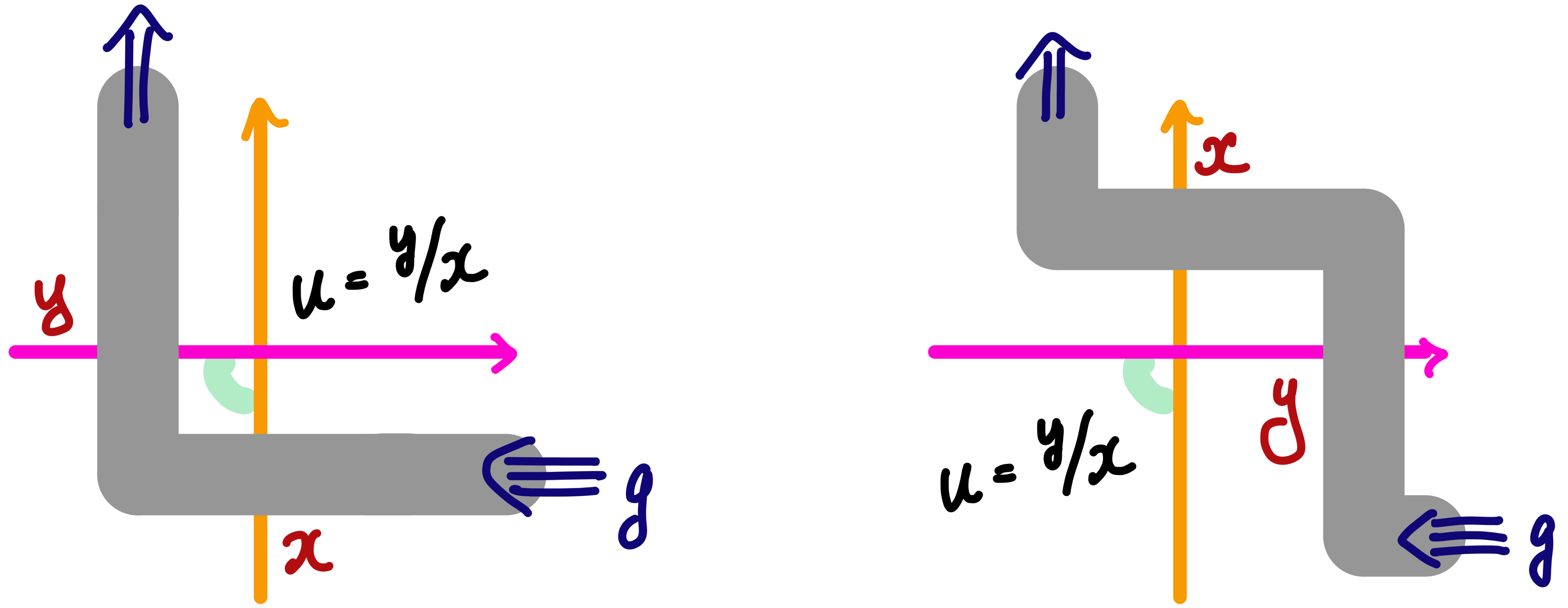
Stationarity for the single-color stochastic six-vertex model



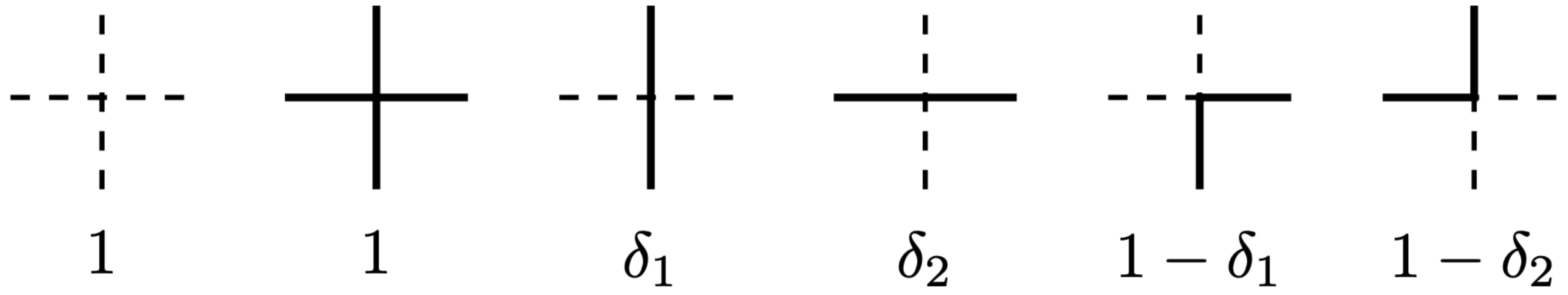
$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



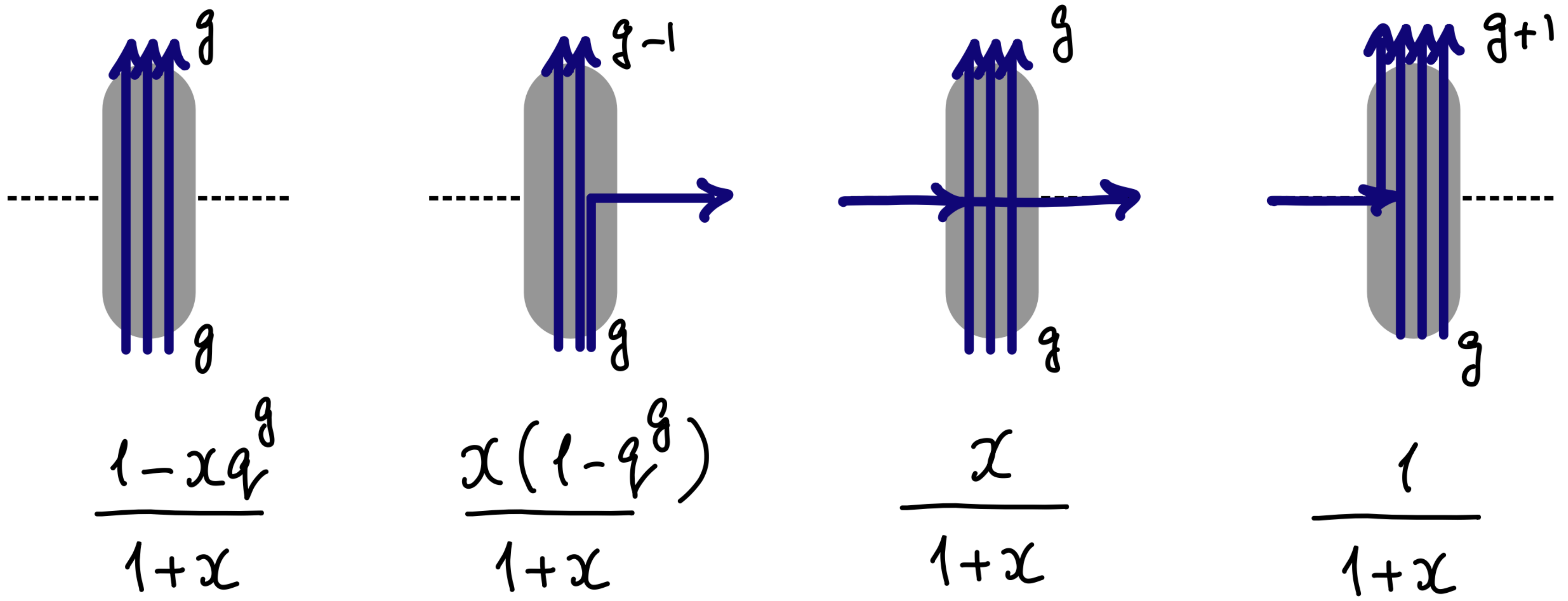
$$\frac{1-xq^g}{1+x} \qquad \frac{x(1-q^g)}{1+x} \qquad \frac{x}{1+x} \qquad \frac{1}{1+x}$$

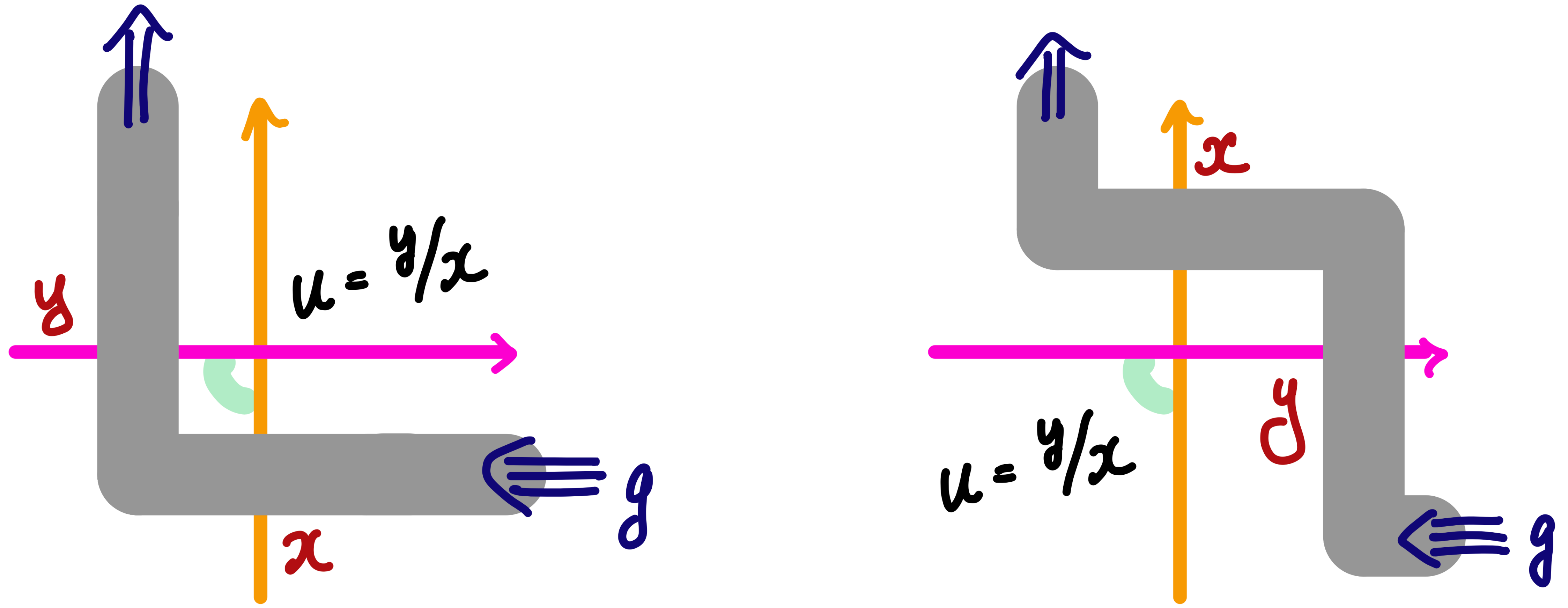


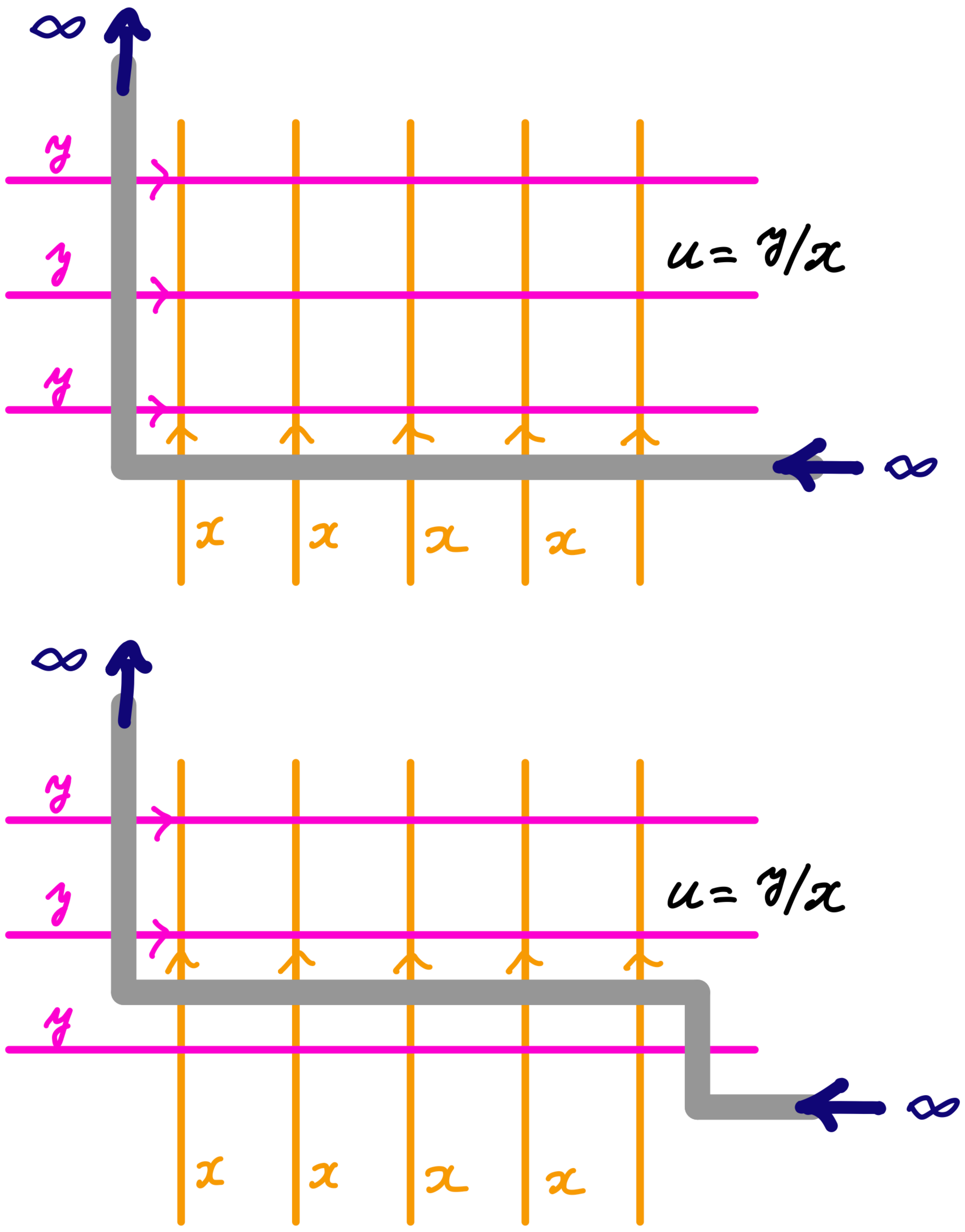
Stationarity for the single-color stochastic six-vertex model



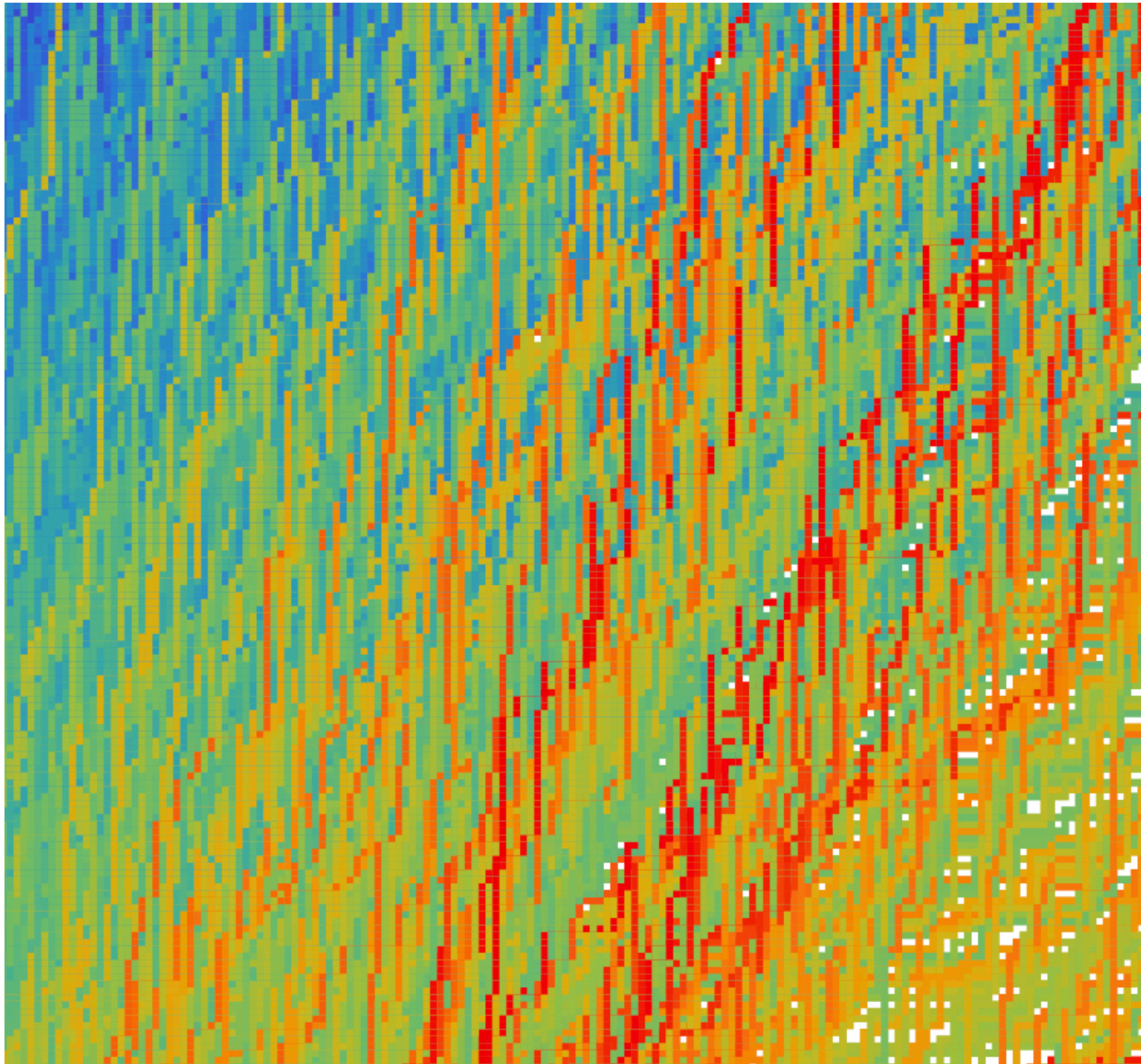
$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$





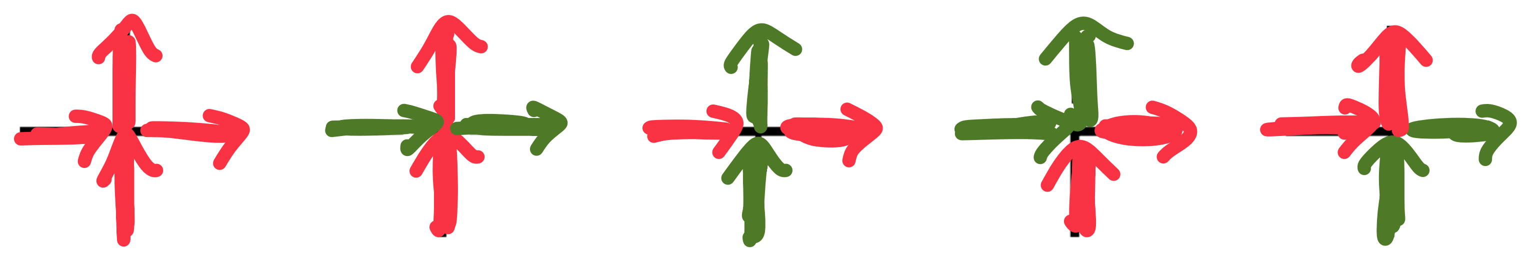


Add color!



Colored stochastic six-vertex model (converges to colored ASEP)

Red > Green

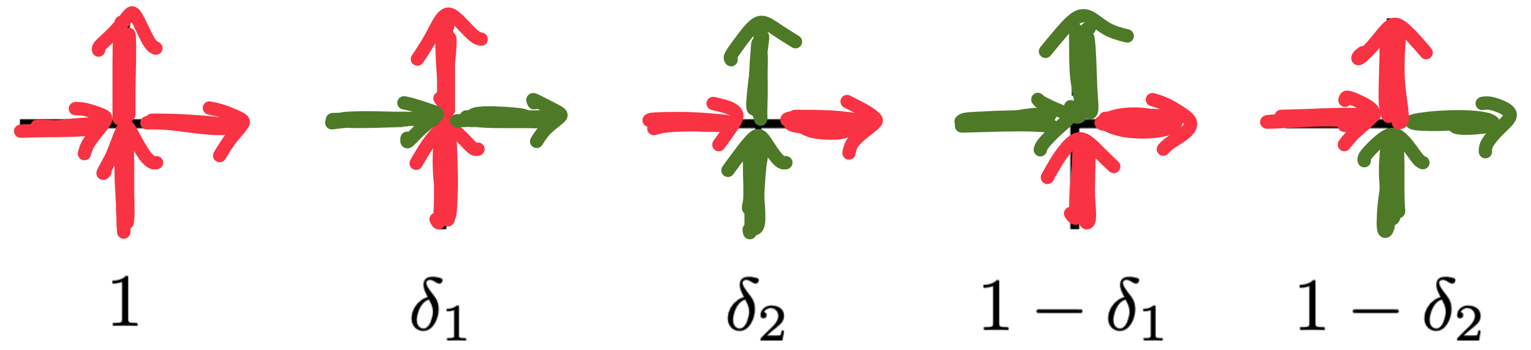


1 δ_1 δ_2 $1-\delta_1$ $1-\delta_2$

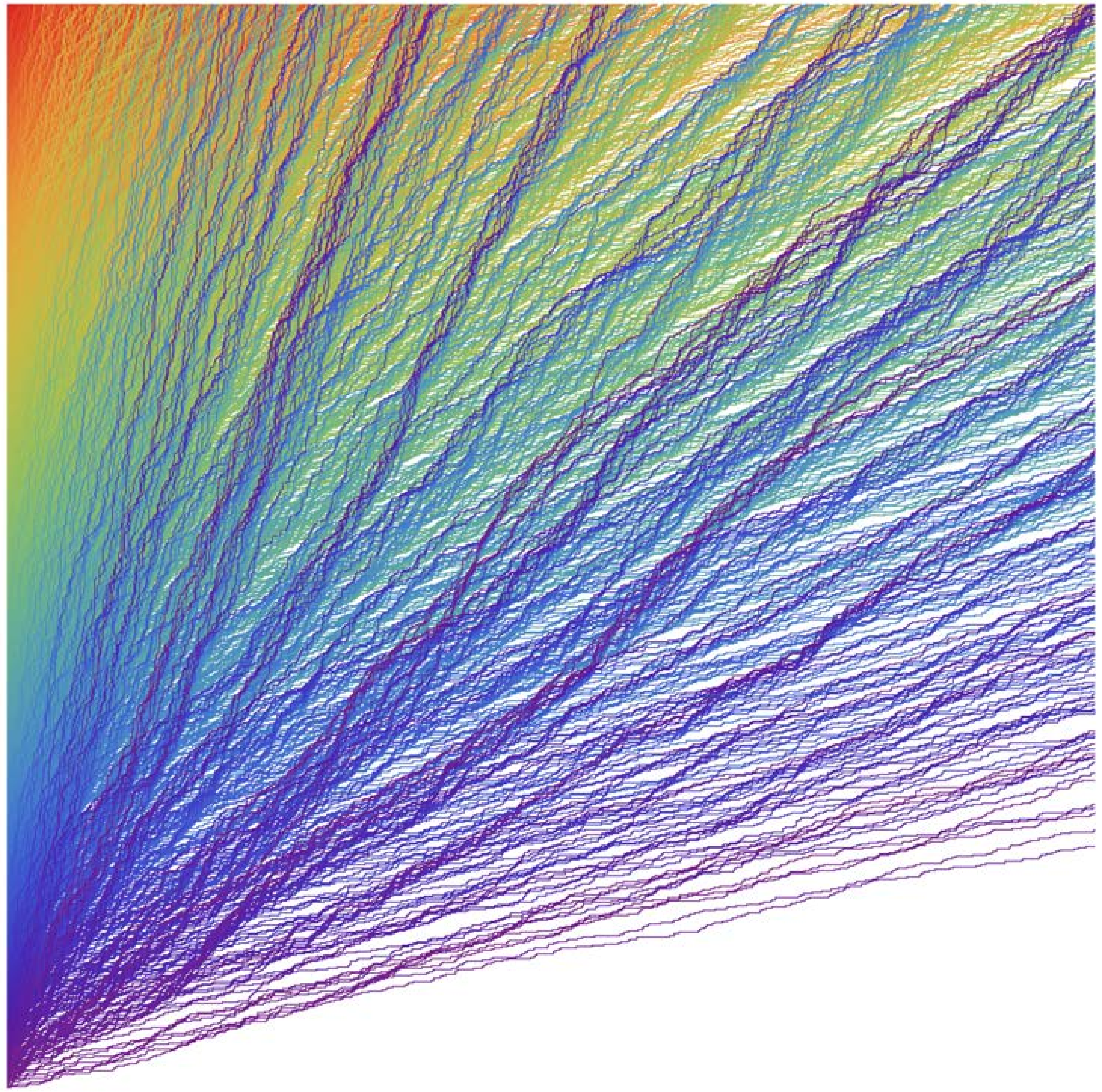
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

Colored stochastic six-vertex model (converges to colored ASEP)

Red > Green

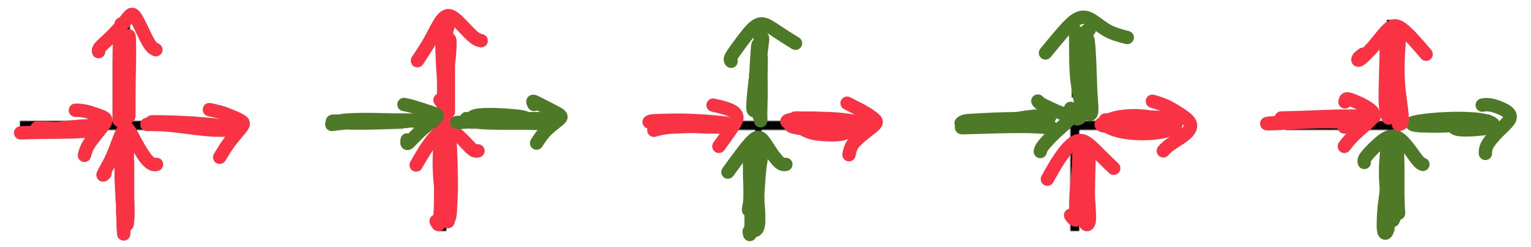


$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



Colored stochastic six-vertex model (converges to colored ASEP)

Red > Green

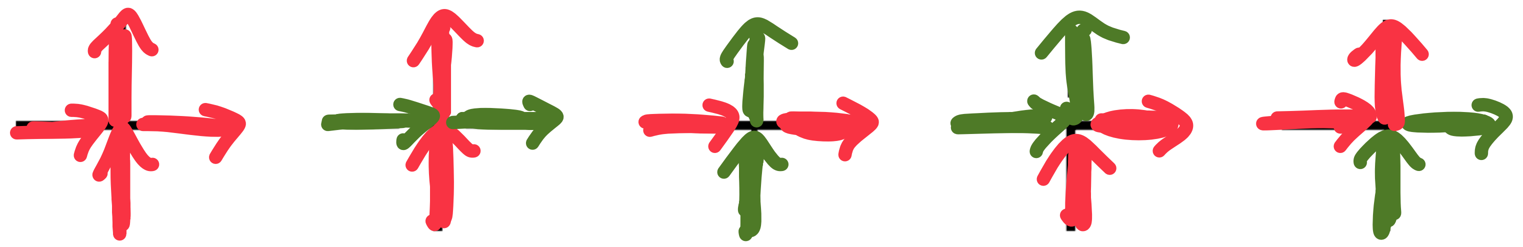


1 $\qquad \delta_1$ $\qquad \delta_2$ $\qquad 1-\delta_1$ $\qquad 1-\delta_2$

$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

Colored stochastic six-vertex model (converges to colored ASEP)

Red > Green

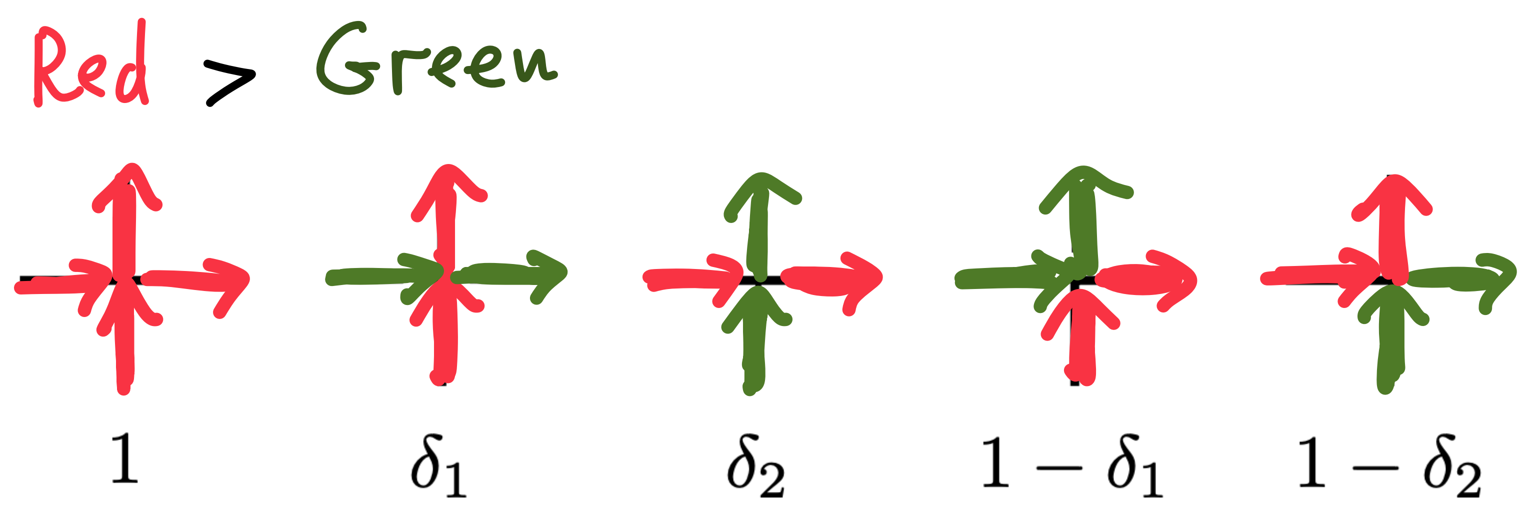


$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

- **Fusion and Yang-Baxter equation.**

Colored stochastic six-vertex model (converges to colored ASEP)



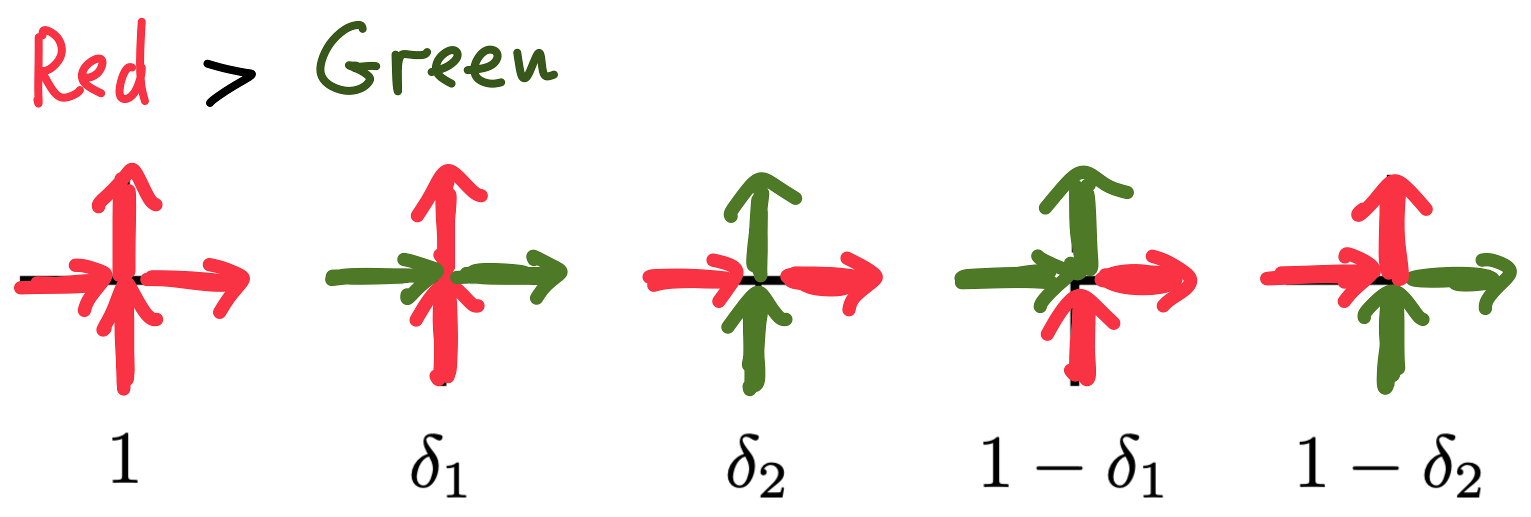
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

- **Fusion and Yang-Baxter equation.**

  Higher spin, higher rank stochastic weights.

  Related to $U_q(\widehat{sl}_{n+1})$**;** $1 \leq k < \ell \leq n$

Colored stochastic six-vertex model (converges to colored ASEP)



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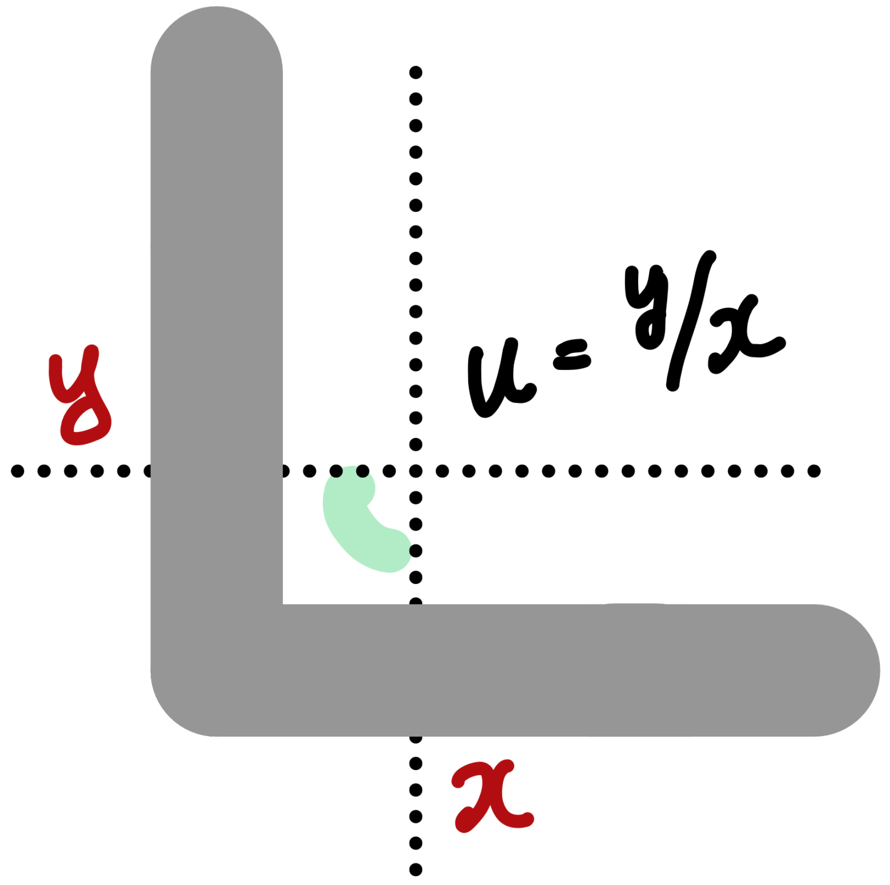
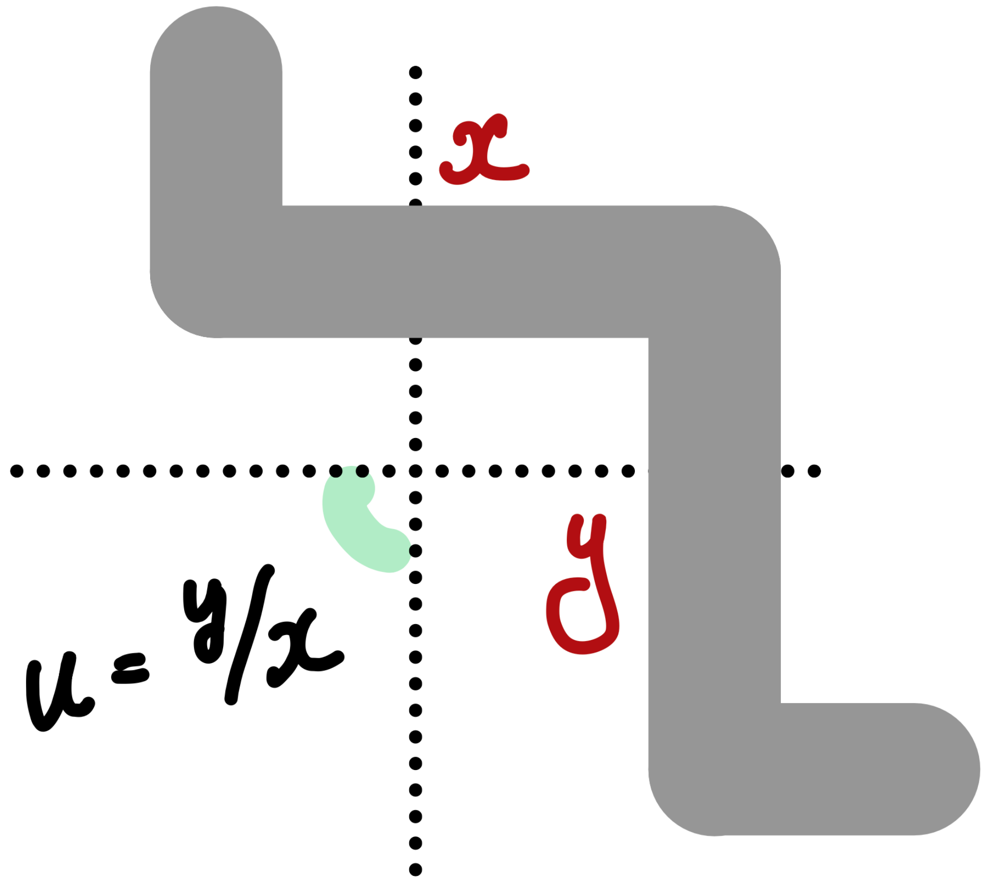
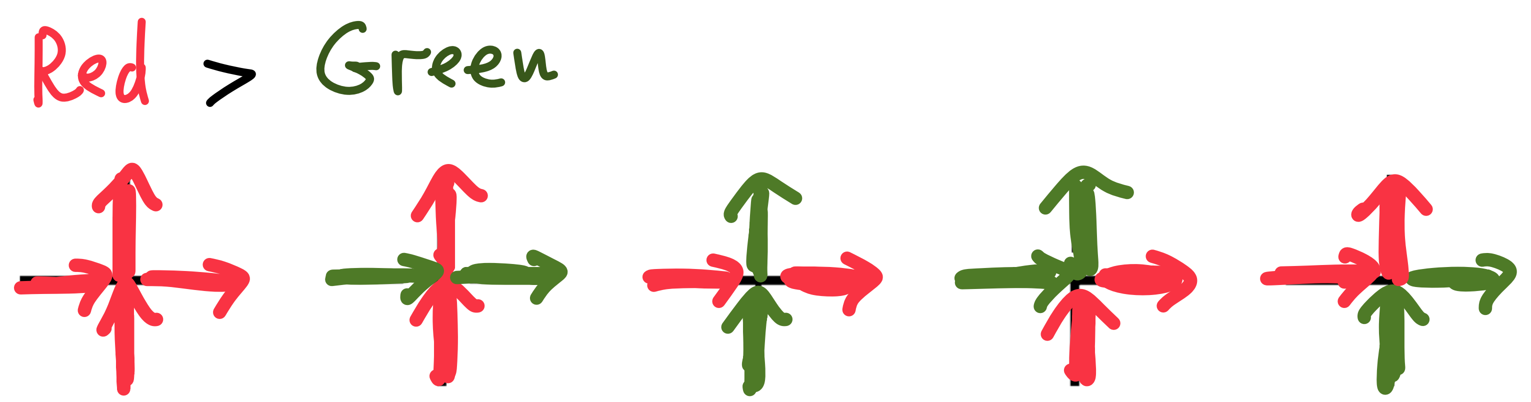




Colored stochastic six-vertex model (converges to colored ASEP)



$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

- **Fusion and Yang-Baxter equation.**

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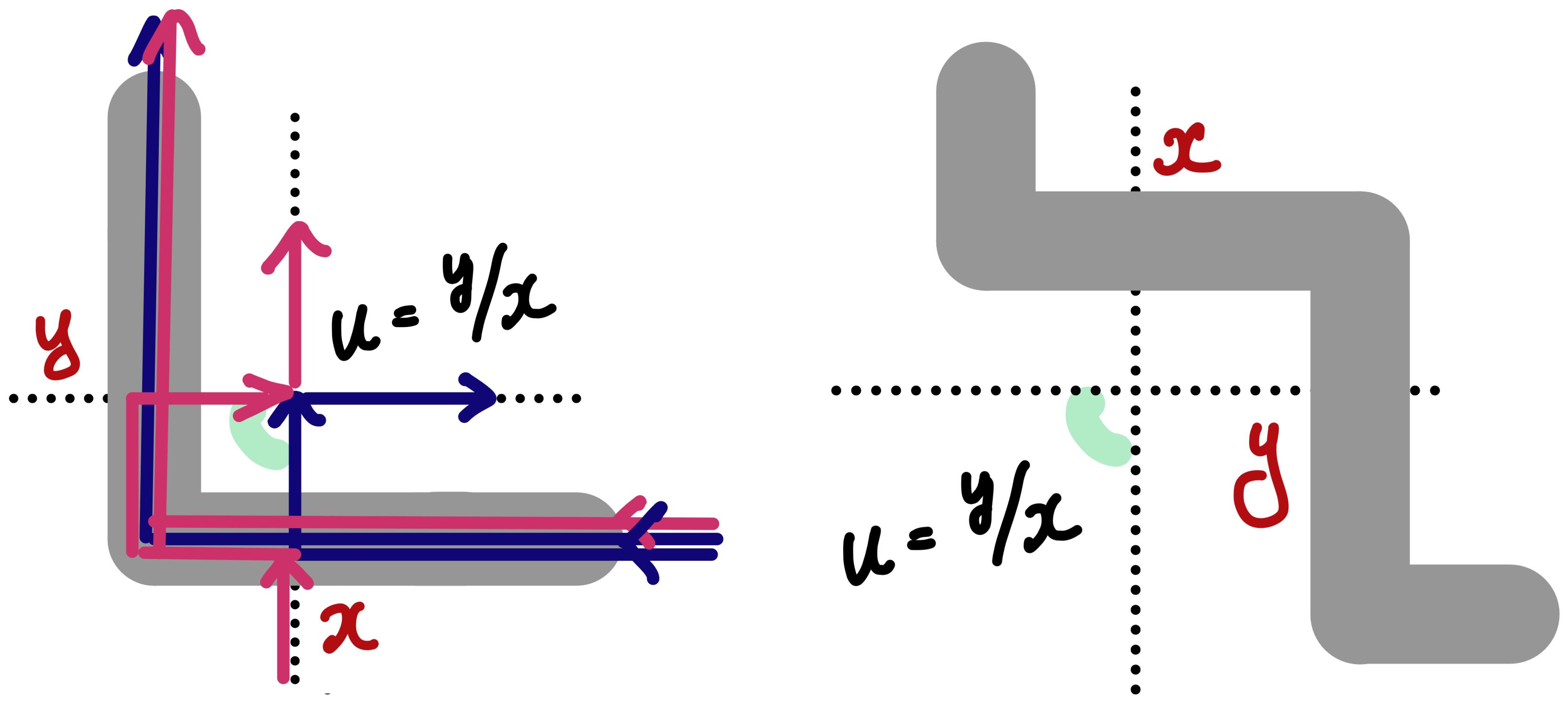
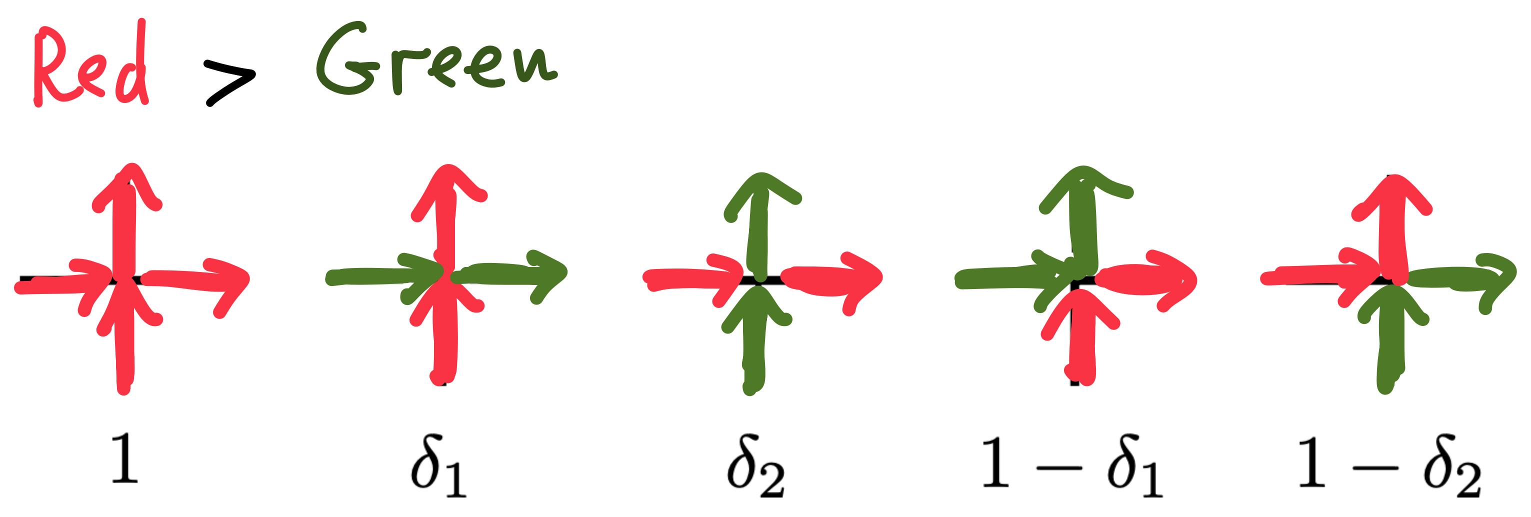


Colored stochastic six-vertex model (converges to colored ASEP)



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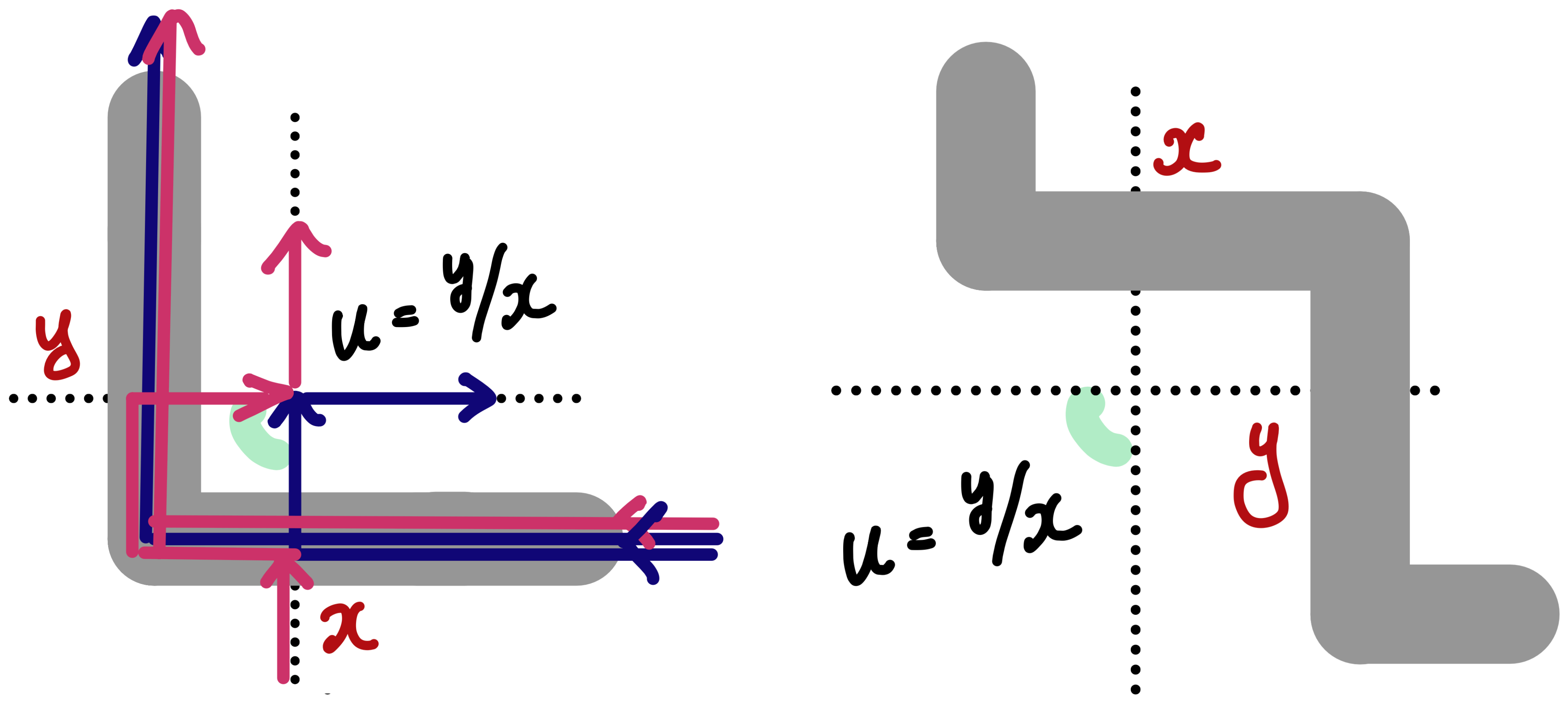
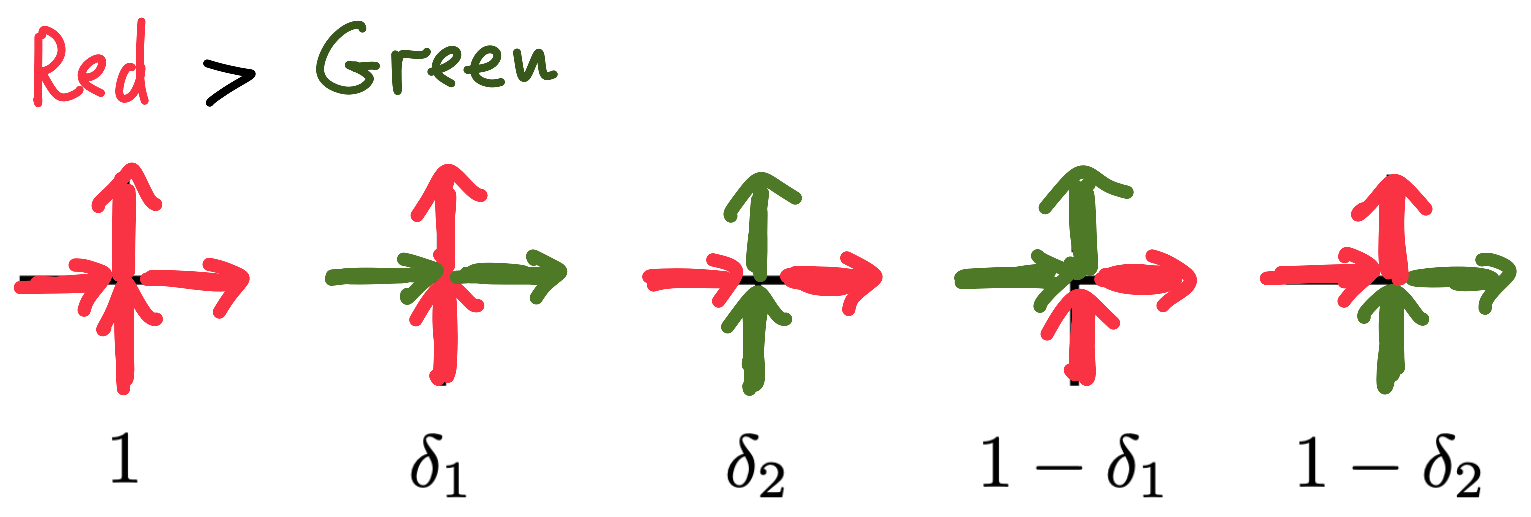


- **Set the number of arrows of a given color** m **to** $+\infty$**.** We get $\mathbb{W}^{(-m)}_{s_m, x_m}$ from the beginning (up to simple factors).

Colored stochastic six-vertex model (converges to colored ASEP)



$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

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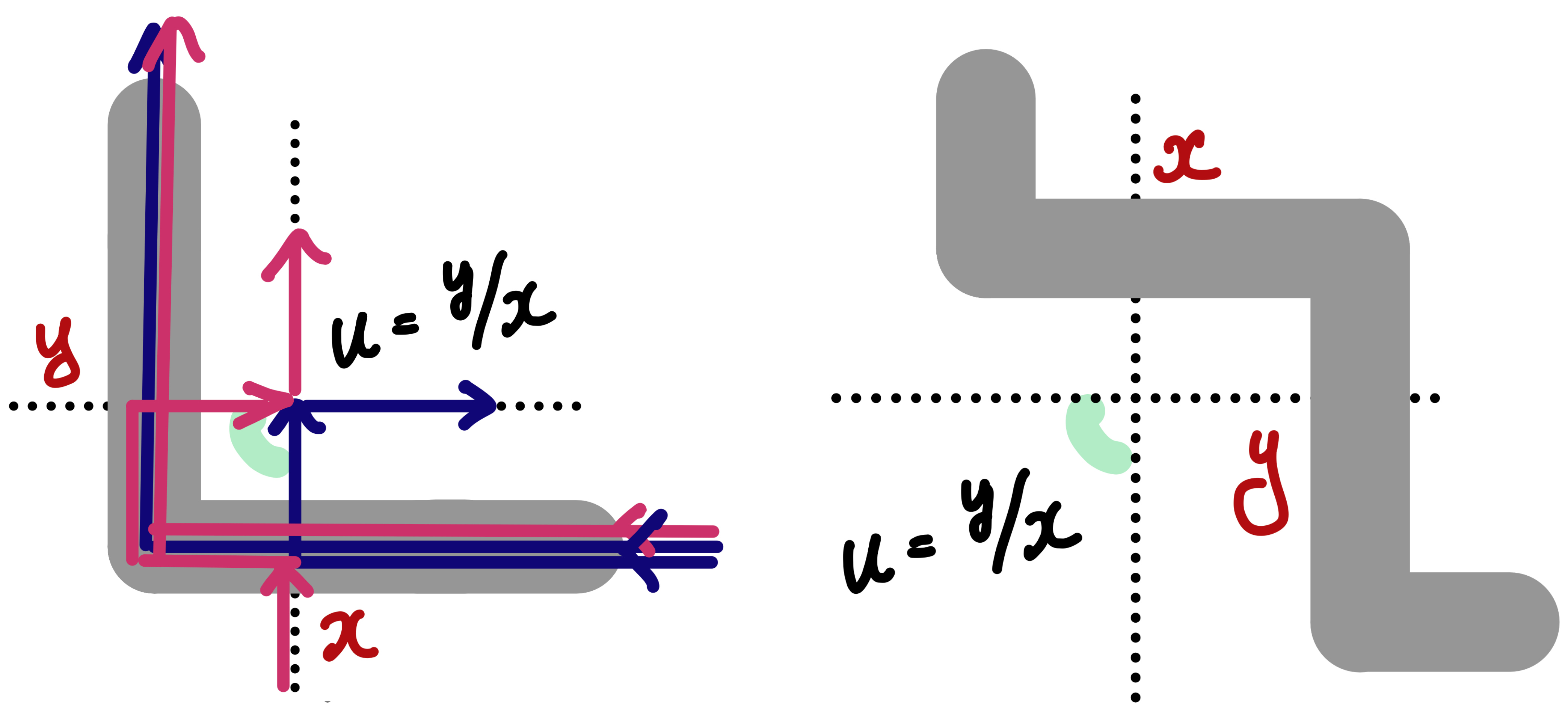
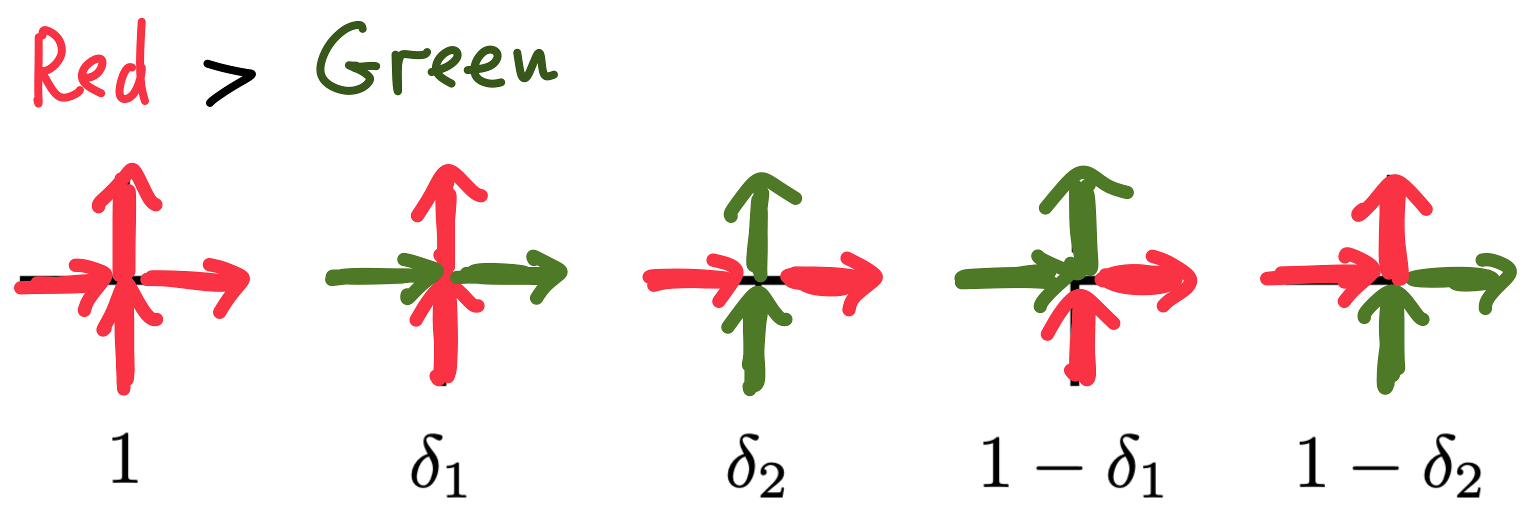


top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$: $\dfrac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$: $\dfrac{(x+\nu q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$: $\dfrac{x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$: $\dfrac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$: $\dfrac{x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$: $\dfrac{\nu(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$

- **Set the number of arrows of a given color** m **to** $+\infty$**.** We get $\mathbb{W}^{(-m)}_{s_m,x_m}$ from the beginning (up to simple factors).

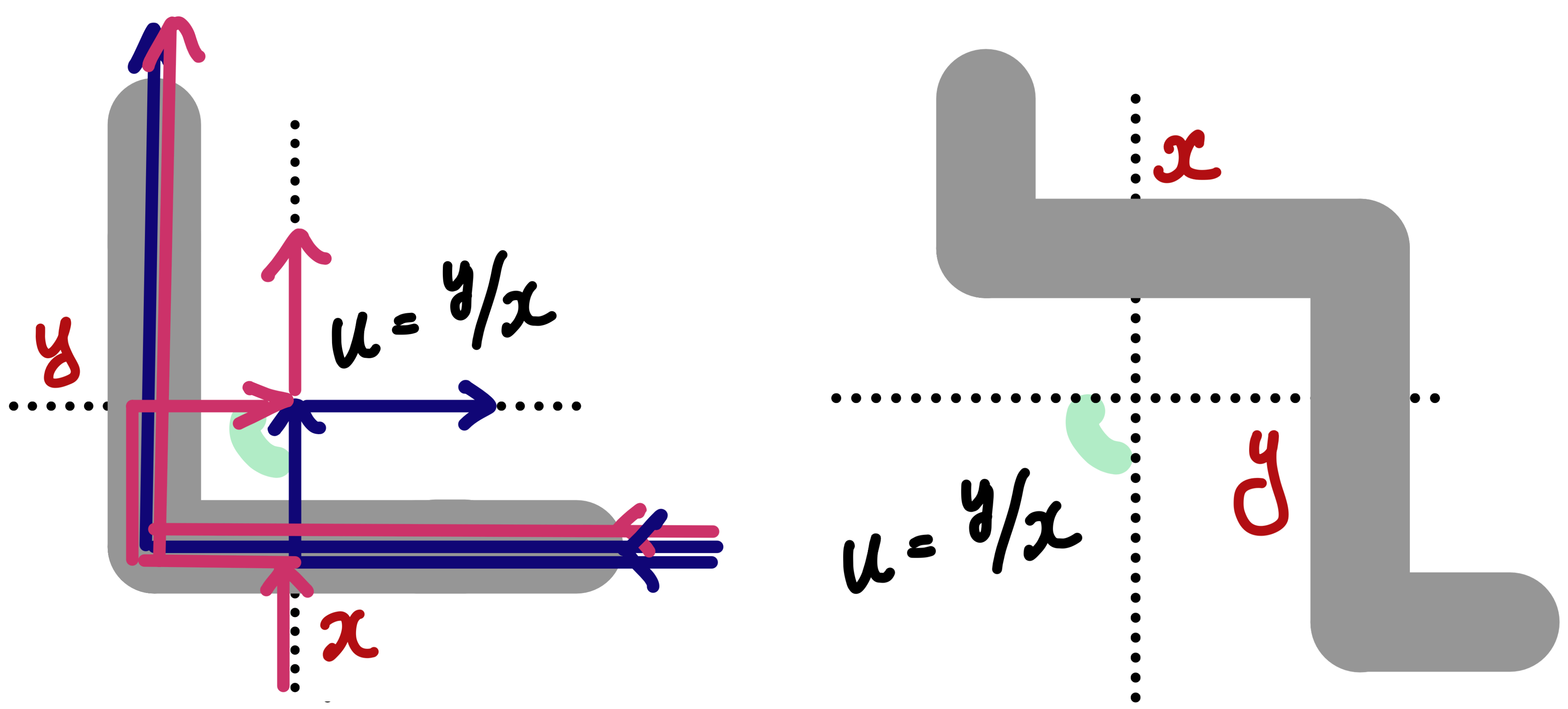
Colored stochastic six-vertex model (converges to colored ASEP)



$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

- **Fusion and Yang-Baxter equation.**

Higher spin, higher rank stochastic weights.
Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$

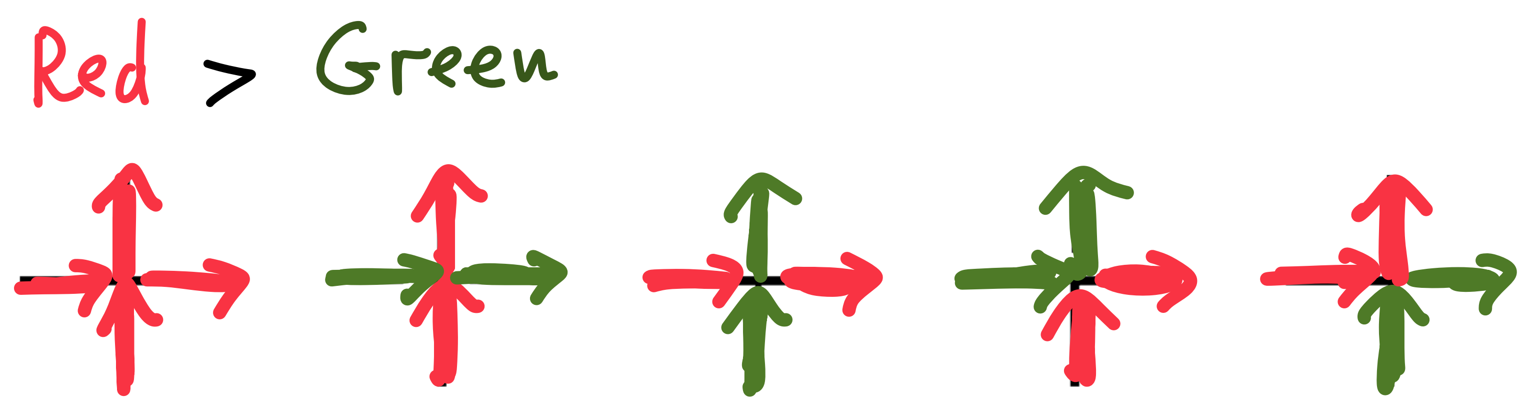


$\begin{array}{ccc} & \mathbf{A} & \\ 0 & + & 0 \\ & \mathbf{A} & \end{array}$ $\dfrac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	$\begin{array}{ccc} & \mathbf{A} & \\ k & + & k \\ & \mathbf{A} & \end{array}$ $\dfrac{(x+\nu q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	$\begin{array}{ccc} & \mathbf{A}_k^- & \\ 0 & + & k \\ & \mathbf{A} & \end{array}$ $\dfrac{x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
$\begin{array}{ccc} & \mathbf{A}_k^+ & \\ k & + & 0 \\ & \mathbf{A} & \end{array}$ $\dfrac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	$\begin{array}{ccc} & \mathbf{A}_{k\ell}^{+-} & \\ k & + & \ell \\ & \mathbf{A} & \end{array}$ $\dfrac{x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	$\begin{array}{ccc} & \mathbf{A}_{\ell k}^{+-} & \\ \ell & + & k \\ & \mathbf{A} & \end{array}$ $\dfrac{\nu\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}}{1+x}$

(Fused weights preserve the incoming/outoing arrows before sending color m to infinity)

- **Set the number of arrows of a given color m to $+\infty$.** We get $\mathbb{W}^{(-m)}_{s_m,x_m}$ from the beginning (up to simple factors).

Colored stochastic six-vertex model (converges to colored ASEP)



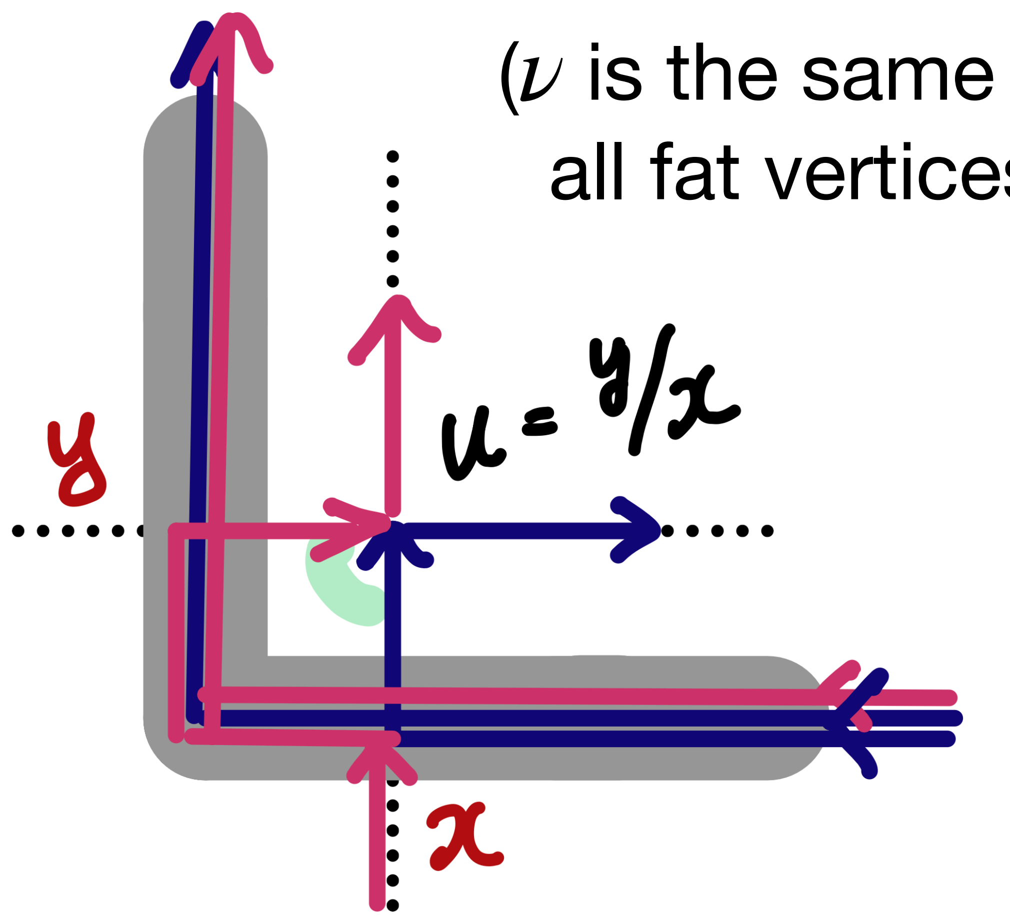
1 $\quad \delta_1$ $\quad \delta_2$ $\quad 1-\delta_1$ $\quad 1-\delta_2$

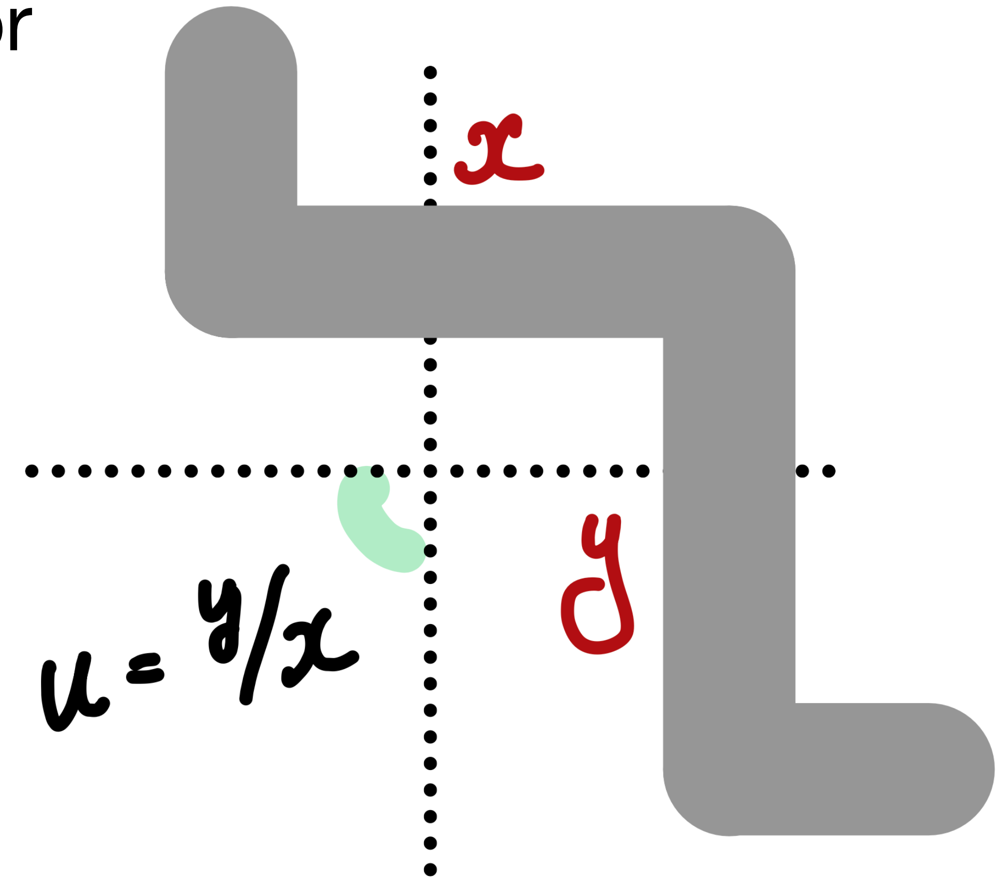
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

- **Fusion and Yang-Baxter equation.**

  Higher spin, higher rank stochastic weights.
  Related to $U_q(\widehat{sl}_{n+1})$**;** $1 \le k < \ell \le n$

(*ν* is the same for all fat vertices)





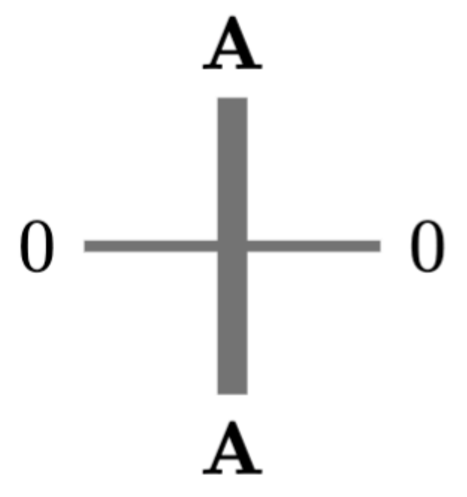
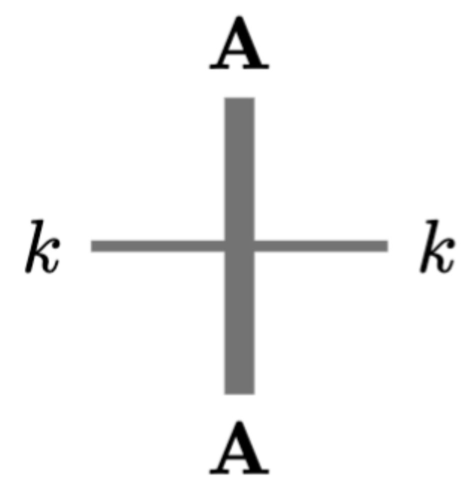
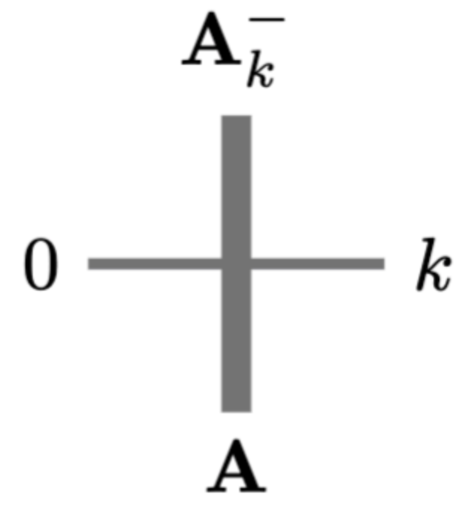
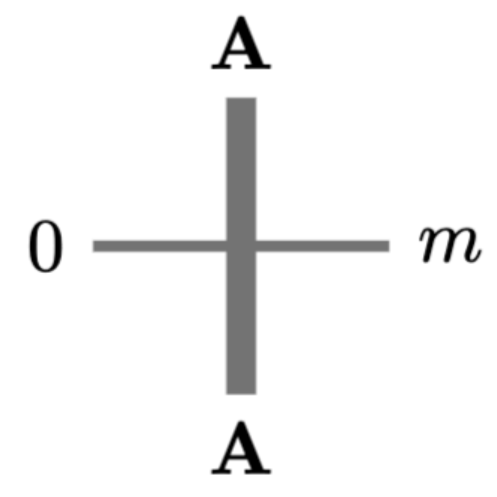
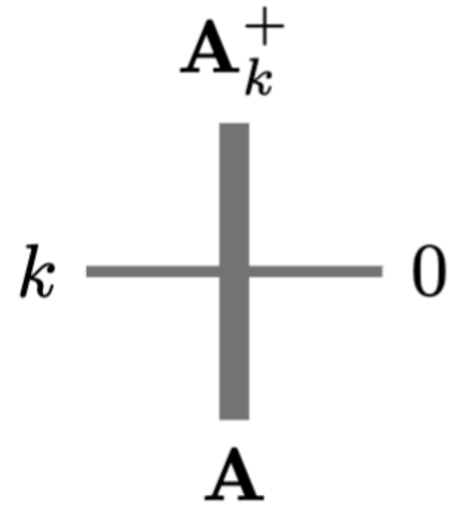
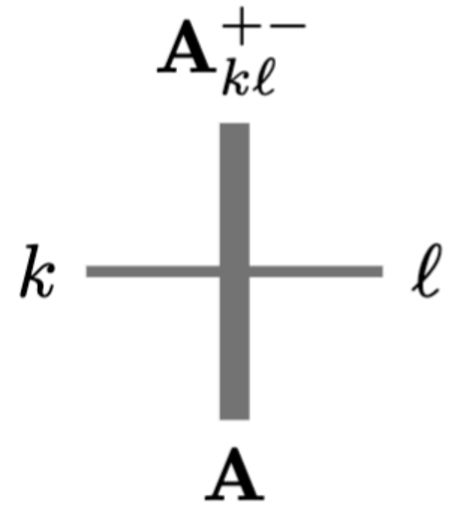
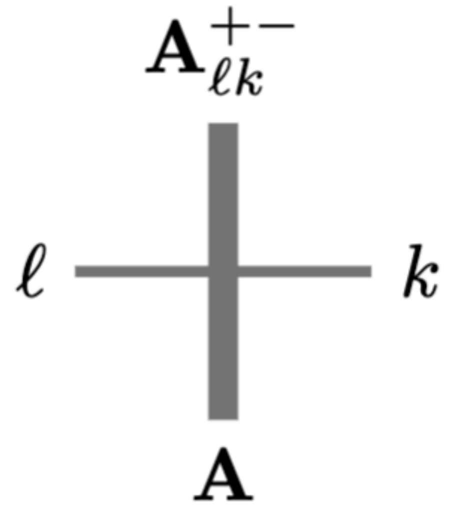
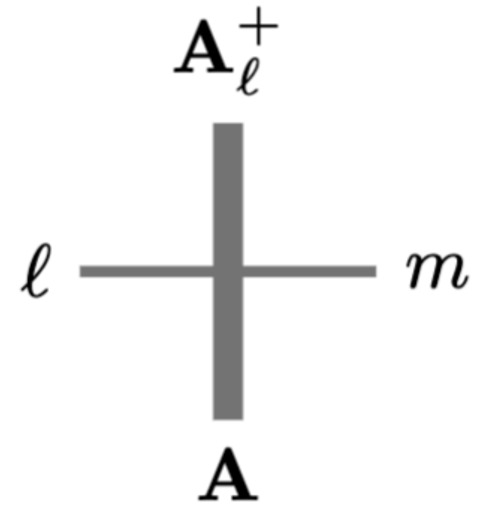
top $\mathbf{A}$, left 0, right 0, bottom $\mathbf{A}$ $\dfrac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}$, left k, right k, bottom $\mathbf{A}$ $\dfrac{(x+\nu q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	top $\mathbf{A}_k^-$, left 0, right k, bottom $\mathbf{A}$ $\dfrac{x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
top $\mathbf{A}_k^+$, left k, right 0, bottom $\mathbf{A}$ $\dfrac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}_{k\ell}^{+-}$, left k, right ℓ, bottom $\mathbf{A}$ $\dfrac{x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	top $\mathbf{A}_{\ell k}^{+-}$, left ℓ, right k, bottom $\mathbf{A}$ $\dfrac{\nu(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$

(Fused weights preserve the incoming/outoing arrows before sending color m to infinity)

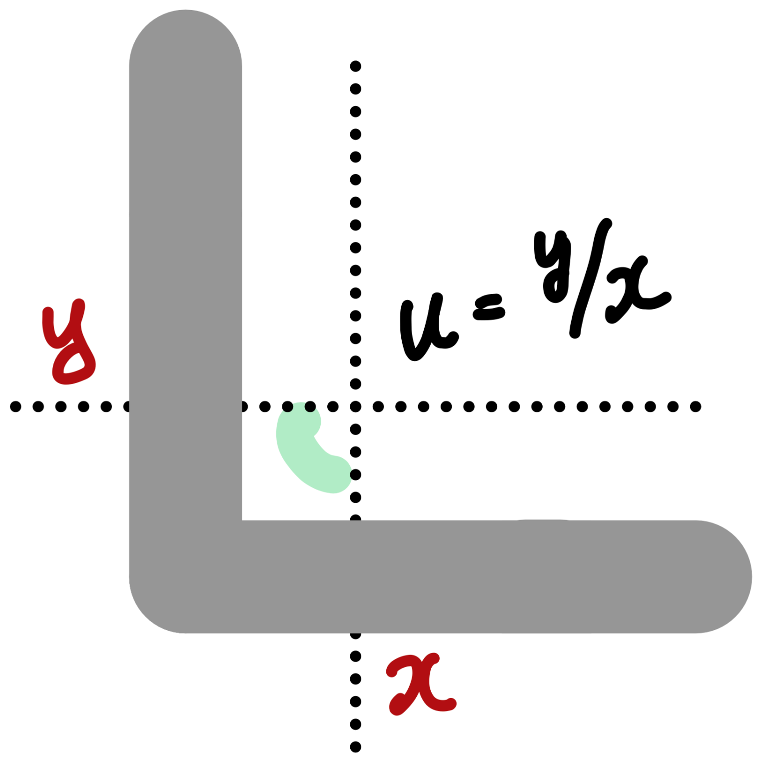
- **Set the number of arrows of a given color** m **to** $+\infty$**.** We get $\mathbb{W}^{(-m)}_{s_m,x_m}$ from the beginning (up to simple factors).

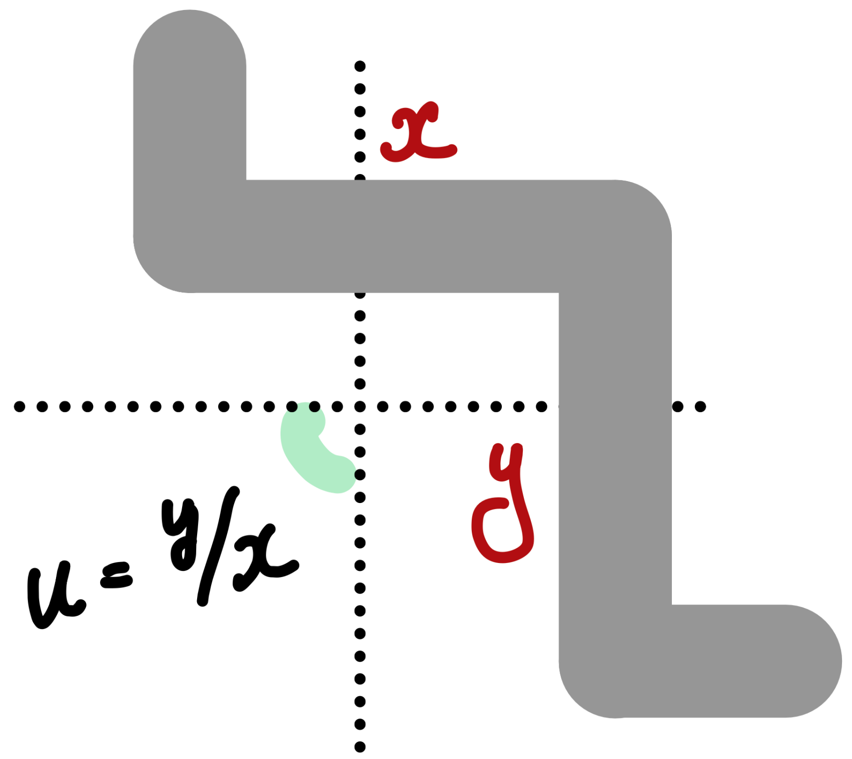
Colored stochastic six-vertex model. Color m goes to infinity

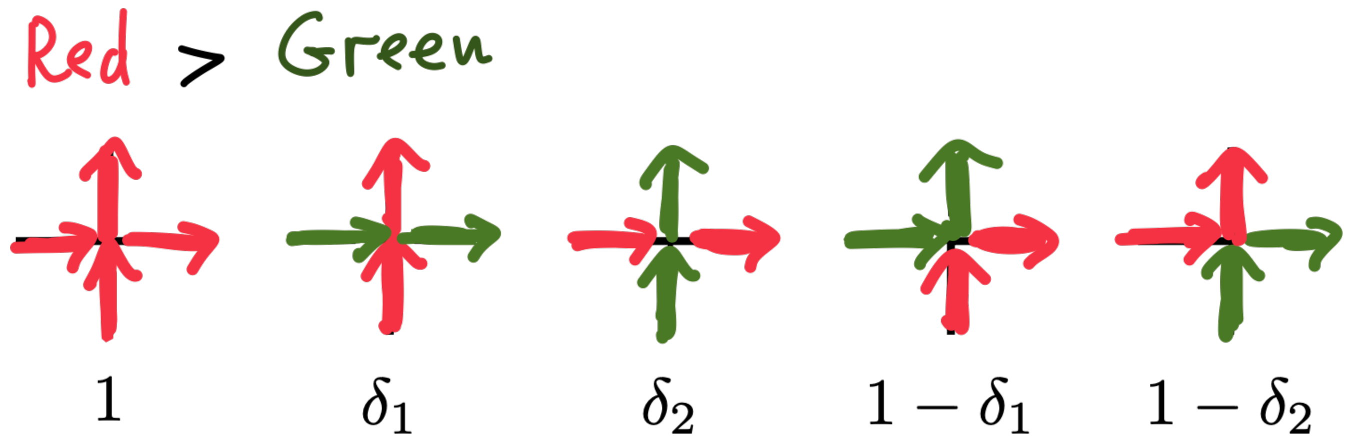
- **Set the number of arrows of a given color m to $+\infty$.** We get $\mathbb{W}_x^{(-m)}$: $\qquad m < k < \ell \leq n$

 $\dfrac{1}{1+x}$	 $\dfrac{x\,q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	 $\dfrac{x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	 $\dfrac{x\,q^{\mathbf{A}_{[m+1,n]}}}{1+x}$
 $\dfrac{1}{1+x}$	 $\dfrac{x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	 0	 0

Colored stochastic six-vertex model. Many colors ⇒ many fat lines

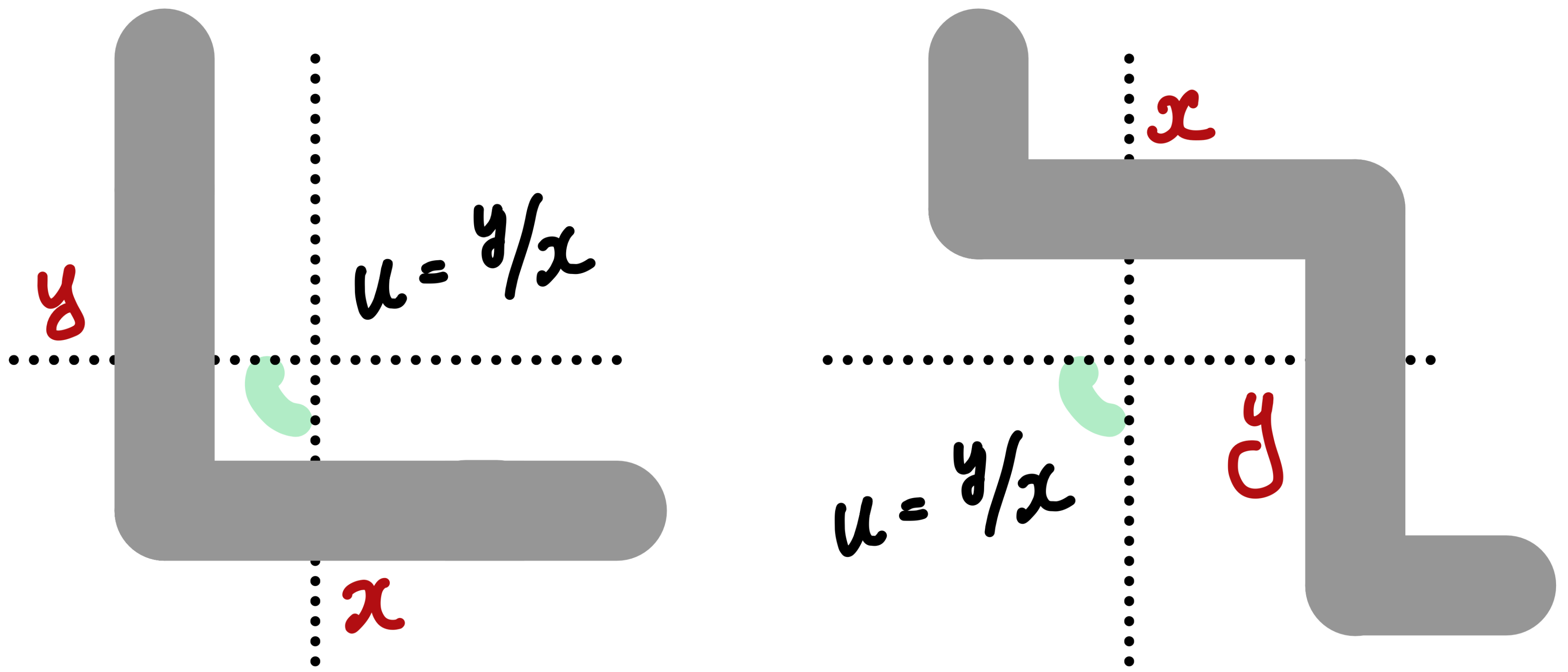


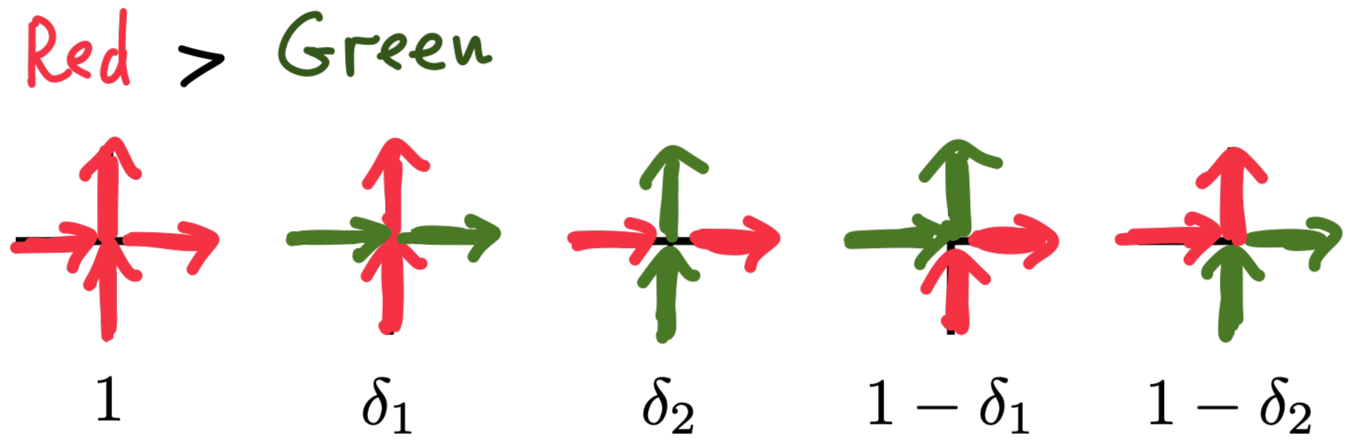




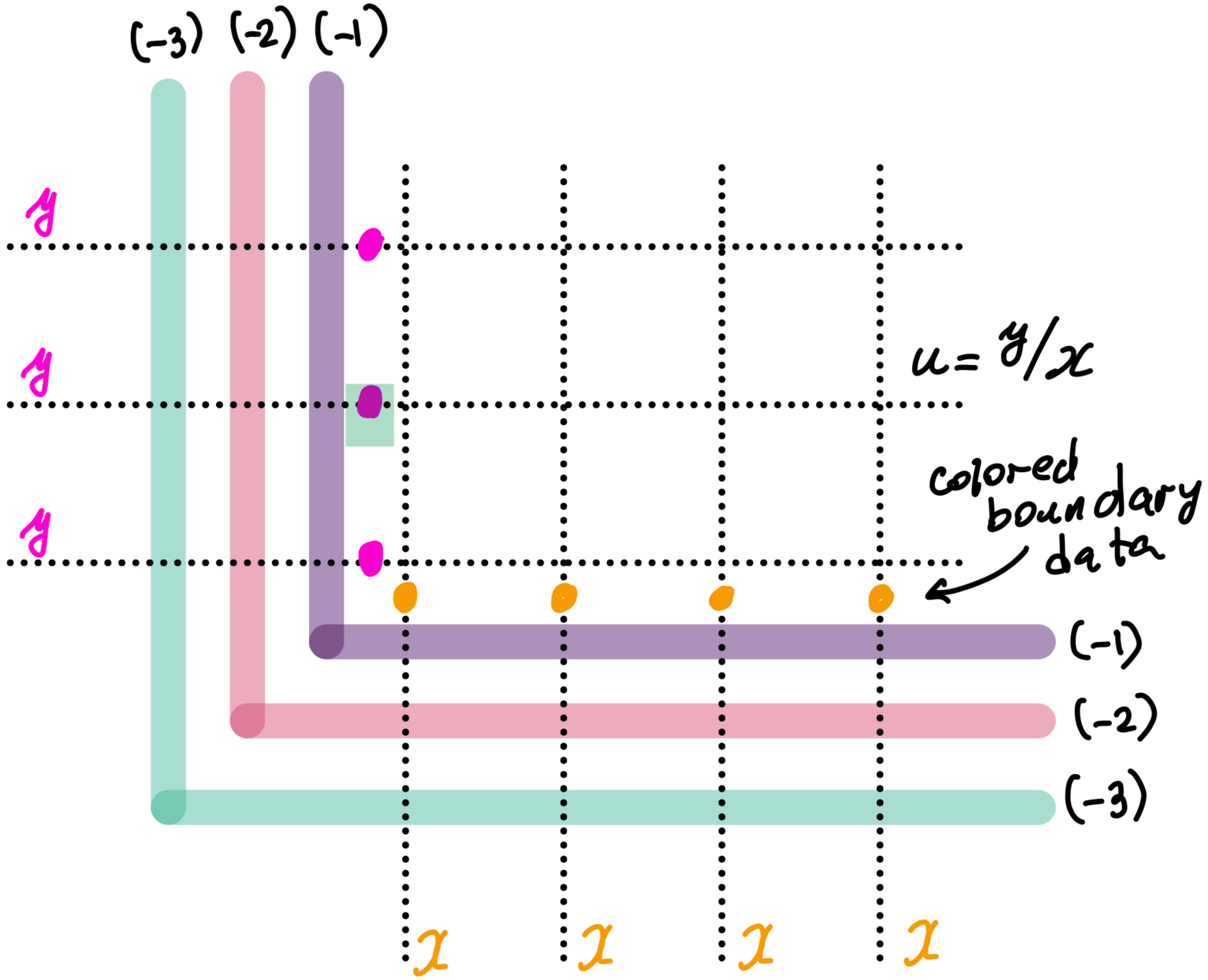
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

Colored stochastic six-vertex model. Many colors ⇒ many fat lines

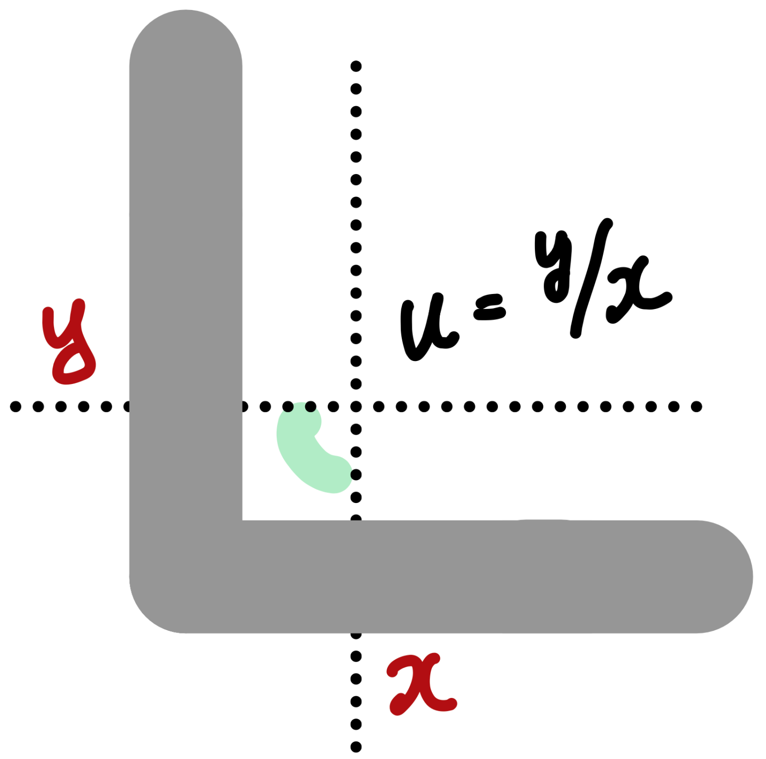


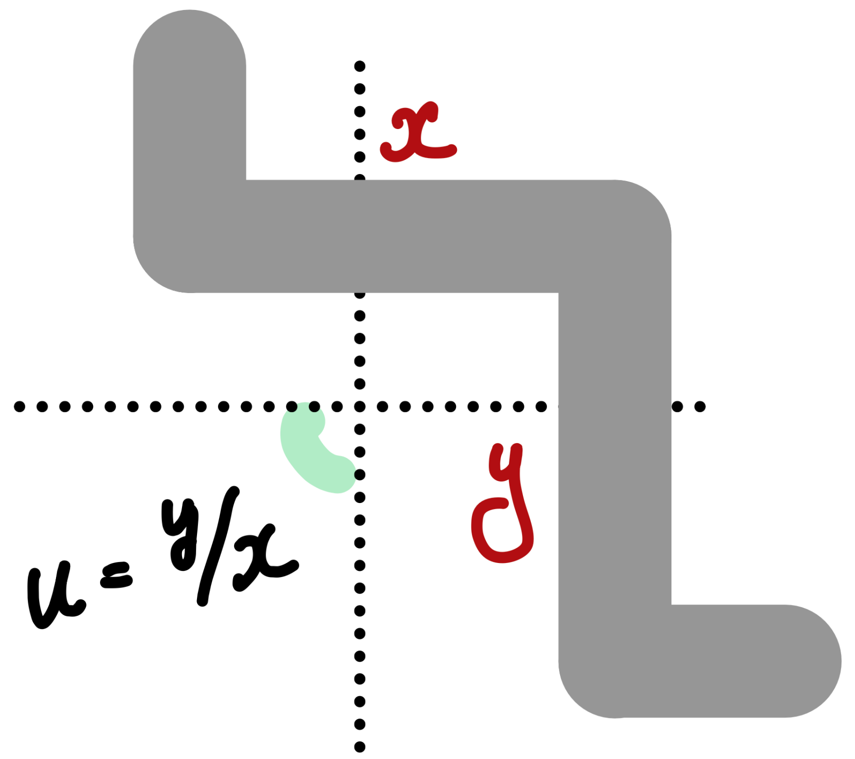


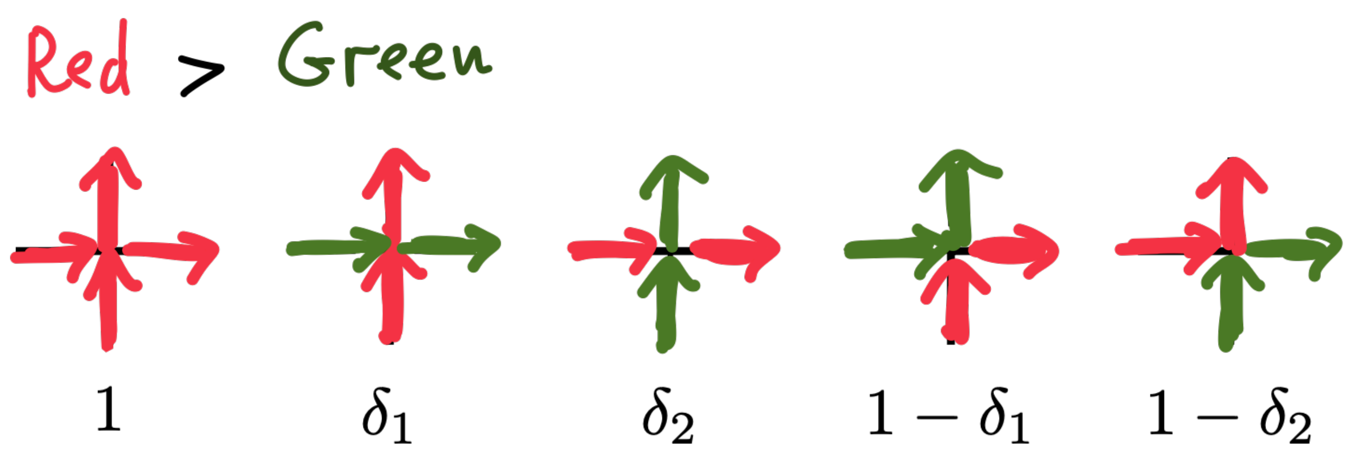
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



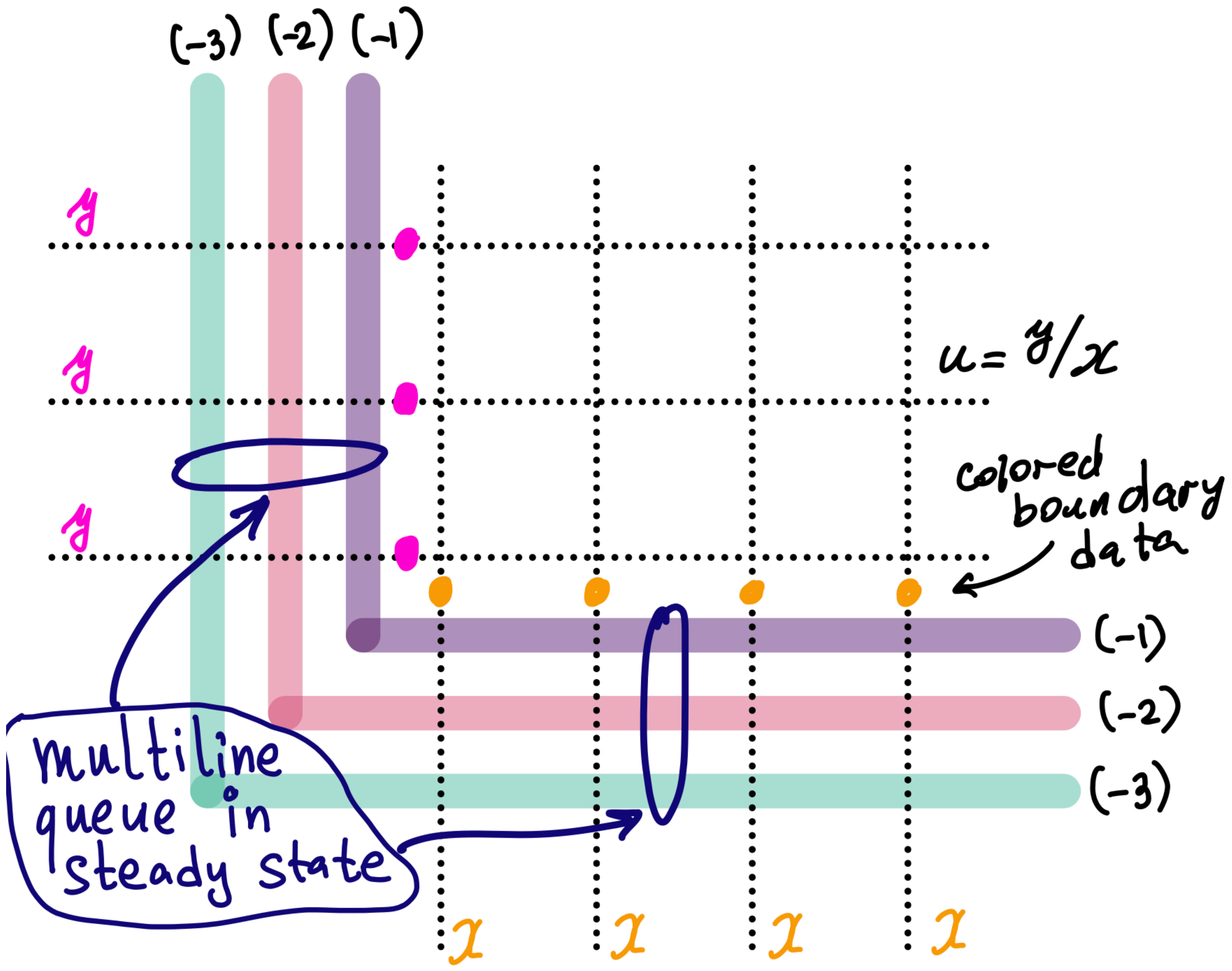
Colored stochastic six-vertex model. Many colors ⇒ many fat lines



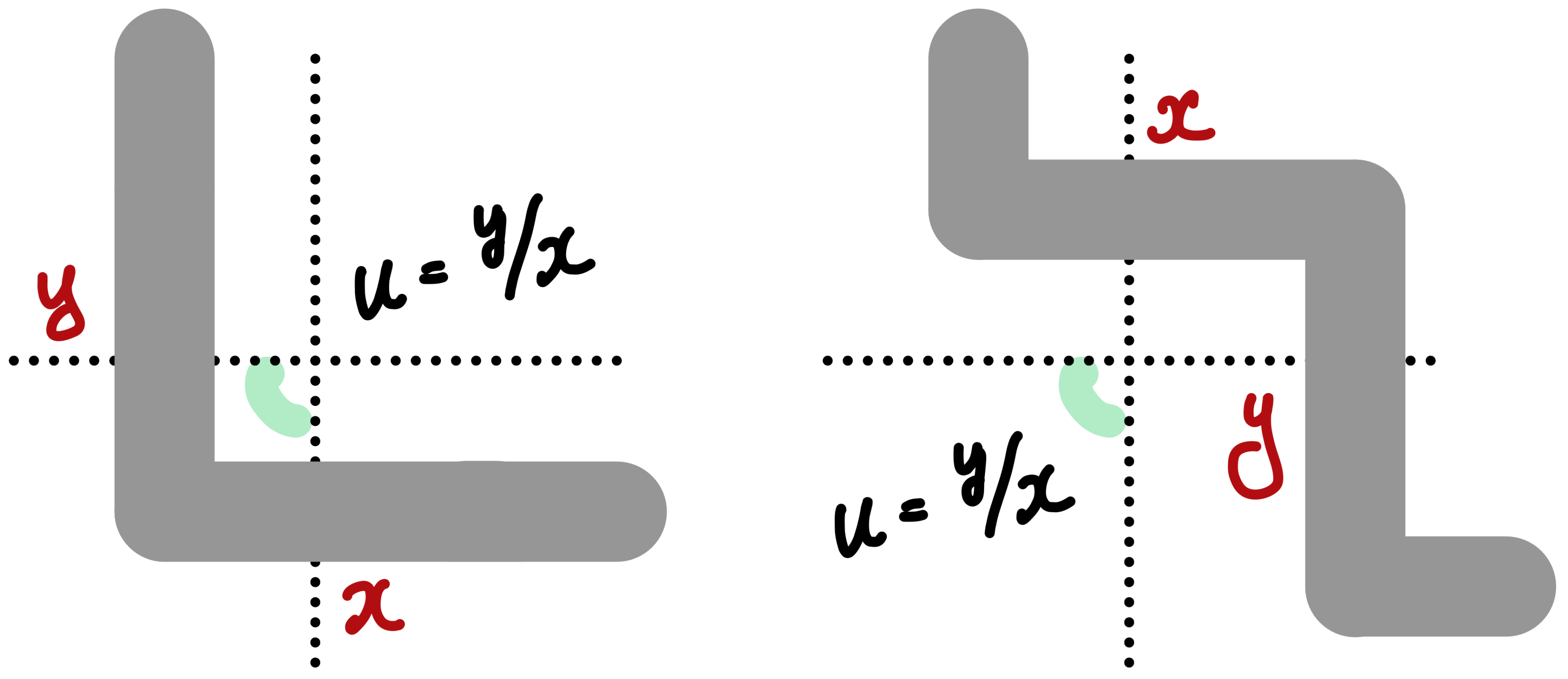


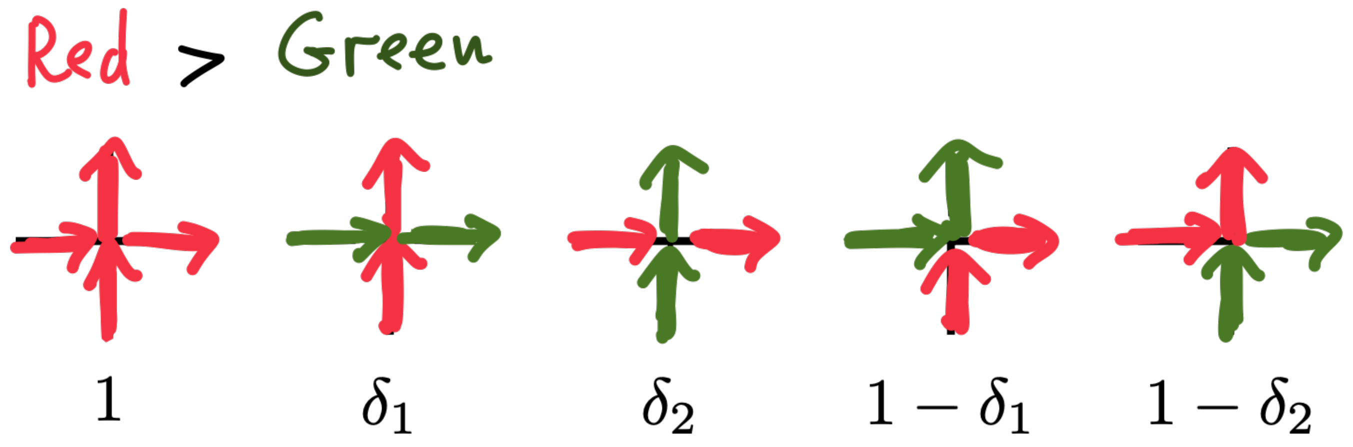


$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

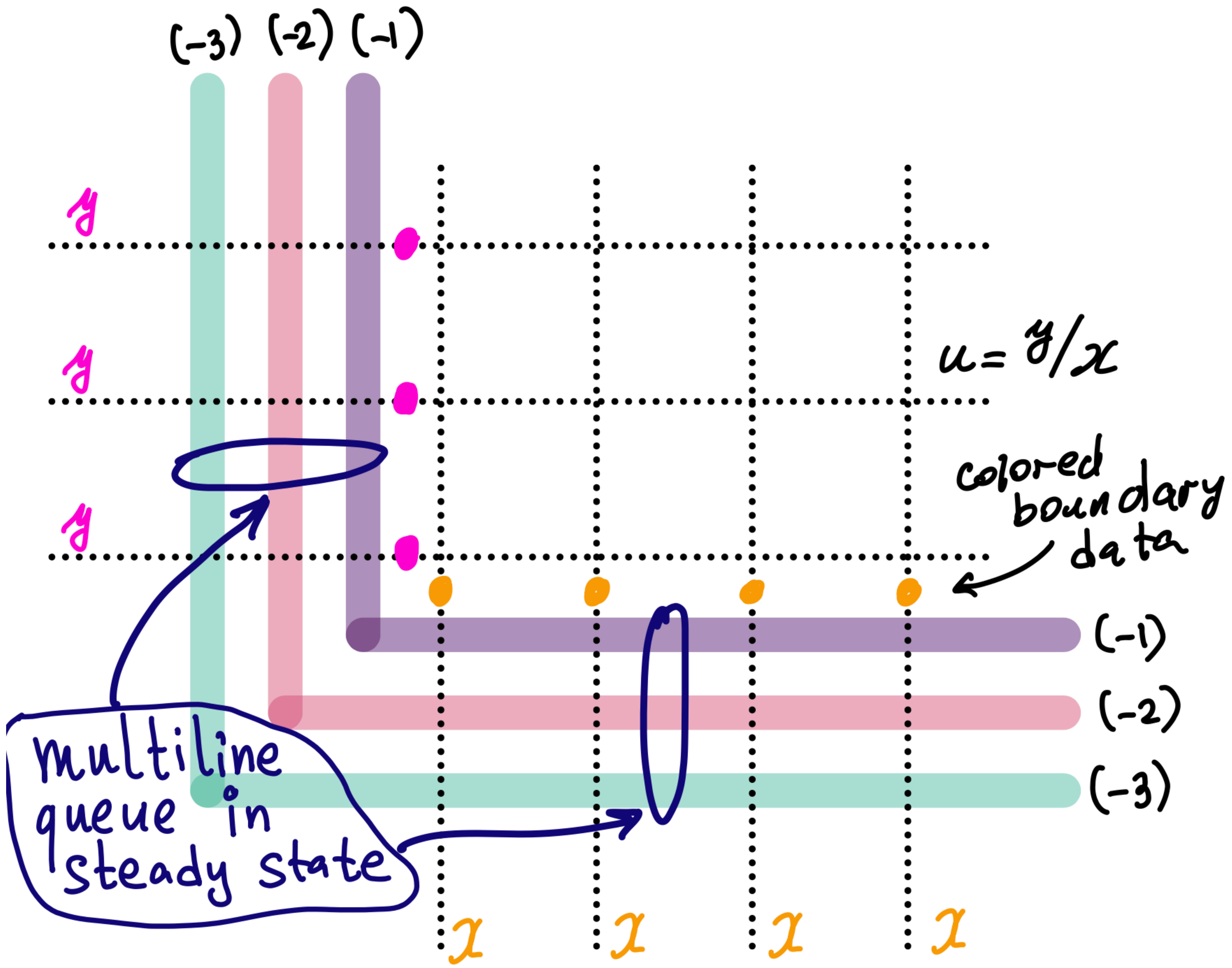


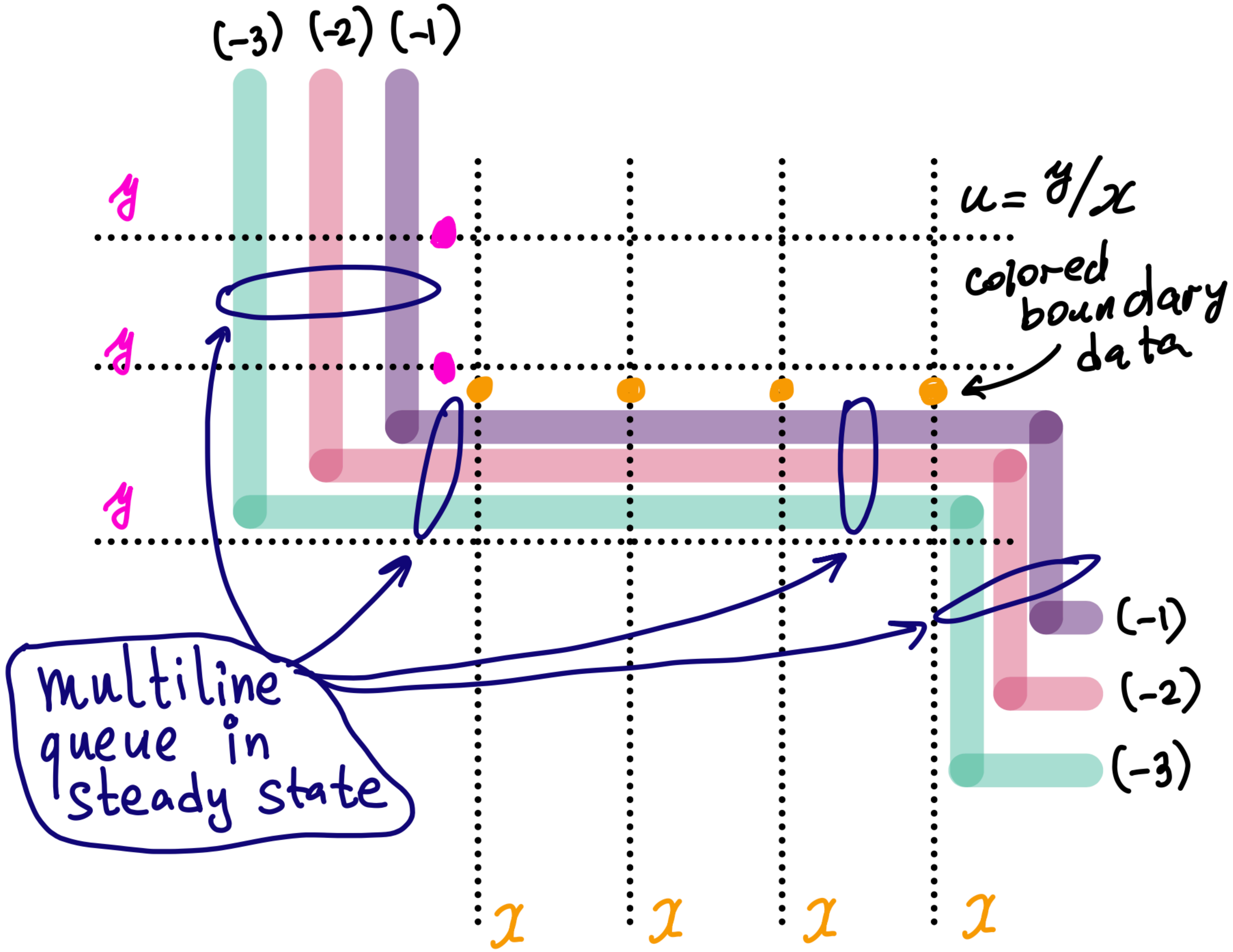
Colored stochastic six-vertex model. Many colors ⇒ many fat lines



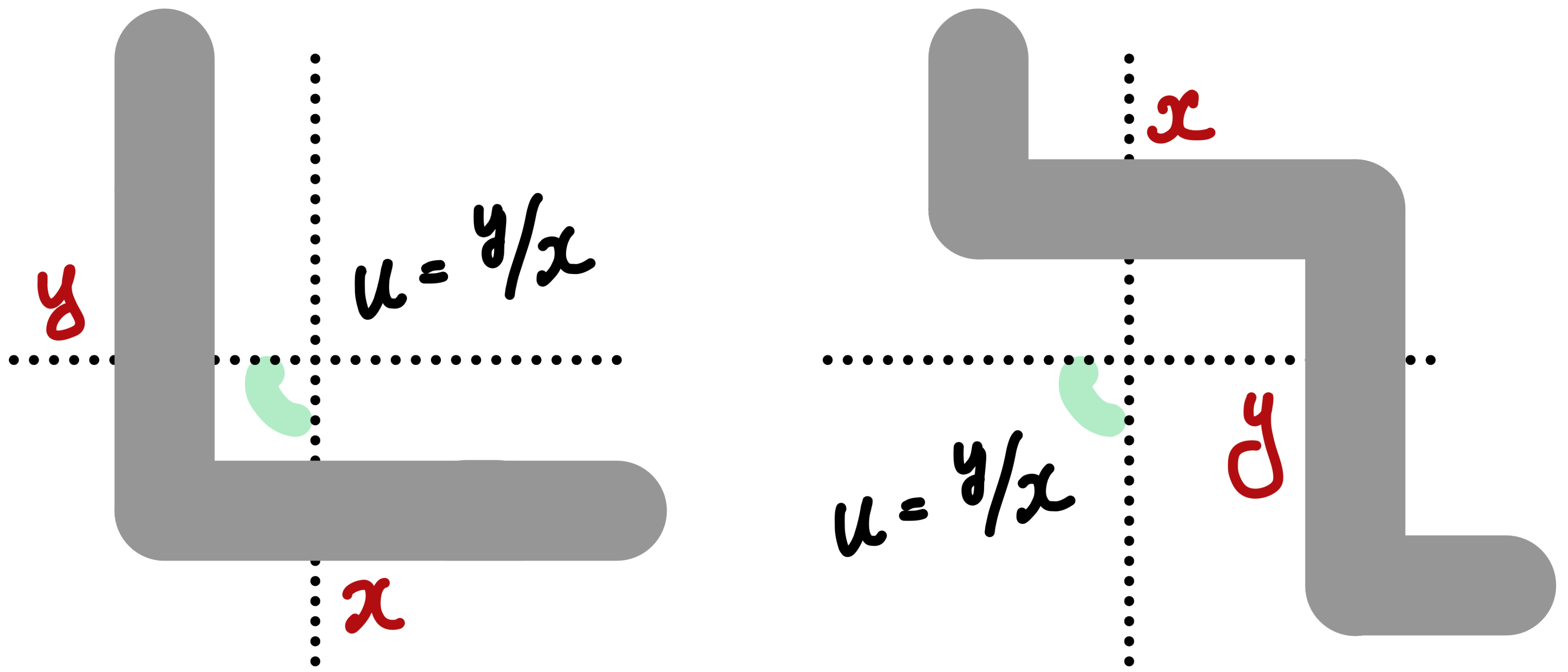


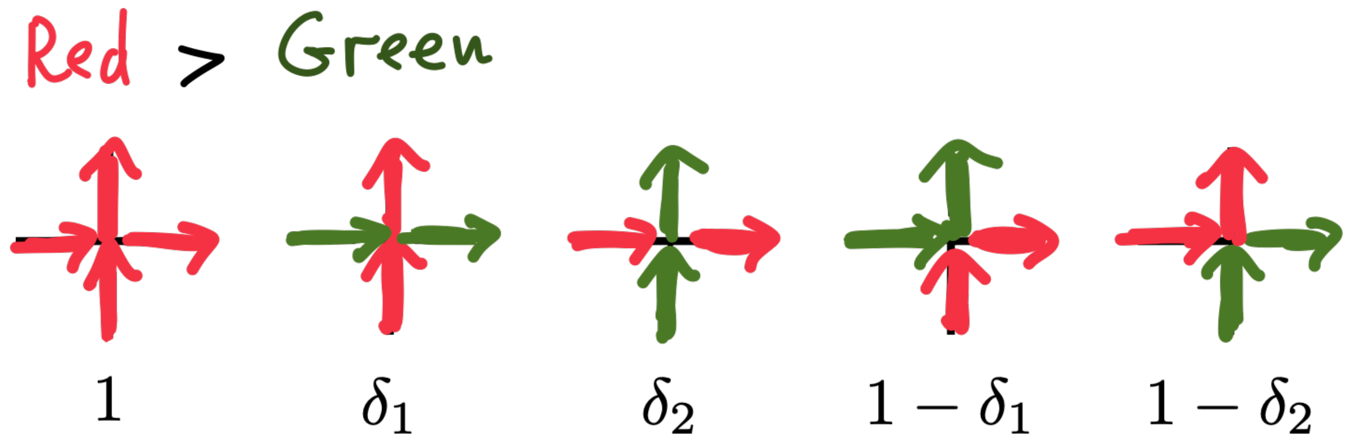
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



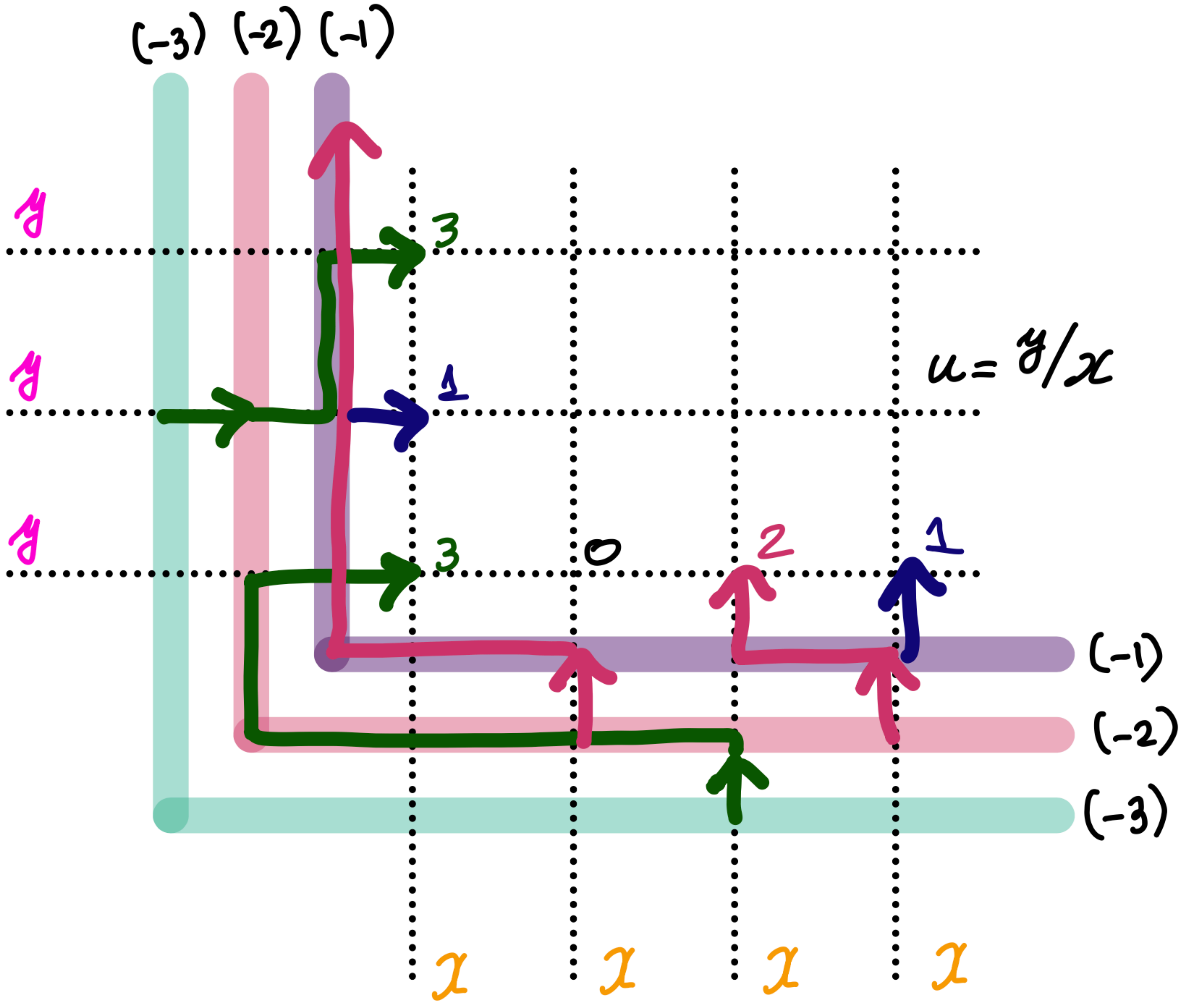


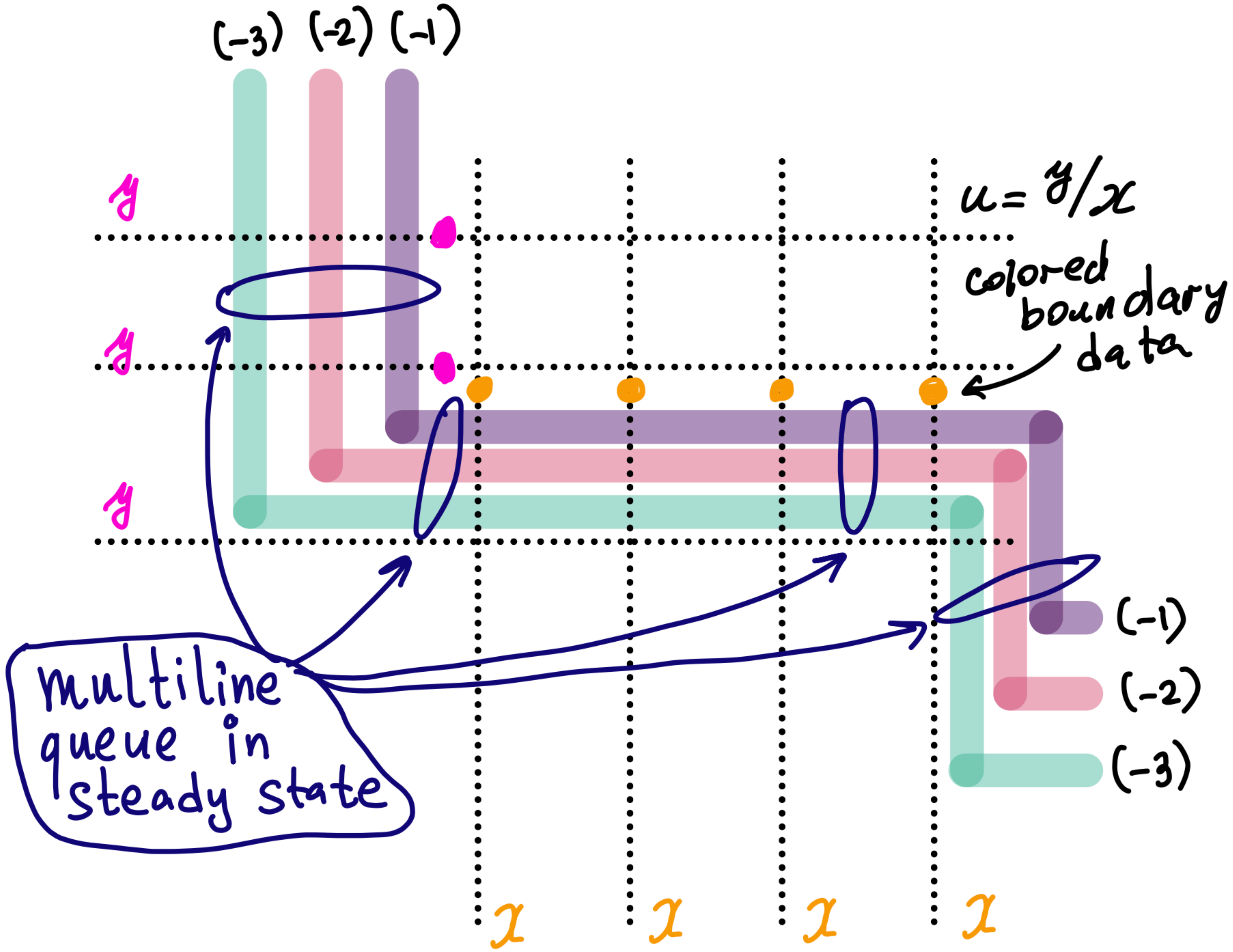
Colored stochastic six-vertex model. Many colors ⇒ many fat lines



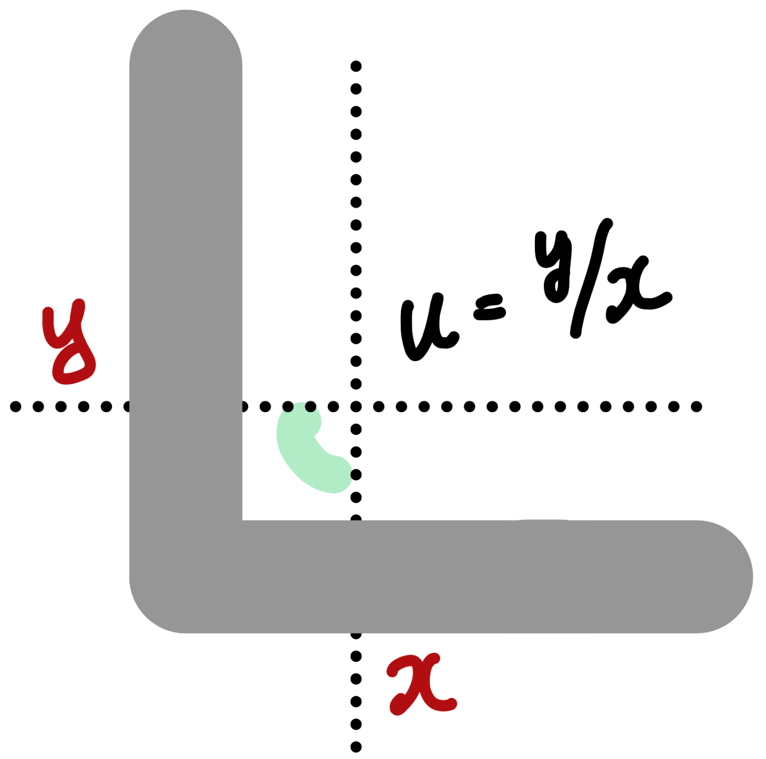


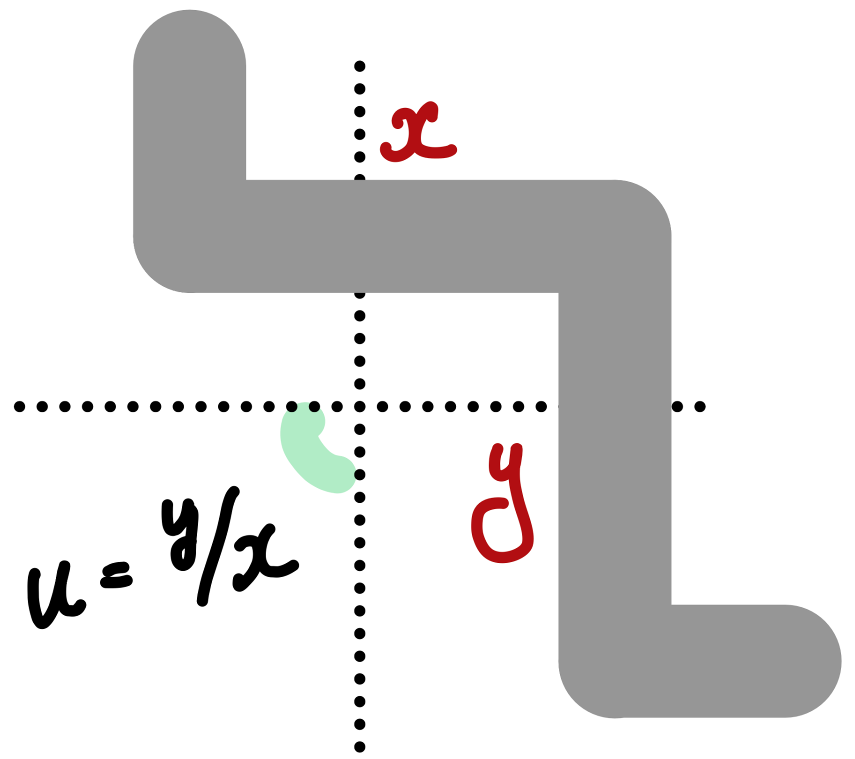
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



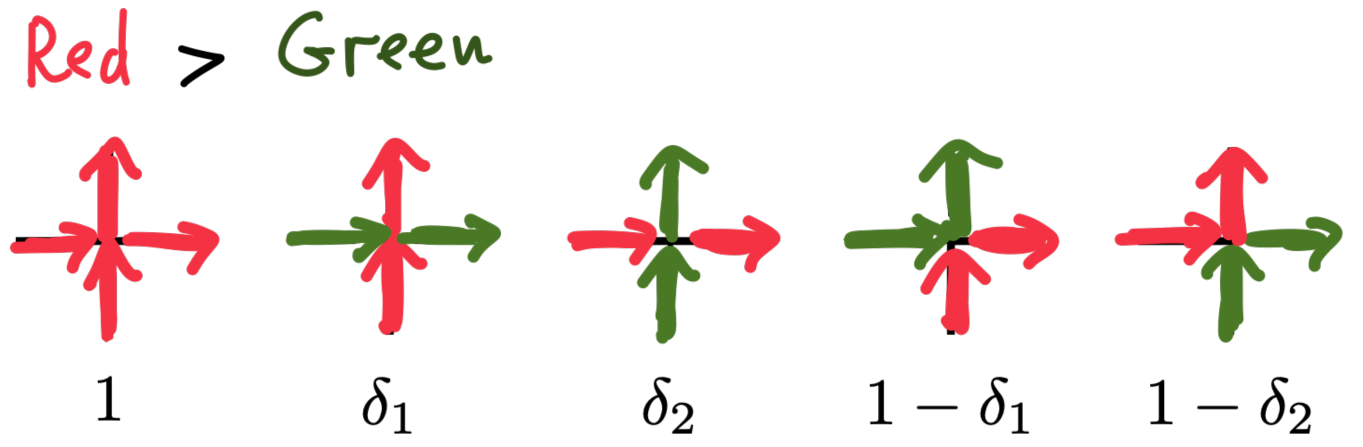


Colored stochastic six-vertex model. Many colors ⇒ many fat lines

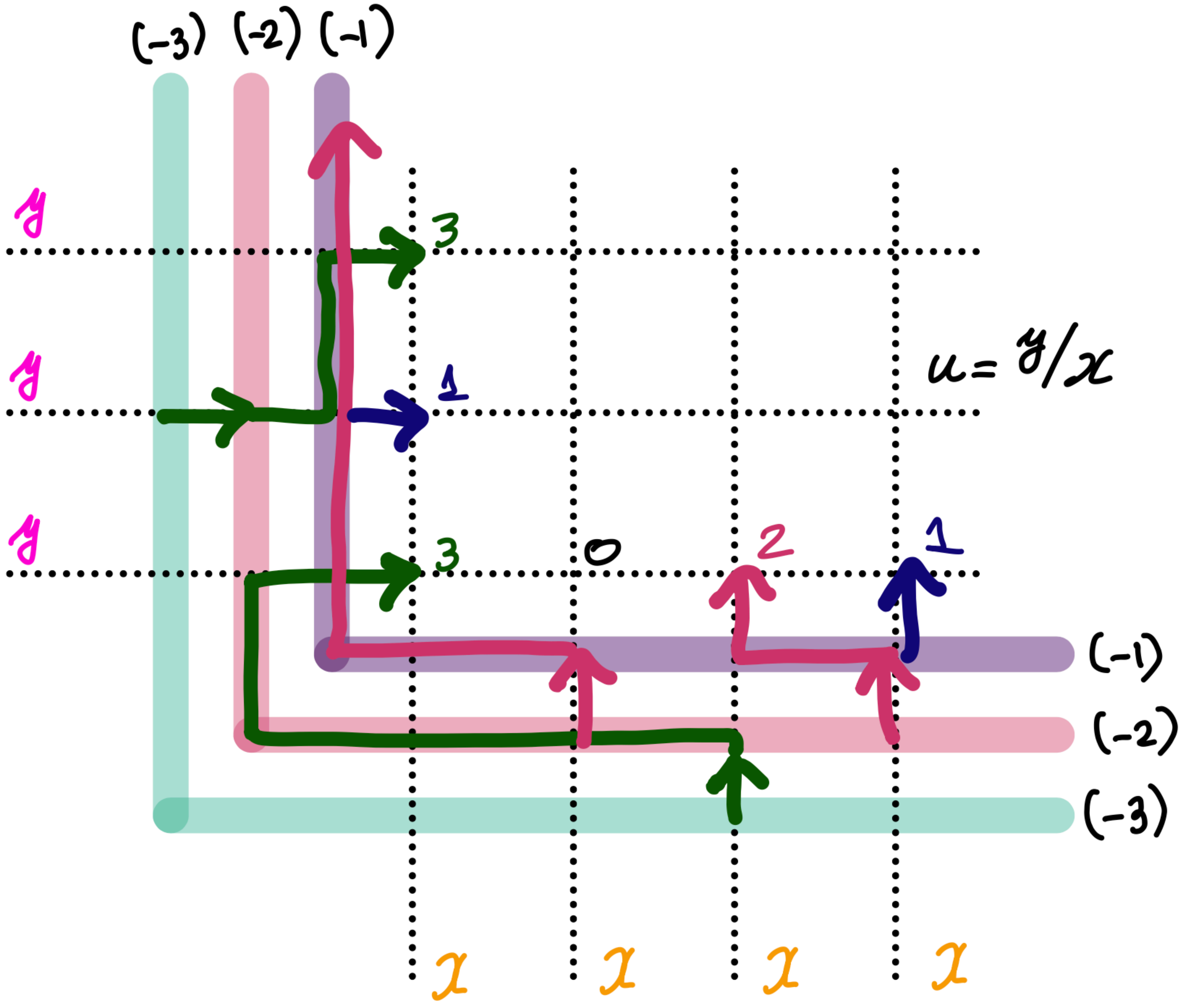


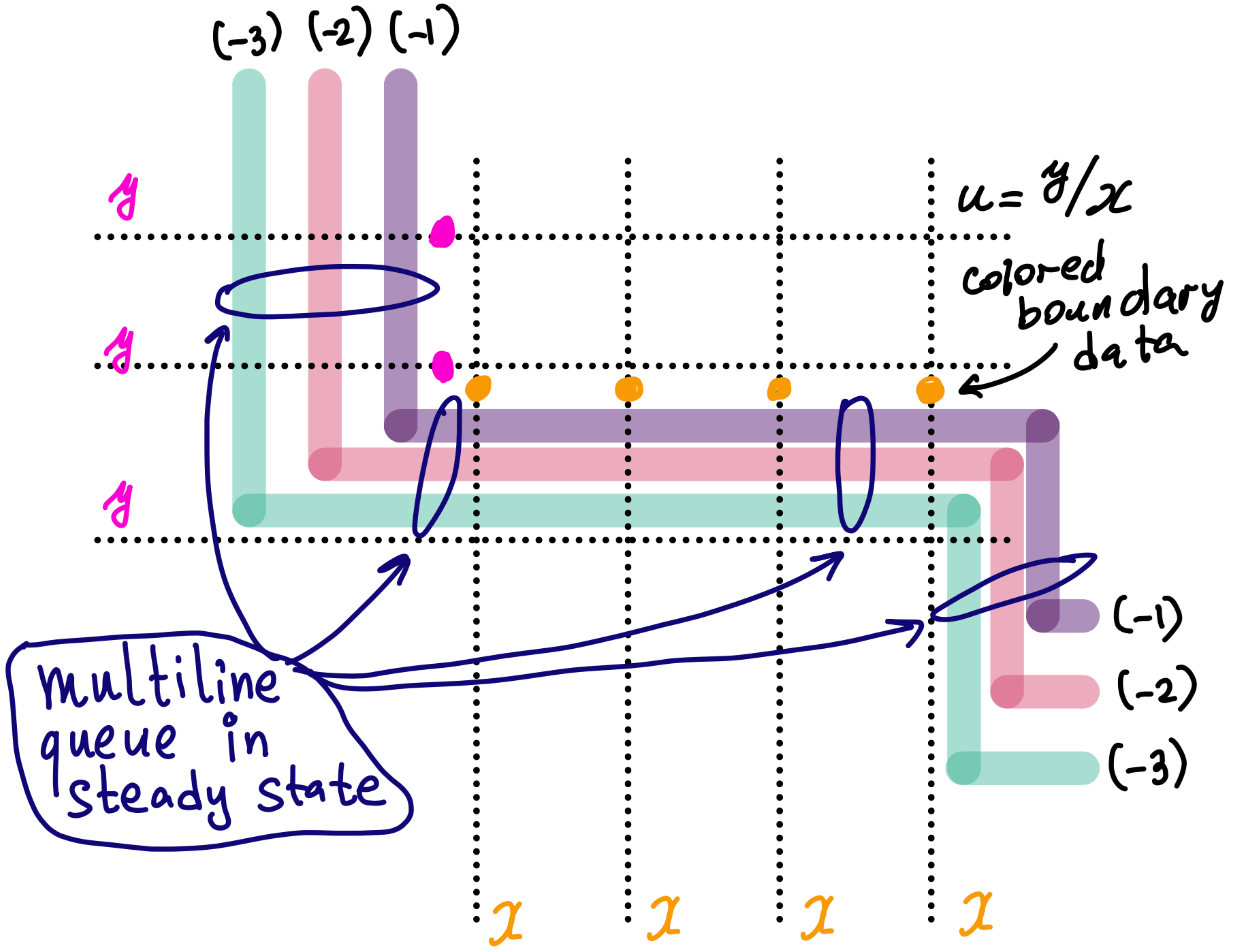


Implies mASEP stationary measure results on the line
+
multiline queues results)

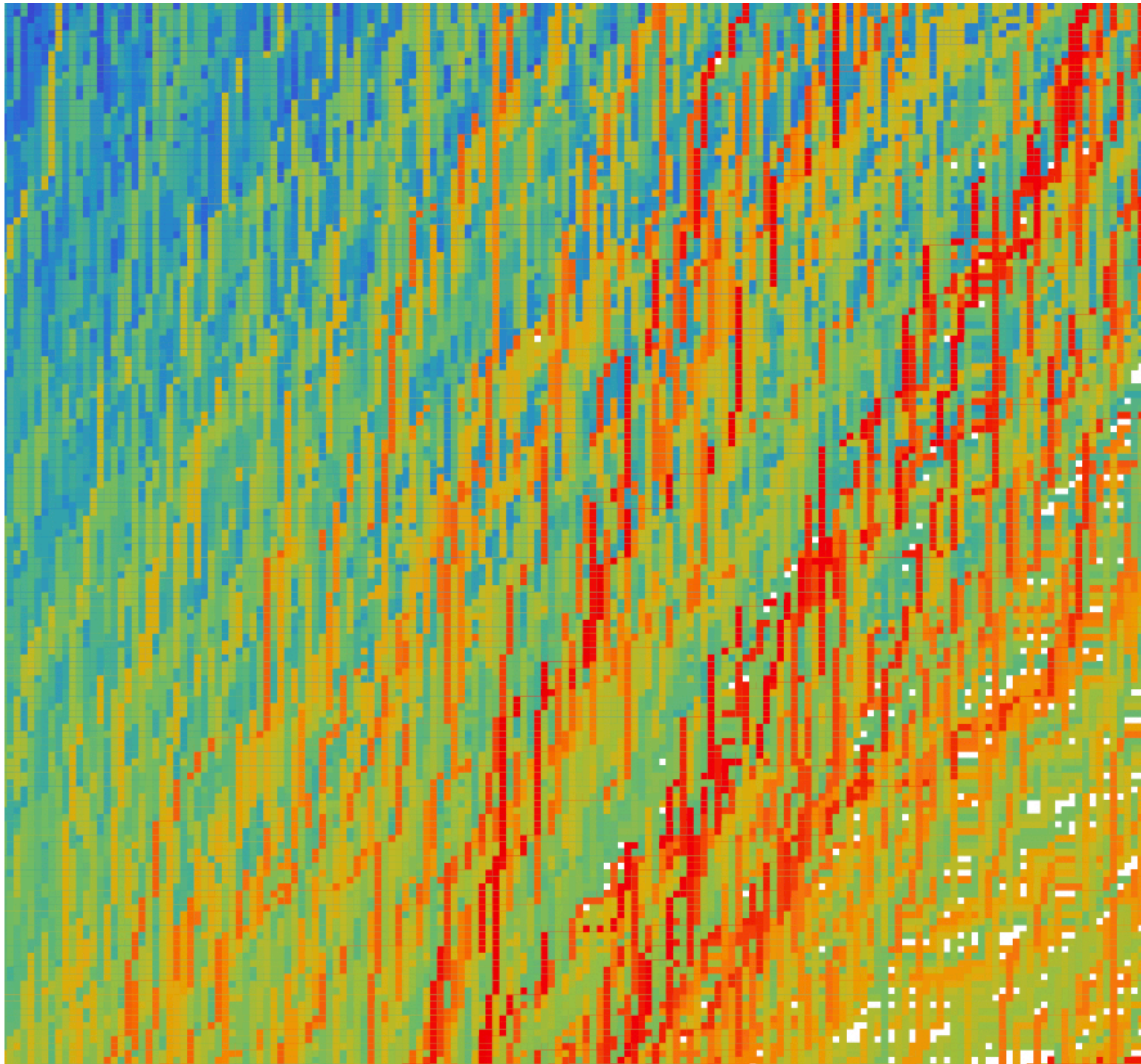


$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$





Go to the ring!

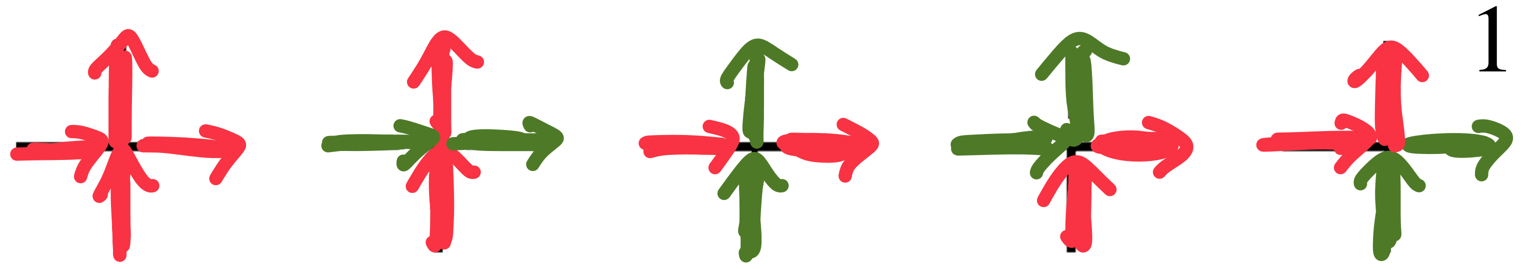


Yang-Baxter equation on the $n \times N$ cylinder

Red > Green

R_u

$$u = \frac{1-\delta_1}{1-\delta_2}$$



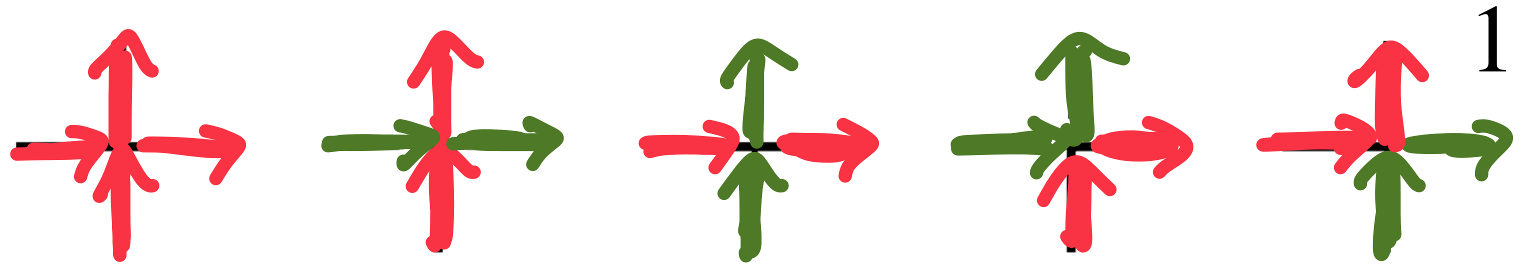
$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

$\mathbf{A}$ $0 \qquad 0$ $\mathbf{A}$ 1	$\mathbf{A}$ $k \qquad k$ $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ $0 \qquad k$ $\mathbf{A}$ $x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ $0 \qquad m$ $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ $k \qquad 0$ $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ $k \qquad \ell$ $\mathbf{A}$ $x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ $\ell \qquad k$ $\mathbf{A}$ $s\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ $\ell \qquad m$ $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Yang-Baxter equation on the $n \times N$ cylinder

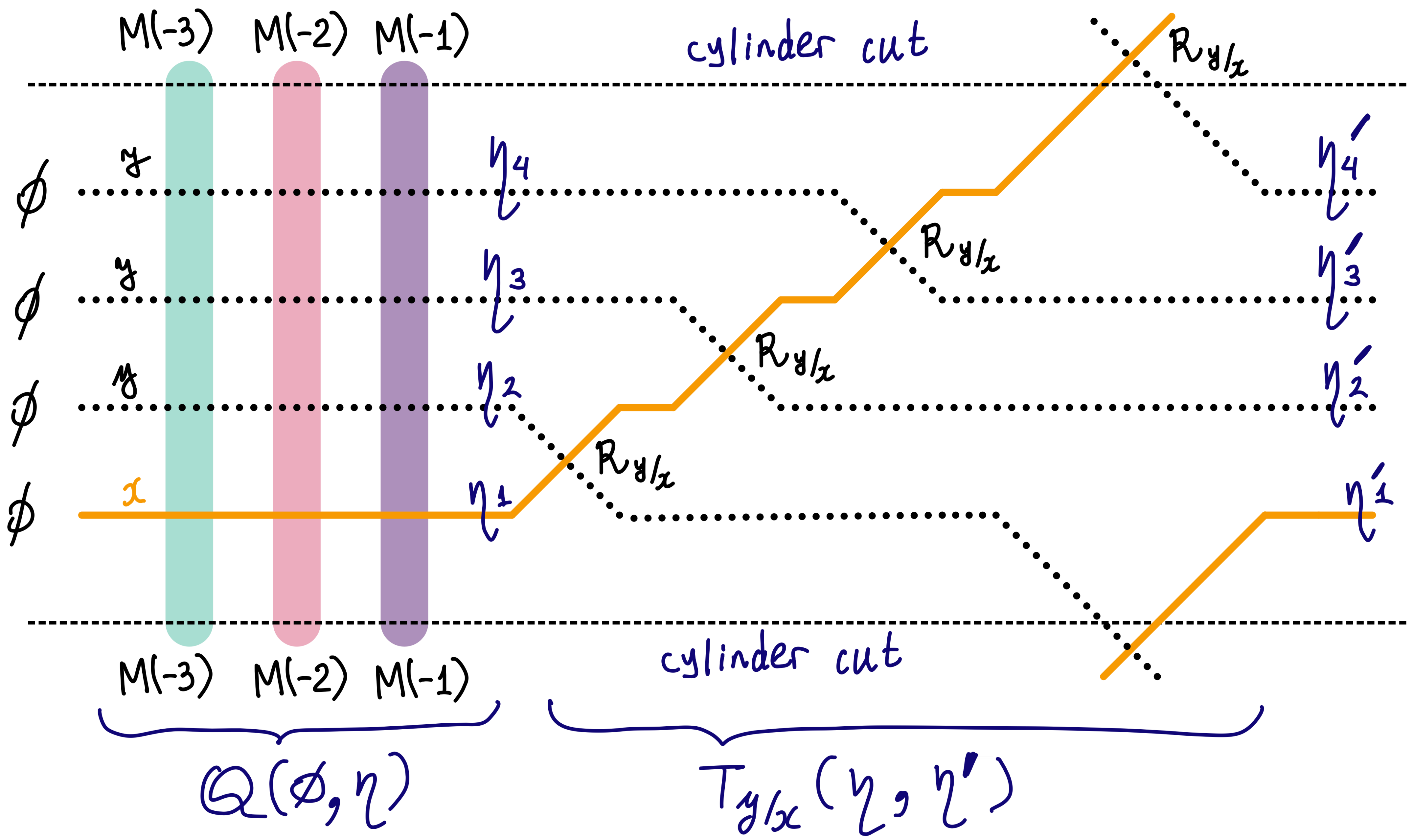
Red > Green $\qquad R_u \qquad u = \dfrac{1-\delta_1}{1-\delta_2}$



$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

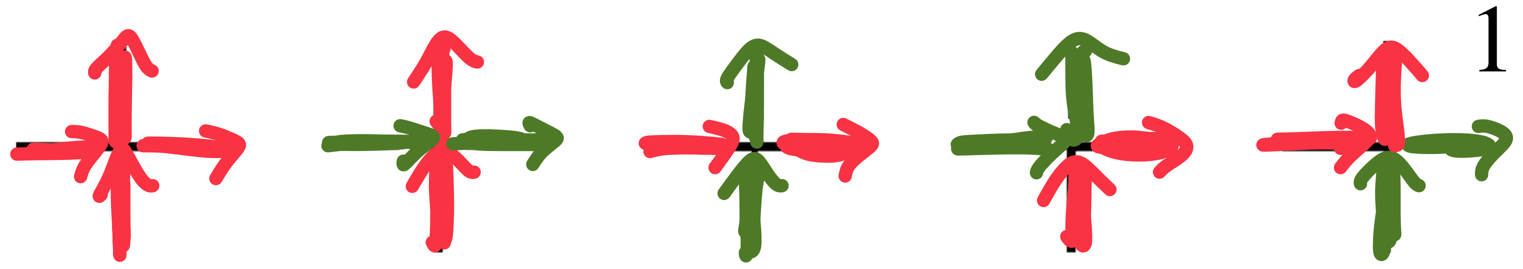
(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

$\mathbf{A}$ 0 ┼ 0 $\mathbf{A}$ 1	$\mathbf{A}$ k ┼ k $\mathbf{A}$ $(x - sq^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ 0 ┼ k $\mathbf{A}$ $x\,(1 - q^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ 0 ┼ m $\mathbf{A}$ $x\, q^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ k ┼ 0 $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ k ┼ ℓ $\mathbf{A}$ $x\,(1 - q^{A_\ell})\, q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ ℓ ┼ k $\mathbf{A}$ $s\,(1 - q^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ ℓ ┼ m $\mathbf{A}$ $s\, q^{\mathbf{A}_{[m+1,n]}}$



Yang-Baxter equation on the $n \times N$ cylinder

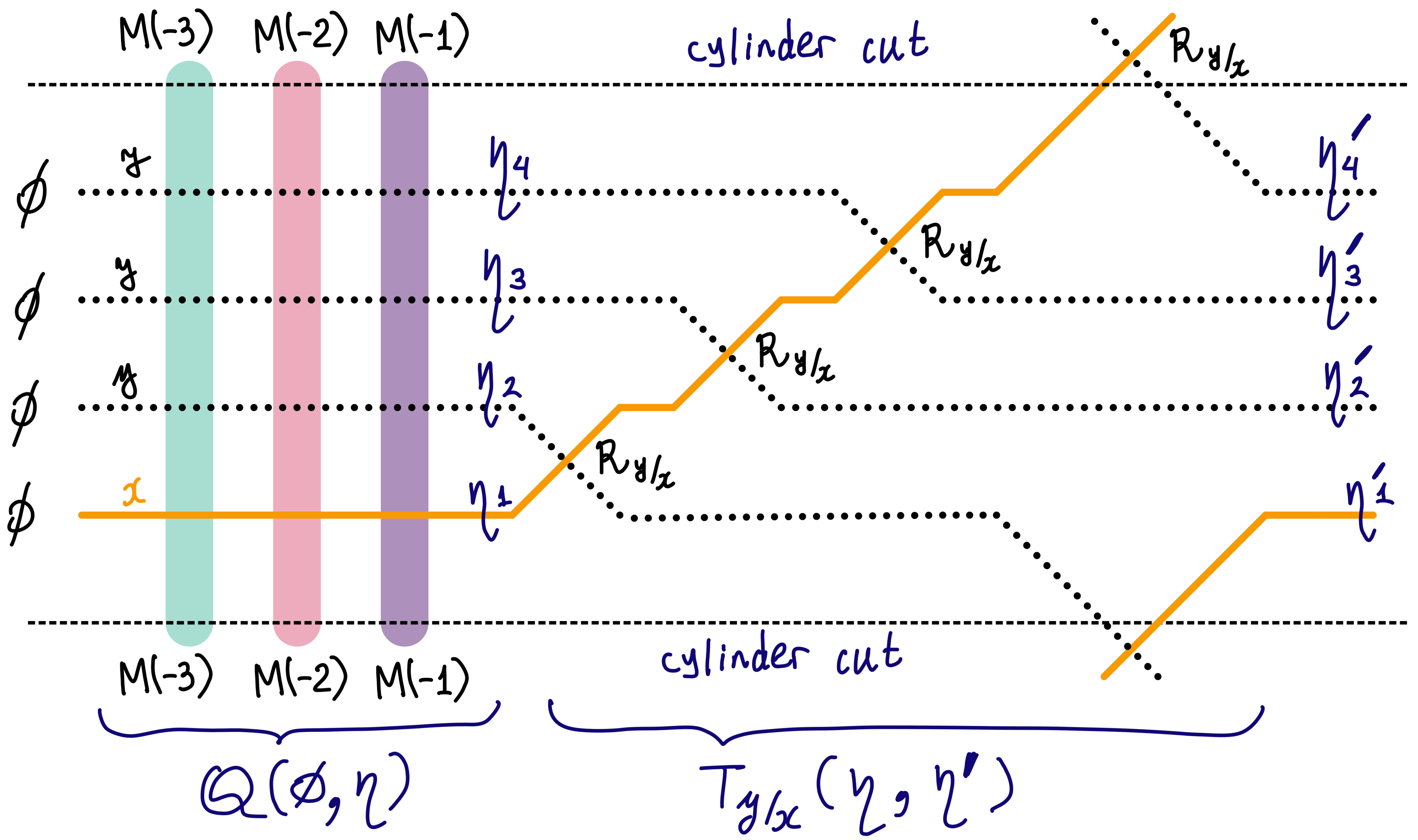
Red > Green $\qquad R_u \qquad u = \dfrac{1-\delta_1}{1-\delta_2}$

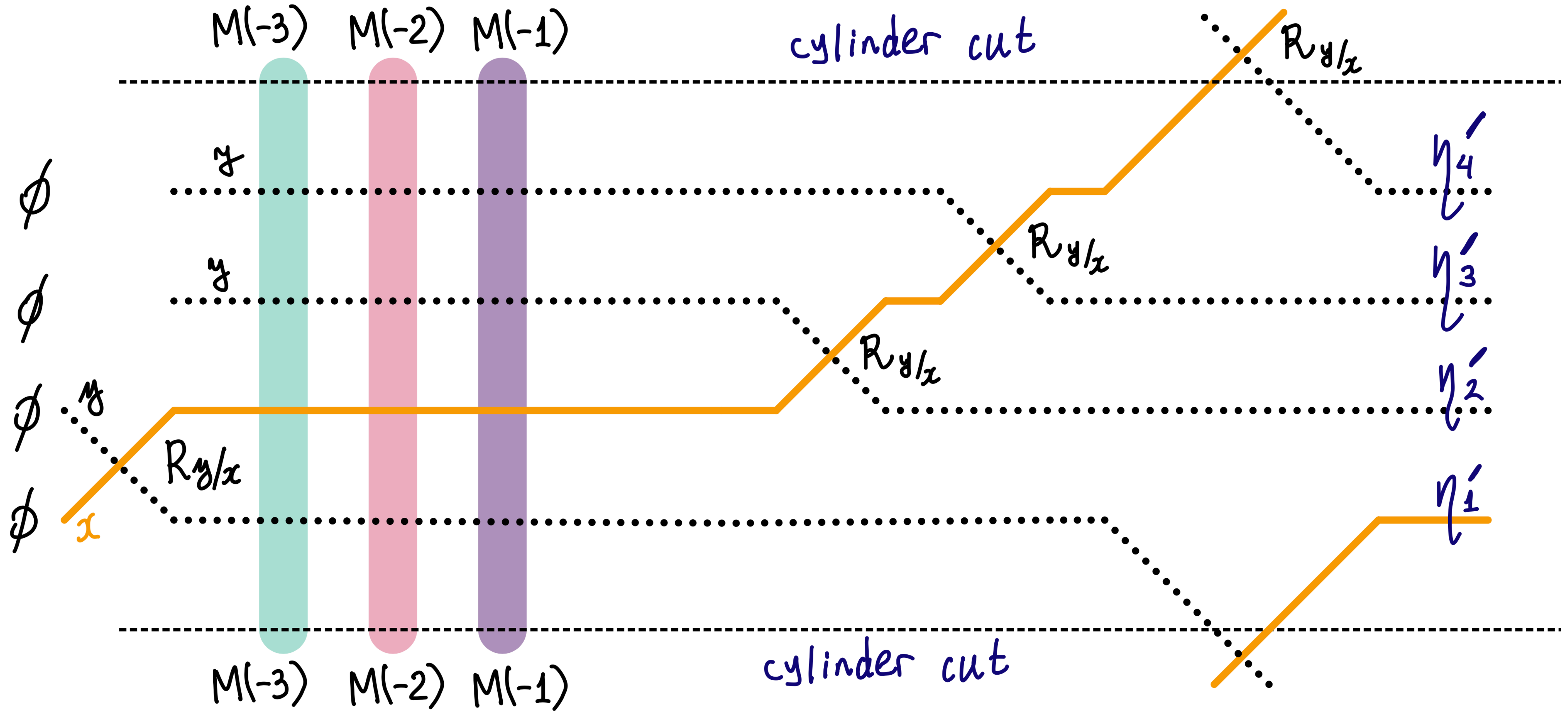


$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

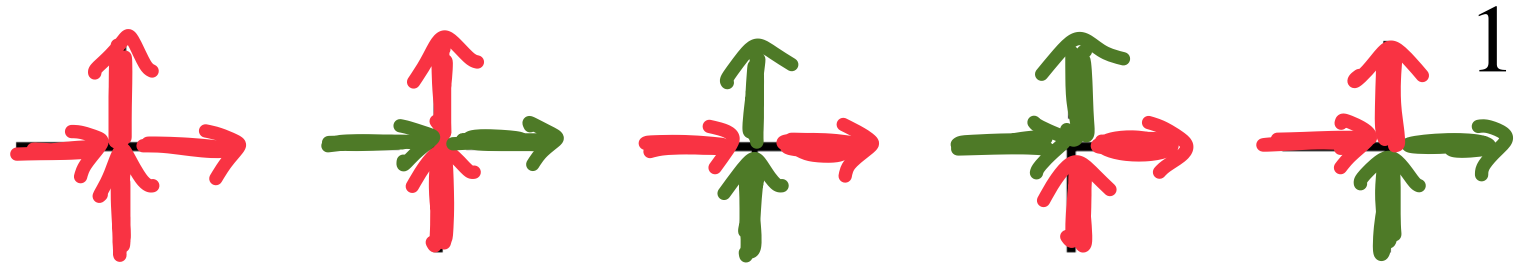
$\mathbf{A}$ 0 ┼ 0 $\mathbf{A}$ 1	$\mathbf{A}$ k ┼ k $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ 0 ┼ k $\mathbf{A}$ $x(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ 0 ┼ m $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ k ┼ 0 $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ k ┼ ℓ $\mathbf{A}$ $x(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ ℓ ┼ k $\mathbf{A}$ $s(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ ℓ ┼ m $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$





Yang-Baxter equation on the $n \times N$ cylinder

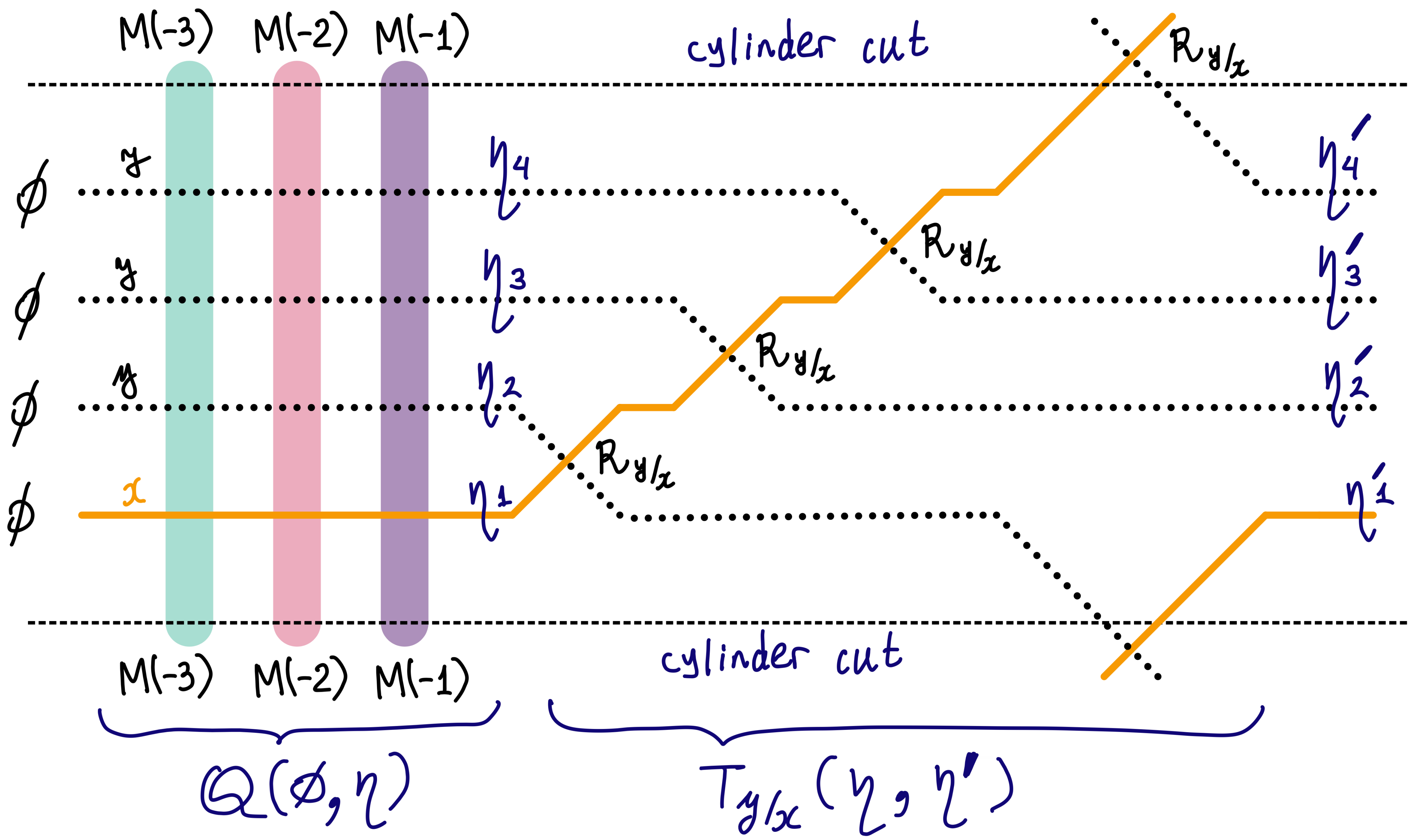
Red > Green $\qquad R_u \qquad u = \dfrac{1-\delta_1}{1-\delta_2}$

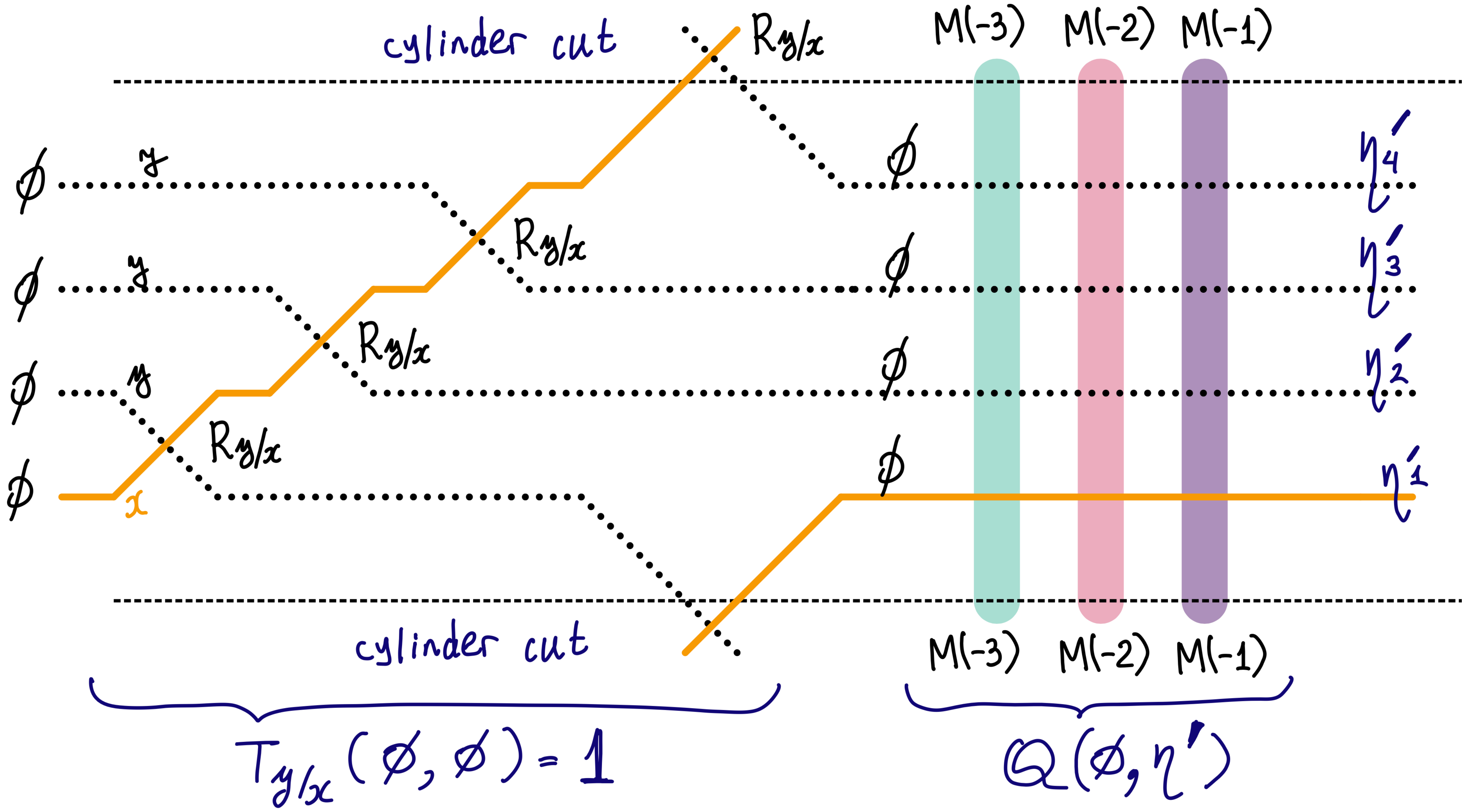


$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

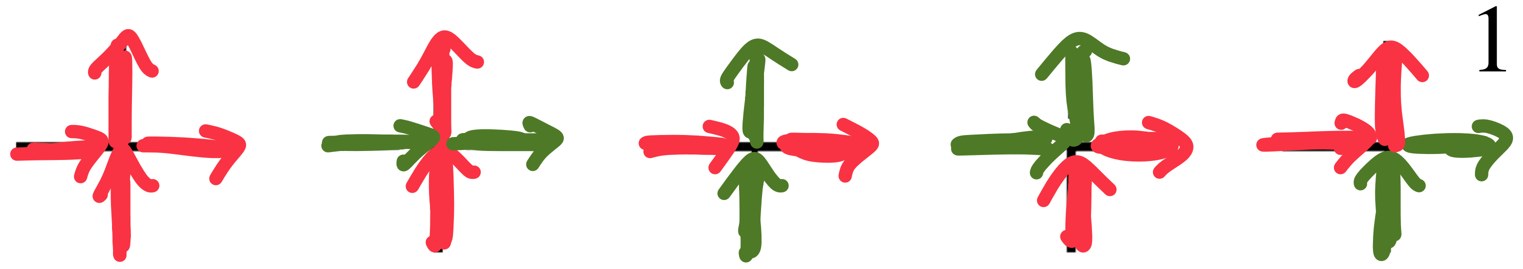
$\begin{array}{ccc} & \mathbf{A} & \\ 0 & + & 0 \\ & \mathbf{A} & \end{array}$ 1	$\begin{array}{ccc} & \mathbf{A} & \\ k & + & k \\ & \mathbf{A} & \end{array}$ $(x - sq^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	$\begin{array}{ccc} & \mathbf{A}_k^- & \\ 0 & + & k \\ & \mathbf{A} & \end{array}$ $x\,(1-q^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	$\begin{array}{ccc} & \mathbf{A} & \\ 0 & + & m \\ & \mathbf{A} & \end{array}$ $x\, q^{\mathbf{A}_{[m+1,n]}}$
$\begin{array}{ccc} & \mathbf{A}_k^+ & \\ k & + & 0 \\ & \mathbf{A} & \end{array}$ 1	$\begin{array}{ccc} & \mathbf{A}_{k\ell}^{+-} & \\ k & + & \ell \\ & \mathbf{A} & \end{array}$ $x\,(1-q^{A_\ell})\, q^{\mathbf{A}_{[\ell+1,n]}}$	$\begin{array}{ccc} & \mathbf{A}_{\ell k}^{+-} & \\ \ell & + & k \\ & \mathbf{A} & \end{array}$ $s\,(1-q^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	$\begin{array}{ccc} & \mathbf{A}_\ell^+ & \\ \ell & + & m \\ & \mathbf{A} & \end{array}$ $s\, q^{\mathbf{A}_{[m+1,n]}}$





Yang-Baxter equation on the $n \times N$ cylinder

Red > Green $\qquad R_u \qquad u = \dfrac{1-\delta_1}{1-\delta_2}$



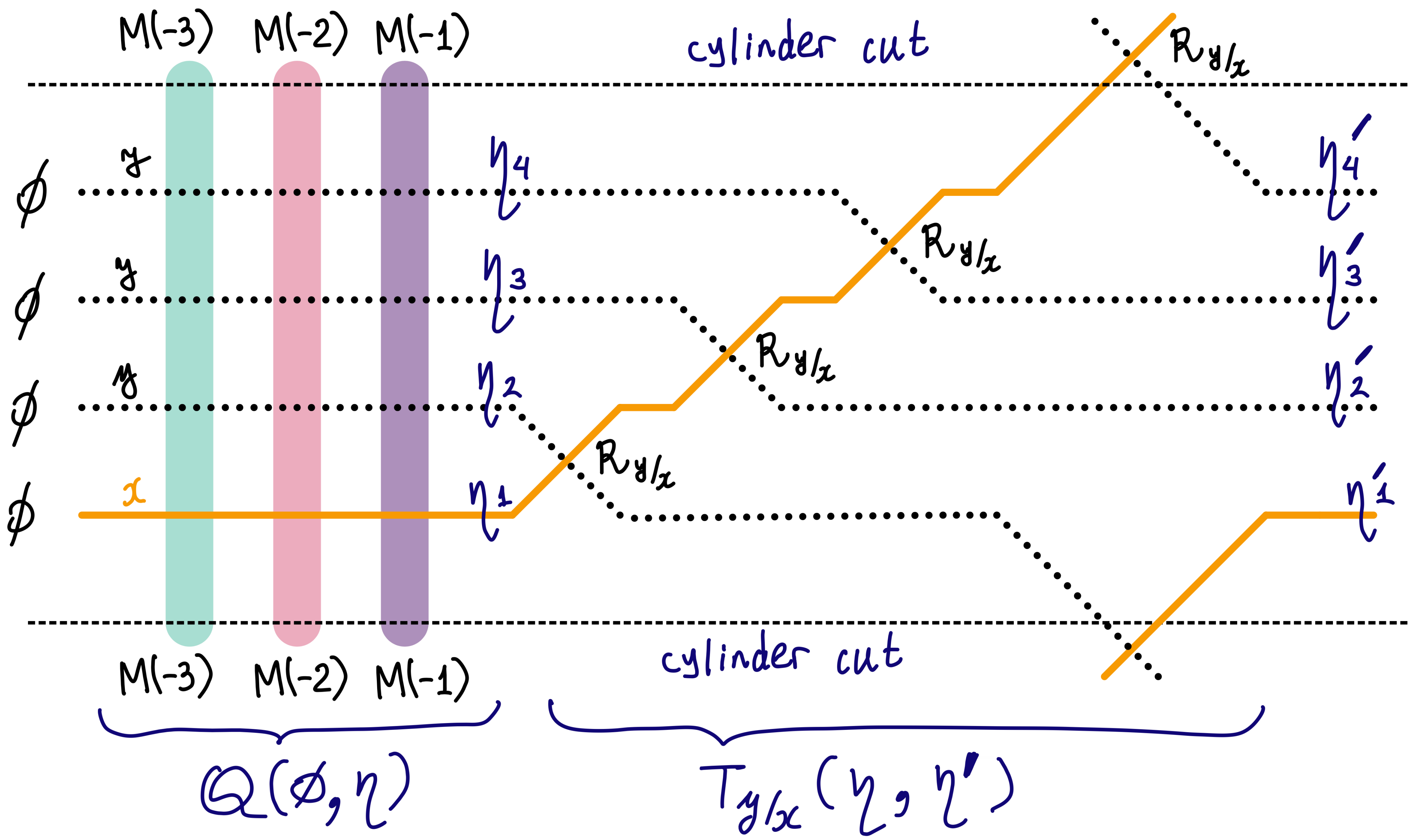
$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

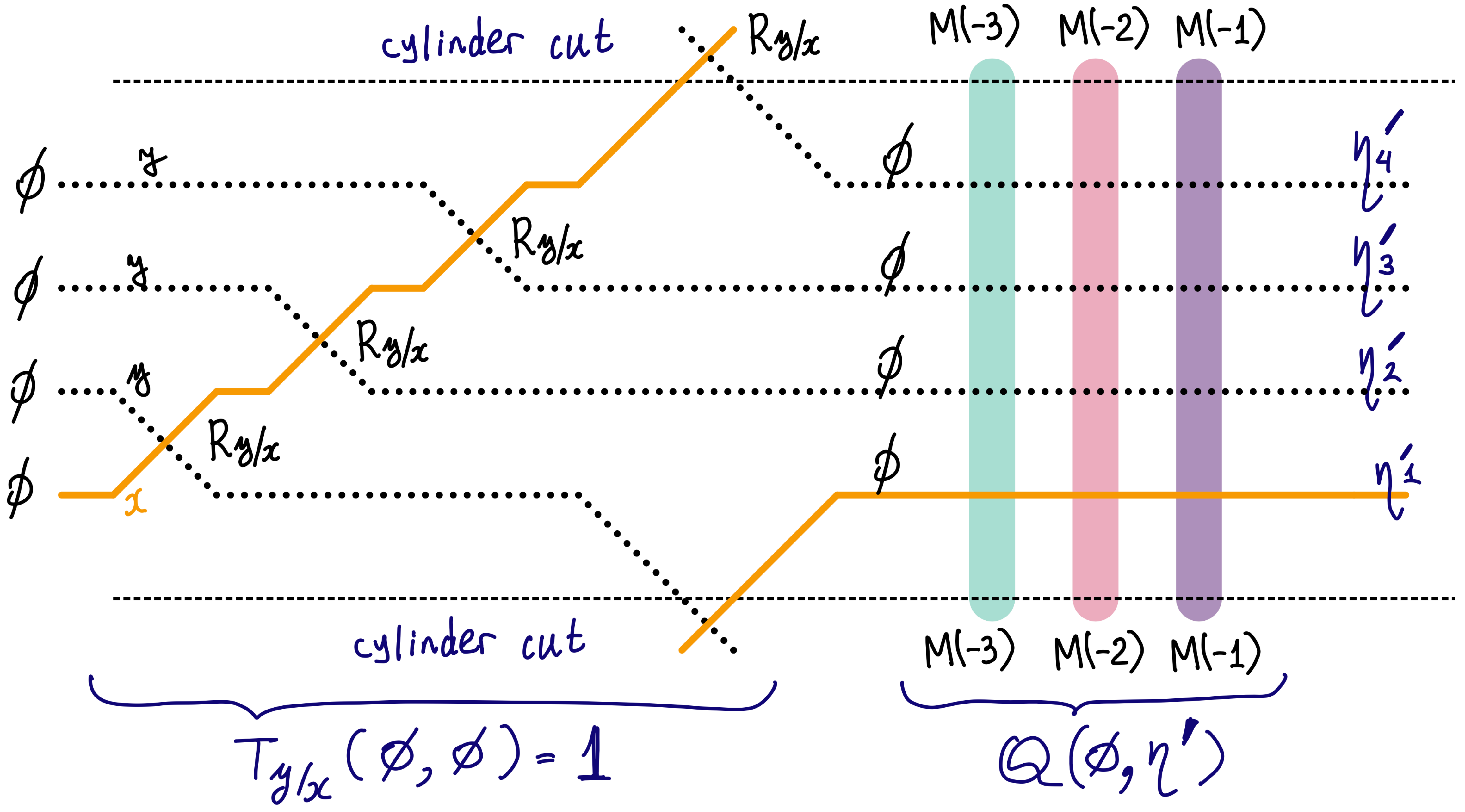
(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

top $\mathbf{A}$, left 0, right 0, bottom $\mathbf{A}$ 1	top $\mathbf{A}$, left k, right k, bottom $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_k^-$, left 0, right k, bottom $\mathbf{A}$ $x(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$, left 0, right m, bottom $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}_k^+$, left k, right 0, bottom $\mathbf{A}$ 1	top $\mathbf{A}_{k\ell}^{+-}$, left k, right ℓ, bottom $\mathbf{A}$ $x(1 - q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}_{\ell k}^{+-}$, left ℓ, right k, bottom $\mathbf{A}$ $s(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_\ell^+$, left ℓ, right m, bottom $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

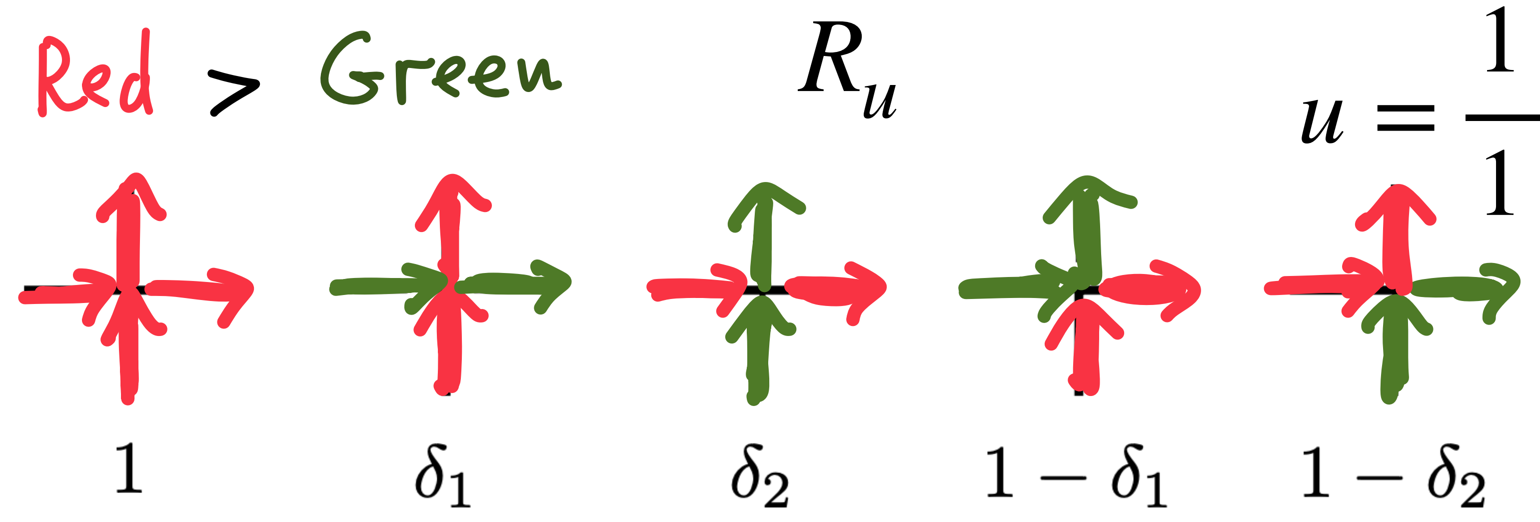
Commutation relation on the cylinder

$$\sum_\eta \mathfrak{Q}(\varnothing,\eta)T_{y/x}(\eta,\eta') = T_{y/x}(\varnothing,\varnothing)\mathfrak{Q}(\varnothing,\eta') = \mathfrak{Q}(\varnothing,\eta')$$





Yang-Baxter equation on the $n \times N$ cylinder



$$u = \frac{1-\delta_1}{1-\delta_2}$$

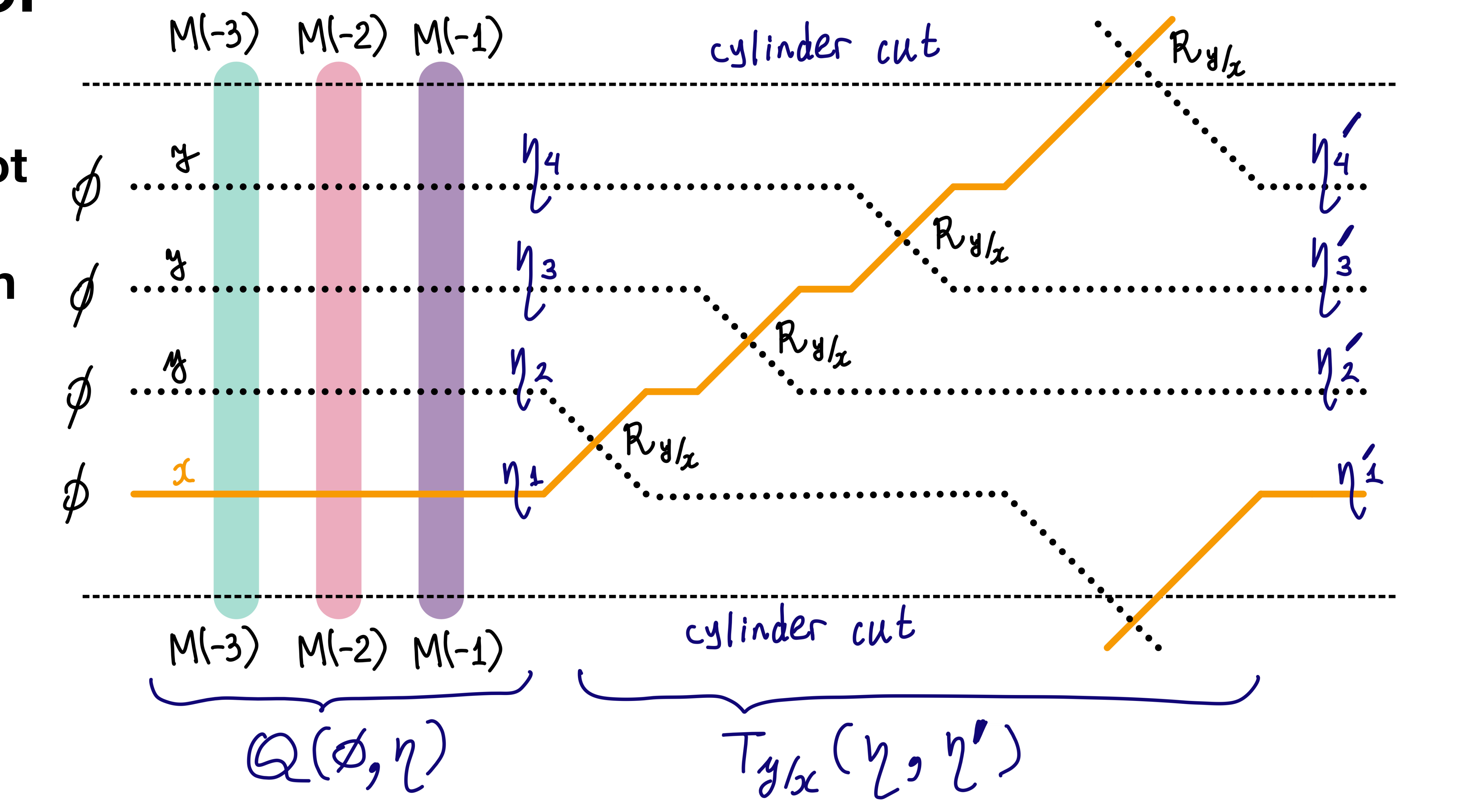
(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

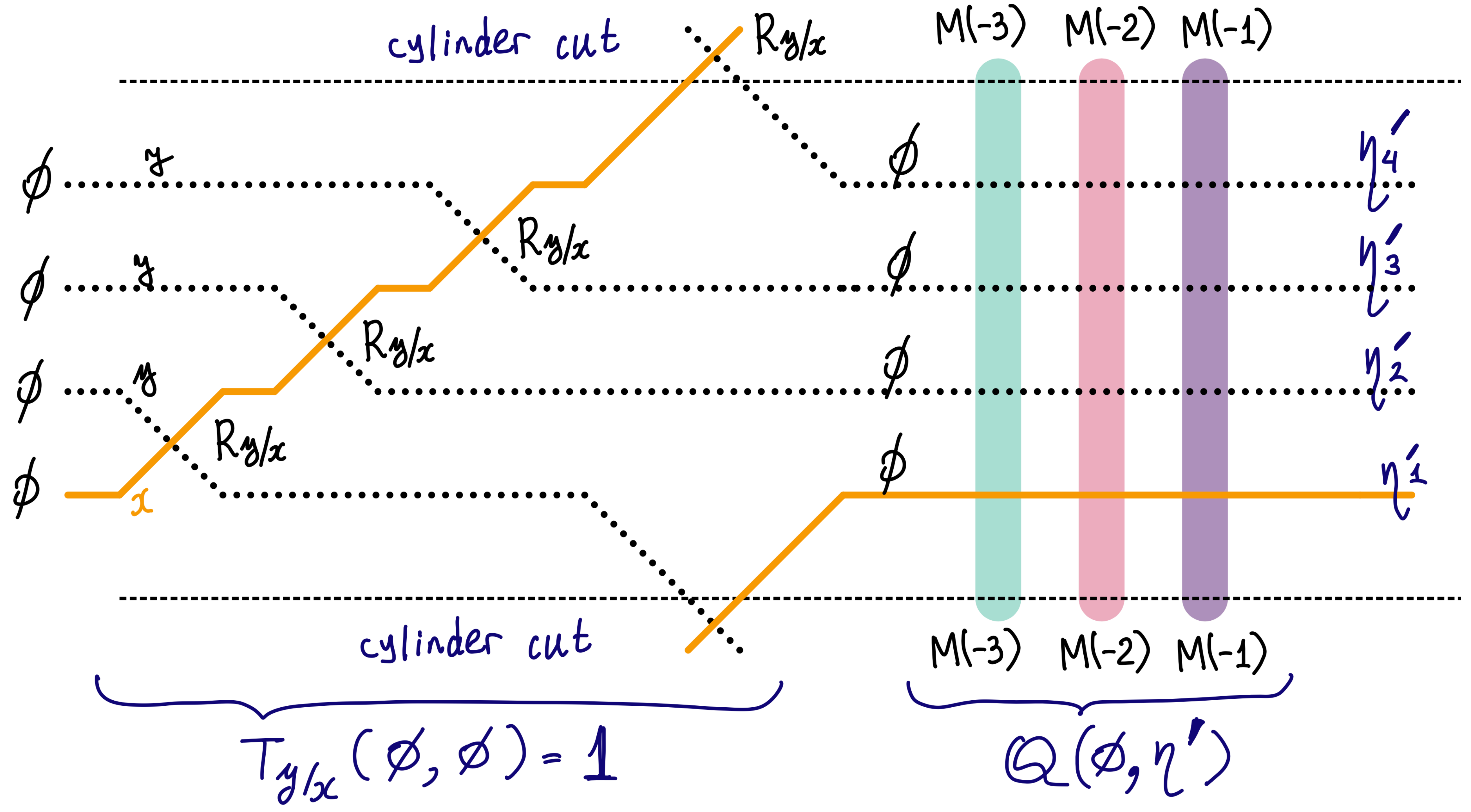
$\mathbf{A}$ $0 \quad 0$ $\mathbf{A}$ 1	$\mathbf{A}$ $k \quad k$ $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ $0 \quad k$ $\mathbf{A}$ $x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ $0 \quad m$ $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ $k \quad 0$ $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ $k \quad \ell$ $\mathbf{A}$ $x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ $\ell \quad k$ $\mathbf{A}$ $s\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ $\ell \quad m$ $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Commutation relation on the cylinder

$$\sum_{\eta} \mathfrak{Q}(\varnothing,\eta)T_{y/x}(\eta,\eta') = T_{y/x}(\varnothing,\varnothing)\mathfrak{Q}(\varnothing,\eta') = \mathfrak{Q}(\varnothing,\eta')$$

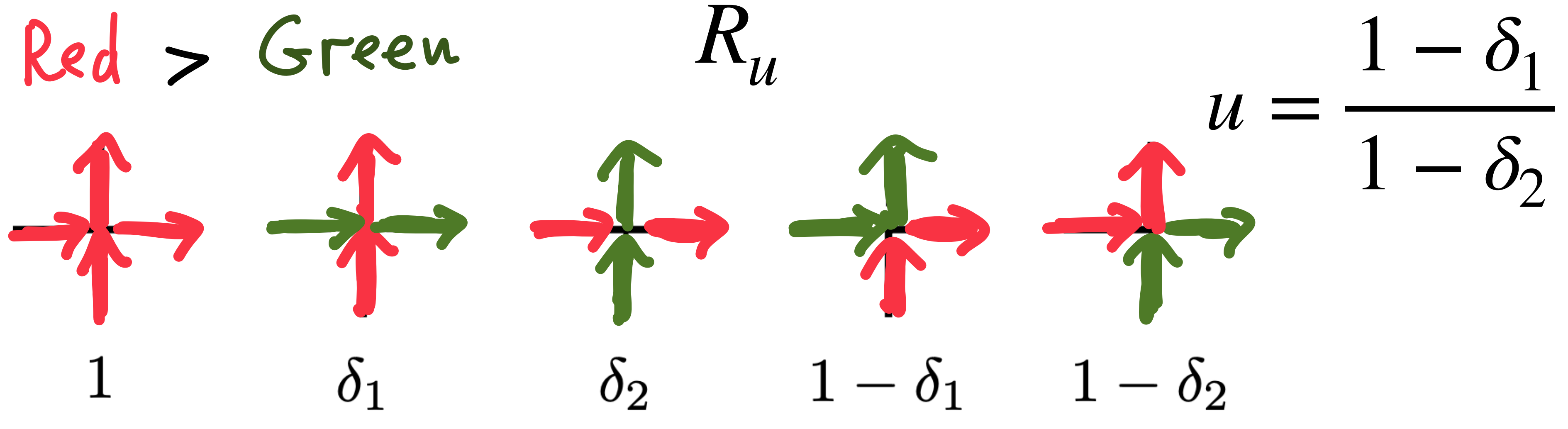
(Bethe Ansatz: construct eigenvalue of T as a partition function)

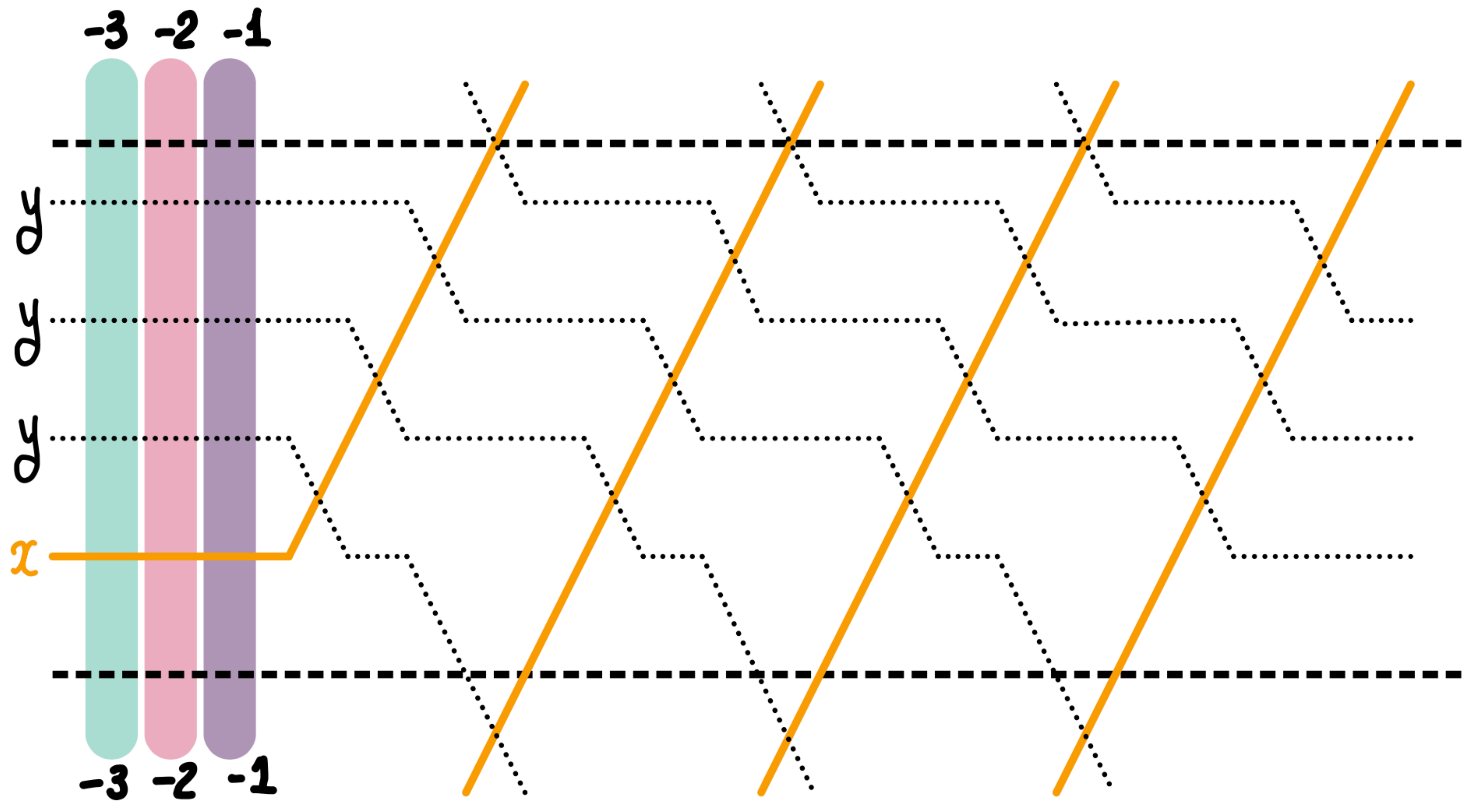




Limit to the mASEP, $y/x = 1 - \epsilon$, continuous time

mASEP limit $\delta_1, \delta_2 \to 0$, $q = \delta_1/\delta_2$





Time $\sim \tau/\epsilon$

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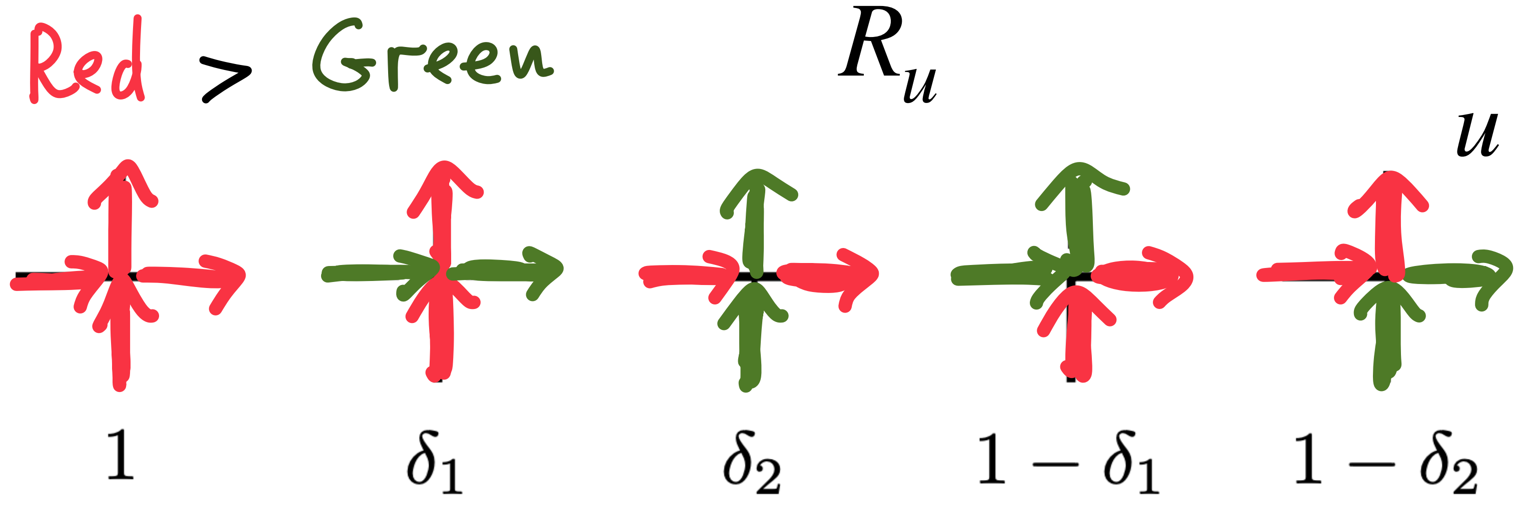
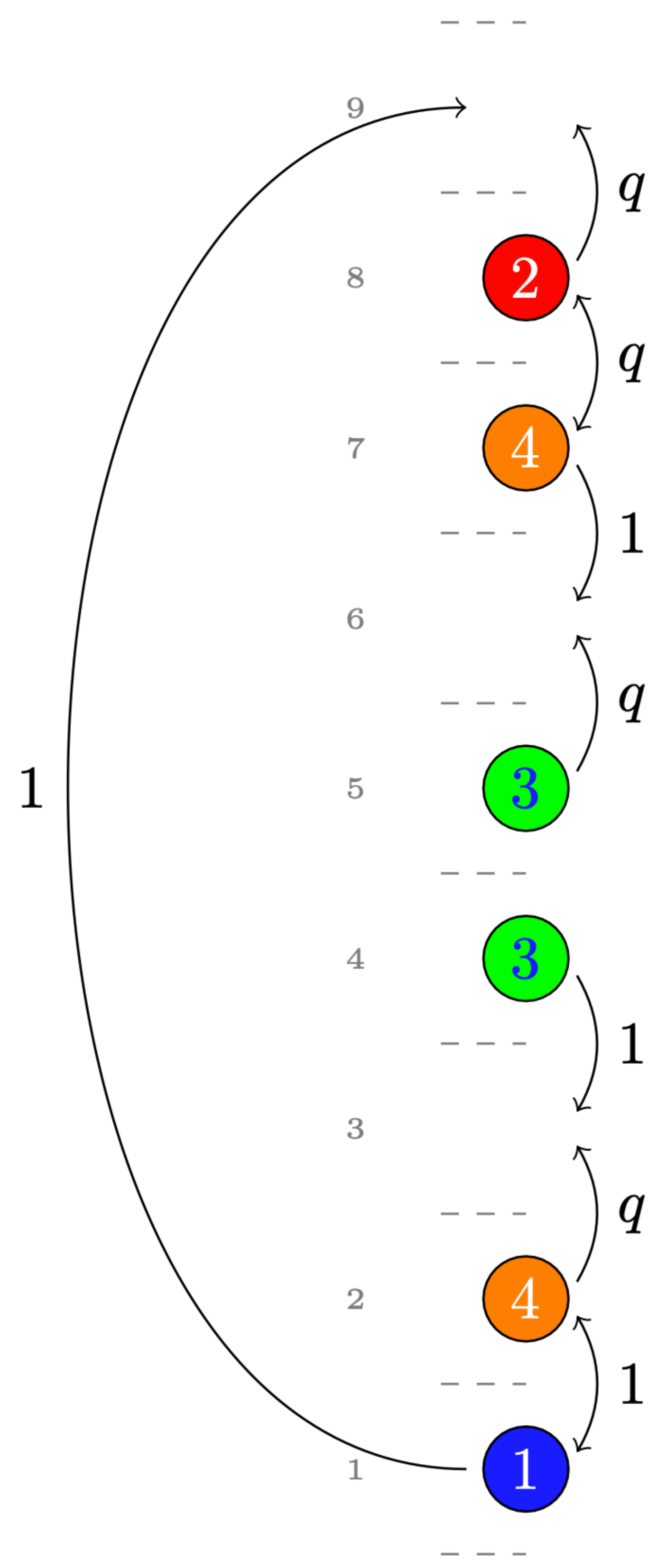
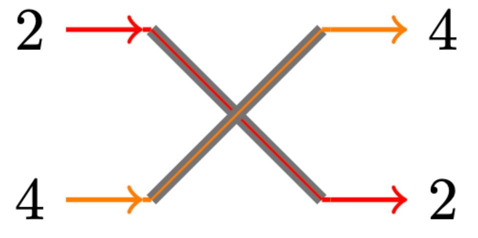


R_u

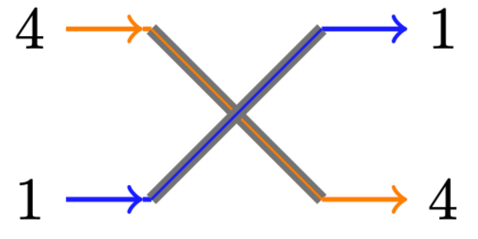
$$u = \frac{1-\delta_1}{1-\delta_2}$$

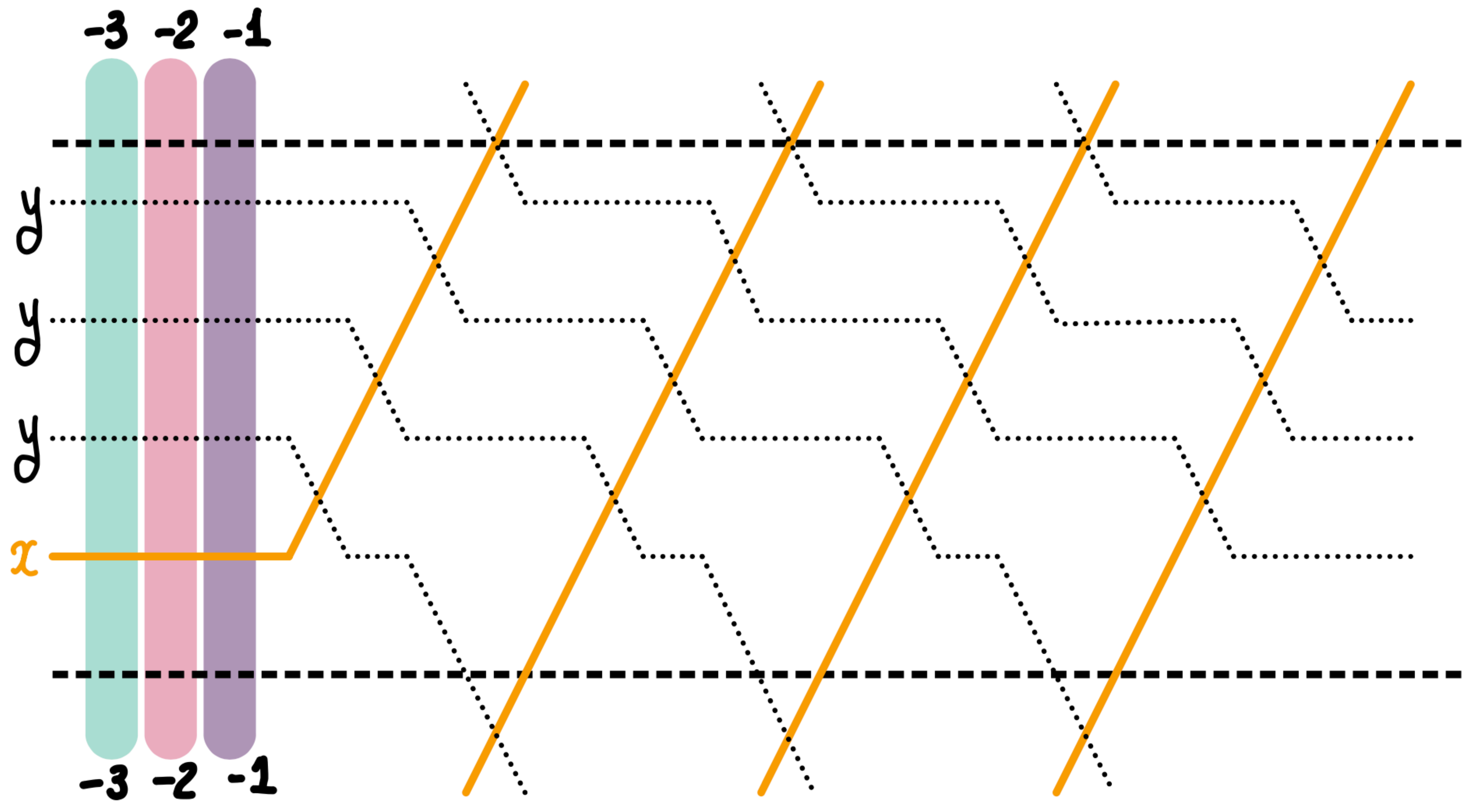


$$\langle 4,2|\check{\mathfrak{R}}_{1-\epsilon}|2,4\rangle = R_{1-\epsilon}(4,2,4,2) = \frac{q\epsilon}{1-q} + O(\epsilon^2)$$



$$\langle 1,4|\check{\mathfrak{R}}_{1-\epsilon}|4,1\rangle = R_{1-\epsilon}(1,4;1,4) = \frac{\epsilon}{1-q} + O(\epsilon^2)$$





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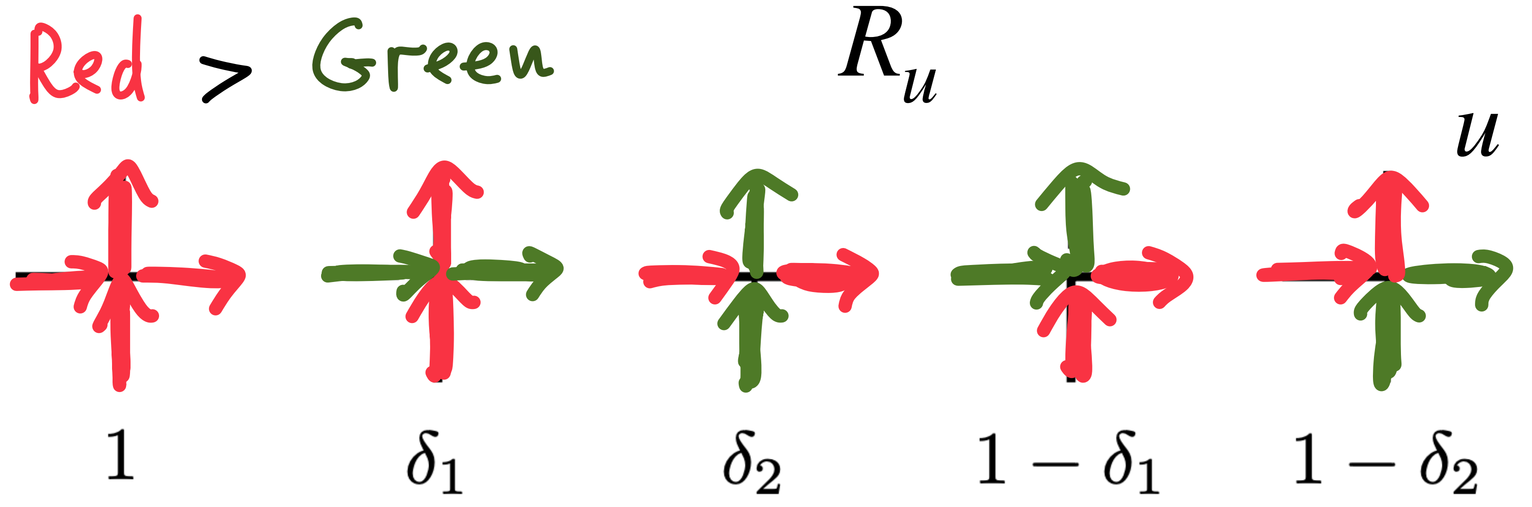

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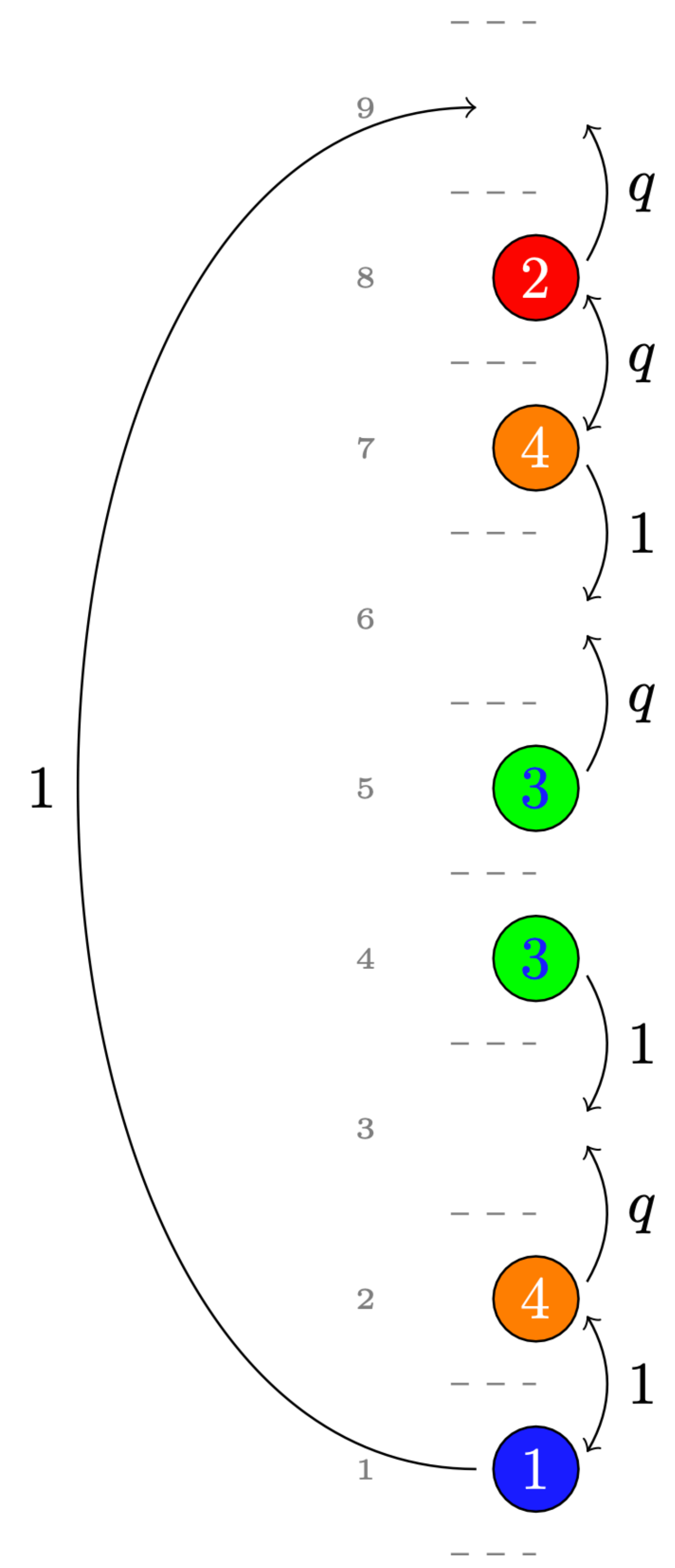

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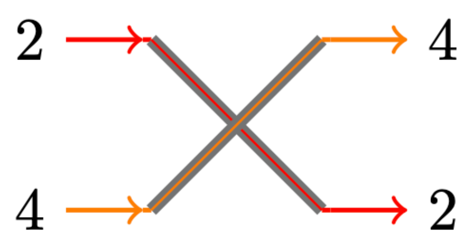

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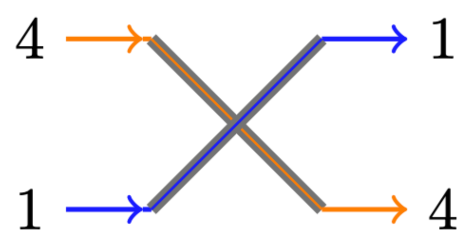
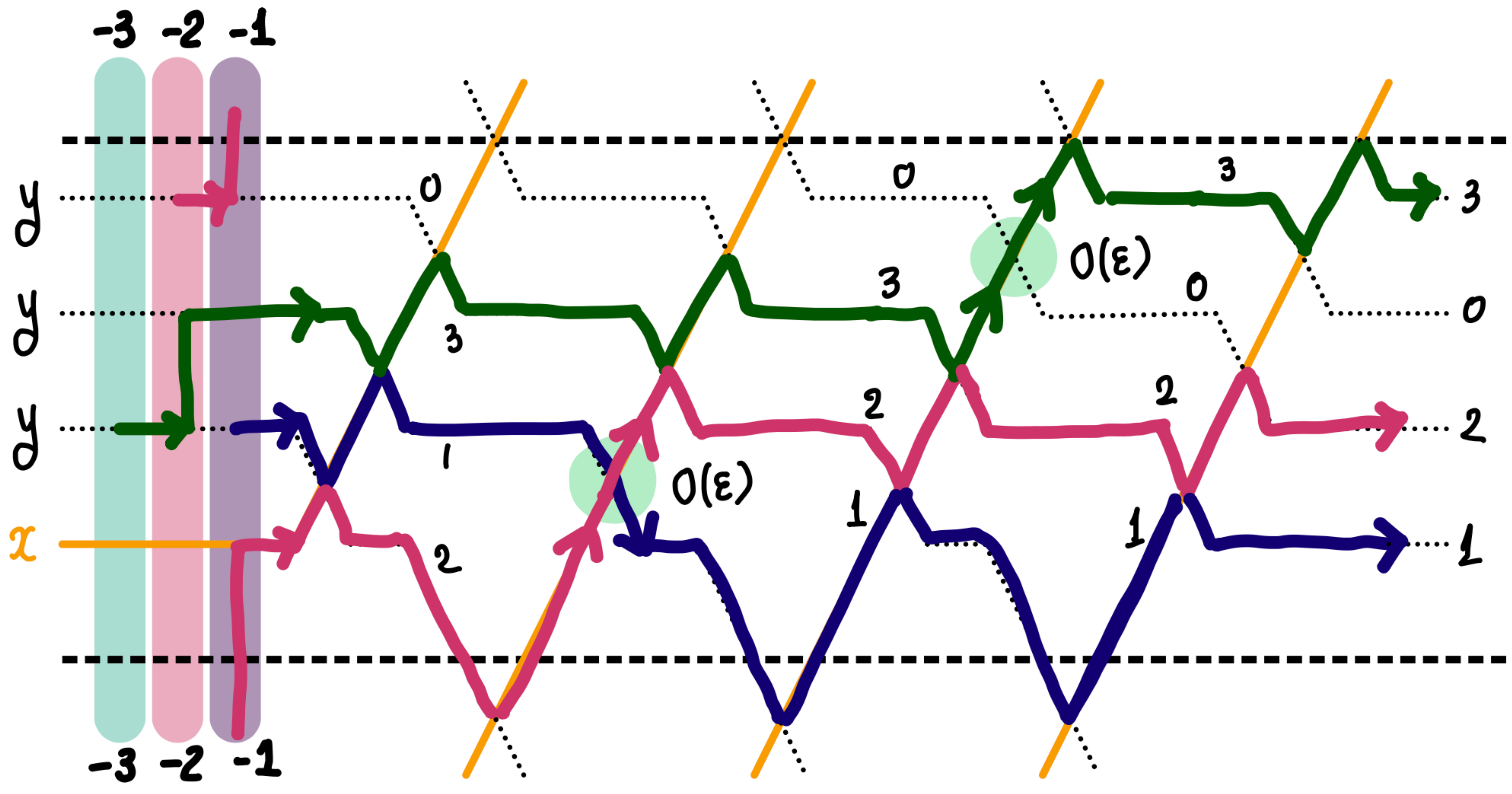
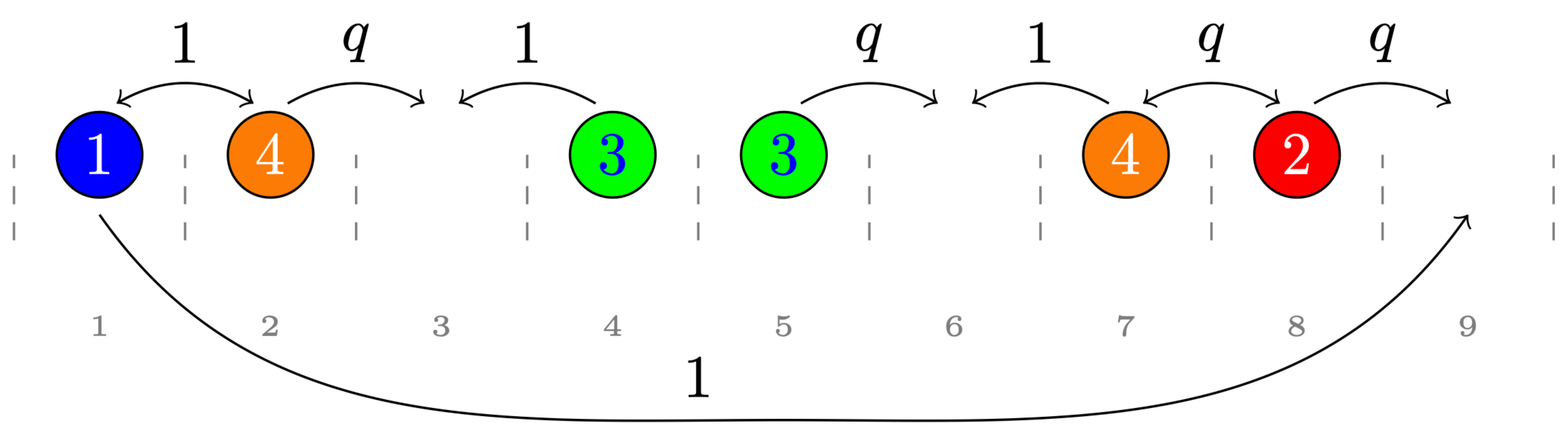




Time $\sim \tau/\epsilon$

Colored ASEP (= multispecies ASEP = mASEP)



- Particles have colors (types) in $\{1,\ldots,n\}$.
- Particles of colors (i_k, i_{k+1}) at adjacent sites $k, k+1$ swap at rate:

$$\mathrm{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

(color n: highest "priority" to move left)

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

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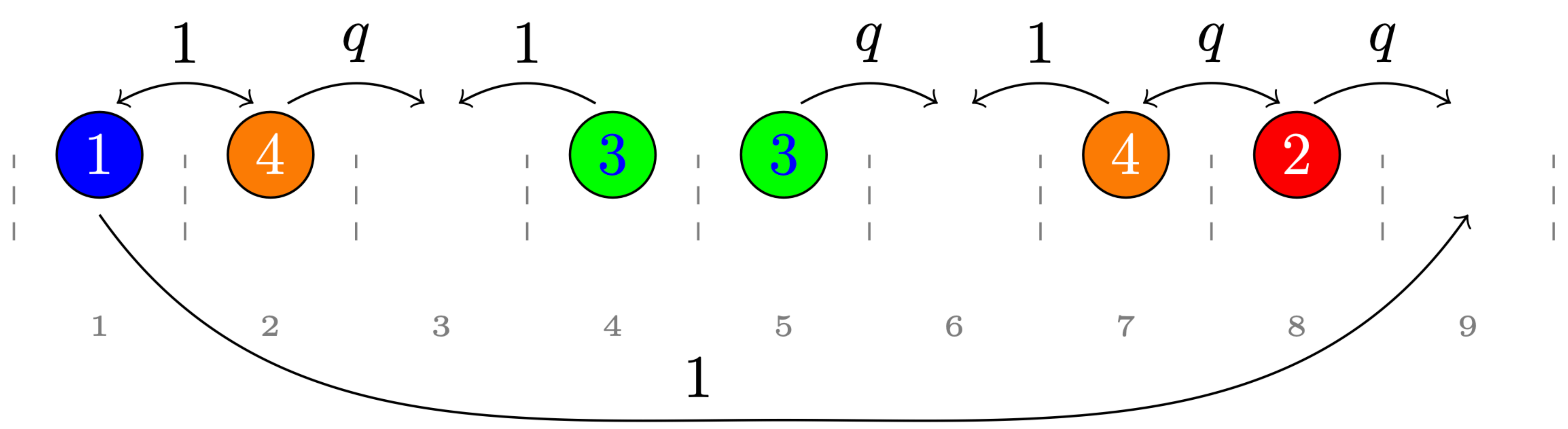

- There is a unique **stationary distribution** $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ in each "sector" $\overrightarrow{N}=(N_1,\ldots,N_n)$, where $\eta_j \in \{0,1,\ldots,n\}$

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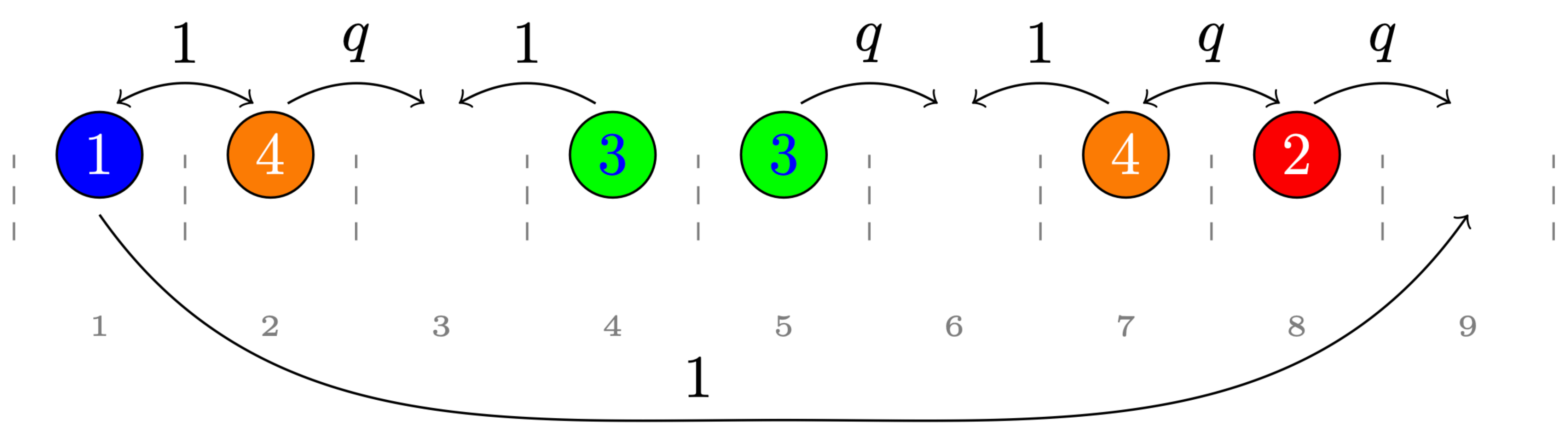

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- For $n = 1$ (single color), it is uniform among all $\binom{N}{N_1}$ configurations

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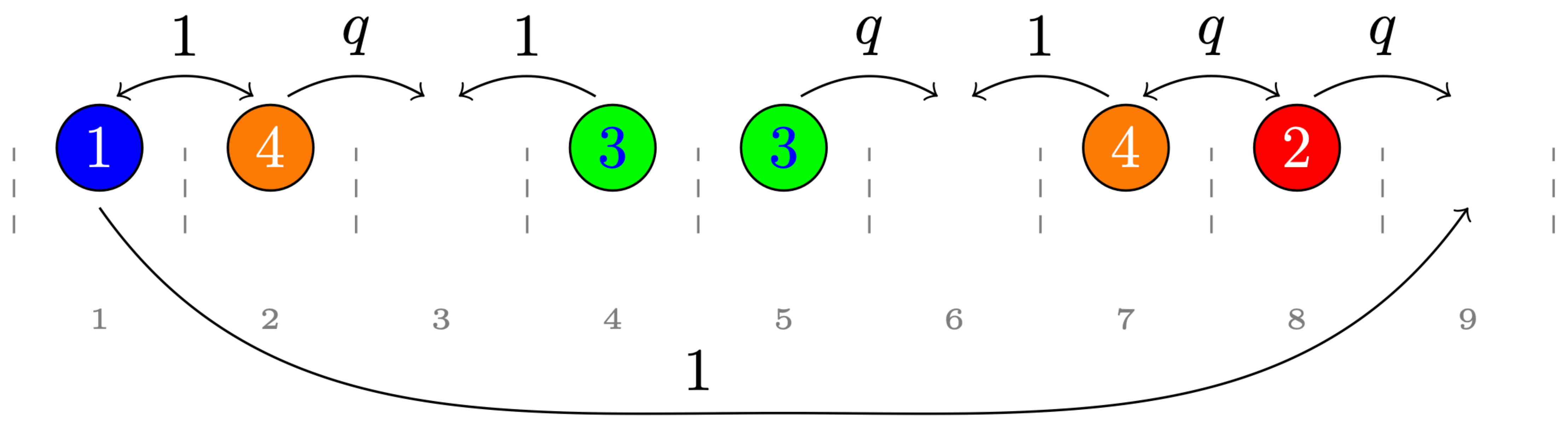

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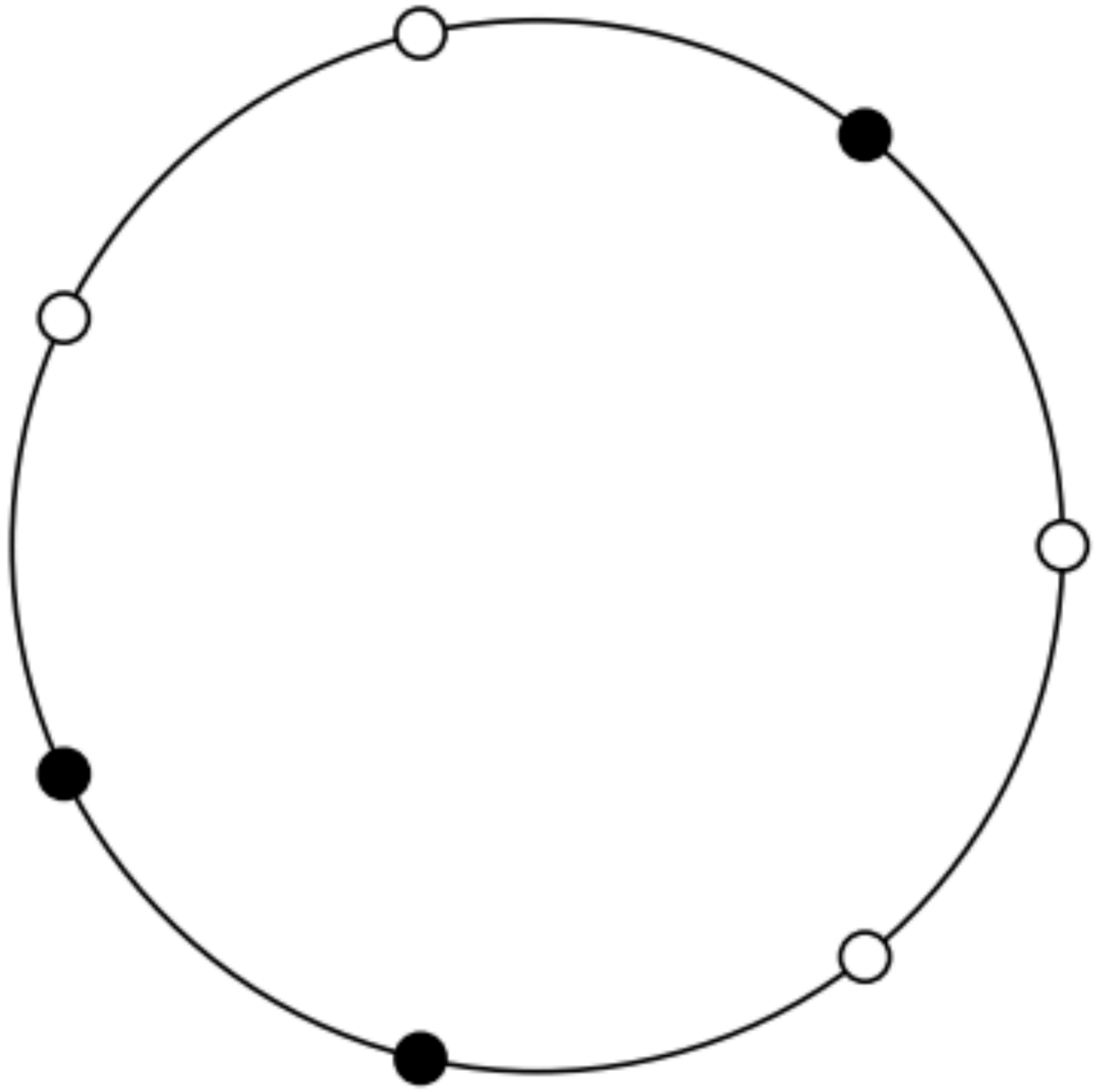
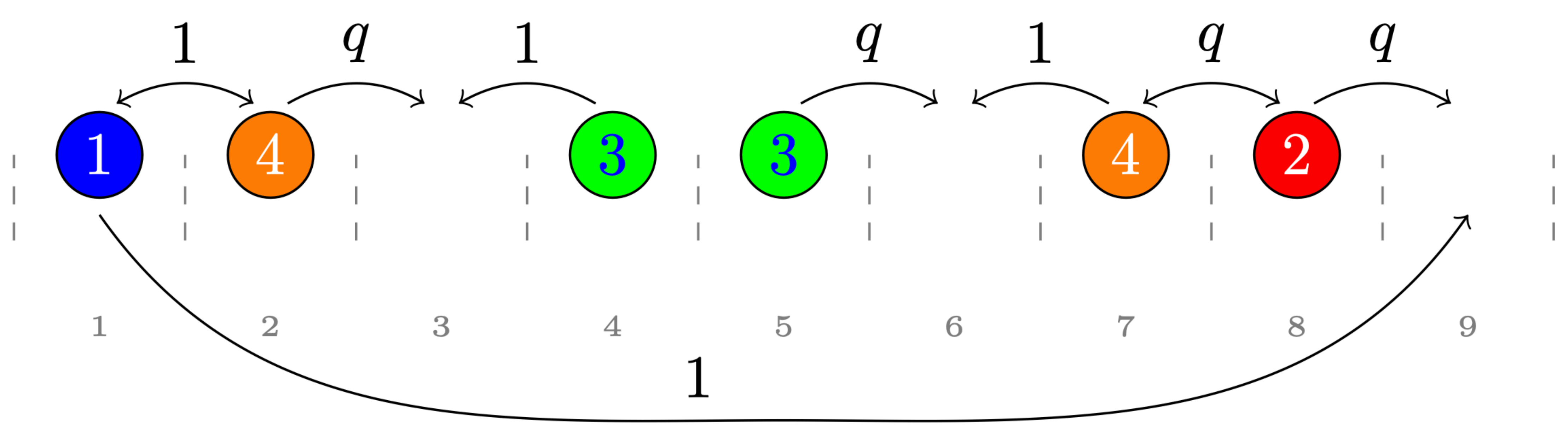


Colored ASEP (= multispecies ASEP = mASEP)



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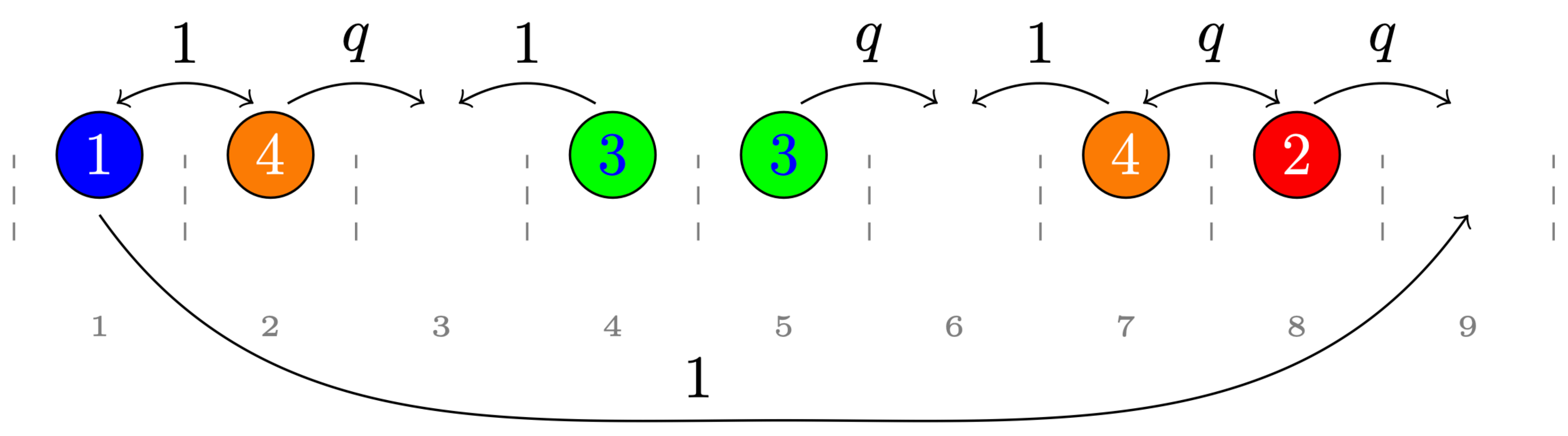

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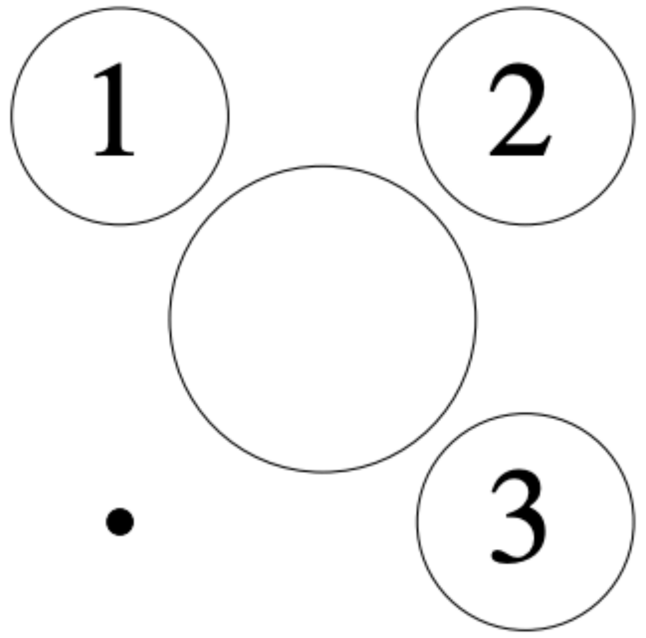
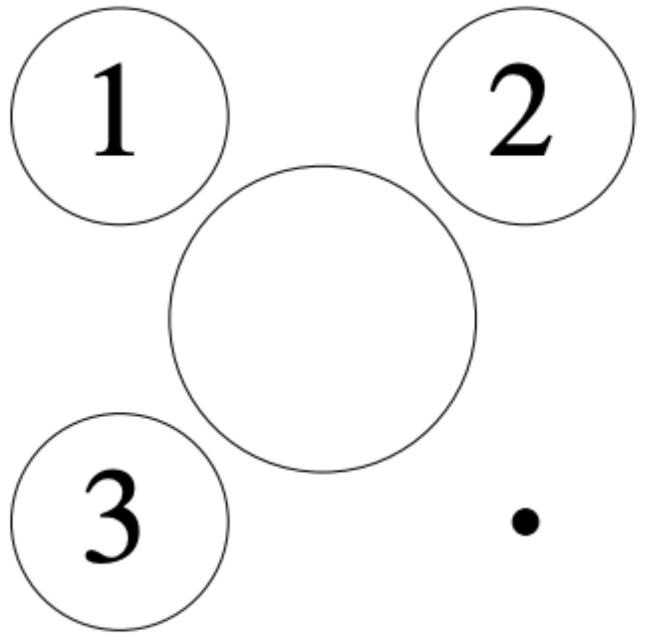
- For $n = 1$ (single color), it is uniform among all $\binom{N}{N_1}$ configurations
- For many colors, nontrivial correlations

Example: Stationary probabilities for 4 sites, “rainbow” configuration

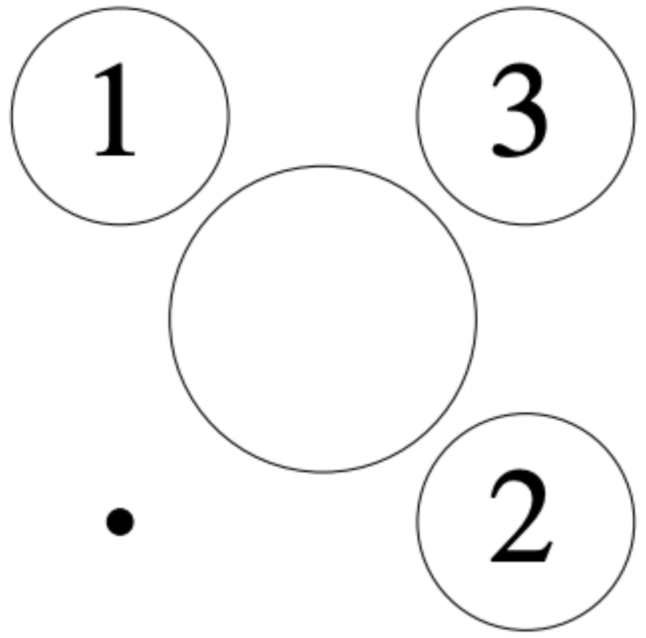
[Martin 2018]



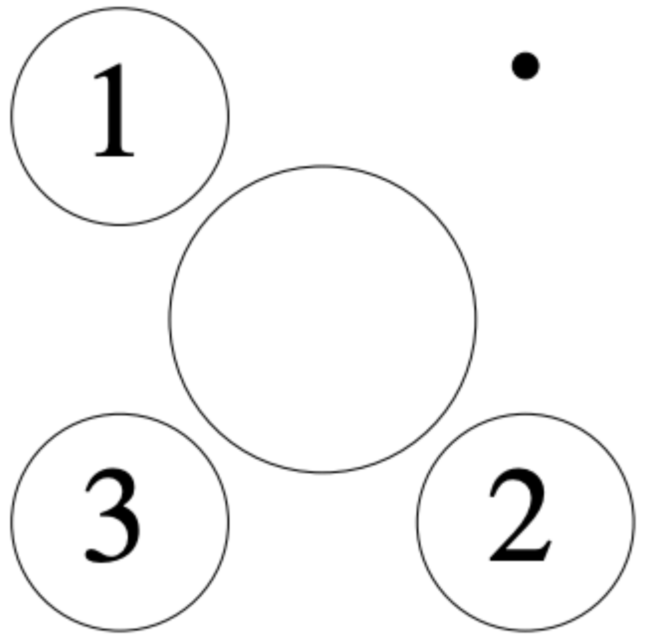
$$\frac{9+7q+7q^2+q^3}{96(1+q)(1+q+q^2)}$$



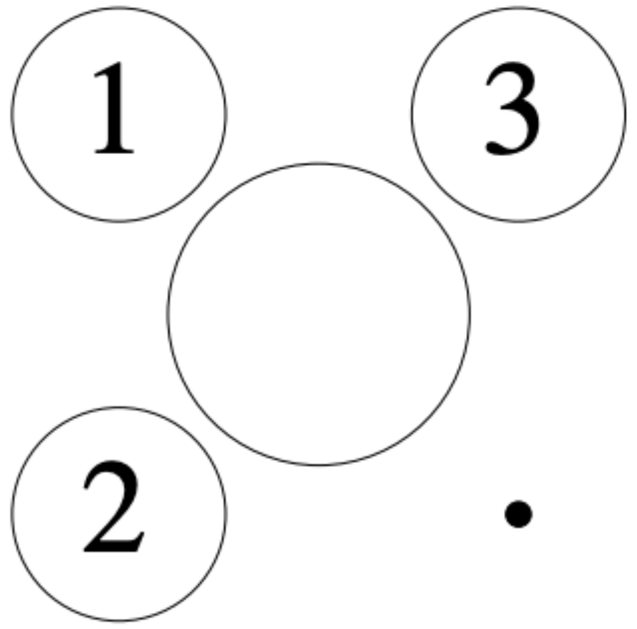
$$\frac{3+9q+9q^2+3q^3}{96(1+q)(1+q+q^2)}$$



$$\frac{3+11q+5q^2+5q^3}{96(1+q)(1+q+q^2)}$$



$$\frac{5+5q+11q^2+3q^3}{96(1+q)(1+q+q^2)}$$



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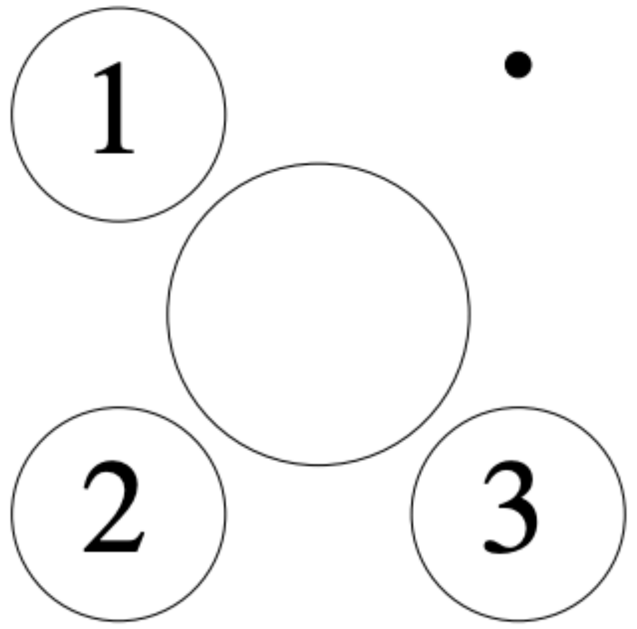
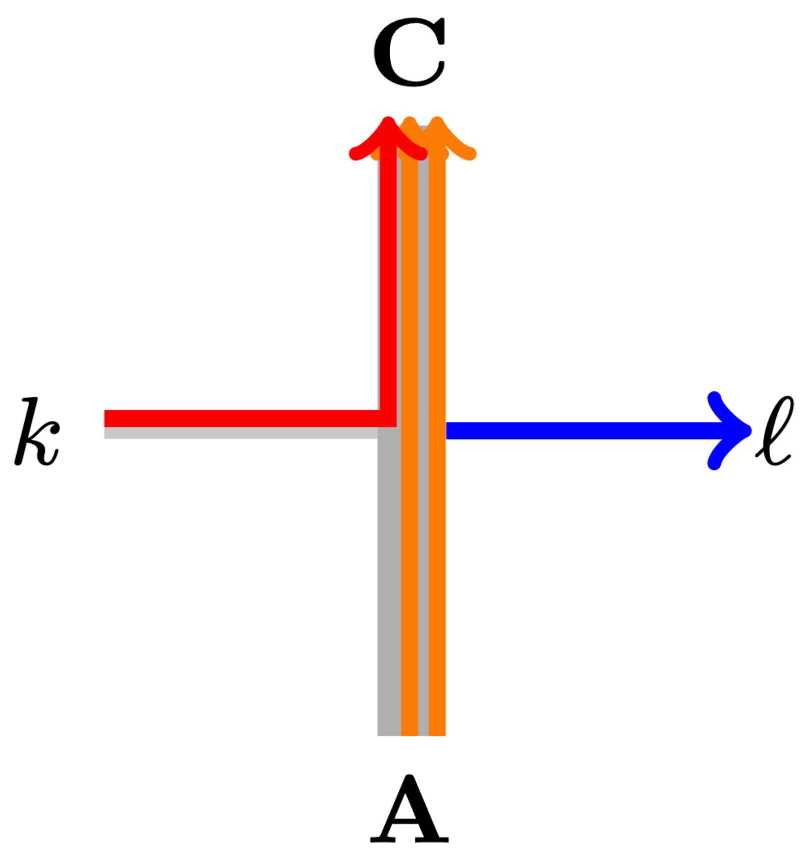


$$\frac{1+7q+7q^2+9q^3}{96(1+q)(1+q+q^2)}$$

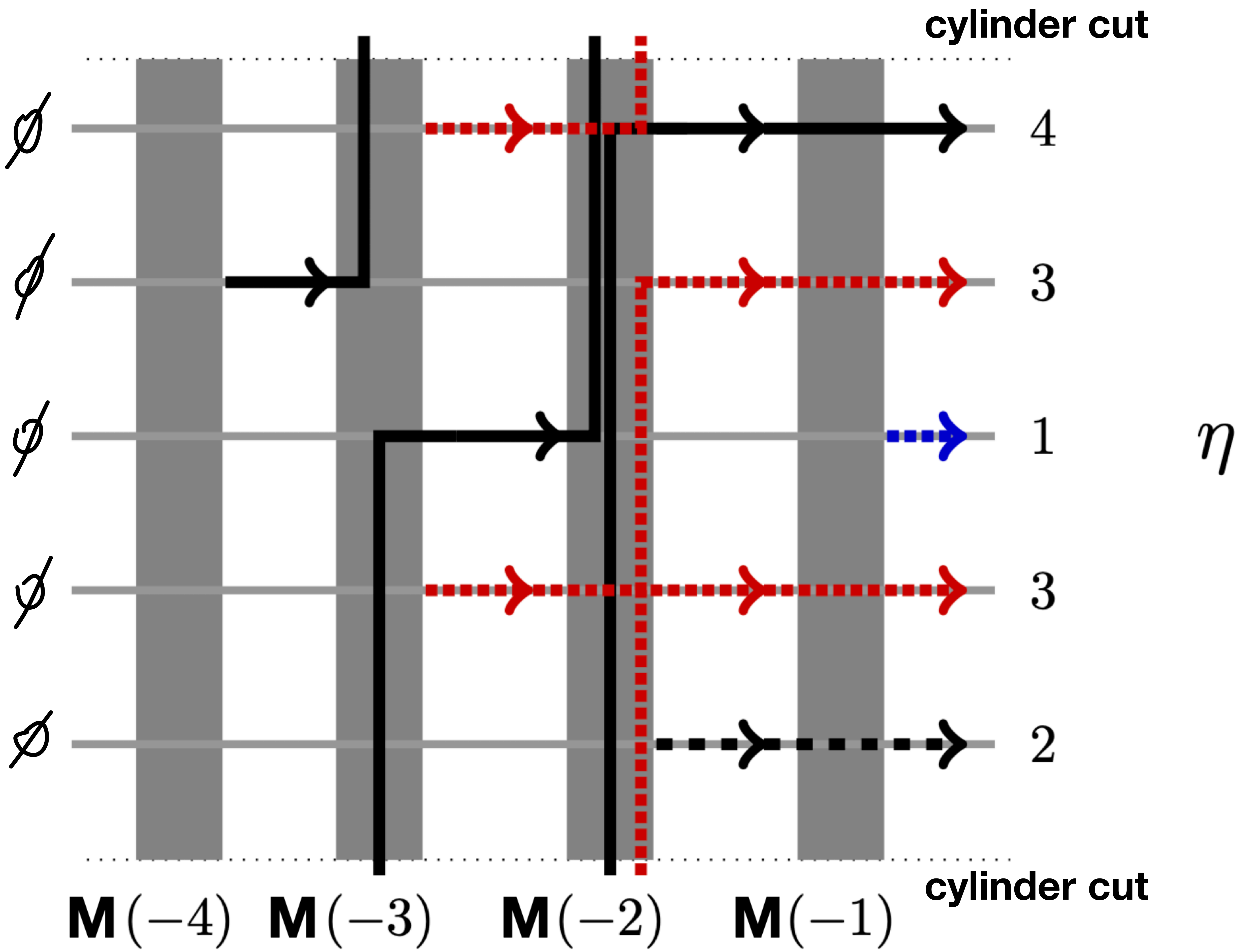
Figure 1.3: Stationary probabilities for a system with 4 sites (numbered clockwise around the ring). The particles are labelled with their type; the single hole may equivalently be seen as a particle of type 4. There are

Stationarity for mASEP on the ring [ANP '23]

- We have a ***vertex model*** on the cylinder $\{-n, -n+1, \ldots, -2, -1\} \times (\mathbb{Z}/N\mathbb{Z})$
- The mASEP configuration $\eta = (\eta_1, \ldots, \eta_N)$ encodes the boundary condition.
- $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1, \ldots, \eta_N)$ is proportional to the ***partition function*** with the boundary η, which involves the summation over the wrappings $\mathbf{M}(-n), \ldots, \mathbf{M}(-1)$. There are infinitely many arrows of color m wrapping around column $(-m)$.
- Weights are denoted by $\mathbb{W}^{(-m)}_{s,x}(\mathbf{A}, k; \mathbf{C}, \ell)$, $\mathbf{A}, \mathbf{C} \in \mathbb{Z}^n_{\geq 0}$, $k, \ell \in \{0,1,\ldots,n\}$.



In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



Stationarity for mASEP on the ring [ANP '23]

Theorem. $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ is proportional to

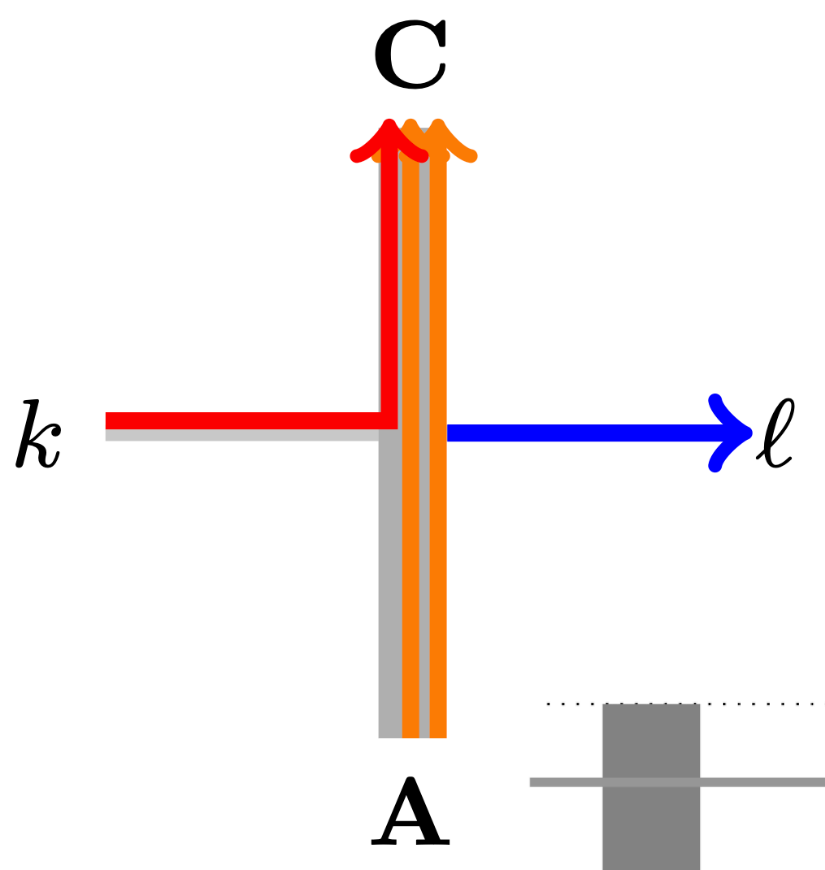
$\sum_{\mathbf{M}(-n),\ldots,\mathbf{M}(-1)} \sum_{\textit{path conf}}$ of products of the weights

$\mathbb{W}^{(-m)}_{s_m,x_m}$ over all $n \times N$ vertices, with boundaries $\varnothing, \eta$.

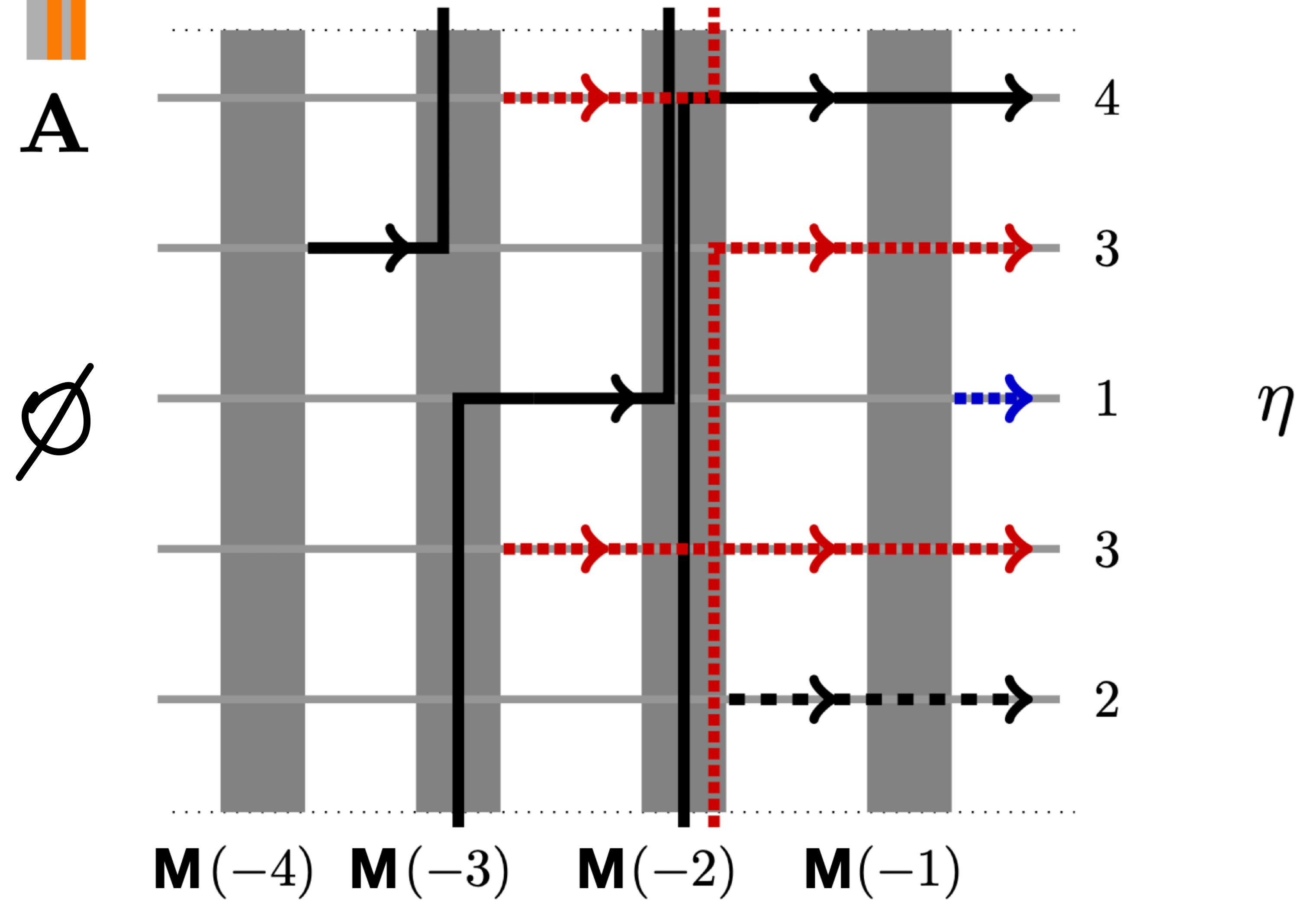
top $\mathbf{A}$, left 0, right 0, bottom $\mathbf{A}$ 1	top $\mathbf{A}$, left k, right k, bottom $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}^-_k$, left 0, right k, bottom $\mathbf{A}$ $x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$, left 0, right m, bottom $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}^+_k$, left k, right 0, bottom $\mathbf{A}$ 1	top $\mathbf{A}^{+-}_{k\ell}$, left k, right ℓ, bottom $\mathbf{A}$ $x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}^{+-}_{\ell k}$, left ℓ, right k, bottom $\mathbf{A}$ $s\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}^+_\ell$, left ℓ, right m, bottom $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \le n$.

These are limits of the fused $U_q(\widehat{\mathfrak{sl}}_n)$ weights to infinitely many arrows.



In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



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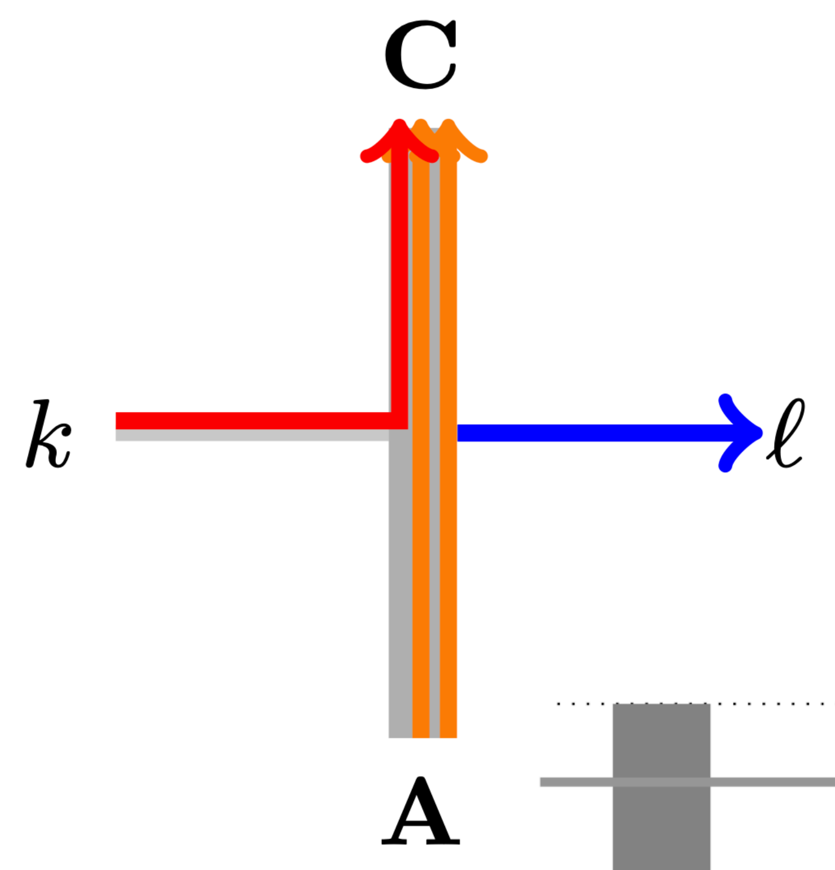
$\mathbf{A}$ 0 — 0 $\mathbf{A}$ 1	$\mathbf{A}$ k — k $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ 0 — k $\mathbf{A}$ $x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ 0 — m $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ k — 0 $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ k — ℓ $\mathbf{A}$ $x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ ℓ — k $\mathbf{A}$ $s\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ ℓ — m $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \leq n$.
These are limits of the fused $U_q(\widehat{\mathfrak{sl}}_n)$ **weights to infinitely many arrows.**

- The sum $path\ conf$ is over path configurations in the $n \times N$ rectangle with boundaries $\varnothing, \mathbf{M}, \eta, \mathbf{M}$.



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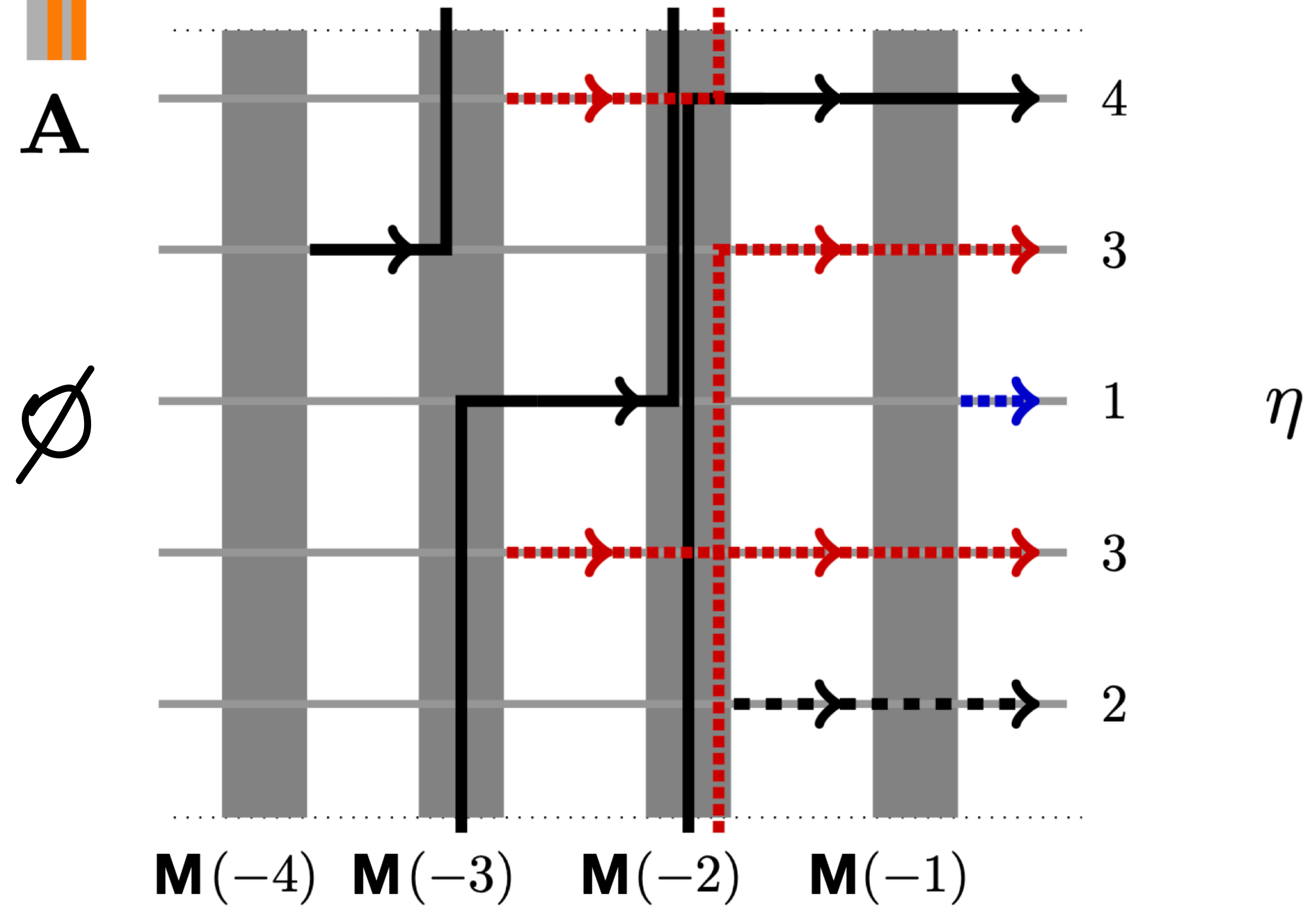

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$\mathbf{A}$ 0 — 0 $\mathbf{A}$ 1	$\mathbf{A}$ k — k $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ 0 — k $\mathbf{A}$ $x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ 0 — m $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ k — 0 $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ k — ℓ $\mathbf{A}$ $x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ ℓ — k $\mathbf{A}$ $s\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ ℓ — m $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \le n$.

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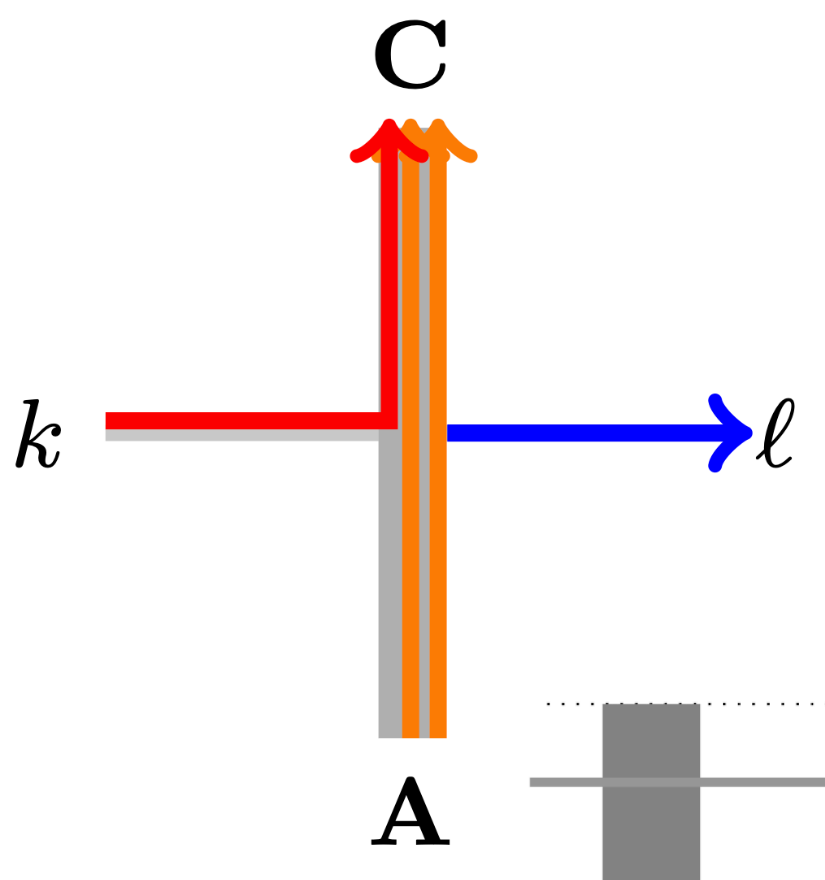
- The sum $path\ conf$ is over path configurations in the $n \times N$ rectangle with boundaries $\varnothing, \mathbf{M}, \eta, \mathbf{M}$.
- Parameters s_m, x_m do not affect $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ on the ring. Positivity is automatic.



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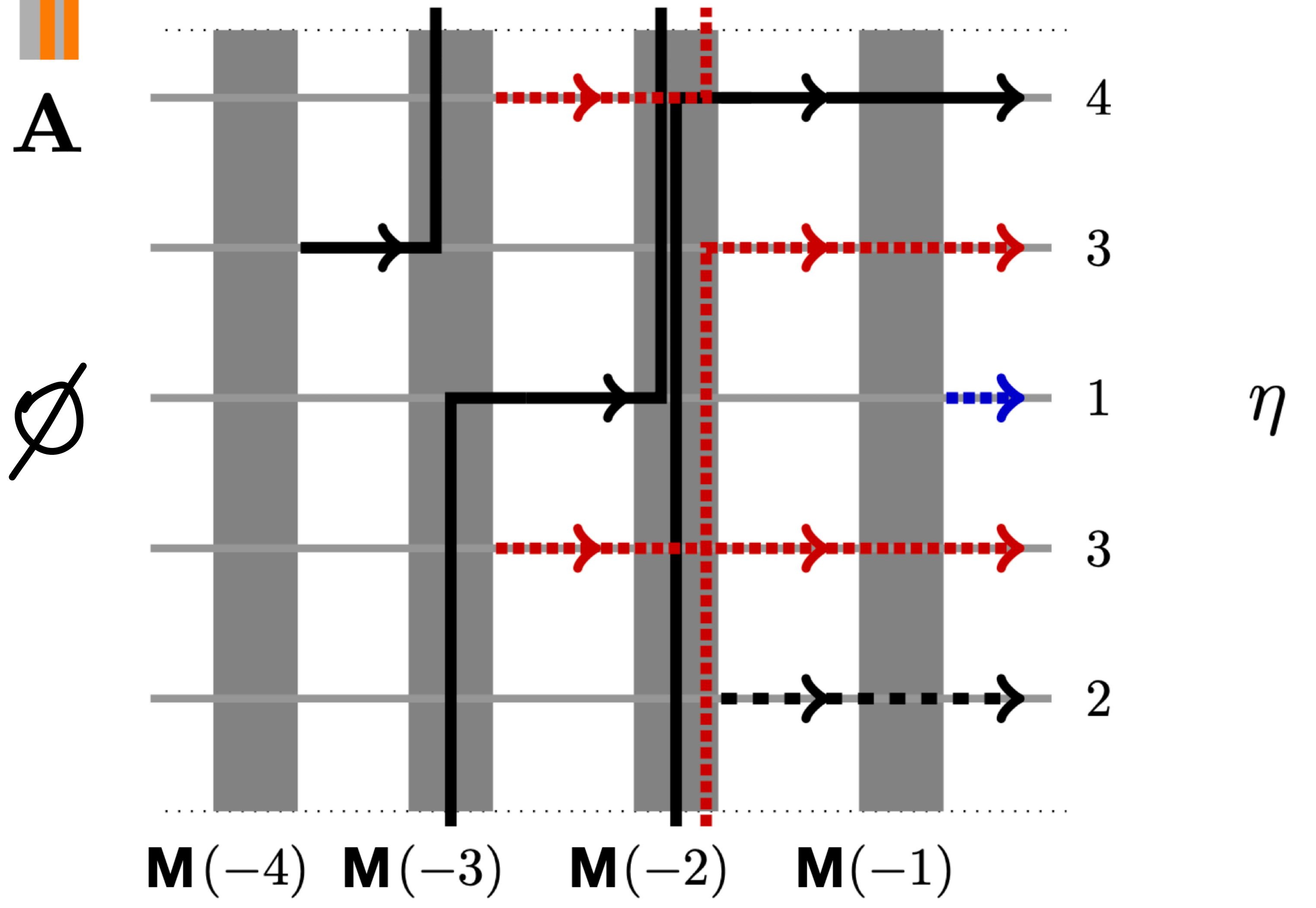

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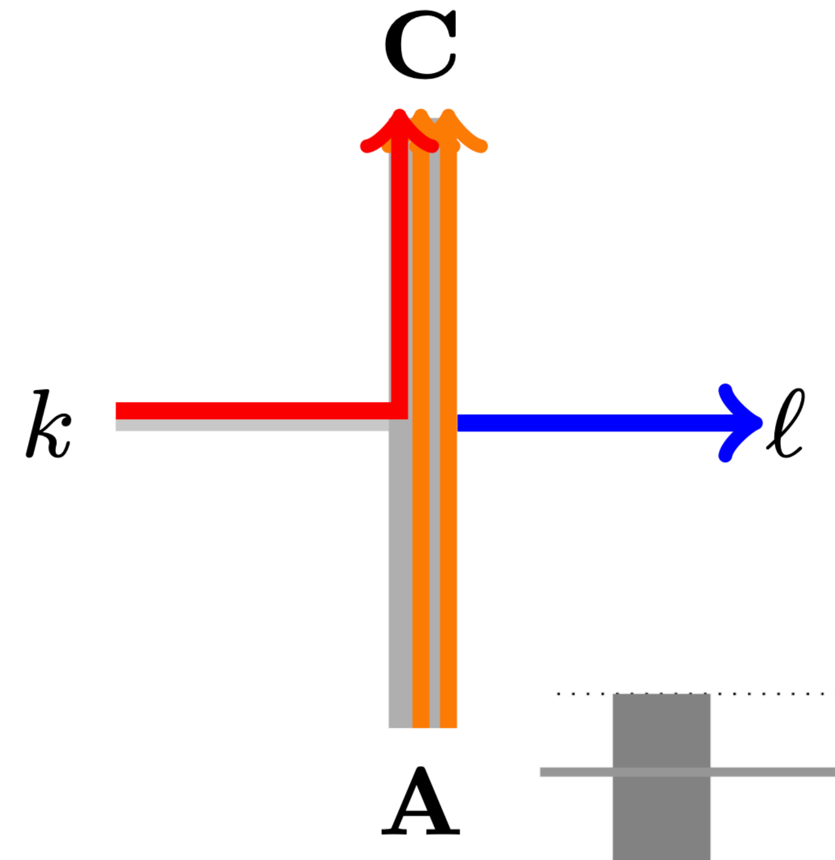
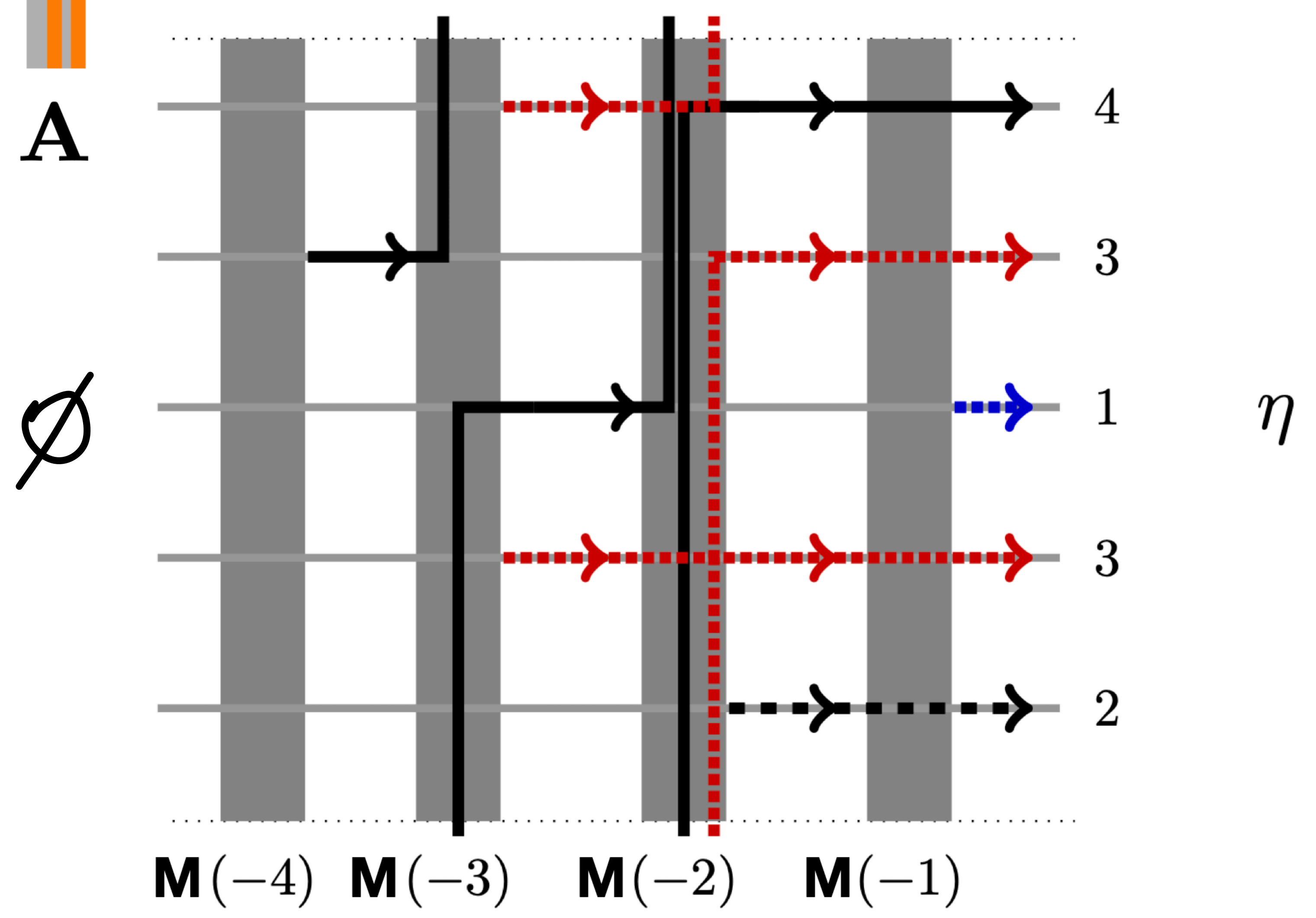
$\mathbf{A}$ $0 \quad 0$ $\mathbf{A}$ 1	$\mathbf{A}$ $k \quad k$ $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}^-_k$ $0 \quad k$ $\mathbf{A}$ $x\,(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ $0 \quad m$ $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}^+_k$ $k \quad 0$ $\mathbf{A}$ 1	$\mathbf{A}^{+-}_{k\ell}$ $k \quad \ell$ $\mathbf{A}$ $x\,(1 - q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}^{+-}_{\ell k}$ $\ell \quad k$ $\mathbf{A}$ $s\,(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}^+_\ell$ $\ell \quad m$ $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \leq n$.
These are limits of the fused $U_q(\widehat{\mathfrak{sl}}_n)$ weights to infinitely many arrows.

- The sum $path\ conf$ is over path configurations in the $n \times N$ rectangle with boundaries $\varnothing, \mathbf{M}, \eta, \mathbf{M}$.
- Parameters s_m, x_m do not affect $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ on the ring. Positivity is automatic.
- Similar result on the line (with fewer parameters for positivity). The remaining parameters are responsible for the color densities.



In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



Conclusions

- Stationary measures for interacting particle systems in different geometries (line, ring, half-space, segment):
  - Motivated by asymptotic phenomena (microscopic characteristics, stationary measures for KPZ equation)
  - Rich algebraic and combinatorial structure (e.g., nonsymmetric Macdonald polynomials)

- We get a Yang-Baxter (vertex) **description of stationarity from first principles**

- Probabilistic consequences follow from **creative placement of Yang-Baxter crosses**, and graphical manipulations.