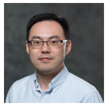
Functional Map and Bases Design via ADMM

Omri Azencot

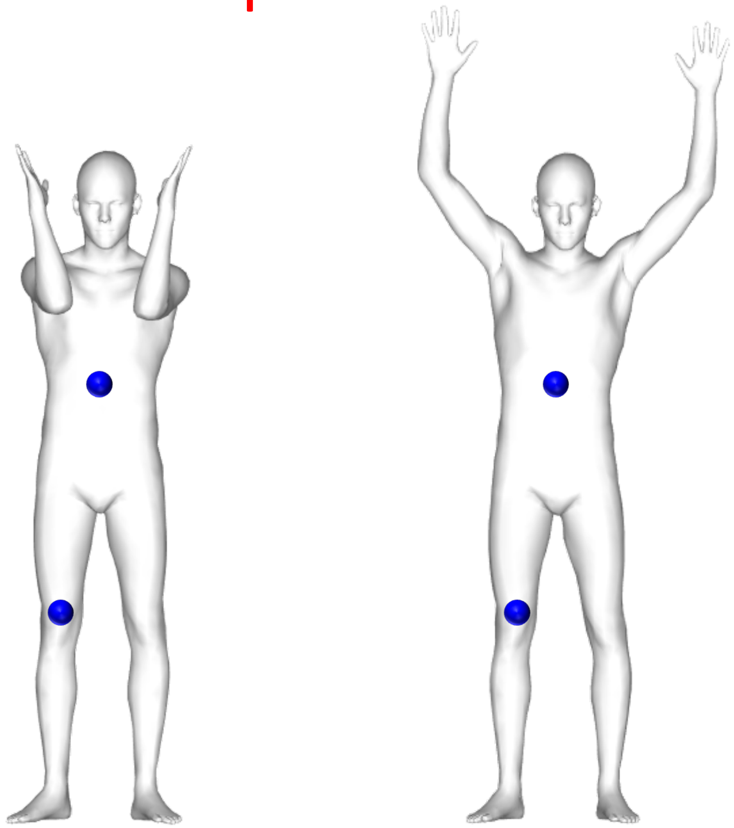
Department of Mathematics

University of California, Los Angeles

Joint work with Rongjie Lai 

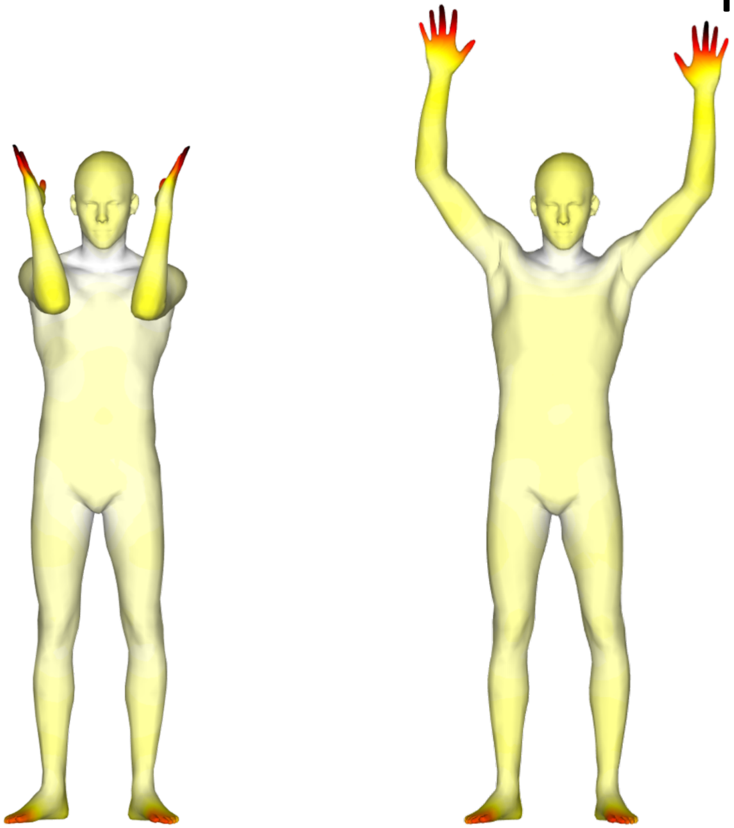
Goal

Compute dense correspondences between shapes



Functional Maps

Linear alignment between functional spaces



Ovsjanikov et al. '12

Functional Maps

Find a matrix C_{21} such that

$$C_{21}F_1 = F_2$$

where we assume

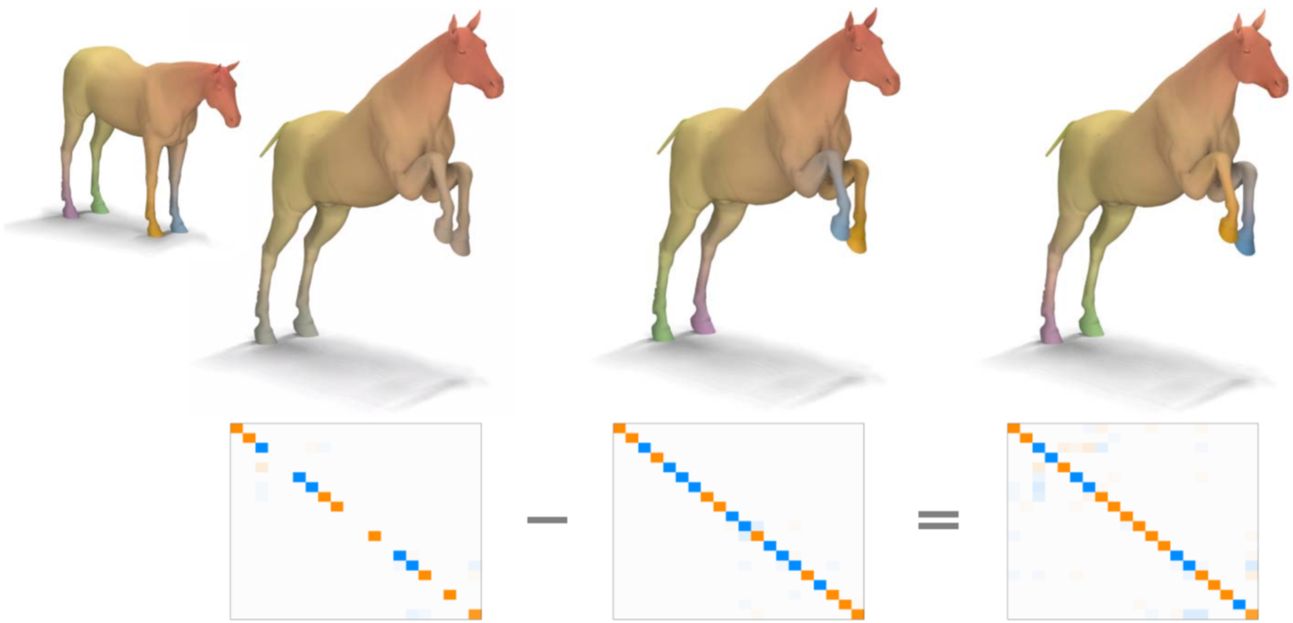
$$C_{21}(f_{1j}) \approx f_{1j} \circ \varphi \approx f_{2j}$$

Ovsjanikov et al. '12

Dimensionality Reduction

Find a matrix C_{21} such that

$$C_{21} B_1^T F_1 \approx B_2^T F_2$$

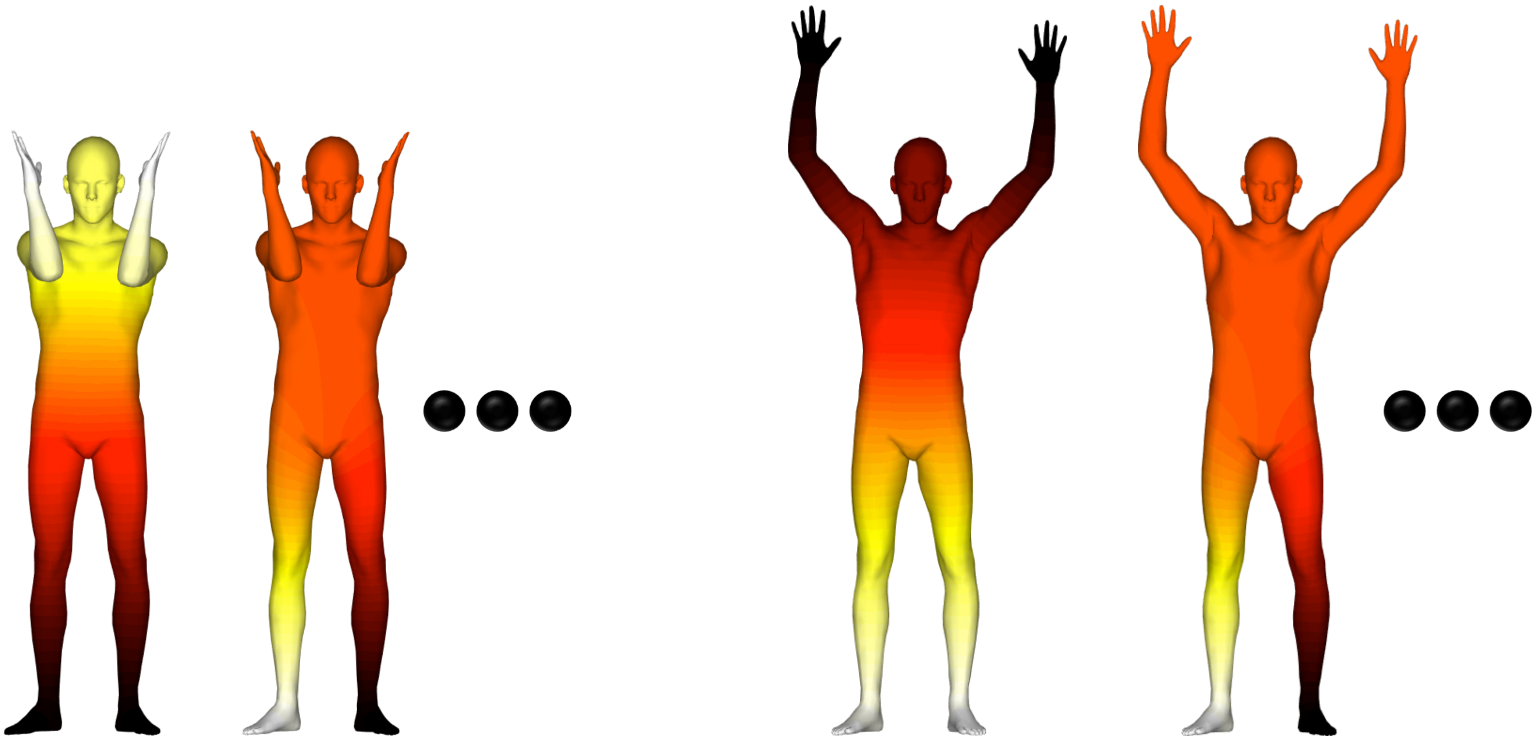


Ovsjanikov et al. '12

$$C_{21} B_1^T F_1 = B_2^T F_2$$

Choice of basis

Eigenfunctions of the Laplace—Beltrami operator



Basis Design

Find bases B_1 and B_2 such that

$$B_1^T F_1 \approx B_2^T F_2$$

Basis Design

Find bases B_1 and B_2 such that

$$\min_{B_1,B_2} \left| B_1^T F_1 - B_2^T F_2 \right|_F^2$$

$$\text{s.t.} \quad B_j^T G_j B_j = I$$

$k \times m_1$

$k \times m_2$

Schonsheck et al. '18

Dimensionality Reduction

Find coefficient matrices Q_1 and Q_2 such that

$$\min_{Q_1,Q_2} \left| Q_1^T B_1^T F_1 - Q_2^T B_2^T F_2 \right|_F^2$$

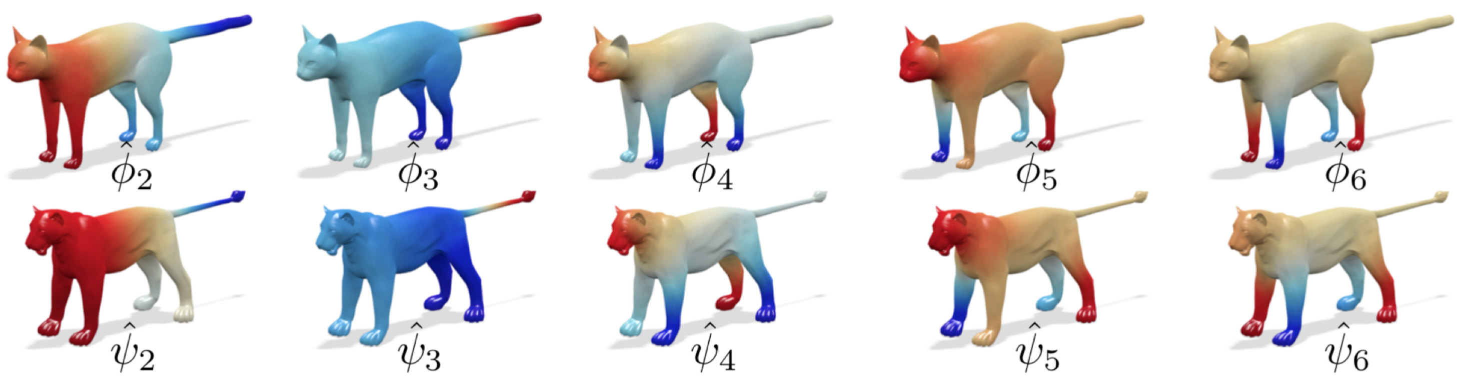
$$\text{s.t.} \quad Q_j^T Q_j = I$$

Coupled Quasi-Harmonic Bases

Find coefficient Q_1 and Q_2 such that

$$\min_{Q_1, Q_2} \left| Q_1^T B_1^T F_1 - Q_2^T B_2^T F_2 \right|_F^2$$

$$\text{s.t.} \quad Q_j^T Q_j = I$$



Kovnatsky et al. '13

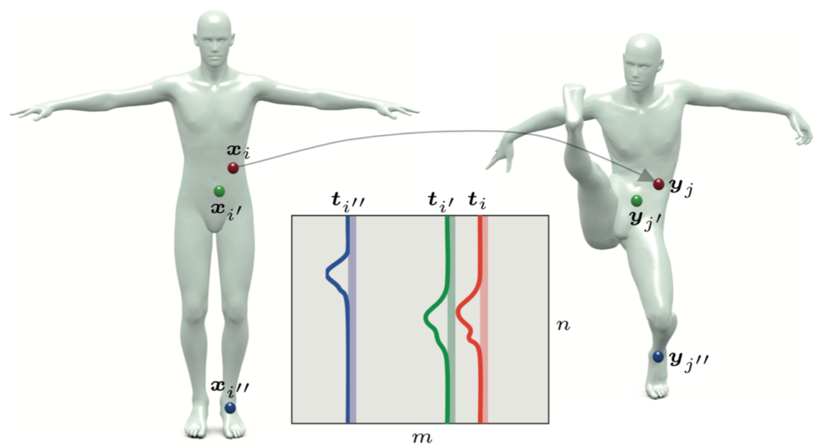
$Q_1^T B_1^T F_1 = Q_2^T B_2^T F_2$

Relation to Matrix Completion

Find a k–rank decomposition of C_{21} such that

$$\min_{Q_1, Q_2} \left| C_{21} B_1^T F_1 - B_2^T F_2 \right|_F^2$$

$$\text{s.t.} \quad C_{21} = Q_2 Q_1^T$$



Kovnatsky et al. ‘15

FM vs. JD

$\boldsymbol{C_{21}} B_1^T F_1 \approx B_2^T F_2$

Use first k basis elements

metric/area
consistency
orientation
…

$\boldsymbol{Q_1^T} B_1^T F_1 \approx \boldsymbol{Q_2^T} B_2^T F_2$

Combines best k basis elements

diagonalize L
sparse p2p
…

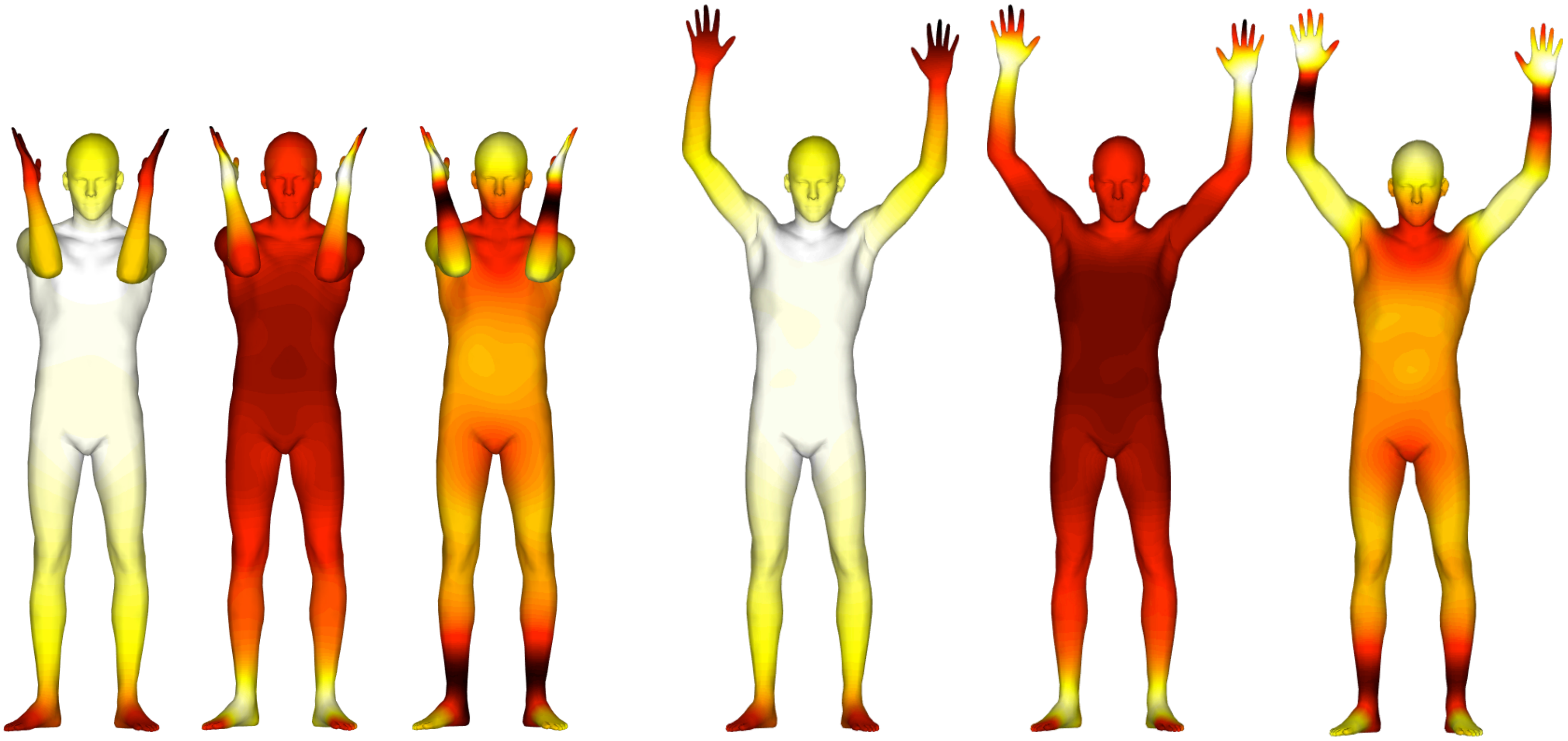
Should we use LB?

Should we use LB?

- Invariant to isometries
- Natural ordering
- Related to Fourier Analysis
- Related to a well studied discrete operator
- Aflalo et al. ‘15, ‘16
- …

Should we use LB?

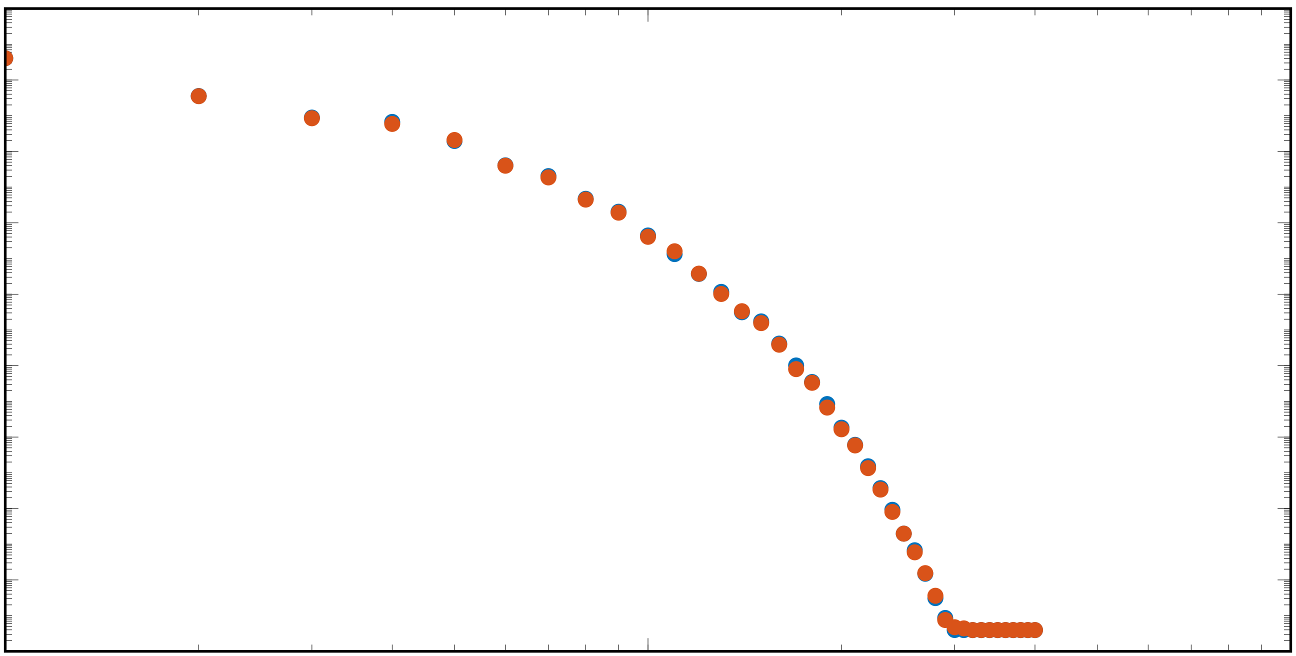
Proper Orthogonal Decomposition (POD) modes:



Proper Orthogonal Decomposition

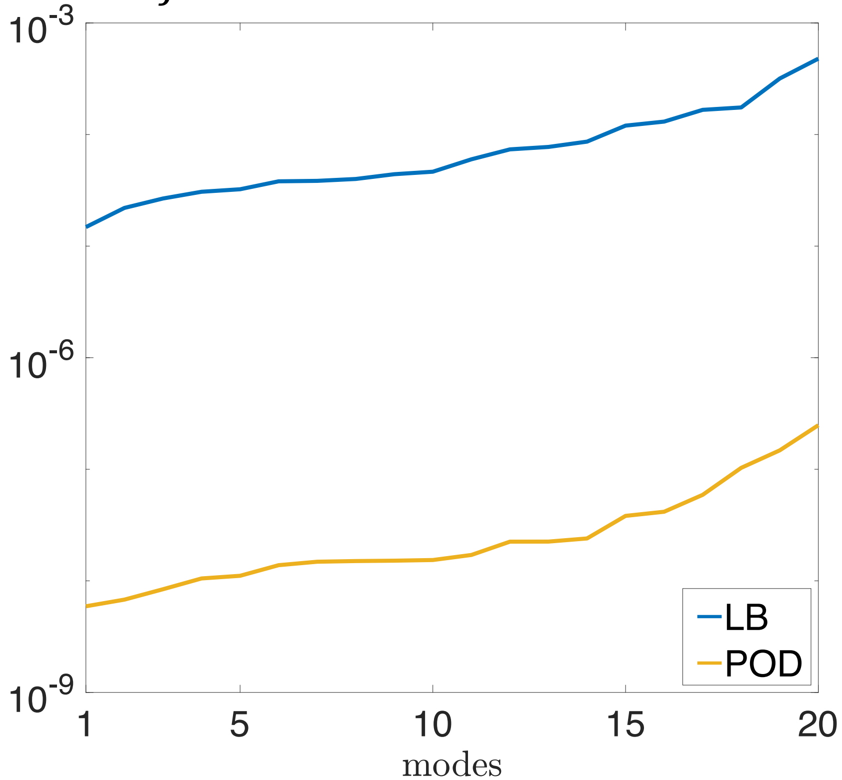
Based on the SVD of a matrix

$$F = USV^*$$



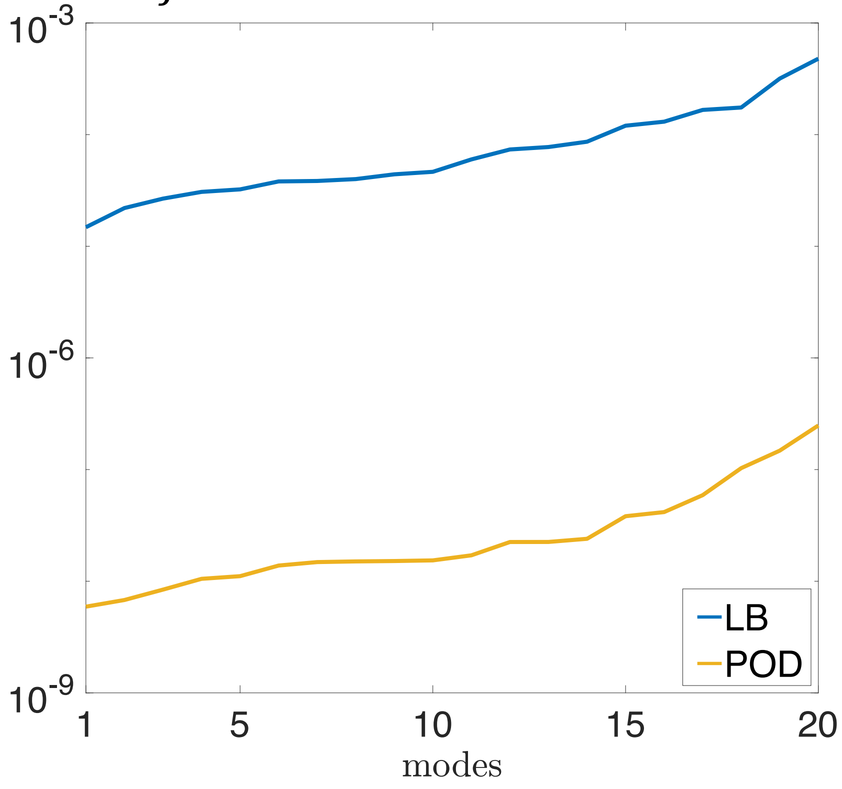
LB vs. POD

$$\sum_j \left(C_{21} B_1^T f_{1j} - B_2^T f_{2j}\right)^2$$

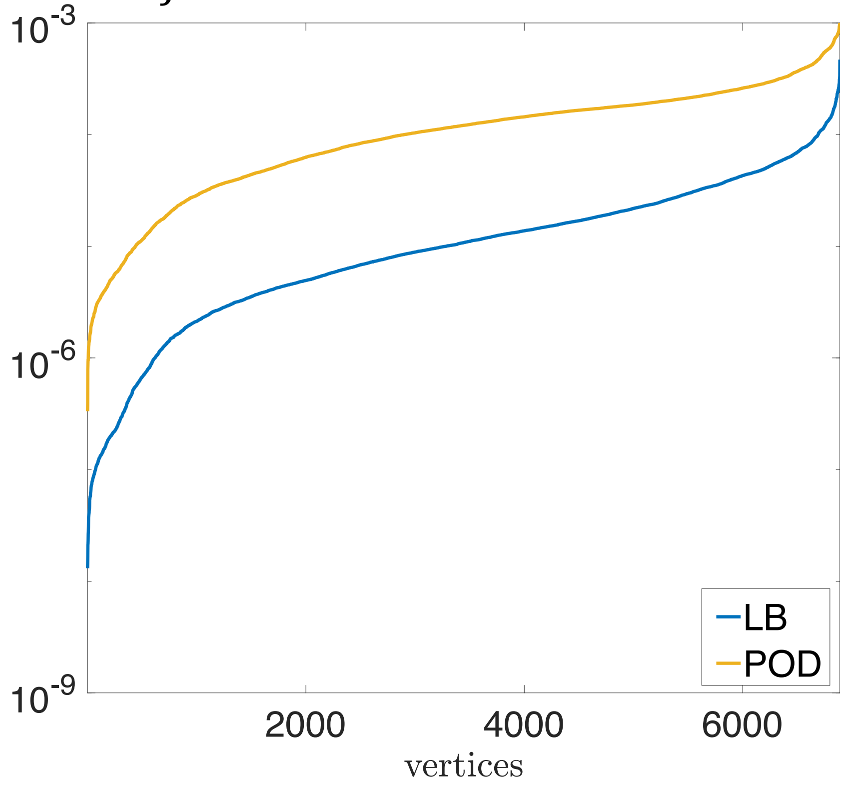


LB vs. POD

$$\sum_j \left(C_{21} B_1^T f_{1j} - B_2^T f_{2j}\right)^2$$



$$\sum_j \left(B_2 C_{21} B_1^T f_{1j} - f_{2j}\right)^2$$



Our Approach

We propose to

1. Solve for C_{21} AND Q_1, Q_2

2. Use POD modes instead of LB

3. Regularize with consistency, smoothness and metric preservation

Our Approach

Find matrices C_{21}, Q_1, Q_2 (C_{12}) that minimize

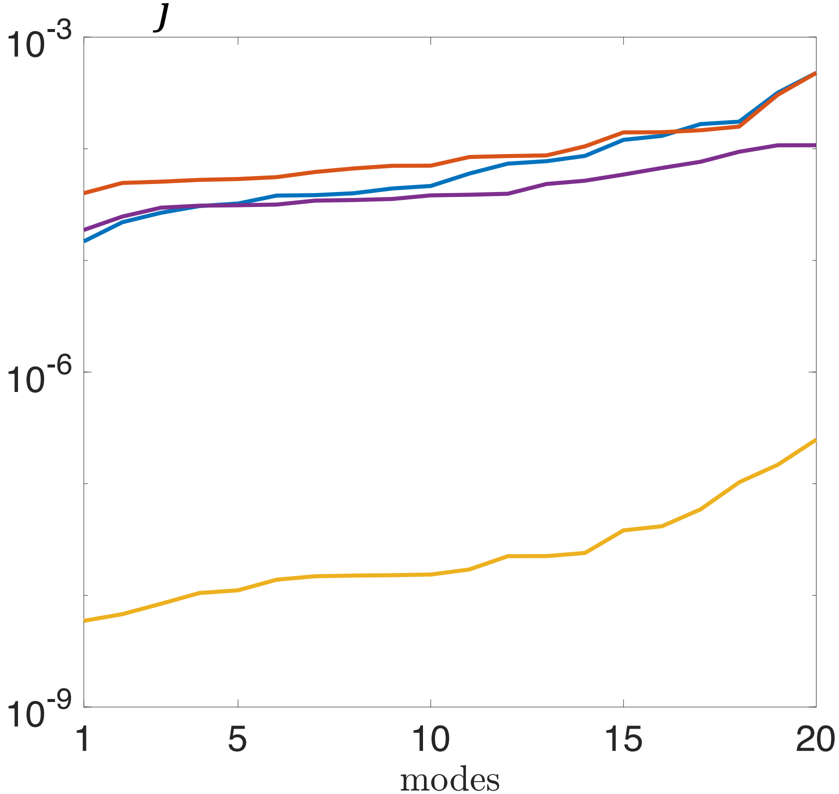
$$\min\left|C_{21}Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2\right|_F^2 +$$

$$\left|Q_1^T U_1^T F_1 - C_{12}Q_2^T U_2^T F_2\right|_F^2 + \varepsilon_{iso} + \varepsilon_{dir}$$

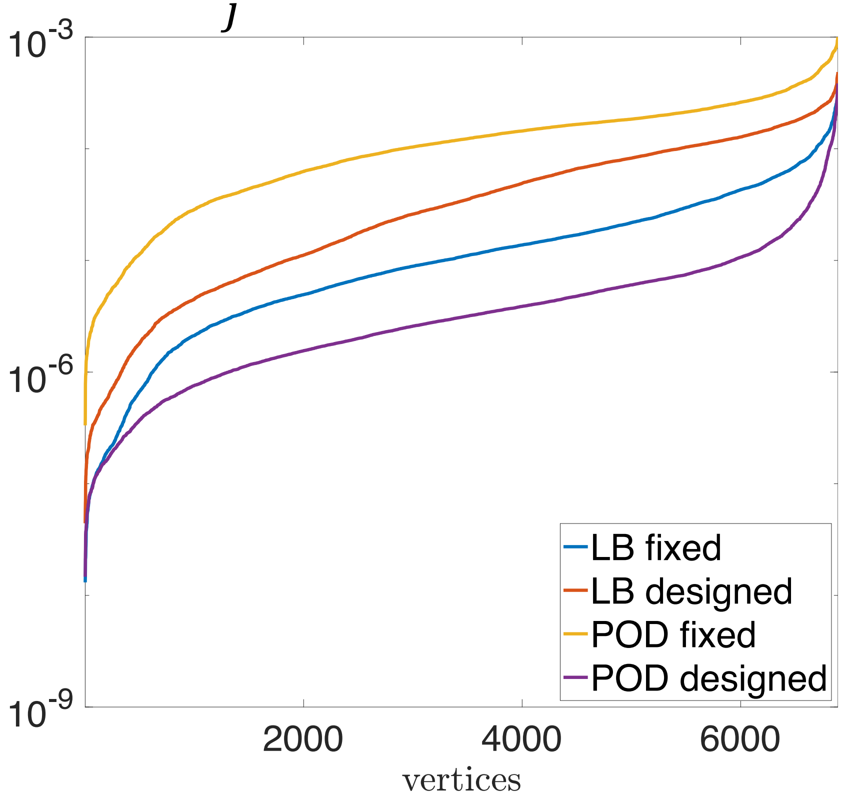
$$s.t. \quad Q_j^T G_j Q_j = I, C_{pq}C_{qp} = I$$

Fixed vs. Designed

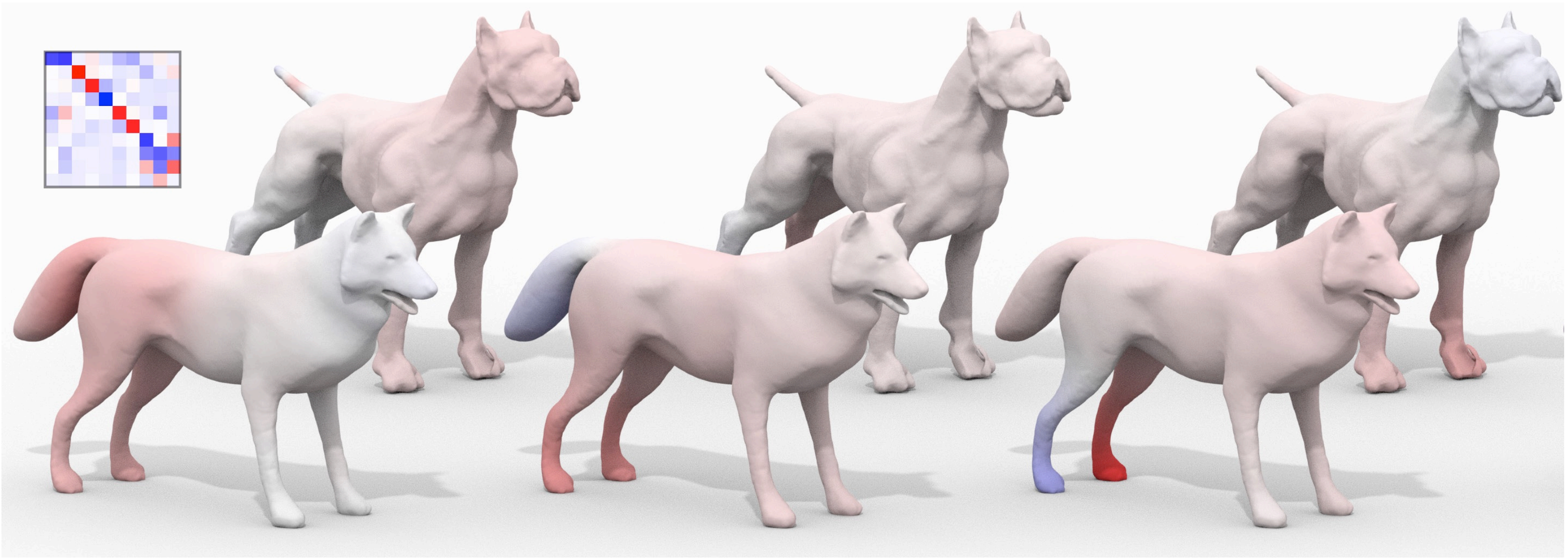
$$\sum_j \left(C_{21} B_1^T f_{1j} - B_2^T f_{2j}\right)^2$$



$$\sum_j \left(B_2 C_{21} B_1^T f_{1j} - f_{2j}\right)^2$$



Functional Map and Bases Design



Numerical Method

Our problem:

$$\min \left| C_{21} Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2 \right|_F^2 \quad s.t.\ Q_j^T G_j Q_j = I$$

Lai et al. '14

Numerical Method

Our problem:

$$\min \left| C_{21} Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2 \right|_F^2 \quad s.t.\ Q_j^T G_j Q_j = I$$

An Alternating Direction Method of Multipliers (ADMM) version:

$$\min \left| C_{21} Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2 \right|_F^2$$
$$s.t.\quad Q_j^T G_j Q_j' = I, Q_j = Q_j{}'$$

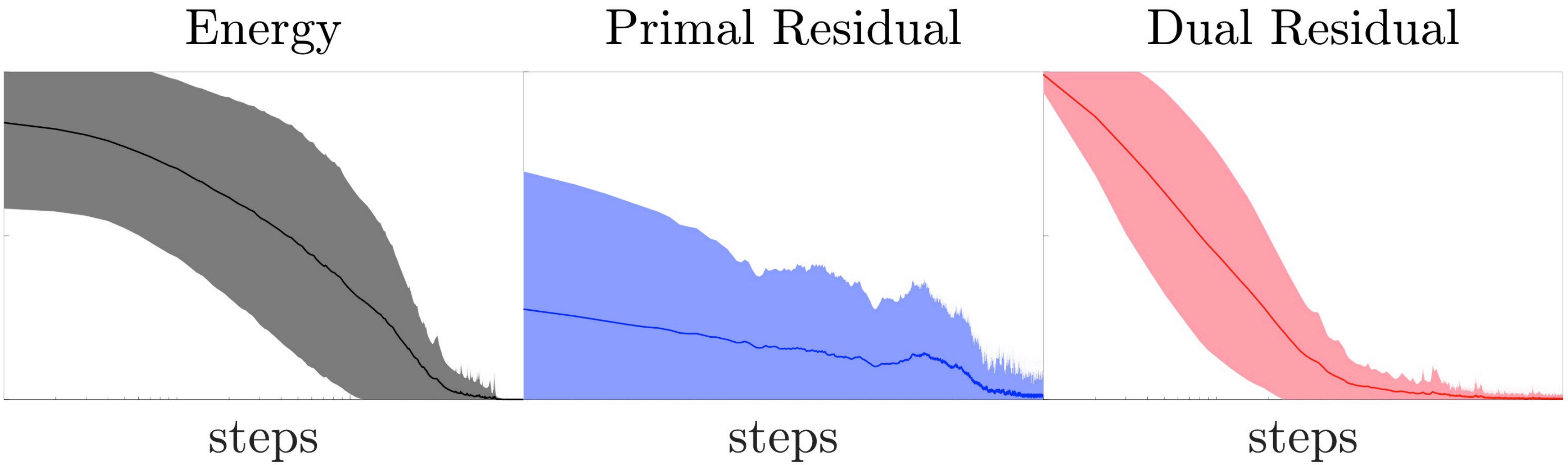
An ADMM Approach

Algorithm:

For k = 0, 1, 2, … do

1. Solve a Sylvester-type equation x 2
2. Solve a linear equation x 3
3. Update the dual variables

Empirical Convergence



Provably Convergent Scheme

We consider the minimization

$$\min \mathcal{G}(\mathcal{X}) + \mathcal{H}(\mathcal{Z})$$
$$s.t.\ \ \mathcal{P}(\mathcal{X}) + \mathcal{Q}(\mathcal{Z}) = 0$$

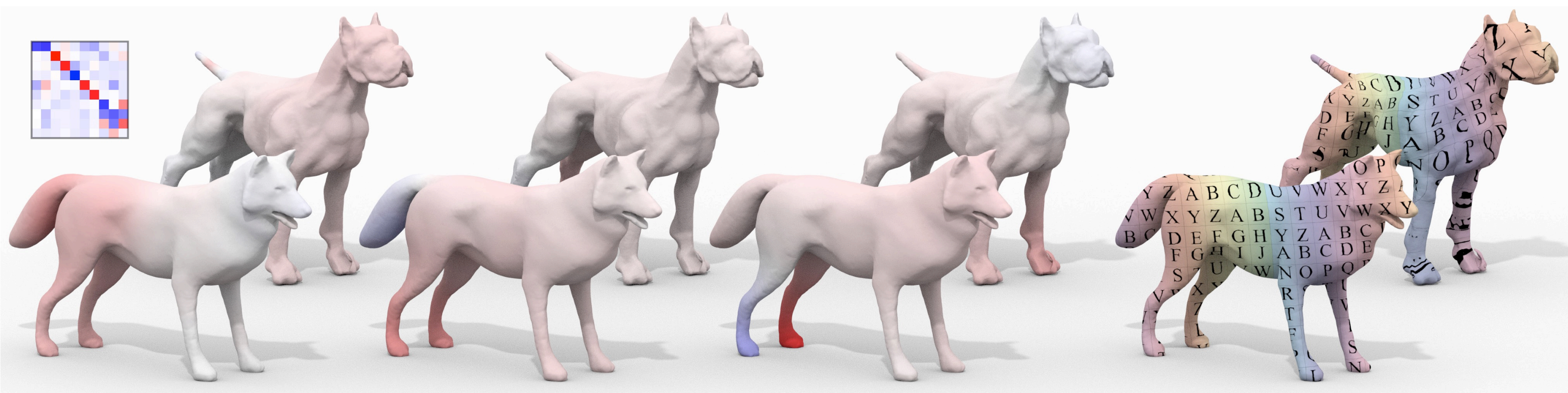
where

$$\mathcal{G}(\mathcal{X}) = \left|C_{21}\tilde{Q}_1^T U_1^T F_1 - \tilde{Q}_2^T U_2^T F_2\right|_F^2$$
$$\mathcal{H}(\mathcal{Z}) = |Z - I|_F^2 + \left|{Q_j}''\right|_F^2 + \left|{\tilde{Q}_j}''\right|_F^2$$
$$\mathcal{P}(\mathcal{X}) = \begin{pmatrix} Q_j^T G_j {Q_j}' \\ Q_j - {Q_j}' \\ Q_j - {\tilde{Q}_j}' \end{pmatrix}, \mathcal{Q}(\mathcal{Z}) = \begin{pmatrix} -Z \\ -{Q_j}'' \\ -{\tilde{Q}_j}'' \end{pmatrix}$$

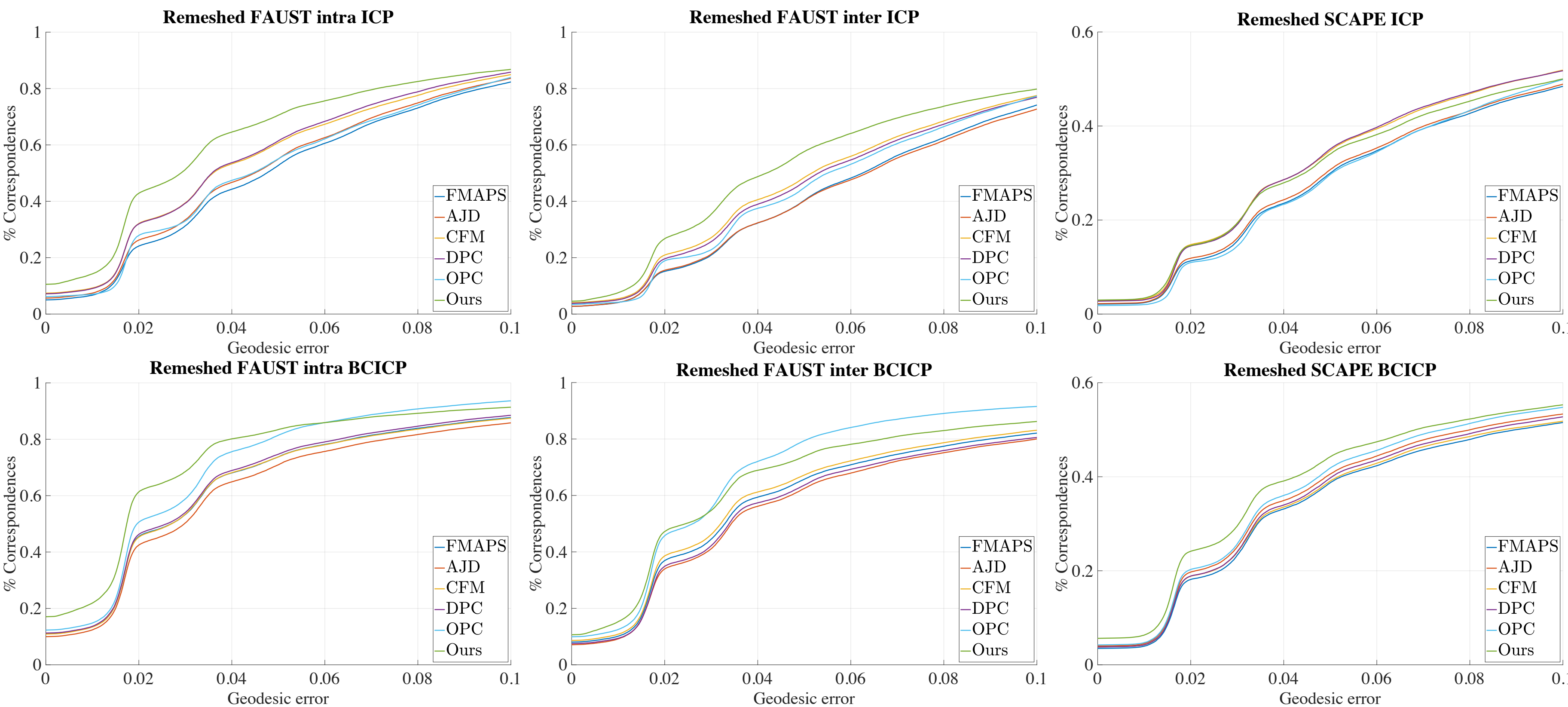
Gao et al. '18

Results

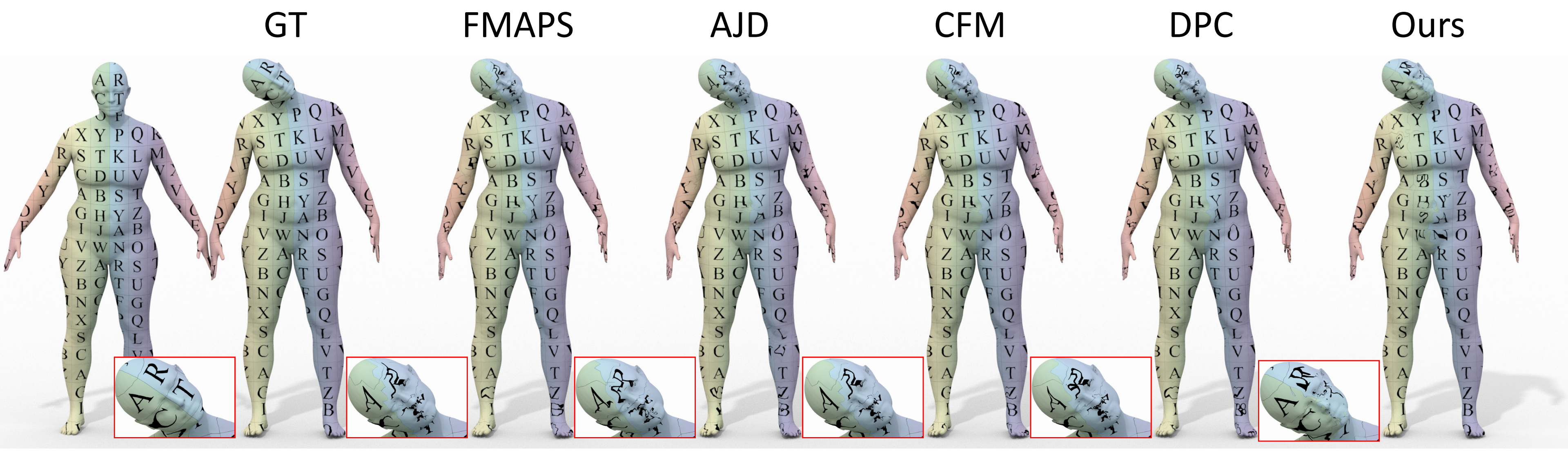
Shape Correspondences



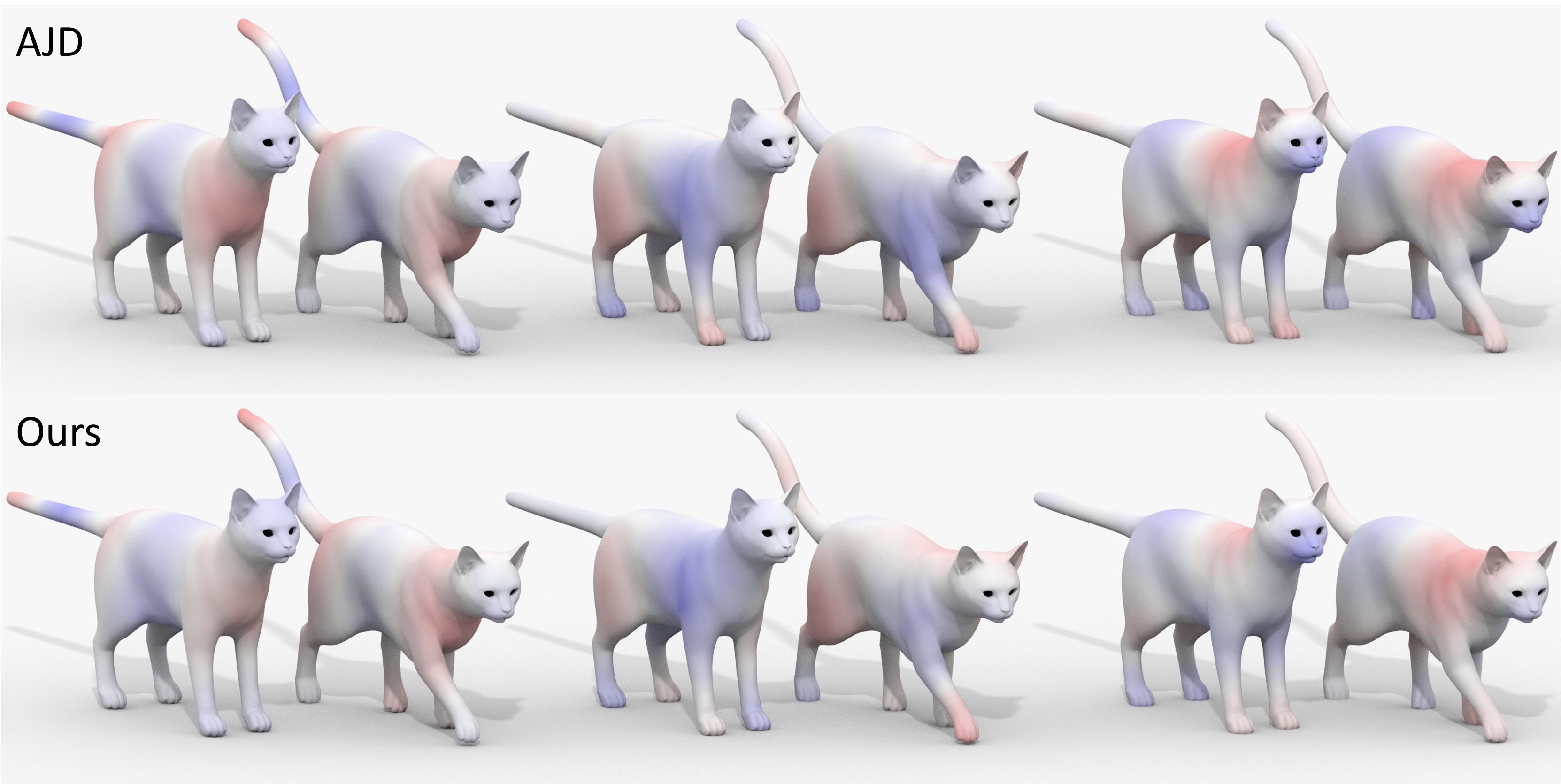
Shape Correspondences



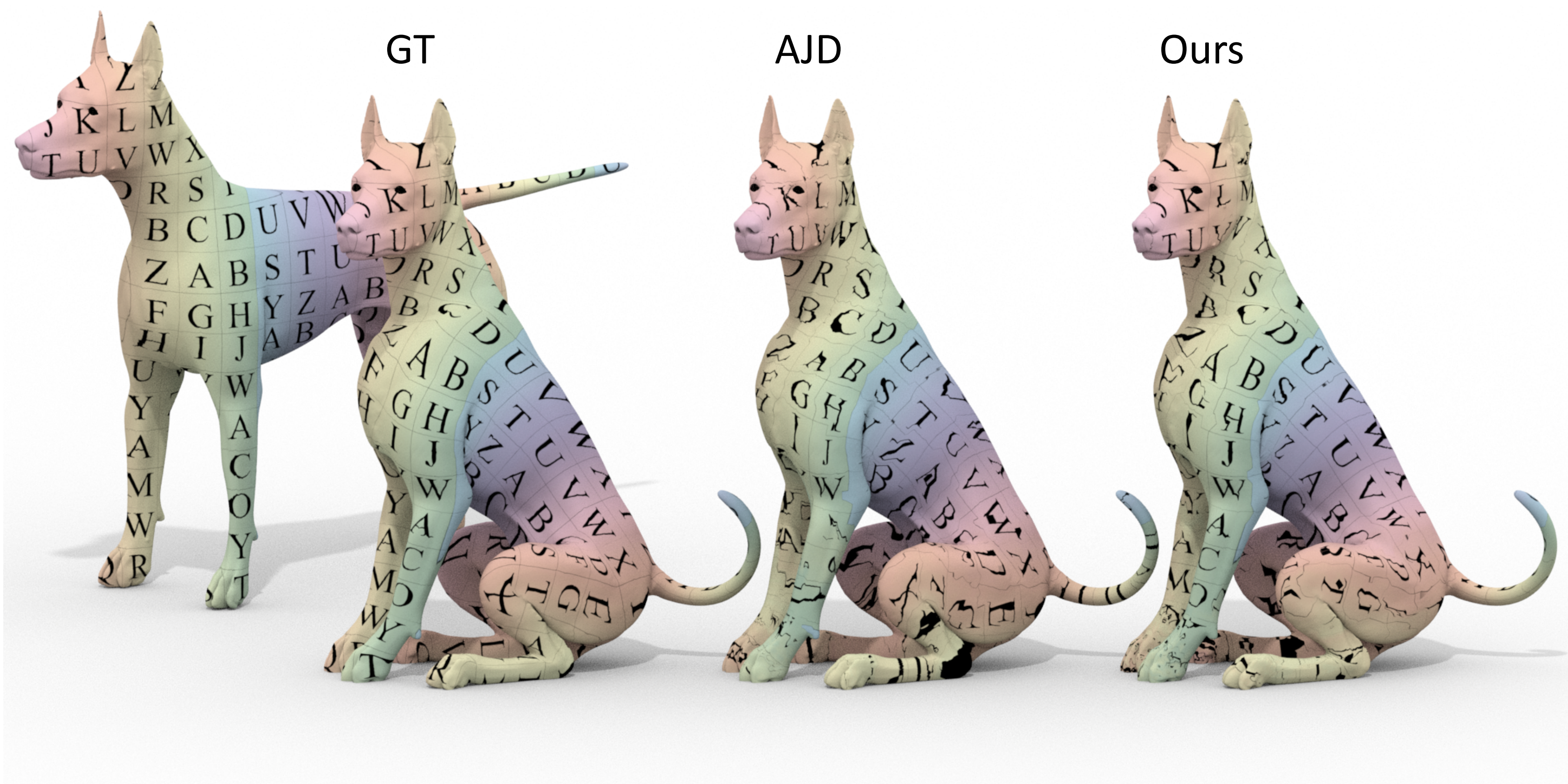
Shape Correspondences



Comparison to AJD

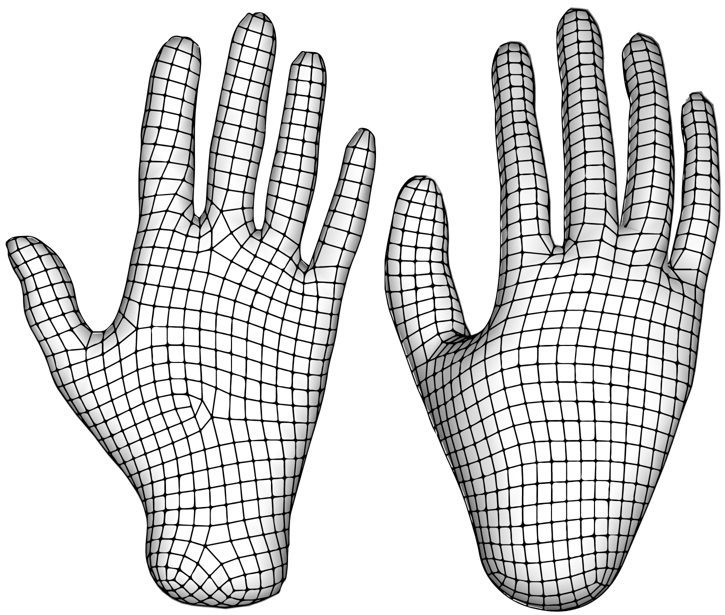


Comparison to AJD

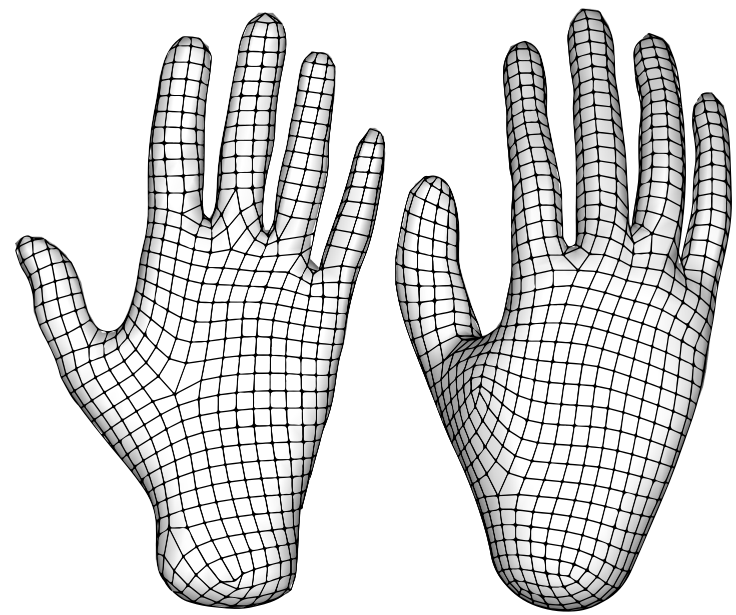


Joint Quadrangulation

Fixed LB



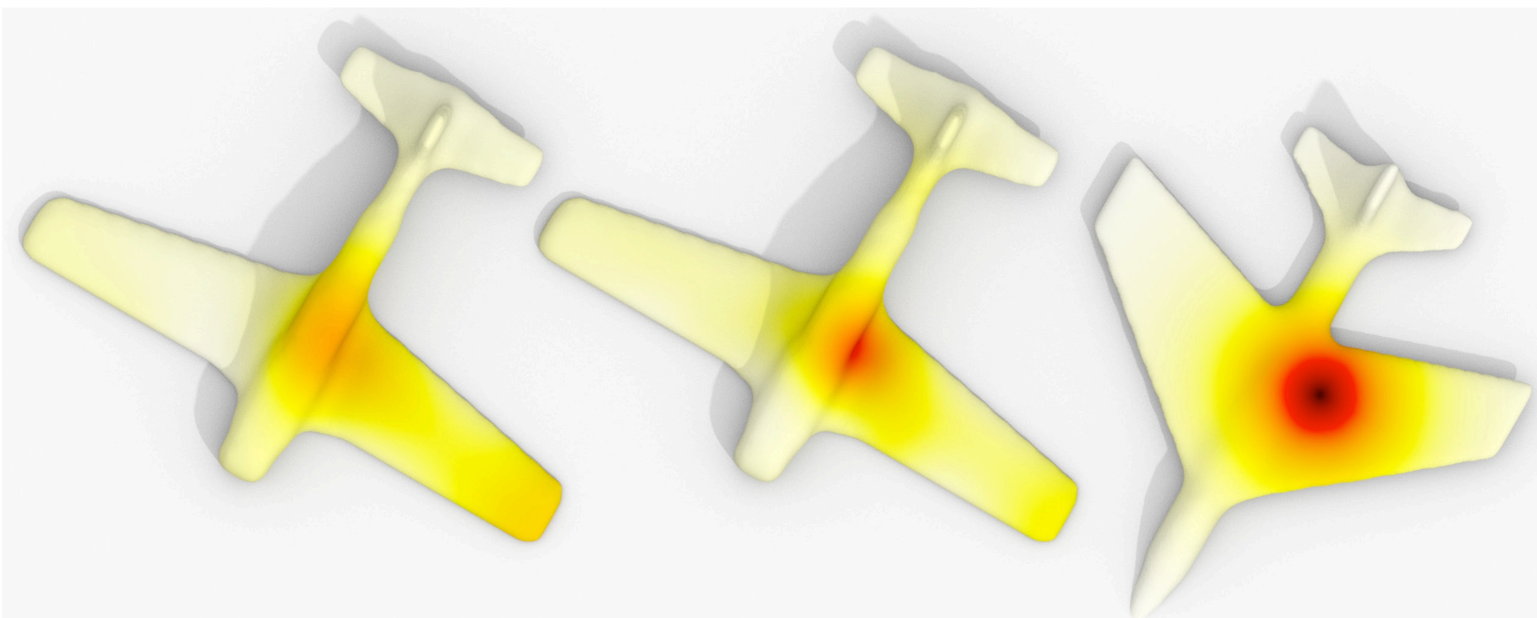
Designed POD



Azencot et al. ‘17

Function Transfer

Standard Transfer　　Product Transfer

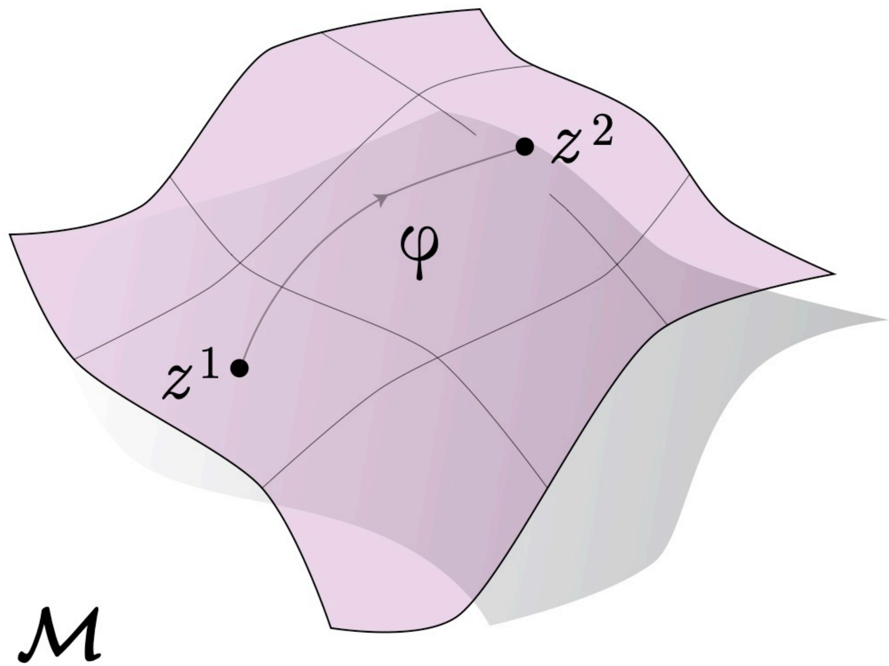


Nogneng et al. '18

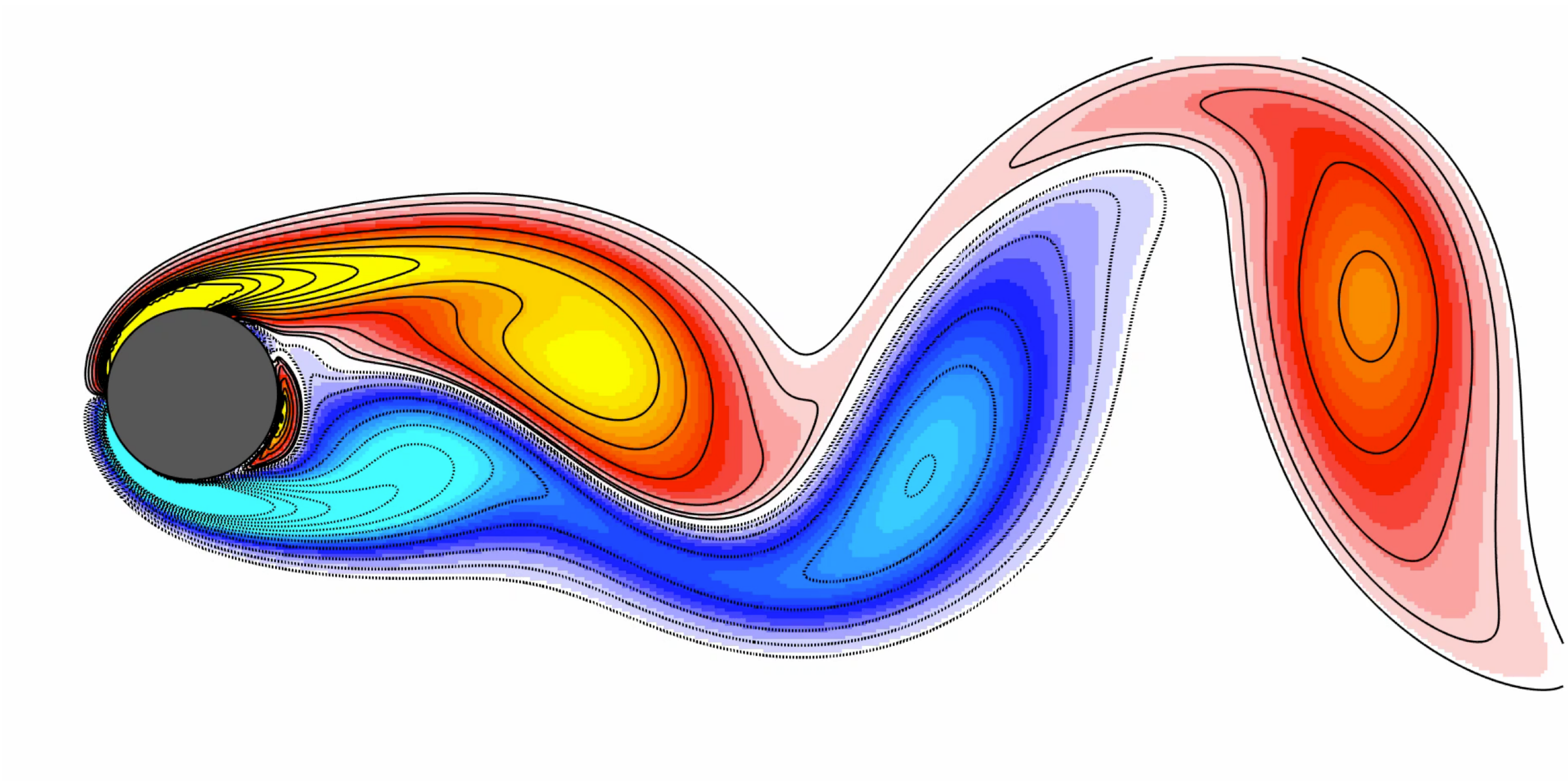
Future Work

- Dependencies between constraints and bases
- Design a basis on a single shape

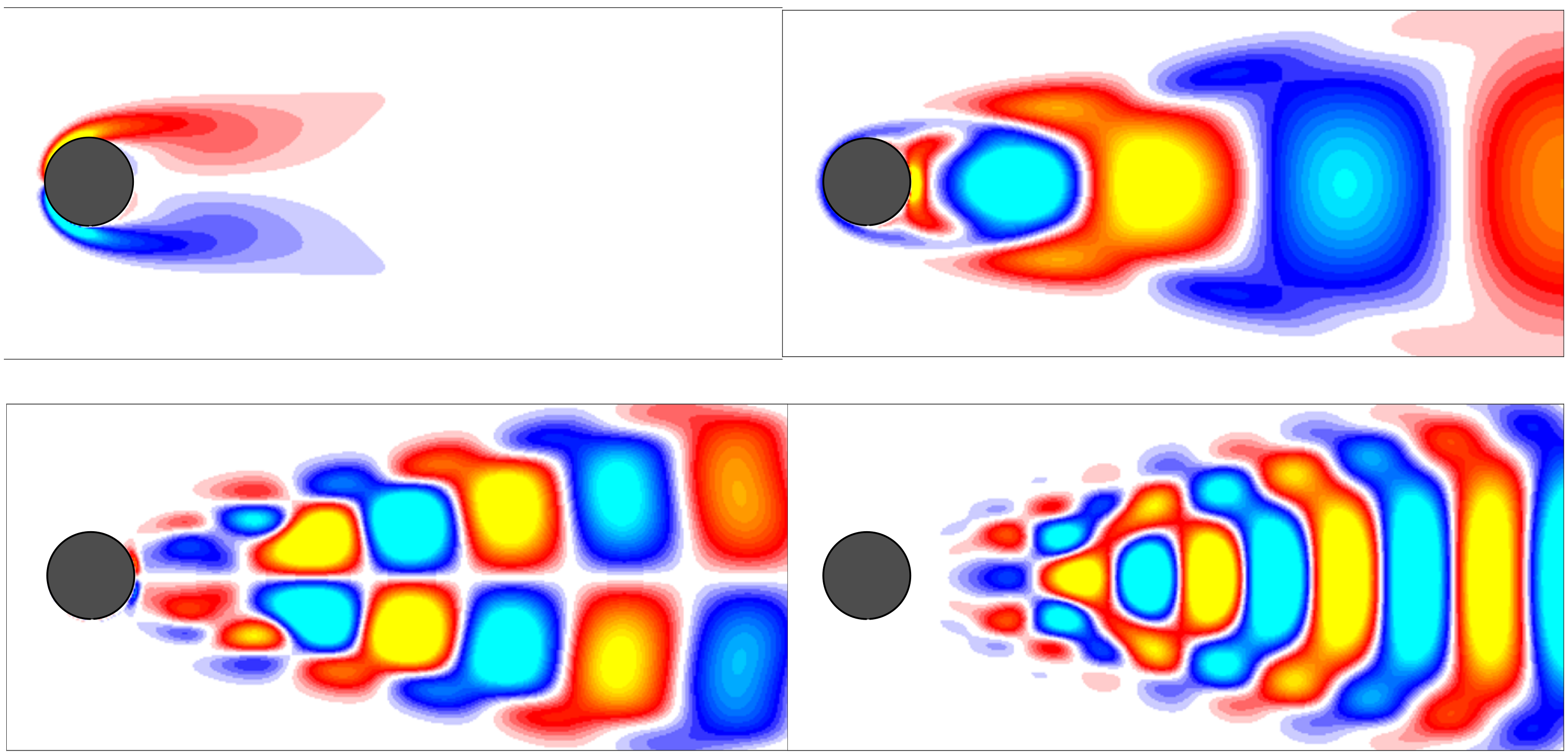
Self Functional Maps



Non Linear Dynamics



Non Linear Dynamics



Azencot, Yin, Bertozzi '19

Gang Reduction in Youth

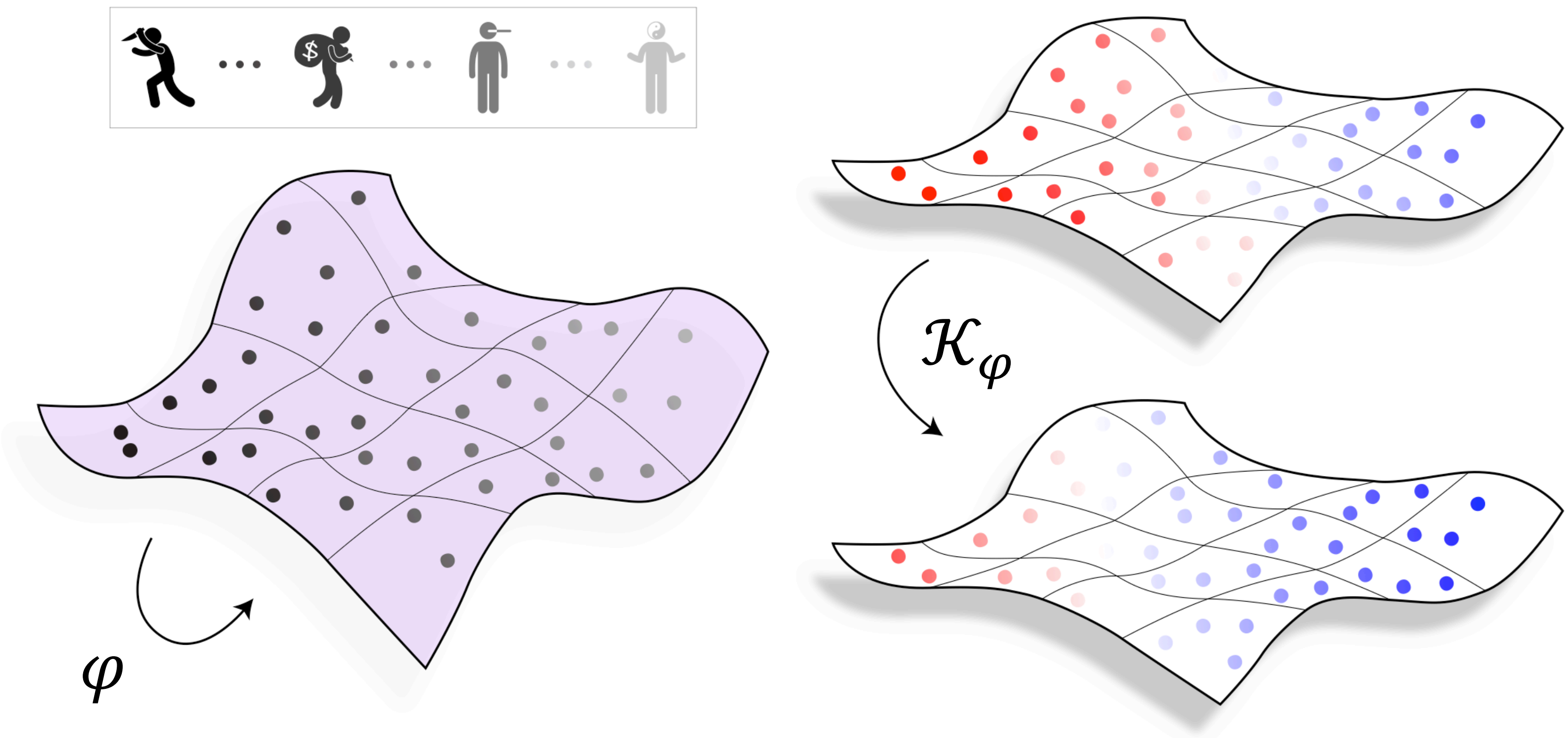
Personality questionnaires taken every 6 months

Q: I get very angry and “lose my temper”

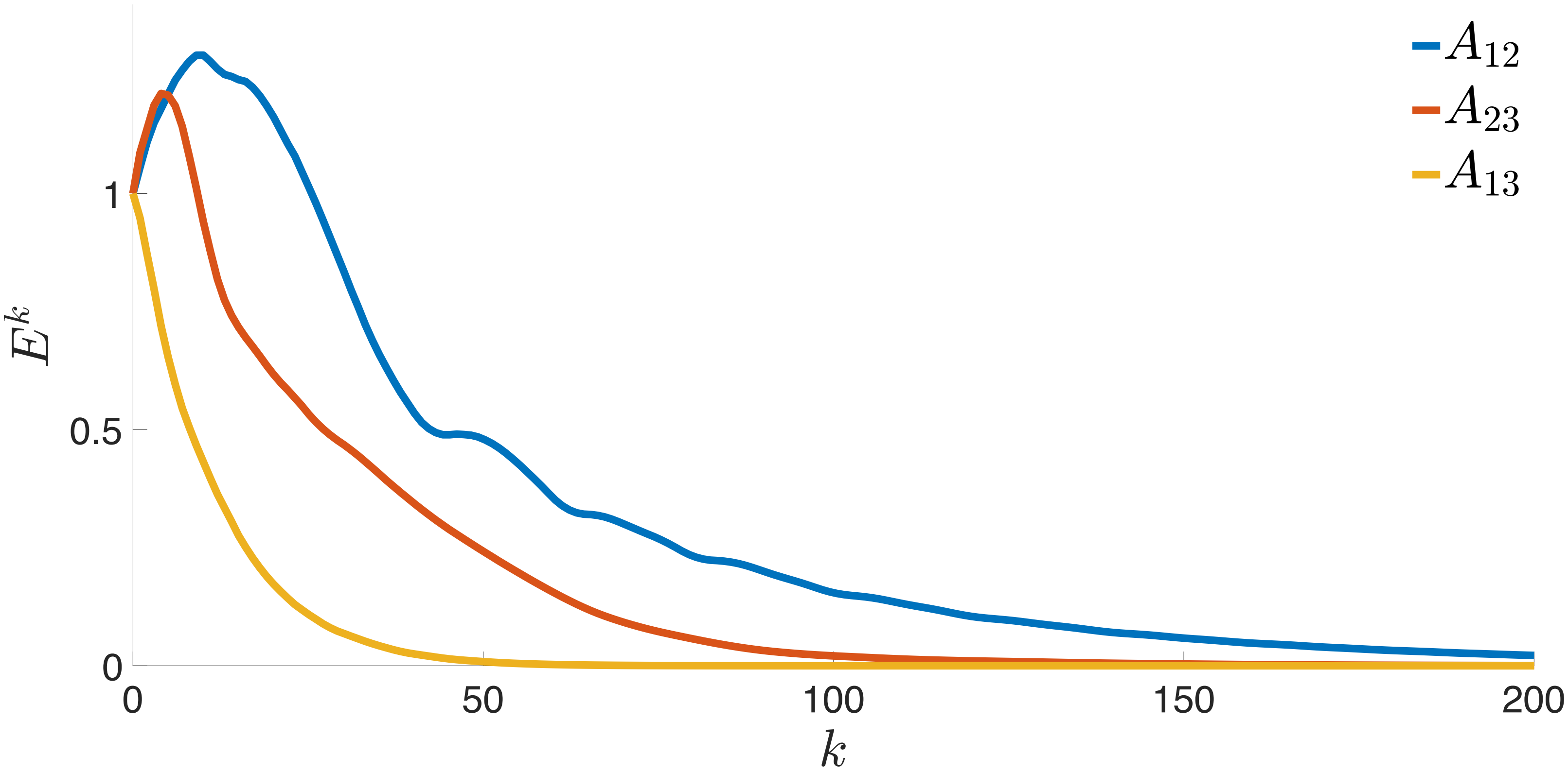
Q: It is okay to beat people up if they hit me first

Q: Attacked someone with a weapon?

Gang Reduction in Youth



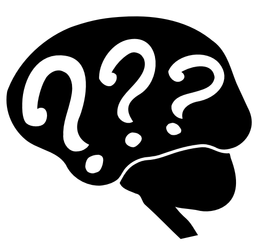
Transient Growth



Conclusions

- Design a basis based on POD modes
- Exploit established regularizers on fmaps
- Solve efficiently via ADMM
- Similar provably convergent scheme

Thank you!



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STEM LEADERSHIP
PROGRAM

European Union's Horizon 2020
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