A Bayesian methodology for systemic risk assessment in financial networks

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The problem

- Consider interbank market as network:
  nodes = banks, directed edges = interbank liabilities.
- In practice, network of interbank liabilities not fully observable.
  Often only total interbank liabilities and assets of every bank available.
- How can one fill in the missing data?
- How can one incorporate additional information about the network structure?
- What are implications for stress testing?

Main contributions

- Development of Bayesian framework (Gibbs sampler) for sampling from distribution of liabilities matrix conditional on its row and column sums.
- Application to systemic risk assessment:
  - Can give probabilities for outcomes of stress tests.
  - Results show limitations of classical approach.
- Show that general monotonicity arguments relating severity of systemic risk to number of edges do not hold in general.
- Code is available as R-package (systemicrisk) on CRAN.

The setting

- Nodes consists of n banks with indices in $\mathcal{N} = \{1, \ldots, n\}$.
- A matrix $L = (L_{ij}) \in \mathbb{R}^{n \times n}$ is called a liabilities matrix if $L_{ij} \geq 0$ $\forall i, j \in \mathcal{N}$ and $L_{ii} = 0$ $\forall i \in \mathcal{N}$.
  L_{ij} represents the nominal liability of bank i to bank j.
- Total nominal interbank liabilities of bank i: $r_i(L) := \sum_{j=1}^{m} L_{ij}$.
- Total nominal interbank assets of bank i: $c_i(L) := \sum_{j=1}^{m} L_{ji}$.

Admissible liabilities matrix

- Interbank liabilities L_{ij} not known; need to be estimated.
- L has $n^2 - n$ unknown entries.
- $n^2 - n - (2n - 1) = n^2 - 3n + 1$ degrees of freedom for L.
- Given $a, l \in [0, \infty)^n$ we refer to an $L \in \mathbb{R}^{n \times n}$ as an admissible liabilities matrix (for a and l) if it is a liabilities matrix and if it satisfies

$$c(L) = a, \quad r(L) = l.$$

Existence of admissible liabilities matrix

Theorem (Existence of an admissible liabilities matrix)

Consider two vectors $a, l \in [0, \infty)^n$ satisfying $A := \sum_{i=1}^n a_i = \sum_{i=1}^n l_i$. Then there exists an admissible liabilities matrix L_{al} for a and l if and only if

$$a_i + l_i \leq A \ \forall i \in \mathcal{N}. \tag{1}$$

- Condition (1) is equivalent to

$$a_i \leq \sum_{\substack{j=1 \\ j \neq i}}^n l_j \quad \forall i \in \mathcal{N}$$

  which requires that the assets of any bank have to be smaller than or equal to the total liabilities of all other banks in the network.

- Proof contains algorithm giving explicit construction of L_{al}.

Example: Network consisting of 3 banks

- Network with $n = 3$ banks.
- 1 degree of freedom for the liabilities matrix.
- Solving the system of linear equations and setting the free parameter to $x = L_{32}$, we find that the general liabilities matrix $L = L(x)$ is

$$L(x) = \begin{pmatrix} 0 & a_2 - x & -a_2 + l_1 + x \\ a_1 - l_3 + x & 0 & l_2 - a_1 + l_3 - x \\ l_3 - x & x & 0 \end{pmatrix}.$$

- Requiring $L_{ij} \geq 0 \; \forall i, j$ leads to a condition on x:

$$\max\{a_2 - l_1, l_3 - a_1, 0\} \leq x \leq \min\{a_2, l_3 - a_1 + l_2, l_3\}.$$

It can be satisfied as long as a, l satisfy condition (1).

The Bayesian framework I

- Probabilistic model for liabilities matrix:
  - Constructs adjacency matrix $\mathcal{A} = (\mathcal{A}_{ij})$;
  - Attaches weights (i.e., liabilities) to existing directed edges.
- With $p_{ij} \in [0,1]\ \forall i \neq j \in \mathcal{N}$, $p_{ii} = 0\ \forall i \in \mathcal{N}$ the model is:

$$\begin{aligned} \mathbb{P}(\mathcal{A}_{ij} = 1) &= p_{ij} \quad \forall i, j \in \mathcal{N}, \\ L_{ij} | \{\mathcal{A}_{ij} = 1\} &\sim \text{Exponential}(\lambda_{ij}) \quad \forall i \neq j \in \mathcal{N}, \\ a = c(L) &\qquad l = r(L). \end{aligned} \tag{2}$$

- We only observe a and l and not L.

The Bayesian framework II

- Main interest: Distribution of $h(L)|l, a$, for some functional h, e.g., default indicators.
- Distribution of $h(L)|l, a$ not available in closed form; can be approximated using MCMC methods (Gibbs sampler).
- Model parameters:
  - $p \in [0,1]^{n \times n}$, where p_{ij} probability of existence of directed edge from i to j,
  - $\lambda \in \mathbb{R}^{n \times n}$, which governs the distribution of weights given that an edge exists.

Gibbs sampling for $L|a, l$

- Key idea of Gibbs sampler: update one or several components of the entire parameter vector by sampling them from their joint conditional distribution given the remainder of the parameter vector.

- By repeating these updating steps one constructs a Markov Chain whose distribution converges to the target distribution.

- Here parameter vector is matrix L:
  - Initialise chain with matrix L that satisfies $r(L) = l$, $c(L) = a$.
  - MCMC sampler produce a sequence of matrices $L^1, L^2, \ldots$.
  - Quantity of interest:

$$\mathbb{E}[h(L)|l, a] \approx \frac{1}{N} \sum_{i=1}^{N} h(L^{i\delta+b}),$$

N number of samples, b burn-in period, $\delta \in \mathbb{N}$ thinning parameter.

Updating components of L

- Need to decide which elements of L need to be updated.
- Need to determine how the new values will be chosen, i.e., need to determine their distribution conditional on remainder of elements of L.

Which elements to update?

- Smallest submatrix of interest for updating: 2 x 2 submatrix.
- Need more general updates than 2 x 2 submatrices.
- Idee: Update on a cycle given by an integer $k \in \{2, \ldots, n\}$ (length of cycle) and mutually disjoint row indices $(i_1, i_2, \ldots, i_k)$ and mutually disjoint column indices $(j_1, j_2, \ldots, j_k)$;
  i.e., update L at indices

$$\eta := ((i_1, j_1), (i_1, j_2), (i_2, j_2), \ldots, (i_k, j_k), (i_k, j_1))$$

  conditional on all other values of L.
- A cycle of length k will contain indices of $2k$ elements of matrix L.

Illustration of updating submatrices

$L_{i_1 j_1}$	$L_{i_1 j_2}$
$L_{i_2 j_1}$	$L_{i_2 j_2}$

	$L_{i_1 j_1}$	$L_{i_1 j_2}$
$L_{i_2 j_3}$		$L_{i_2 j_2}$
$L_{i_3 j_3}$	$L_{i_3 j_1}$	

	$L_{i_1 j_1}$	$L_{i_1 j_2}$	
		$L_{i_2 j_2}$	$L_{i_2 j_3}$
$L_{i_3 j_4}$			$L_{i_3 j_3}$
$L_{i_4 j_4}$	$L_{i_4 j_1}$		

How to choose the cycle?

- Sample cycle length k from discrete distribution which assumes that probability of selecting a cycle of length k is $2^{n-k}/(2^{n-1}-1)$ for $k \in 2, \ldots, n$.
- Hence, for large n we roughly select $k = 2$ with probability $1/2$ and $k = 3$ with probability $1/4$.
- After that we sample $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ uniformly from $\mathcal{N}$ without replacement.

How to update?

- Want to update L at $\eta = ((i_1, j_1), (i_1, j_2), (i_2, j_2), \ldots, (i_k, j_k), (i_k, j_1))$ conditional on all other values of L.
- L_η: vector containing the $2k$ elements of L along the cycle;
- Let $g : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$,

$$g(x) = (x_1 + x_2, x_2 + x_3, \ldots, x_{2k-1} + x_{2k}, x_{2k} + x_1) \quad \forall x \in \mathbb{R}^{2k}.$$

- Let $\tilde{L}_\eta$ have the same distribution as L_η. Want distribution of $\tilde{L}_\eta$ conditional on $g(\tilde{L}_\eta) = g(L_\eta)$.
- The only $\tilde{L}_\eta$ satisfying this are

$$\tilde{L}_{\eta_i} = L_{\eta_i} + (-1)^{i+1}\Delta,$$

where $\Delta \in \mathbb{R}$.

Sampling Δ

- As the elements of $\tilde{L}$ are nonnegative, we need

$$\Delta \geq -L_{\eta_i} \text{ for } i \text{ odd} \qquad \text{and} \qquad \Delta \leq L_{\eta_i} \text{ for } i \text{ even.}$$

Hence,

$$\Delta \in [\Delta_{\mathrm{low}}, \Delta_{\mathrm{up}}] := \left[-\min_{i \text{ odd}} L_{\eta_i}, \min_{i \text{ even}} L_{\eta_i} \right].$$

- The unconditional density of $\tilde{L}_\eta$ (with respect to the sum of the Lebesgue measure and the counting measure at 0) is

$$f(\tilde{L}_\eta) = \prod_{i=1}^{2k} \Big((1 - p_{\eta_i}) \mathbb{I}(\tilde{L}_{\eta_i} = 0) + p_{\eta_i} \mathbb{I}(\tilde{L}_{\eta_i} > 0) \lambda_{\eta_i} \exp(-\lambda_{\eta_i} \tilde{L}_{\eta_i}) \Big).$$

- Can work out the the conditional density on $g(\tilde{L}_\eta) = g(L_\eta)$ analytically.

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49.9
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47.1	49.8	91.2	100	66.5	54.9

	0	14.7	32.3	0	0
0		19.5	11	0	19.5
23.5	20.3		20.5	20.5	6.4
23.5	20.3	23.5		23	9.5
0	9.1	19.5	18.4		19.5
0	0	14	17.9	23	

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47.1	49.8	91.2	100	66.5	54.9

	0	14.7	32.3	0	0	47
0		19.5	11	0	19.5	49.9
23.5	20.3		20.5	20.5	6.4	91.3
23.5	20.3	23.5		23	9.5	99.9
0	9.1	19.5	18.4		19.5	66.5
0	0	14	17.9	23		54.9

47.1	49.8	91.2	100	66.5	54.9

	0	$\boxed{13.9}$	32.3	0	$\boxed{0.8}$		47
0		19.5	11	0	19.5		49.9
23.5	20.3		20.5	20.5	6.4		91.3
23.5	20.3	23.5		23	9.5		99.9
0	9.1	$\boxed{20.3}$	18.4		$\boxed{18.6}$		66.5
0	0	14	17.9	23			54.9

47.1	49.8	91.2	100	66.5	54.9

	0	13.9	32.3	0	0.8		47
0		$\boxed{19.5}$	11	0	$\boxed{19.5}$		49.9
23.5	20.3		$\boxed{20.5}$	20.5	$\boxed{6.4}$		91.3
23.5	20.3	23.5		23	9.5		99.9
0	9.1	$\boxed{20.3}$	$\boxed{18.4}$		18.6		66.5
0	0	14	17.9	23			54.9

47.1	49.8	91.2	100	66.5	54.9

	0	13.9	32.3	0	0.8	47
0		$\boxed{35.5}$	11	0	$\boxed{3.5}$	49.9
23.5	20.3		$\boxed{4.5}$	20.5	$\boxed{22.5}$	91.3
23.5	20.3	23.5		23	9.5	99.9
0	9.1	$\boxed{4.3}$	$\boxed{34.4}$		18.6	66.5
0	0	14	17.9	23		54.9
47.1	49.8	91.2	100	66.5	54.9	

	0	13.9	32.3	0	0.8
0		35.5	11	0	3.5
23.5	20.3		4.5	20.5	22.5
23.5	20.3	23.5		23	9.5
0	9.1	4.3	34.4		18.6
0	0	14	17.9	23	

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0		35.5	11	0	3.5
23.5	20.3		4.5	20.5	22.5
23.5	20.3	23.5		23	9.5
0	9.1	4.3	34.4		18.6
0	0	14	17.9	23	

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0		35.5	11	0	3.5		49.9
23.5	20.3		4.5	20.5	22.5		91.3
23.5	20.3	23.5		23	9.5		99.9
0	9.1	4.3	34.4		18.6		66.5
0	0	14	17.9	23			54.9

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	0	13.9	32.3	0	0.8		47
0		35.5	11	0	3.5		49.9
23.5	30.9		4.5	9.9	22.5		91.3
23.5	9.8	23.5		33.6	9.5		99.9
0	9.1	4.3	34.4		18.6		66.5
0	0	14	17.9	23			54.9

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	0	13.9	32.3	0	0.8	47
$\boxed{0}$		$\boxed{35.5}$	11	0	3.5	49.9
23.5	30.9		4.5	9.9	22.5	91.3
$\boxed{23.5}$	9.8	$\boxed{23.5}$		33.6	9.5	99.9
0	9.1	4.3	34.4		18.6	66.5
0	0	14	17.9	23		54.9

47.1	49.8	91.2	100	66.5	54.9

	0	13.9	32.3	0	0.8	47
5.4		30.1	11	0	3.5	49.9
23.5	30.9		4.5	9.9	22.5	91.3
18.1	9.8	28.9		33.6	9.5	99.9
0	9.1	4.3	34.4		18.6	66.5
0	0	14	17.9	23		54.9

47.1	49.8	91.2	100	66.5	54.9

	0	$\boxed{13.9}$	32.3	$\boxed{0}$	0.8		47
5.4		30.1	11	0	3.5		49.9
23.5	30.9		4.5	9.9	22.5		91.3
18.1	9.8	$\boxed{28.9}$		$\boxed{33.6}$	9.5		99.9
0	9.1	4.3	34.4		18.6		66.5
0	0	14	17.9	23			54.9

47.1	49.8	91.2	100	66.5	54.9

	0	$\boxed{1.2}$	32.3	$\boxed{12.7}$	0.8
5.4		30.1	11	0	3.5
23.5	30.9		4.5	9.9	22.5
18.1	9.8	$\boxed{41.7}$		$\boxed{20.9}$	9.5
0	9.1	4.3	34.4		18.6
0	0	14	17.9	23	

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	0	1.2	32.3	12.7	0.8		47
5.4		30.1	11	0	3.5		49.9
23.5	30.9		4.5	9.9	22.5		91.3
18.1	9.8	41.7		20.9	9.5		99.9
0	9.1	4.3	34.4		18.6		66.5
0	0	14	17.9	23			54.9

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	$\boxed{0}$	$\boxed{1.2}$	32.3	12.7	0.8
5.4		30.1	11	0	3.5
23.5	30.9		4.5	9.9	22.5
18.1	9.8	41.7		20.9	9.5
0	9.1	4.3	34.4		18.6
0	$\boxed{0}$	$\boxed{14}$	17.9	23	

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	0	1.2	$\boxed{32.3}$	$\boxed{12.7}$	$\boxed{0.8}$		47
5.4		30.1	$\boxed{11}$	$\boxed{0}$	3.5		49.9
23.5	30.9		4.5	9.9	22.5		91.3
$\boxed{18.1}$	9.8	$\boxed{41.7}$		20.9	9.5		99.9
$\boxed{0}$	9.1	4.3	34.4		$\boxed{18.6}$		66.5
0	0	$\boxed{14}$	17.9	$\boxed{23}$			54.9

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	$\boxed{0}$	1.2	$\boxed{32.3}$	12.7	0.8
5.4		30.1	11	0	3.5
$\boxed{23.5}$	30.9		$\boxed{4.5}$	9.9	22.5
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5.4		30.1	11	0	3.5
23.5	30.9		4.5	9.9	22.5
13.8	9.8	46		20.9	9.5
4.3	9.1	0	34.4		18.6
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23.5	30.9		4.5	9.9	22.5		91.3
13.8	9.8	46		20.9	9.5		99.9
4.3	9.1	0	34.4		18.6		66.5
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	0	1.2	32.3	12.7	0.8
5.4		30.1	2.5	8.4	3.5
23.5	30.9		12.9	1.5	22.5
13.8	9.8	46		20.9	9.5
4.3	9.1	0	34.4		18.6
0	0	14	17.9	23	

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$\boxed{5.4}$		30.1	2.5	$\boxed{8.4}$	3.5		49.9
23.5	30.9		12.9	1.5	22.5		91.3
13.8	9.8	46		20.9	9.5		99.9
$\boxed{4.3}$	$\boxed{9.1}$	0	34.4		18.6		66.5
0	$\boxed{0}$	14	17.9	$\boxed{23}$			54.9

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	0	1.2	32.3	12.7	0.8		47
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23.5	30.9		12.9	1.5	22.5		91.3
13.8	9.8	46		20.9	9.5		99.9
8.1	5.3	0	34.4		18.6		66.5
0	3.8	14	17.9	19.2			54.9

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	0	1.2	32.3	12.7	0.8
1.6		30.1	2.5	12.3	3.5
$\boxed{23.5}$	$\boxed{30.9}$		12.9	1.5	22.5
13.8	9.8	46		20.9	9.5
$\boxed{8.1}$	$\boxed{5.3}$	0	34.4		18.6
0	3.8	14	17.9	19.2	

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1.6		30.1	2.5	12.3	3.5
31.7	22.8		12.9	1.5	22.5
13.8	9.8	46		20.9	9.5
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47.1	49.8	91.2	100	66.5	54.9

	0	$\boxed{9.7}$	32.3	$\boxed{4.2}$	0.8
$\boxed{10.1}$		$\boxed{21.5}$	2.5	12.3	3.5
31.7	22.8		12.9	1.5	22.5
$\boxed{5.3}$	9.8	46		$\boxed{29.4}$	9.5
0	13.4	0	34.4		18.6
0	3.8	14	17.9	19.2	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8
10.1		21.5	2.5	12.3	3.5
31.7	22.8		12.9	1.5	22.5
5.3	9.8	46		29.4	9.5
0	13.4	0	34.4		18.6
0	3.8	14	17.9	19.2	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8		47
10.1		21.5	2.5	12.3	3.5		49.9
31.7	22.8		12.9	1.5	22.5		91.3
5.3	9.8	46		29.4	9.5		99.9
0	13.4	0	34.4		18.6		66.5
0	3.8	14	17.9	19.2			54.9
47.1	49.8	91.2	100	66.5	54.9		

	0	9.7	32.3	4.2	0.8
10.1		21.5	2.5	12.3	3.5
31.7	22.8		12.9	1.5	22.5
5.3	9.8	46		29.4	9.5
0	13.4	0	34.4		18.6
0	3.8	14	17.9	19.2	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8		47
10.1		21.5	2.5	12.3	3.5		49.9
31.7	22.8		12.9	1.5	22.5		91.3
5.3	9.8	46		29.4	9.5		99.9
0	13.4	0	34.4		18.6		66.5
0	3.8	14	17.9	19.2			54.9

47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8
10.1		21.5	2.5	12.3	3.5
31.7	22.8		12.9	1.5	22.5
$\boxed{5.3}$	9.8	46		$\boxed{29.4}$	9.5
0	13.4	0	34.4		18.6
$\boxed{0}$	3.8	14	17.9	$\boxed{19.2}$	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8		47
10.1		21.5	2.5	12.3	3.5		49.9
31.7	22.8		12.9	1.5	22.5		91.3
2.8	9.8	46		31.9	9.5		99.9
0	13.4	0	34.4		18.6		66.5
2.5	3.8	14	17.9	16.6			54.9

47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8	47
10.1		21.5	2.5	12.3	3.5	49.9
31.7	22.8		12.9	1.5	22.5	91.3
2.8	9.8	46		31.9	9.5	99.9
0	13.4	0	34.4		18.6	66.5
2.5	3.8	14	17.9	16.6		54.9
47.1	49.8	91.2	100	66.5	54.9	

	0	9.7	32.3	4.2	0.8		47
10.1		21.5	2.5	12.3	3.5		49.9
31.7	22.8		12.9	1.5	22.5		91.3
2.8	9.8	46		31.9	9.5		99.9
0	13.4	0	34.4		18.6		66.5
2.5	3.8	14	17.9	16.6			54.9
47.1	49.8	91.2	100	66.5	54.9		

	0	9.7	32.3	4.2	0.8
10.1		21.5	2.5	12.3	3.5
31.7	22.8		12.9	1.5	22.5
$\boxed{2.8}$	9.8	46		31.9	$\boxed{9.5}$
$\boxed{0}$	13.4	0	34.4		$\boxed{18.6}$
2.5	3.8	14	17.9	16.6	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8
10.1		21.5	2.5	12.3	3.5
31.7	22.8		12.9	1.5	22.5
$\boxed{2.8}$	9.8	46		31.9	$\boxed{9.5}$
$\boxed{0}$	13.4	0	34.4		$\boxed{18.6}$
2.5	3.8	14	17.9	16.6	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8
10.1		$\boxed{21.5}$	$\boxed{2.5}$	12.3	3.5
31.7	22.8		12.9	1.5	22.5
2.8	9.8	46		31.9	9.5
0	$\boxed{13.4}$	0	$\boxed{34.4}$		18.6
2.5	$\boxed{3.8}$	$\boxed{14}$	17.9	16.6	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8
10.1		20.2	3.8	12.3	3.5
31.7	22.8		12.9	1.5	22.5
2.8	9.8	46		31.9	9.5
0	14.7	0	33.2		18.6
2.5	2.6	15.2	17.9	16.6	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8
10.1		20.2	3.8	12.3	3.5
31.7	22.8		12.9	1.5	22.5
2.8	9.8	46		31.9	9.5
0	14.7	0	33.2		18.6
2.5	2.6	15.2	17.9	16.6	

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47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8		47
10.1		20.2	3.8	12.3	3.5		49.9
31.7	22.8		12.9	1.5	22.5		91.3
2.8	9.8	46		31.9	9.5		99.9
0	7.1	7.6	33.2		18.6		66.5
2.5	10.2	7.6	17.9	16.6			54.9

47.1	49.8	91.2	100	66.5	54.9

	0	9.7	32.3	4.2	0.8		47
10.1		20.2	3.8	12.3	3.5		49.9
31.7	22.8		12.9	1.5	22.5		91.3
2.8	9.8	46		31.9	9.5		99.9
0	7.1	7.6	33.2		18.6		66.5
2.5	10.2	7.6	17.9	16.6			54.9
47.1	49.8	91.2	100	66.5	54.9		

	$\boxed{4.4}$	$\boxed{5.3}$	32.3	4.2	0.8	47
10.1		20.2	3.8	12.3	3.5	49.9
31.7	22.8		12.9	1.5	22.5	91.3
2.8	9.8	46		31.9	9.5	99.9
0	$\boxed{2.6}$	$\boxed{12}$	33.2		18.6	66.5
2.5	10.2	7.6	17.9	16.6		54.9

47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8
10.1		20.2	3.8	12.3	3.5
31.7	22.8		12.9	1.5	22.5
2.8	9.8	46		31.9	9.5
0	2.6	12	33.2		18.6
2.5	10.2	7.6	17.9	16.6	

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47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8		47
10.1		20.2	3.8	12.3	3.5		49.9
$\boxed{20.1}$	22.8		12.9	$\boxed{13}$	22.5		91.3
2.8	9.8	46		$\boxed{20.4}$	$\boxed{21.1}$		99.9
$\boxed{11.5}$	2.6	12	33.2		$\boxed{7.1}$		66.5
2.5	10.2	7.6	17.9	16.6			54.9

47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8
10.1		20.2	3.8	12.3	3.5
20.1	22.8		12.9	13	22.5
2.8	9.8	46		20.4	21.1
11.5	2.6	12	33.2		7.1
2.5	10.2	7.6	17.9	16.6	

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47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8	47
$\boxed{6.4}$		20.2	3.8	12.3	$\boxed{7.2}$	49.9
20.1	22.8		12.9	13	22.5	91.3
2.8	9.8	46		20.4	21.1	99.9
11.5	2.6	12	$\boxed{36.9}$		$\boxed{3.3}$	66.5
$\boxed{6.3}$	10.2	7.6	$\boxed{14.2}$	16.6		54.9

47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8	47
$\boxed{6.4}$		20.2	$\boxed{3.8}$	12.3	7.2	49.9
20.1	22.8		$\boxed{12.9}$	$\boxed{13}$	22.5	91.3
$\boxed{2.8}$	9.8	46		$\boxed{20.4}$	21.1	99.9
11.5	2.6	12	36.9		3.3	66.5
6.3	10.2	7.6	14.2	16.6		54.9
47.1	49.8	91.2	100	66.5	54.9	

	4.4	5.3	32.3	4.2	0.8		47
6.5		20.2	3.6	12.3	7.2		49.9
20.1	22.8		13	12.9	22.5		91.3
2.6	9.8	46		20.5	21.1		99.9
11.5	2.6	12	36.9		3.3		66.5
6.3	10.2	7.6	14.2	16.6			54.9

47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8	47
6.5		20.2	3.6	12.3	7.2	49.9
20.1	22.8		13	12.9	22.5	91.3
2.6	9.8	46		20.5	21.1	99.9
11.5	2.6	12	36.9		3.3	66.5
6.3	10.2	7.6	14.2	16.6		54.9

47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8
6.5		20.2	$\boxed{0}$	$\boxed{15.9}$	7.2
20.1	22.8		13	12.9	22.5
2.6	9.8	46		20.5	21.1
11.5	2.6	12	36.9		3.3
6.3	10.2	7.6	$\boxed{17.8}$	$\boxed{13}$	

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47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8
6.5		20.2	0	15.9	7.2
20.1	22.8		13	12.9	22.5
2.6	9.8	46		20.5	21.1
11.5	2.6	12	36.9		3.3
6.3	10.2	7.6	17.8	13	

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47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8
6.5		20.2	0	15.9	7.2
$\boxed{19.4}$	$\boxed{23.6}$		13	12.9	22.5
2.6	9.8	46		20.5	21.1
$\boxed{12.3}$	$\boxed{1.9}$	12	36.9		3.3
6.3	10.2	7.6	17.8	13	

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47.1	49.8	91.2	100	66.5	54.9

	4.4	5.3	32.3	4.2	0.8
6.5		20.2	0	15.9	7.2
19.4	23.6		13	12.9	22.5
2.6	9.8	46		20.5	21.1
12.3	1.9	12	36.9		3.3
6.3	10.2	7.6	17.8	13	

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47.1	49.8	91.2	100	66.5	54.9

	$\boxed{0.4}$	5.3	32.3	$\boxed{8.2}$	0.8
6.5		20.2	$\boxed{4}$	$\boxed{11.9}$	7.2
19.4	$\boxed{27.6}$		$\boxed{9}$	12.9	22.5
2.6	9.8	46		20.5	21.1
12.3	1.9	12	36.9		3.3
6.3	10.2	7.6	17.8	13	

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47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	32.3	8.2	0.8
6.5		20.2	4	11.9	7.2
19.4	27.6		9	12.9	22.5
2.6	9.8	46		20.5	21.1
12.3	1.9	12	36.9		3.3
6.3	10.2	7.6	17.8	13	

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47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	27.2	13.3	0.8		47
6.5		25.3	4	6.8	7.2		49.9
19.4	27.6		9	12.9	22.5		91.3
2.6	9.8	46		20.5	21.1		99.9
12.3	1.9	7	42		3.3		66.5
6.3	10.2	7.6	17.8	13			54.9

47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	27.2	13.3	0.8
6.5		25.3	4	6.8	7.2
19.4	$\boxed{27.6}$		9	12.9	$\boxed{22.5}$
2.6	9.8	46		20.5	21.1
12.3	$\boxed{1.9}$	7	42		$\boxed{3.3}$
6.3	10.2	7.6	17.8	13	

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47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	27.2	13.3	0.8
6.5		25.3	4	6.8	7.2
19.4	29.5		9	12.9	20.6
2.6	9.8	46		20.5	21.1
12.3	0	7	42		5.2
6.3	10.2	7.6	17.8	13	

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47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	27.2	13.3	0.8		47
6.5		25.3	4	6.8	7.2		49.9
19.4	29.5		9	12.9	20.6		91.3
2.6	9.8	46		20.5	21.1		99.9
12.3	0	7	42		5.2		66.5
6.3	10.2	7.6	17.8	13			54.9

47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	27.2	13.3	0.8	47
6.5		25.3	4	6.8	7.2	49.9
19.4	28.1		10.4	12.9	20.6	91.3
2.6	9.8	44.6		20.5	22.4	99.9
12.3	1.4	7	42		3.9	66.5
6.3	10.2	9	16.5	13		54.9

47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	27.2	13.3	0.8
6.5		25.3	4	6.8	7.2
19.4	28.1		10.4	12.9	20.6
2.6	9.8	44.6		20.5	22.4
12.3	1.4	7	42		3.9
6.3	10.2	9	16.5	13	

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47.1	49.8	91.2	100	66.5	54.9

	0.4	5.3	27.2	13.3	0.8	47
6.5		25.3	4	6.8	7.2	49.9
19.4	$\boxed{33.1}$		10.4	12.9	$\boxed{15.6}$	91.3
2.6	9.8	44.6		20.5	22.4	99.9
12.3	1.4	$\boxed{2.1}$	42		$\boxed{8.8}$	66.5
6.3	$\boxed{5.3}$	$\boxed{13.9}$	16.5	13		54.9
47.1	49.8	91.2	100	66.5	54.9	

Balance sheets and fundamental defaults

- Balance sheet of bank i:

Assets		Liabilities	
external assets	$a_i^{(e)}$	external liabilities	$l_i^{(e)}$
interbank assets	$a_i := c_i(L)$	interbank liabilities	$l_i := r_i(L)$
		net worth	$w_i := w_i(L, a_i^{(e)}, l_i^{(e)})$

$$w_i := w_i(L, a_i^{(e)}, l_i^{(e)}) := a_i^{(e)} + c_i(L) - l_i^{(e)} - r_i(L).$$

- Stress tests: apply proportional shock $s \in [0, 1]^n$ to external assets; shocked external assets are $s_i a_i^{(e)}\ \forall i$.

- Fundamental defaults: $\mathbb{D}_0(L, a^{(e)}, l^{(e)}, s) := \{i \mid w_i(L, s_i a_i^{(e)}, l_i^{(e)}) < 0\}$

- Fundamental defaults can be checked from balance sheet aggregates without needing to know the whole matrix L!

- To check for contagious defaults we need to know L.

Deriving contagious defaults via clearing [Rogers & V. (2013)], [Eisenberg & Noe(2001)]

- $l^{\mathrm{all}} := l^{(e)} + r(L)$ the vector of total liabilities.
- Relative liabilities matrix $\Pi \in \mathbb{R}^{n \times n}$,

$$\Pi_{ij} := \begin{cases} L_{ij}/l_i^{\mathrm{all}}, & \text{if } l_i^{\mathrm{all}} > 0, \\ 0, & \text{if } l_i^{\mathrm{all}} = 0. \end{cases}$$

- A clearing vector for the financial system with shock realisation s is a vector $c^*(s) \in [0, l^{\mathrm{all}}]$ such that $c^*(s) = \Phi(c^*(s))$,

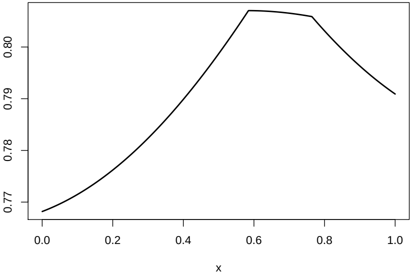
$$\Phi(c(s))_i := \begin{cases} l_i^{\mathrm{all}}, & \text{if } \quad l_i^{\mathrm{all}} \leq s_i a_i^{(e)} + \sum_{j=1}^n c_j(s)\Pi_{ji}, \\ \alpha s_i a_i^{(e)} + \beta \sum_{j=1}^n c_j(s)\Pi_{ji}, & \text{else.} \end{cases}$$

Example: Three banks cont'd

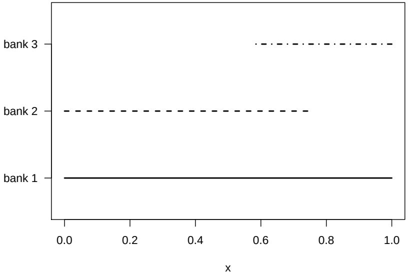
- Assume that $r(L)_i = c(L)_i = 1 \, \forall i = 1, 2, 3$.
- Then with $x \in [0, 1]$ liabilities matrix is:

$$L(x) = \begin{pmatrix} 0 & 1-x & x \\ x & 0 & 1-x \\ 1-x & x & 0 \end{pmatrix}.$$

- External assets and liabilities: $a^{(e)} = \left(\frac{1}{2}, \frac{5}{8}, \frac{3}{4}\right)^\top$, $l^{(e)} = \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)^\top$
- Net worth $w = \left(-1, \frac{2}{8}, \frac{1}{4}\right)^\top$. Only bank 1 in fundamental default.
- What happens to bank 2 and 3 for different network structures, i.e., for different $x \in [0, 1]$?



(a) Sum of clearing payments / sum of total liabilities



(b) Default range

Some comments on example with three banks

- Just from marginals a, l we cannot predict outcomes for bank 2 and bank 3.
- Standard KL-method would give $L(1/2)$. Indeed contagion can be underestimated by the KL method.
- Network with fewest number of edges: either $x = 0$ or $x = 1$, i.e., the circular networks.
  Here: sparsest network consistent with the marginals is not the network with the highest number of defaults.

Empirical example - data

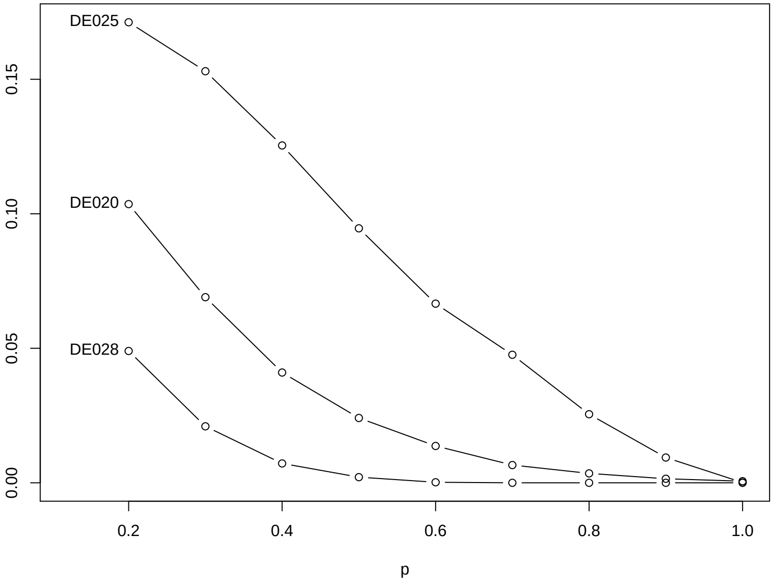
Balance sheet data (in million Euros) from banks in the EBA 2011 stress test:

Bank code	Bank	$a^{(e)} + a$	a	w
DE017	DEUTSCHE BANK AG	1,905,630	47,102	30,361
DE018	COMMERZBANK AG	771,201	49,871	26,728
DE019	LANDESBANK BADEN-WURTTEMBERG	374,413	91,201	9,838
DE020	DZ BANK AG	323,578	100,099	7,299
DE021	BAYERISCHE LANDESBANK	316,354	66,535	11,501
DE022	NORDDEUTSCHE LANDESBANK -GZ-	228,586	54,921	3,974
DE023	HYPO REAL ESTATE HOLDING AG	328,119	7,956	5,539
DE024	WESTLB AG, DUSSELDORF	191,523	24,007	4,218
DE025	HSH NORDBANK AG, HAMBURG	150,930	4,645	4,434
DE027	LANDESBANK BERLIN AG	133,861	27,707	5,162
DE028	DEKABANK DEUTSCHE GIROZENTRALE	130,304	30,937	3,359

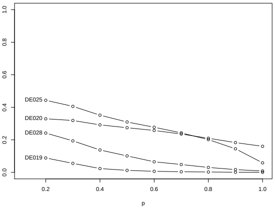
Stress testing

- We apply a deterministic shock to external assets of all 11 banks in the network by considering the shocked external assets $s_i a_i^{(e)}$ with $s_i = 0.97 \, \forall i \in \mathcal{N}$.
- Shock causes fundamental default of 4 banks: DE017, DE022, DE023, DE024.
- Apply new methodology to determine the posteriori default probabilities for remaining 7 banks.

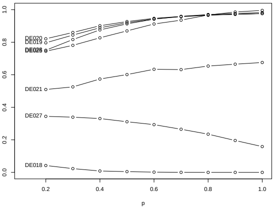
Default probabilities of banks as a function of p.



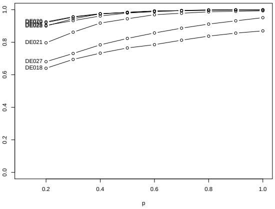
Default probabilities for clearing with default costs



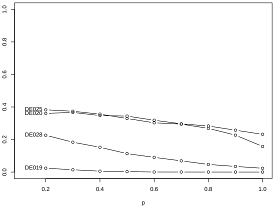
(c) Clearing with $\alpha = 1, \beta = 0.95$



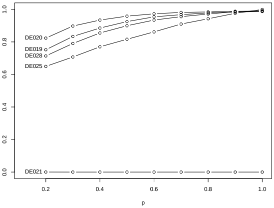
(d) Clearing with $\alpha = 1, \beta = 0.7$



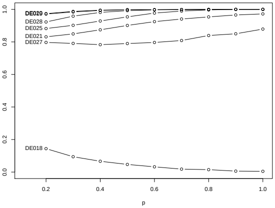
(e) Clearing with $\alpha = 1, \beta = 0.5$



(f) Clearing with $\alpha = 0.99, \beta = 1$

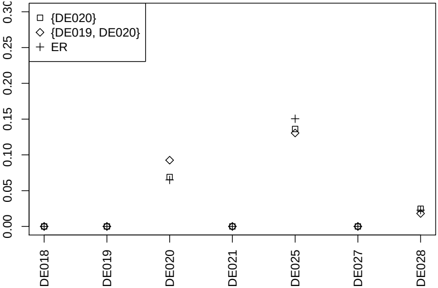


(g) Clearing with $\alpha = 0.95, \beta = 1$

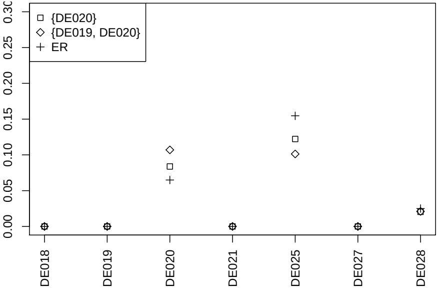


(h) Clearing with $\alpha = 0.9, \beta = 1$

Effect of tiering (interbank assets) on default probabilities

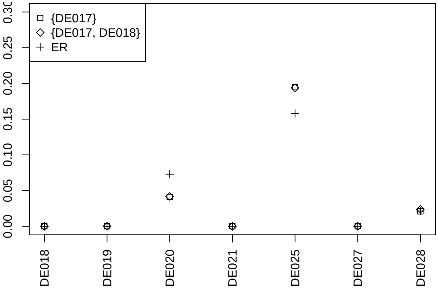


(i) $p^{(ER)} = 0.3,\ p^{(\ell)} = 0.4$

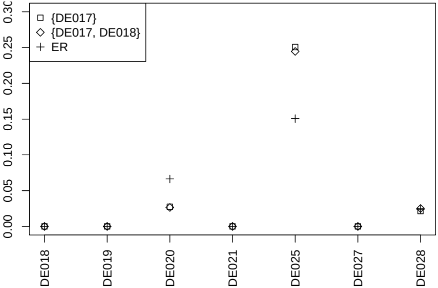


(j) $p^{(ER)} = 0.3,\ p^{(\ell)} = 0.5$

Effect of tiering (total assets) on default probabilities



(k) $p^{(ER)} = 0.3,\ p^{(\ell)} = 0.4$



(l) $p^{(ER)} = 0.3,\ p^{(\ell)} = 0.5$

Mean out-degree of banks, i.e., $\mathbb{E}[\sum_j \mathcal{A}_{ij} \mid a, l]$, for different p^{ER} in the Erdős-Rényi network

	l	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
DE020	99936	3.50	4.40	5.40	6.20	6.90	7.60	8.30	9.00	10.00
DE019	91314	3.30	4.20	5.10	6.00	6.70	7.50	8.20	8.90	10.00
DE021	66494	2.90	3.70	4.70	5.50	6.40	7.20	8.00	8.80	10.00
DE022	54907	2.70	3.50	4.40	5.30	6.10	7.00	7.80	8.80	10.00
DE018	49864	2.60	3.40	4.30	5.10	6.00	6.90	7.80	8.70	10.00
DE017	46989	2.50	3.30	4.20	5.10	5.90	6.80	7.70	8.70	10.00
DE028	30963	2.20	2.80	3.60	4.50	5.40	6.30	7.30	8.40	10.00
DE027	27679	2.10	2.70	3.50	4.30	5.20	6.10	7.10	8.30	10.00
DE024	23971	1.90	2.60	3.30	4.10	5.00	5.90	7.00	8.20	10.00
DE023	8023	1.40	1.80	2.30	2.80	3.50	4.30	5.40	6.90	10.00
DE025	4841	1.20	1.50	1.90	2.40	2.90	3.60	4.60	6.10	10.00

Summary

- Development of Bayesian framework (Gibbs sampler) for sampling from distribution of liabilities matrix conditional on its row and column sums.
- Can be used for stress tests using empirical data.
- Can be used as a simulation tool to analyse heterogeneous networks.
- Can incorporate additional information such as expert views etc. on the network structure.
- R package (systemicrisk) available from CRAN.

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