Contagion! The Spread of Systemic Risk in Financial Networks

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Quotation (Dickens, *David Copperfield*)

Annual income twenty pounds, annual expenditure nineteen nineteen six, result happiness. Annual income twenty pounds, annual expenditure twenty pounds ought and six, result mis- ery. The blossom is blighted, the leaf is withered, the god of day goes down upon the dreary scene, and—and, in short, you are for ever floored.

The Goal of the Book

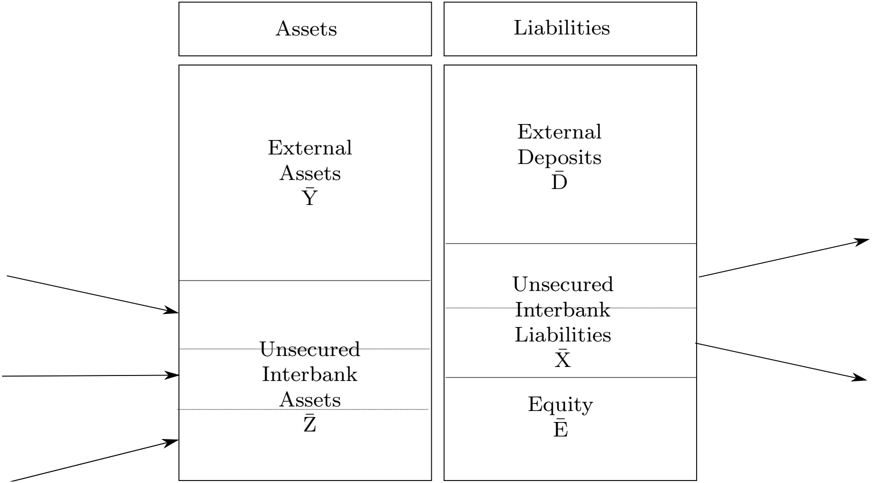
Main Aim

To crystallize a stream of systemic risk research that focuses on the basic modelling structure that ensures some kind of mathematical tractability, while allowing a great deal of both reality and complexity in the actual finance network specification.

Basic Modelling Strategy

1. Propose a definition of Random Financial Network of N banks, that gives a minimal description of the system at any time.
2. Propose a Cascade Mechanism.
3. Hit the system with a random shock at time 0 and determine the evolution of the resultant cascade.
4. Compare to exact $N \to \infty$ analytics that arise by "extending the logic of percolation theory".

Eisenberg-Noe 2001 Model: Balance Sheets



EN2001 Assumptions

1. External debt $\bar{\mathrm{D}}$ is senior to interbank debt $\bar{\mathrm{X}}$;
2. All interbank debt is of equal seniority;
3. There are no losses due to bankruptcy charges.

Also:

1. Exposure $\bar{\Omega}_{vw}$: what v owes w.
2. Equity and Default buffers: $E_v = \max(0, \bar{\Delta}_v)$.
3. Limited Liability: A bank is defaulted at cascade step n if and only if $\Delta_v^{(n)} \leq 0$.

EN2001 Default Buffer Mapping

1. If $\Delta_w^{(n)}$ denotes the default buffer after n cascade steps, then

$$\Delta_w^{(n)} = \Delta_w^{(0)} - \sum_v \bar{\Omega}_{vw}\left(1 - h(\Delta_v^{(n-1)}/\bar{\mathrm{X}}_v)\right)$$

2. Threshold function

$$h(x) \quad = \max(x+1, 0) - \max(x, 0)$$

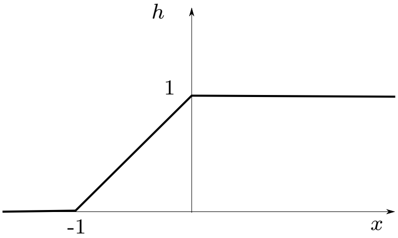
   determines fractional recovered value of defaulted interbank assets.

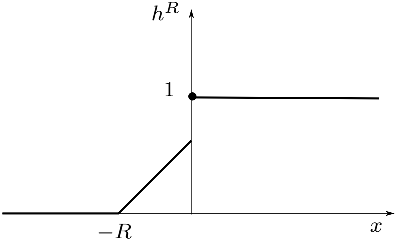
3. As $n \to \infty$, buffers $\Delta_w^{(n)}$ converge to unique fixed point $\Delta^+ = \{\Delta_v^+\}$ of solvency cascade mapping.

Gai-Kapadia 2010 Default Model

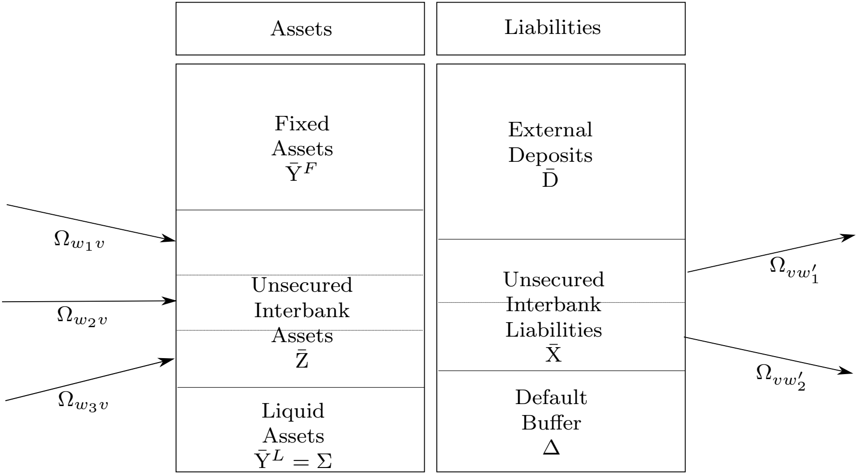
1. GK assumes zero recovery at default and is formally identical to EN2001, but with threshold function h replaced by $\tilde{h}(x) = \mathbf{1}_{x>0}$.
2. Partial recovery at default can be modelled by

$$h^R(x) = Rh(x/R) + (1-R)\tilde{h}(x) \ .$$





Illiquidity Cascades: Balance Sheets



Illiquidity Cascade: Gai-Kapadia 2010b

1. At time 0, some banks experience deposit withdrawals that deplete their liquidity buffer $\Sigma_v := Y_v^L$ (allowing it to go negative).
2. Bank v with $\Sigma_v \leq 0$ reacts by hoarding liquidity; its debtor banks $w \in \mathcal{N}_v^+$ each receive a liquidity shock.
3. Under 100% hoarding, cascade mapping at step n is

$$\Sigma_v^{(n)} = \Sigma_v^{(0)} - \sum_{w \in \mathcal{N}_v^+} \bar{\Omega}_{vw}(1 - \tilde{h}(\Sigma_w^{(n-1)}/\bar{Z}_w))$$

4. Formally identical to GK 2010 Default Cascade under interchange of assets and liabilities.

Illiquidity Cascade 2: Seung Hwan Lee 2013 Model

1. At time 0, banks experience deposit withdrawals $\Delta d_v \geq 0$.
2. These are paid immediately first by liquid assets $\bar{\mathrm{Z}} + \bar{\mathrm{Y}}^L$, then fixed assets $\bar{\mathrm{Y}}^F$.
3. Debtor banks receive liquidity shocks;
4. Each bank v has initial liquidity buffer $\Sigma_v^{(0)} = -\Delta d_v \leq 0$
5. After $n-1$ cascade steps, then

$$\Sigma_w^{(n)} = \Sigma_w^{(0)} - \sum_v \bar{\Omega}_{wv} \left(1 - h(\Sigma_v^{(n-1)}/\bar{\mathrm{Z}}_v)\right)$$

6. Mathematically identical to a restricted version of EN 2001!

Asset Fire Sale Cascades

(c.f. Cifuentes et al 2005 and Caccioli et al 2012)

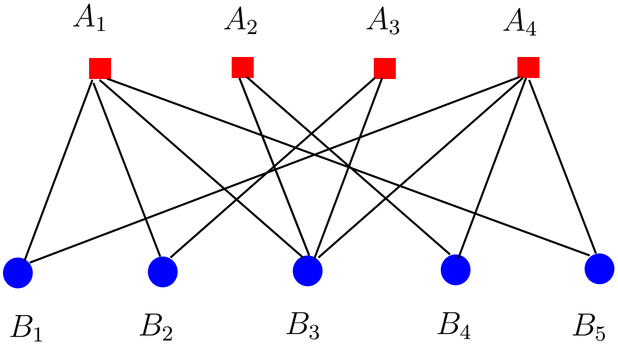


Figure: A bipartite graph with 5 banks (blue nodes) co-owning 4 assets (red nodes).

Asset Fire Sales

Banks $v \in \mathcal{N} = \{1, 2, \ldots, N\}$, Assets $a \in \mathcal{M} = \{1, 2, \ldots, M\}$. Let $\bar{s}_{av}$ be amount of asset a held by bank v.

On the nth cascade step:

1. When default buffer $\Delta_v^{(n)}$ hits a threshold, v begins to liquidate assets.
2. Amount $s_{av}^{(n)}$ of asset a held by bank v after n cascade steps is determined by $\Delta_v^{(n)}$.
3. The new mark-to-market price is determined by the total amount sold through an inverse demand function

$$p_a^{(n+1)} = d_a^{-1}\Big(\sum_v (\bar{s}_{av} - s_{av}^{(n)})\Big)$$

4. Banks mark-to-market to compute their new buffers $\Delta_v^{(n+1)}$.

Asset Fire Sale Cascades

1. Complex cascades result even with no interbank sector $\bar{\Omega} = 0$.
2. Each blue node v is governed by a buffer variable $\Delta_v^{(n)}$
3. Each red node a is governed its price $p_a^{(n)}$, which can be considered as a buffer variable.
4. One buffer per node!
5. Global cascades can start either in banks or in assets: once it starts it doesn't matter much where it started.

Single Buffer Models

1. In all these models, each node's behaviour, and hence the cascade itself, is determined by a single buffer Δ_v, Σ_v or p_a.
2. Single buffer models can easily account for multiple thresholds of behaviour.

A Double Buffer Model

1. In more complex models, banks' behaviour is determined by two or more buffers.
2. HCCMS 2013 introduces a double cascade model of illiquidity and insolvency, intertwining two buffers Δ_v, Σ_v, that combines the essence of both [GK, 2010a] default cascade and [GK, 2010b] liquidity cascade.

Question

What effect does a bank's behavioural response to liquidity stress have on the probable level of eventual defaults in entire system?

Essential Modelling Assumptions

To have analytical results, not just Monte Carlo simulation experiments, should assume locally tree-like independence properties (LTI):

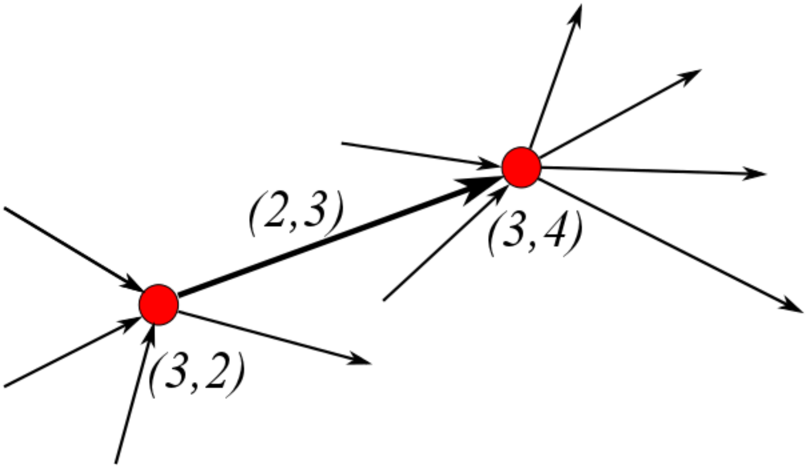
1. Network of bank counterparties is large, sparse and not too heterogeneous.
2. Interbank exposures and bank characteristics are families of random variables with hierarchical dependence.
3. Cascade mechanisms that model bank behaviour are "compatible" with network structure.

Definition: Random Financial Networks

Three layers of mathematical structure:

1. Skeleton: random (directed) graph $(\mathcal{N}, \mathcal{E})$ with banks $v \in \mathcal{N}$ and edges or links $\ell = (vw) \in \mathcal{E}$ represent a non-negligible counterparty relation (or directed interbank exposure).
2. Conditioned on skeleton $(\mathcal{N}, \mathcal{E})$: Random buffers Δ_v for each bank.
3. Finally, conditioned on skeleton and balance sheets: Random exposures Ω_ℓ for each link $\ell = (w, v) \in \mathcal{E}$.

Directed Graphs: 2 Nodes and 1 Edge



Directed Configuration Model

1. Skeleton $(\mathcal{N}, \mathcal{E})$ is a directed configuration (LT) random graph with specified node and edge degree distribution matrices $\{P_{jk}, Q_{kj}\}$.
2. $P_{jk} = \mathbb{P}[v \in \mathcal{N}_{jk}]$ where $\mathcal{N}_{jk}$ is the set of nodes with $\deg^-(v) = j, \deg^+(v) = k$.
3. $Q_{kj} = \mathbb{P}[\ell \in \mathcal{E}_{kj}]$.
4. Marginals: $P_k^+ = \sum_j P_{jk}, P_j^- = \sum_k P_{jk}$ and $Q_k^+ = \sum_j Q_{kj}, Q_j^- = \sum_k Q_{kj}$.
5. P and Q must be consistent:

$$\begin{aligned} z &:= \sum_{jk} j P_{jk} = \sum_{jk} k P_{jk} \\ Q_k^+ &= k P_k^+ / z, \quad Q_j^- = j P_j^- / z \ . \end{aligned}$$

Note: when bivariate distribution $Q \neq Q^+ Q^-$ the network is called assortative.

Assortative Configuration Construction

Theorem

In the assortative configuration graph construction with probabilities P, Q, the following convergence properties hold as $N \to \infty$.

1. *The fraction of type (k, j) edges in the matching sequence $(k_l, j_\ell)_{\ell\in[L]}$ concentrates with high probability around the nominal edge distribution Q_{kj}.*
2. *For any fixed finite number S, the first S edges $\ell, \ell \in [S]$ have degree sequence $(k_l, j_\ell)_{\ell\in[S]}$ that converge in distribution to $(\hat{k}_l, \hat{j}_\ell)_{\ell\in[S]}$, an independent sequence of identical Q distributed random variables.*

Other Potential Random Graph Constructions

1. Preferential attachment models;
2. Preferential attachment and detachment models;
3. Inhomogeneous Random Graphs: each bank $v \in \mathcal{N} = \{1, \dots, N\}$ is independently assigned a random type $t_v \sim F$ with values in type space $\mathcal{T}$; Conditioned on $\{t_v\}$, edges $(w, v) \in \mathcal{E}$ are drawn independently with probability

$$\frac{K(u(t_v), u(t_w))}{N - 1 + K(u(t_v), u(t_w))}$$

where $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is symmetric and $u : \mathcal{T} \to \mathbb{R}_+$.

Percolation Logic

1. A large sparse random graph typically has few short closed cycles: call it locally tree-like.
2. Implies all sums over counterparties are almost (conditionally) independent: local tree-like independence (LTI).
3. Vulnerable edge: a directed edge $(w, v) \in \mathcal{E}$ such that $\Omega_{wv} \geq \Delta_v$.
4. Percolation Logic: large scale cascades starting from a single seed default are only possible if the directed subgraph $\mathcal{E}_V$ of vulnerable edges has a large strongly connected component Giant-SCC.
5. Because of the LTI property, size $|\text{GSCC}|$ can be analyzed exactly in the limit $N \to \infty$.

Zoology of Components of Directed Graphs

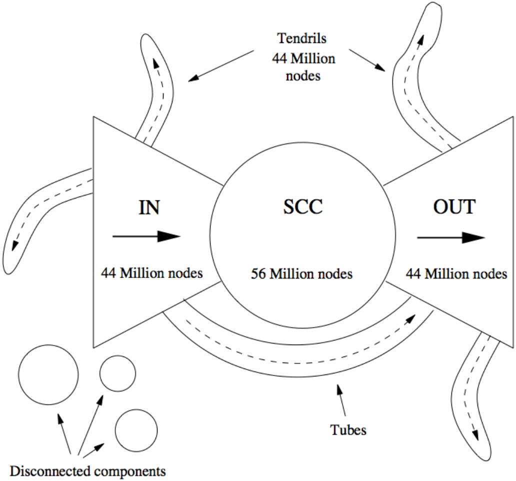


Figure: The connected components of the World Wide Web in 1999. (Source: Broder et al 2000.)

Percolation Theory on Random Graphs $\mathcal{G}$

Question

What is the size of the largest connected cluster $\mathcal{C}$ in $\mathcal{G} = (\mathcal{N}, \mathcal{E})$?

Theorem (Molloy-Reed 2000)

Let $\mathcal{G}^{(N)}$, $N := |\mathcal{N}| \to \infty$ be a "well-behaved" random graph sequence with the locally tree-like (LT) property. Then

$$\mathbb{E}[|\mathcal{C}|] \overset{N\to\infty}{=} N(1 - g(\xi^*))[1 + o(1)]$$

Here $\xi^* = \frac{g'(\xi^*)}{g'(1)}$ and $g(x) := \sum_k \mathbb{P}[k_v = k]x^k$ is the generating function of the asymptotic degree distribution.

Configuration Graphs and Inhomogeneous Random Graphs have the LT property.

Percolation Condition

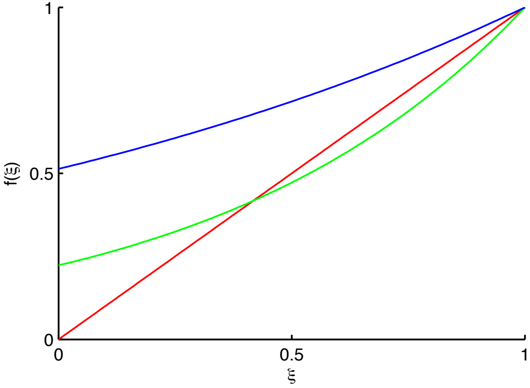


Figure: Supercritical case is found for the green curve, which has a non-trivial fixed point $\xi^* < 1$. The blue curve has only the trivial fixed point at $\xi^* = 1$, and corresponds to a sub-critical random graph.

Cascade Dynamics = Bootstrap Percolation?

1. Bootstrap Percolation is a dynamic version of percolation introduced in 1979 by Chalupa, Leath and Reich for magnetic systems on regular lattices.
2. It follows the growth of connected clusters of nodes $v \in \mathcal{N}$ that become "activated" when the number of its active neighbours exceeds a threshold.
3. Exact analytic asymptotics are sometimes possible on LTI networks.
4. Watts' 2002 Information Cascade Model is a basic example of Bootstrap Percolation.

Extended Gai-Kapadia RFN

1. Skeleton graph is a directed configuration random graph with $\{P_{jk}, Q_{kj}\}$.
2. Conditionally on the skeleton, buffers $\bar{\Delta}_v$ are a collection of independent non-negative random variables with

$$\mathbb{P}[\bar{\Delta}_v \leq x | v \in \mathcal{N}_{jk}] = D_{jk}(x), \ x \geq 0 \ . \tag{1}$$

3. Conditionally on the skeleton, exposures $\bar{\Omega}_\ell$ form a collection of independent positive random variables, independent as well from the default buffers $\bar{\Delta}_v$ with

$$\begin{aligned} W_{kj}(x) &= \mathbb{P}[\bar{\Omega}_\ell \leq x | \ell \in \mathcal{E}_{kj}], \\ w_{kj}(x) &= W'_{kj}(x) \ , \end{aligned} \tag{2}$$

Extended Gai-Kapadia Model

1. GK 2010 zero recovery default mechanism.
2. Let $\mathcal{D}^n \subset \mathcal{N}$ be the set of defaulted banks after n cascade steps.
3. Initial default probabilities

$$p^{(0)}_{jk} = \mathbb{P}[v \in \mathcal{D}^0 | v \in \mathcal{N}_{jk}] = D_{jk}(0) \ .$$

4. After n cascade steps: $p^{(n)}_{jk} = \mathbb{P}[v \in \mathcal{D}^n | v \in \mathcal{N}_{jk}]$.
5. $$\{v \in \mathcal{D}^n\} = \{\Delta_v \le \sum_{w \in \mathcal{N}^-_v} \Omega_{wv} \mathbf{1}(w \in \mathcal{D}^{n-1})\}$$

Zero Recovery Cascade Mapping Theorem

Theorem

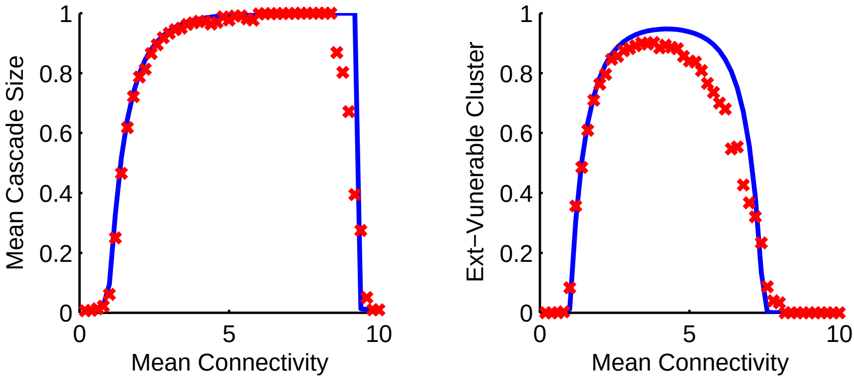
Consider the LTI sequence of GK financial networks $(N, P, Q, \bar{\Delta}, \bar{\Omega})$. Then the following formulas hold with high probability as $N \to \infty$:

1. *The quantities $p_{jk}^{(n)}, \pi_k^{(n)}$ satisfy the recursive formulas*

$$\begin{aligned}
p_{jk}^{(n)} &= \langle D_{jk}, (\widetilde{w}_j^{(n-1)})^{*j} \rangle \ , \\
\pi_k^{(n)} &= \sum_{j'} p_{j'k}^{(n)} P_{j'|k} \ , \\
\tilde{w}_j^{(n-1)}(x) &= \sum_{k'} Q_{k'|j} \left((1 - \pi_{k'}^{(n-1)}) \delta_0(x) + \pi_{k'}^{(n-1)} w_{k'j}(x) \right) \ .
\end{aligned}$$

2. *$\vec{\pi}^{(n)}$ are a vector valued function $G(\vec{\pi}^{(n-1)})$, explicit in terms of RFN specification $(N, P, Q, \bar{\Delta}, \bar{\Omega})$.*

Mean Cascade Size and Cascade Frequency



(a) (b)

Future Directions

1. Develop LTI algorithms for a variety of full and partial recovery cascade mechanisms, such as EN 2001, liquidity hoarding, fire sale models and their extensions.
2. Investigate limits of the LTI approximation.
3. Explore RFN cascade models with IRG skeletons with community and multiplex structure.
4. Calibrate joint distributions of balance sheets and exposures from systemic risk databases.
5. Justify bank crisis behaviour assumptions using the theory of global games (see Morris and Shin).

Thanks

. . . to many friends, colleagues and collaborators for sharing ideas and insight into an exciting field. Enjoy the conference!