Limit Order Books

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Standard Assumptions in Finance

Black-Scholes theory

- Price given by a **single** number
- **infinite** liquidity
  - one **can** buy or sell **any** quantity **at** this price
  - with **NO IMPACT** on the asset price
- Fixes to account for liquidity frictions
  - Transaction Costs (**Constantinides, Davis, Paras, Zariphopoulou, Shreve, Soner**, ...)
  - liquidity $\sim$ transaction cost (**Cetin-Jarrow-Protter**)

Not satisfactory for

- Large trades (over short periods)
- High Frequency Trading

Need **Market Microstructure**

- e.g. understand how are buy and sell orders executed?

New Markets

- **Quote Driven Markets**
  - **Market Maker** or **Dealer** centralizes **buy** and **sell** orders and **provides liquidity** by setting bid and ask quotes.
    Ex: **NYSE** specialist system
- **Order Driven Markets**
  - electronic platforms aggregate all available orders in a

    Limit Order Book (LOB)

    Ex: NASDAQ, NYSE, ICE, BATS, CHX, LSE, ...
- Same stock traded on **several venues**
- **Price discovery** made difficult as most instruments can be traded off market without printing the trade to a publicly accessible data source
  - **Competition** between markets leads to **lower fees** and **smaller tick sizes**
- Creation of **Dark Pools**
- Increase in updating **frequency** of **order books**

High Frequency Trading

Speculative figures – **Sound plausible**

- HFT accounts for 60 – 75% of all share volume.
- 10% of that is **predatory** $\approx$ 600 million shares per day
- At \$0.01-\$0.02 per share, predatory HFT is profiting \$6-\$12 million a day or \$1.5-\$3 billion e year

Algorithmic Trading – **Source of concern**

- Moving computing facilities **closer** to trading platform (**latency**)
- Relying on / competing with **Benchmark Tracking** execution algorithms

Pros & Cons

Pros

- Smaller tick size;
- HF traders provide extra liquidity
- Dark pools reduce trade execution costs from price impact
- Markets are more efficient

Cons

- Expensive technological **arms race**
- **Dark** trading incentivizes **price manipulation**, **fishing** and **predatory trading**
- Little or no oversight possible by humans (e.g. **flash crash**) & increased **systemic risk**
- HF trading algorithms do not use **economic fundaments** (e.g. value & profitability of a firm)

Some Highly Publicized Mishaps

Flash Crash of May 6, 2010

- Dow Jones IA plunged about 1000 points (recovered in minutes)
- Biggest one-day point decline (998.5 points)
  - At 2:32 pm a mutual fund program started to sell 75,000 E-Mini S&P 500 contracts ($\approx$ 4.1 billion USD) at an execution rate of 9%
  - HT trading programs were among the buyers: **quickly bought and resold** contracts to each other
  - **hot-potato** volume effect, combined sales drove the E-mini price down 3% in just 4 minutes

Other Notable Crashes

- Associated Press' Twitter account **hack**
  - **White House** bombed
  - **President Obama** injured
  - **DJIA lost 140 points** and recovered in minutes
- Several **mini flash crashes** on NASDAQ in 2012

Limit Order Book (LOB)

List of all the waiting **buy** and **sell** orders

- Prices are multiple of the tick size
- For a given price, orders are arranged in a **First-In-First-Out** (FIFO) stack
- At each time t
  - The **bid** price B_t is the price of the highest **waiting buy order**
  - The **ask** price A_t is the price of the lowest **waiting sell order**
- The state of the order book is modified by **order book events**:
  - **limit orders**
  - **market orders**
  - **cancelations**
- **consolidated order book**: If the stock is traded in several venues, one aggregates over all (**visible**) trading venues.
- Here, **little** or **no** discussion of **pools**

The Role of a LOB

- Crucial in high frequency finance: explains **transaction costs**.
- ***Liquidity providers*** post trading intentions: Bids and Offers.
- ***Liquidity takers*** execute certain orders: adverse selection.

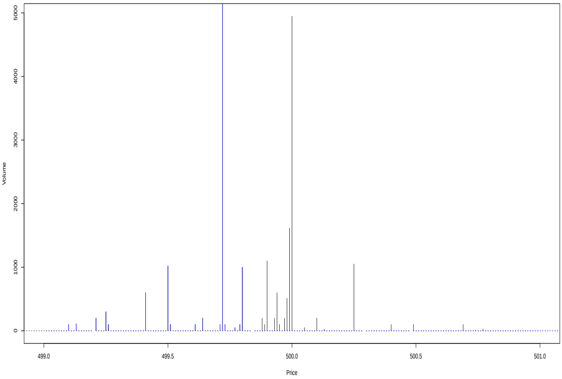
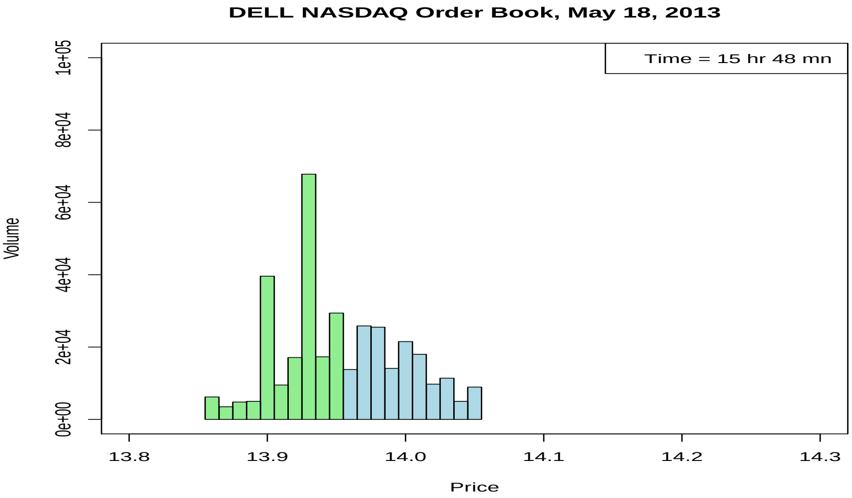


Figure : Snapshot of Apple order book at 8:43 (NASDAQ)

DELL Limit Order Book on May 18, 2013



Limit Orders

A **limit order** sits in the order book until it is

- either executed against a **matching** market order
- or it is **canceled**

A **limit order**

- may be executed very quickly if it corresponds to a price near the bid and the ask
- may take a long time if
  - the market price moves away from the requested price
  - the requested price is too far from the bid/ask.
- can be canceled at any time

Typically, a **limit order** waits for a **match**

- transaction **cost** is **known**
- execution **time** is **uncertain**

Market Orders

A **market order** is an order to buy/sell a certain quantity of the asset at the **best available price** in the book.

- Agents can put a **market order** that, for a buy (resp. sell) order,
  - the first share(s) will be traded at the ask (resp. bid) price
  - the remaining one(s) will be traded some ticks upper (resp. lower)

  in order to fill the order size.
- The ask (resp. bid) price is then modified accordingly.
- When either the bid or ask queue is **depleted** by
  - market orders
  - cancelations

  the price is **updated** up or down to the next level of the order book.

Typically a **market order** consumes the cheapest limit orders

- **immediate execution** (if the book is filled enough)
- **price** per share instead **uncertain** (depends upon the order size)

Cancellations

- Agents can put a **cancellation** of x orders in a given queue reduces the queue size by x
- When either the bid or ask **queue is depleted** by market orders and cancelations, the **price moves** up or down to the next level of the order book.

LOB Dynamics Summary

- Actual trades come in two forms
- Agents can put a **limit order** and wait that this order matches another one
  - transaction cost is known
  - execution time is uncertain
- Agents can put a **market order** that consumes the cheapest limit orders in the book
  - immediate execution (if the book is filled enough)
  - price per share instead depends on the order size

  For a buy (resp. sell) order, the first share will be traded at the ask (resp. bid) price while the last one will be traded some ticks upper (resp. lower) in order to fill the order size. The ask (resp. bid) price is then modified accordingly.
- Agents can put a **cancellation** of x orders in a given queue reduces the queue size by x
- When either the bid or ask queue is depleted by market orders and cancelations, the price moves up or down to the next level of the order book.

Impact of Large Market Order Fills

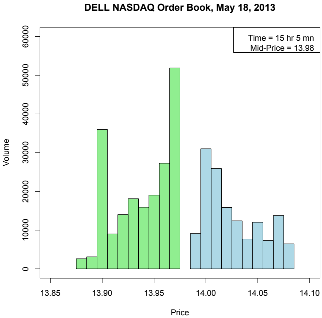
- Current **mid-price** $p_{mid} = (p_{Bid} + p_{Ask})/2 = 13.98$
- Fill size $N = 76015$ (e.g. buy)
- n_1 shares available at best bid p_1, n_2 shares at price $p_2 > p_1$, $\dots$
- n_k shares at price $p_k > p_{k-1}$ $N = n_1 + n_2 + n_3 + \cdots + n_k$
- **colorblueTransaction cost**

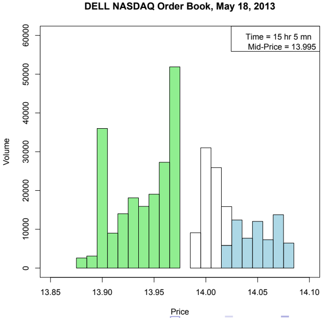
  $$n_1 p_1 + n_2 p_2 + \cdots + n_k p_k = 1064578$$

- **Effective price**

  $$p_{eff} = \frac{1}{N}(n_1 p_1 + n_2 p_2 + \cdots + n_k p_k) = 14.00484$$

- New **mid-price** $p_{mid} = 13.995$





A LOB Idiosyncrasy: Hidden Liquidity

- Some exchanges (e.g. NASDAQ & NYSE) allow **Hidden Orders**
- Made **visible** to the broader market **after being executed**
- **Controversial**
  - barrier to the implementation of a fully transparent market
  - impediment to price discovery and information dissemination

Results of First Empirical Analyzes

- Encourage **fishing**
- After it is **revealed** that a hidden order was executed
  - rash increase of order placement inside the bid-ask after
- HF Traders **divided** in two groups
  - Traders try to take advantage of the remaining hidden liquidity
  - Traders try to steal execution priority from the fully hidden orders

”Partially Hidden” Orders: Iceberg Orders

- **Dark liquidity** posted inside the LOB
- Two components: the **shown quantity** and the **hidden remainder**
- Order queued with the **lit** part of the LOB, only the shown quantity is visible
- When the order reaches the front of the queue, **only the display quantity is filled**
- Trade (price & quantity filled) **revealed**
- **hidden part** put at the back of the queue
- Sometimes **extra execution fee** charged by the exchange

Dark Pools / Crossing Networks

- **Electronic engine** that matches buy and sell orders without routing them to lit exchanges
- **Raison d'être:** move large amounts without impacting the price (no need for iceberg orders)
- **Run by private brokerages**
  - Ex: Liquidnet, Pipeline, ITG's Posit, Goldman's SIGMA X.
  - Participants submit (wish) lists of orders to a matching engine
  - Matched orders are executed at the midpoint of the bid-ask spread.
  - **Pros:** trade at mid-point can be better than on a lit market
  - **Cons:** May have to wait a long time for a match to occur
- Regulated by SEC (in the US) as **Alternative Trading Systems**
  - Little or no public disclosure
  - Not much has been done to increase *transparency*
- Trading on dark pools $\approx$ 32% of trades in 2012 (!)

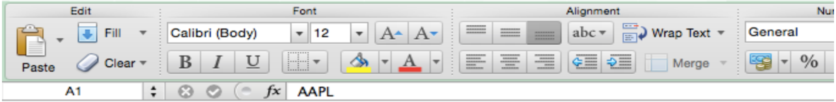
Limit Order Book Data: NASDAQ ITCH

- Large **binary** file
- **Message ID**: unique identifier
- **Market events**: emergency halts, resumptions, ...
- **Time stamp**: number of seconds (in nanoseconds) since midnight
- **Security symbol**
- **Market**: NYSE, MYSE Amex, NYSE Arca, NASDAQ Capital Market, NASDAQ Global Market, BATS, ...
- **Indicators**: order imbalance, near price, price variation, ...
- **MPID** (Market Participant ID) rarely given
- **Order Code**: see next

ITCH Message Codes for Orders

- **A** - Add order
- **C** - Execution with price improvement
- **D** - Full order deletion
- **E** - Execution at listed price
- **F** - Add order with Market participant ID
- **P** - Execution of a non-displayed order (non-cross)
- **Q** - Cross trade (bid/ask overlap or at market open/close)
- **UD** - Delete order as part of a replace order (U).
- **UA** - Add order as part of a replace order (U). Follows UD.
- **X** - Partial order cancellation (remove specified # of shares)

Example of a Processed File



	A	B	C	D	E	F	G	H	I	J
136	AAPL	34200.6297	A	9876615	100	391	B	404.84	405	
137	AAPL	34200.6309	A	9876860	200	403.5	B	404.84	405	
138	AAPL	34200.634	A	9877547	100	403.5	B	404.84	405	
139	AAPL	34200.6859	Q	0	156060	405		404.84	405	
140	AAPL	34200.6859	A	1117772	1	404.9	B	404.84	405	
141	AAPL	34200.6859	A	3326930	25	405	S	404.9	405	
142	AAPL	34200.6859	F	2021313	20	404.9	B	404.9	405	UBSS
143	AAPL	34200.6859	A	3328043	35	405	S	404.9	405	
144	AAPL	34200.6859	A	1655778	20	404.84	B	404.9	405	
145	AAPL	34200.6859	A	3328806	15	405	S	404.9	405	
146	AAPL	34200.6859	F	7859038	100	404.81	B	404.9	405	ATDF
147	AAPL	34200.6859	A	3329014	26	405	S	404.9	405	
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149	AAPL	34200.6859	A	3331024	5	405	S	404.9	405	
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153	AAPL	34200.6859	A	4983600	150	405	S	404.9	405	
154	AAPL	34200.6859	A	4050945	5	404.75	B	404.9	405	
155	AAPL	34200.6859	A	5436522	10	405	S	404.9	405	
156	AAPL	34200.6859	A	3330195	17	404.69	B	404.9	405	
157	AAPL	34200.6859	F	6548347	50	405	S	404.9	405	ATDF
158	AAPL	34200.6859	A	8536495	1	405	S	404.9	405	
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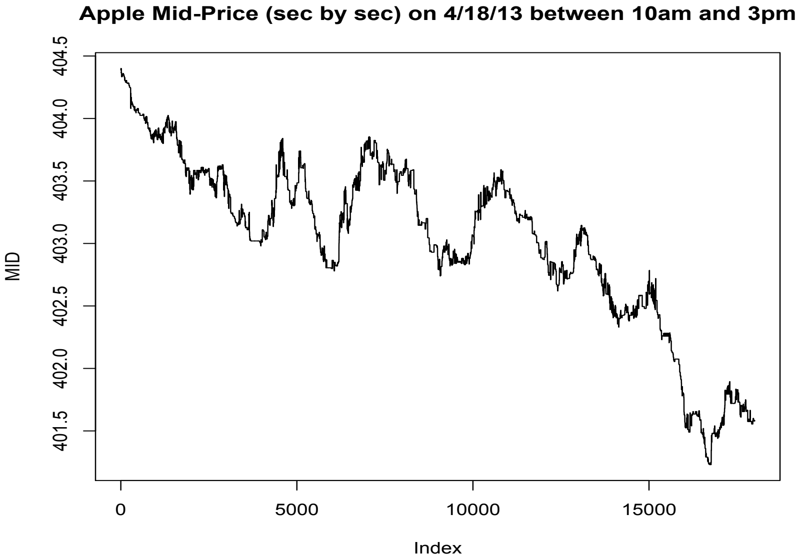
160	AAPL	34200.6859	A	1629298	10	405.3	S	404.9	405	
161	AAPL	34200.6859	F	4620020	30	405.36	S	404.9	405	UBSS
162	AAPL	34200.6859	A	7397081	167	405.38	S	404.9	405	
163	AAPL	34200.6859	F	1902664	15	405.5	S	404.9	405	ATDF
164	AAPL	34200.6939	E	1117772	1	404.9	B	404.9	405	
165	AAPL	34200.6939	E	2021313	20	404.9	B	404.9	405	


Apple Mid-Price (sec by sec) on 4/18/13 between 10am and 3pm
MID
404.5
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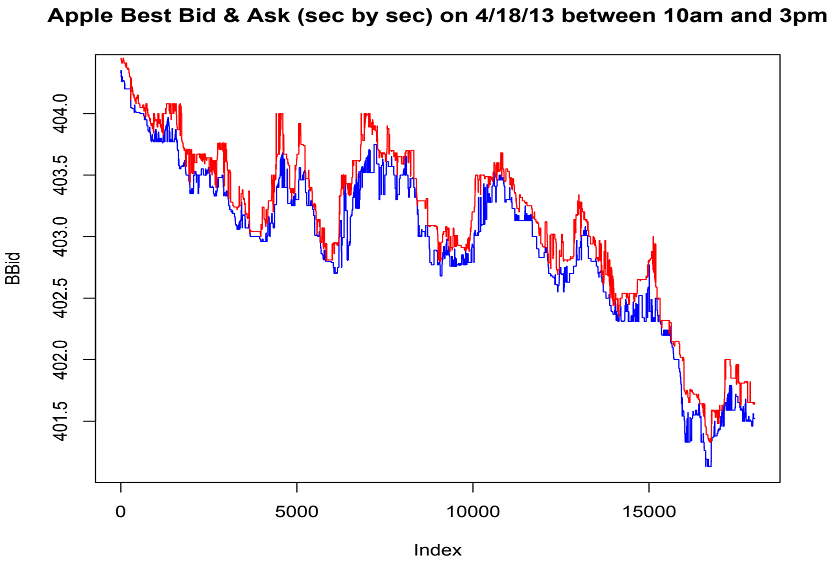
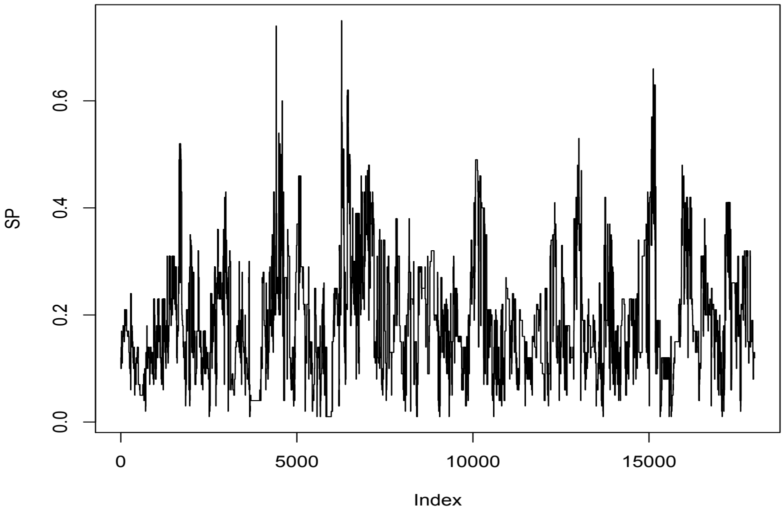
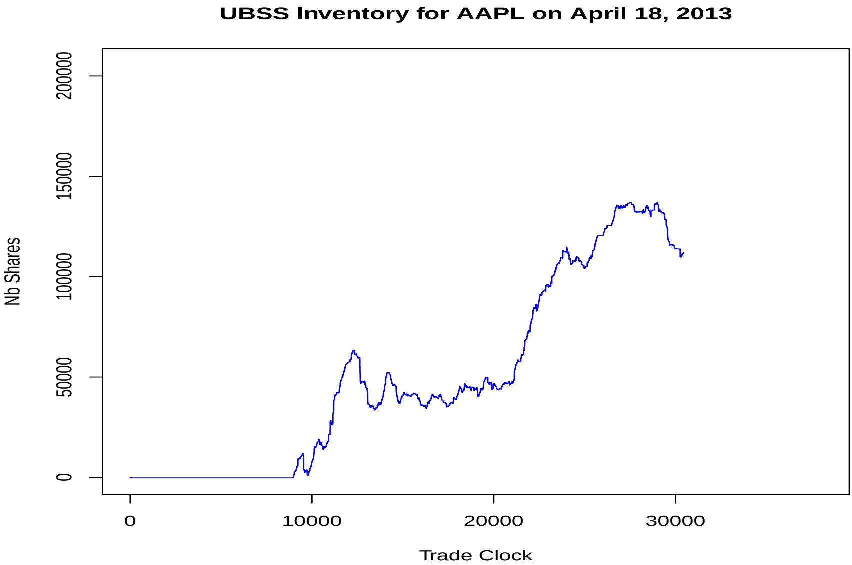

Apple Best Bid & Ask (sec by sec) on 4/18/13 between 10am and 3pm
BBid
404.0
403.5
403.0
402.5
402.0
401.5
0
5000
10000
15000
Index

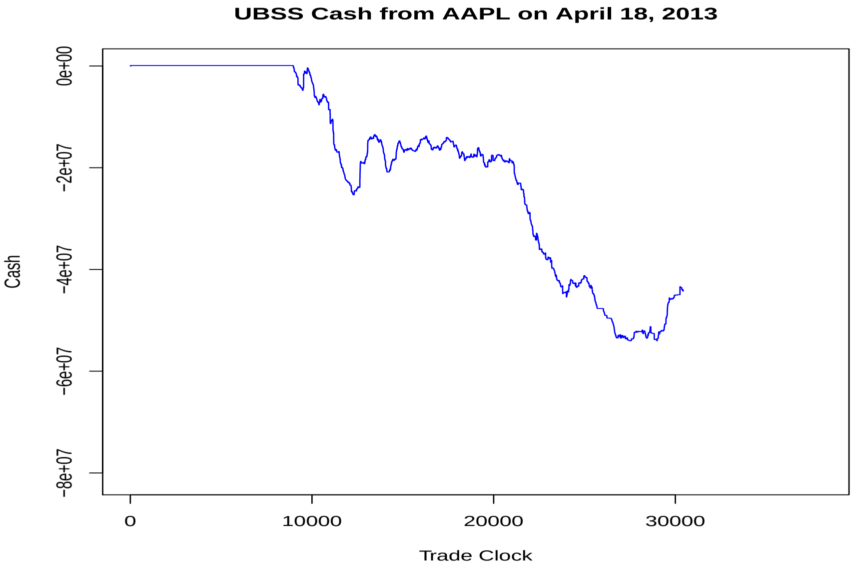
Apple SPREAD (sec by sec) on 4/18/13 between 10am and 3pm



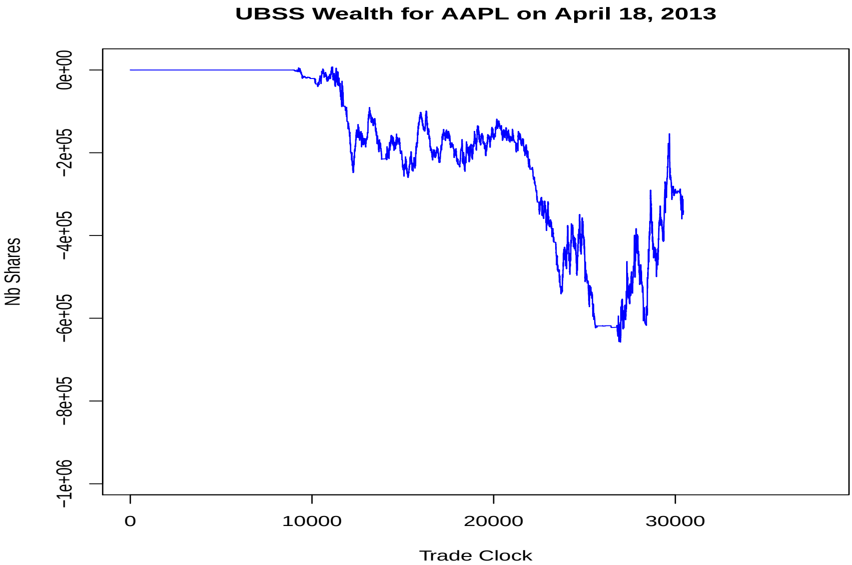
UBSS Inventory Accumulated in the Sun Shine on April 18, 2013



UBSS Cash Accumulated in the Sun Shine on April 18, 2013.



UBSS Wealth Accumulated in the Sun Shine on April 18, 2013



Order Book Models

Roughly speaking, LOB is a set of two histograms (**Bids** and **Asks**)
Reduced form model: Markov process $(O_t)_t$ on a large state space of order books $\mathcal{O}$.

- Early models
  - e.g. **Smith-Farmer-Guillemot-Krishnamurthy** (SFGK) Model
  - **Market orders** (**buys** and **sells**) arrive according to a **Poisson process** with rate $\mu/2$
  - **Cancellation** of existing limit orders: outstanding limit orders **"die"** at a rate ν
- More modern versions
  - use **Hawkes processes** instead of Poisson processes

Another Model Capturing Stylized Facts

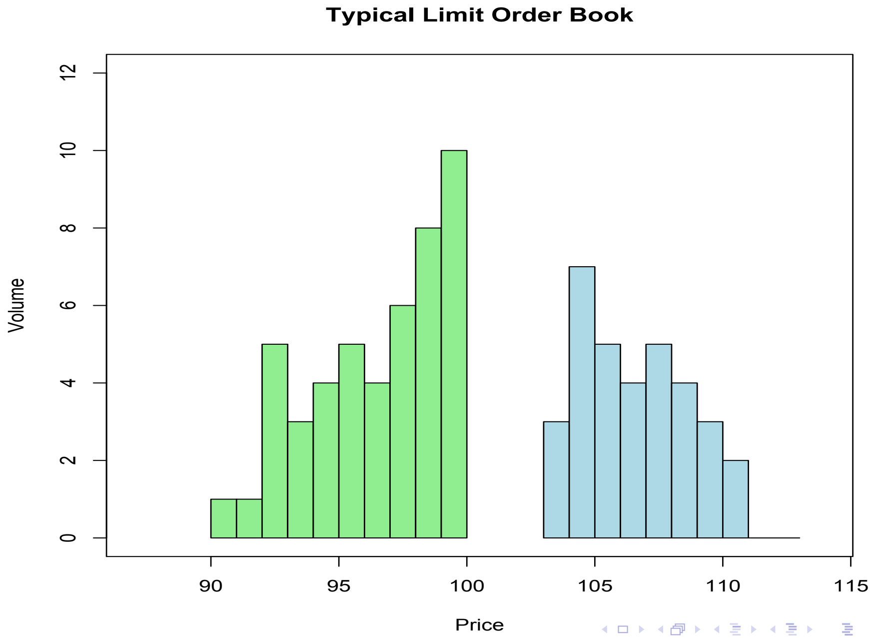
Cont-Stoikov-Talreja

- $\mathcal{P} = \{1, 2, \cdots, n\}$ **price grid** in multiples of price tick
- LOB at time t $O(t) = (O_1(t), O_2(t), \cdots, O_n(t))$
  - $|O_p(t)|$ is the number of outstanding limit orders at price p
  - There are $-O_p(t)$ bid orders at price p if $O_p(t) < 0$
  - There are $O_p(t)$ ask orders at price p if $O_p(t) > 0$
- Admissible **state space**

$$\mathcal{O} = \Big\{ O \in \mathbb{Z}^n;\ \exists 1 \le k \le \ell \le n,\ O_p < 0 \text{ for } p \le k,$$
$$O_p = 0 \text{ for } k < p < \ell,\ O_p > 0 \text{ for } \ell \le p \Big\}$$

- **Ask price** at time t:
  $P_A(t) := (n+1) \wedge \inf\{p;\ 1 \le p \le n,\ O_p(t) > 0\}$
- **Bid price** at time t:
  $P_B(t) := 0 \vee \sup\{p;\ 1 \le p \le n,\ O_p(t) < 0\}$
- **Mid-price** $\tilde{P}(t) = \frac{1}{2}[P_A(t) + P_B(t)]$
- **Bid-Ask spread** $\tilde{S}(t) = P_A(t) - P_B(t)$

A Typical State of the LOB



LOB Dynamics

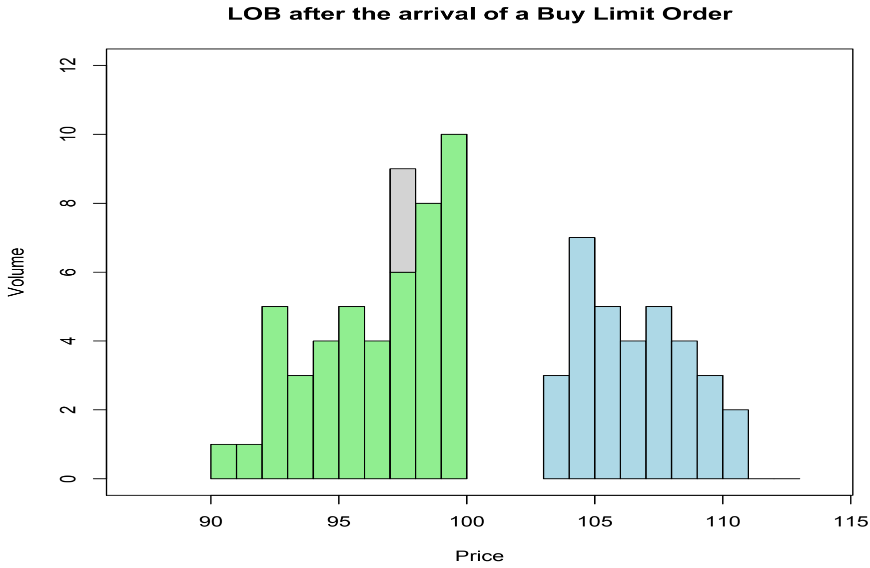
- For the sake of **simplicity**, we assume that the changes to the LOB happen

  one share at a time!

- We review the **events** causing the LOB state **transitions**
- Convenient Notation $O^{p\pm 1}$ as a transition from O

$$O_i^{p\pm 1} = \begin{cases} O_i & \text{if } i \neq p \\ O_i \pm 1 & \text{if } i = p \end{cases}$$

Limit buy order at price level $p < P_B(t)$



Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p-1}$

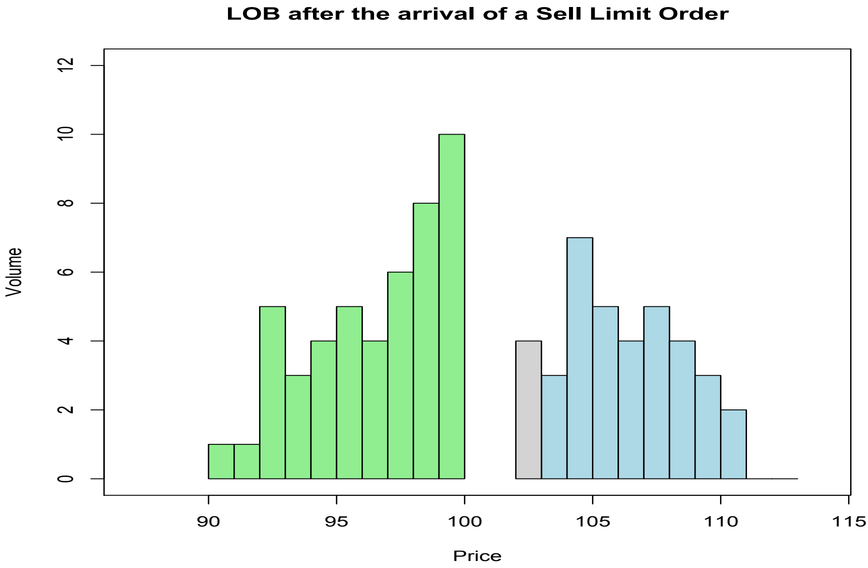
Limit buy order at price level $p > P_B(t)$

Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p-1}$

Limit sell order at price level $p = P_A(t)$

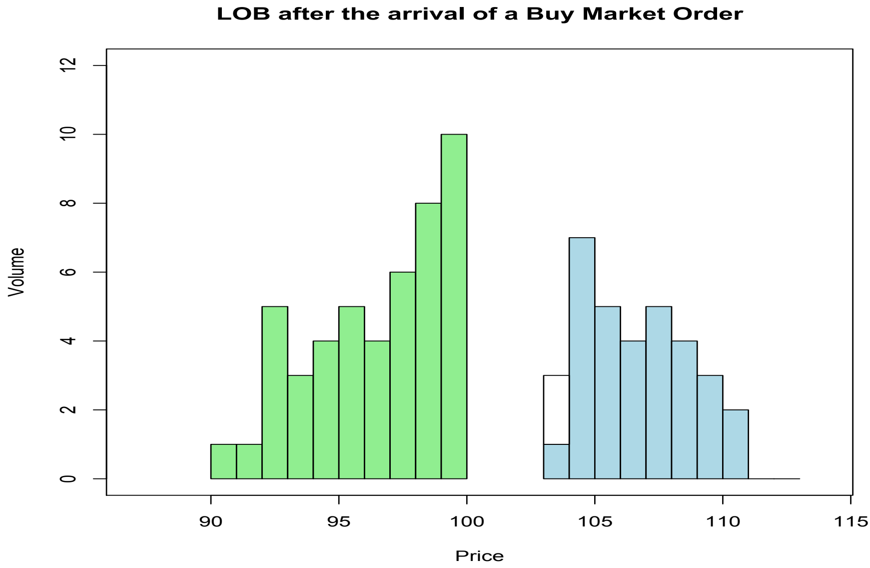
Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$

Limit sell order at price level $p < P_A(t)$



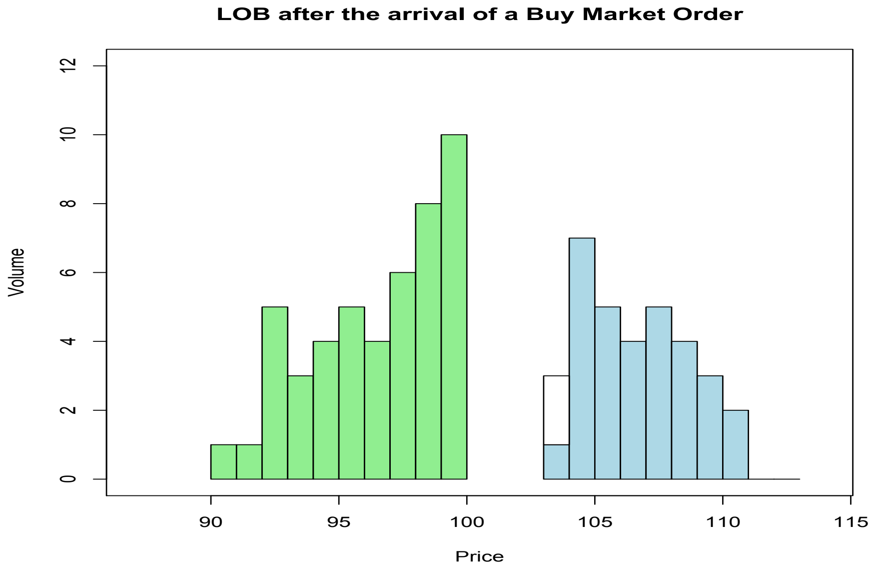
Increases the quantity at level p: $O(t) \hookrightarrow O(t)^{p+1}$

Market buy order



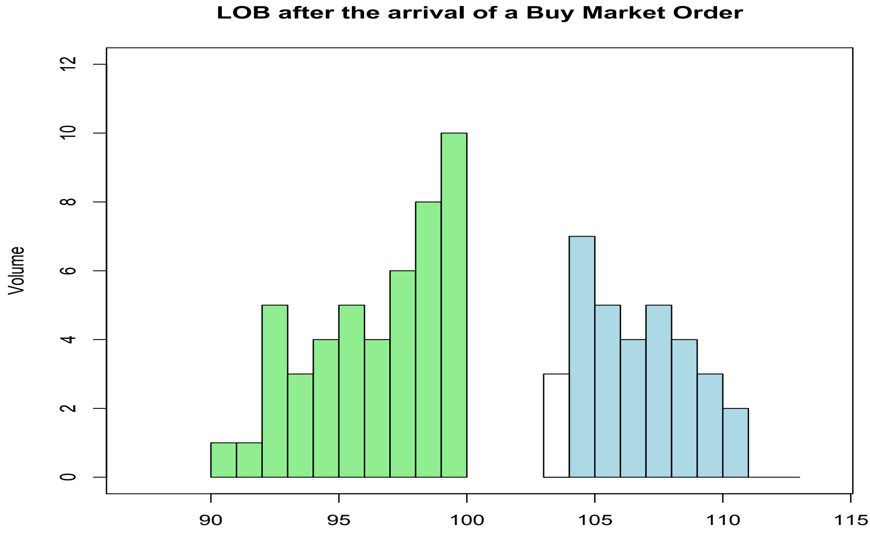
Decreases the quantity at the ask price: $O(t) \hookrightarrow O(t)^{P_A(t)-1}$

Cancellation of a Limit Sell Order at price $p = P_A(t)$



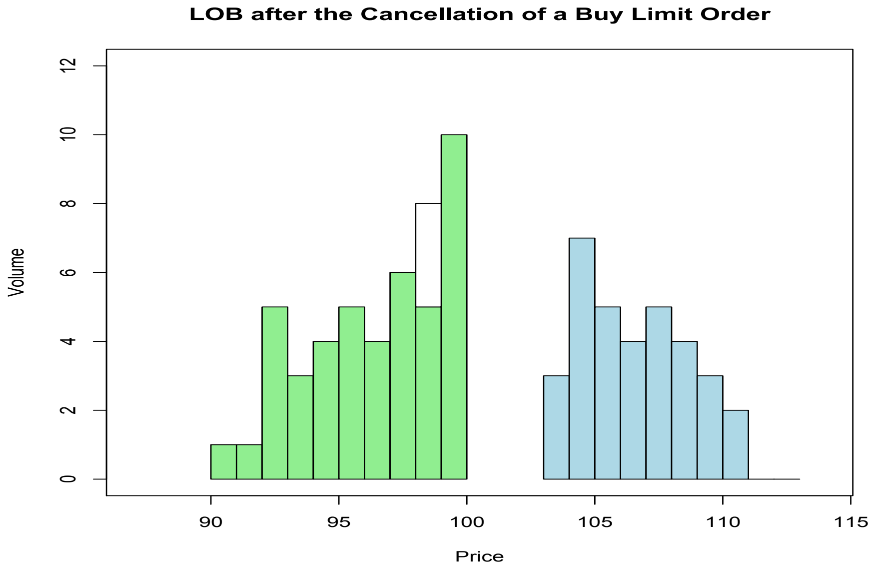
Decreases the quantity offered at that price.

Cancellation of a Limit Sell Order or Market Buy



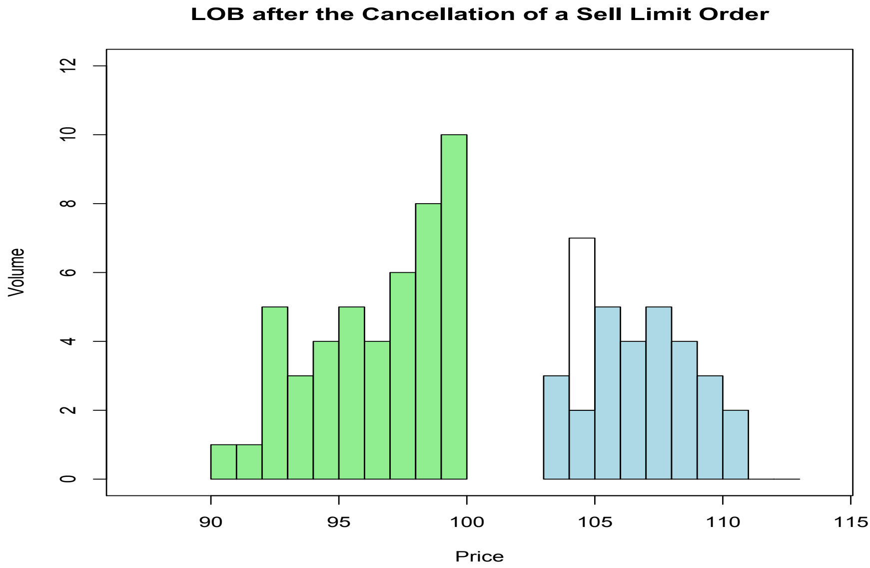
Decreases the quantity at the ask price: $O(t) \hookrightarrow O(t)^{P_A(t)-1}$. Now the Ask price $P_A(O)$ changes

Cancellation of a limit buy order at price $p < P_B(t)$



Decreases the quantity available at price p

Cancellation of a limit sell order at price $p > P_A(t)$



Decreases the quantity available at price p

Practical Assumptions

- Limit buy (respectively sell) orders arrive at a distance of i ticks from the opposite best quote at independent, exponential times with rate $\lambda(i) = Ki^{-\beta}$ for some $K > 0$ and $\beta > 0$
- Market buy (respectively sell) orders arrive at independent, exponential times with constant rate μ
- Cancellations of limit orders at a distance of i ticks from the opposite best quote occur at a rate proportional to the number of outstanding orders: If the number of outstanding orders at that level is x, then the cancellation rate is $\theta(i)x$.
- The above events are mutually independent.

Summary

Under these assumptions, $\underline{O} = [O(t)]_{t \geq 0}$ is a continuous-time Markov chain with state space $\mathcal{O}$ and transition rates:

- $O \hookrightarrow O^{p-1}$ with rate $\lambda(P_A(t) - p)$ for $p < P_A(t)$
- $O \hookrightarrow O^{p-1}$ with rate $\theta(p - P_B(t))|O_p|$ for $p > P_B(t)$
- $O \hookrightarrow O^{p+1}$ with rate $\lambda(p - P_B(t))$ for $p > P_B(t)$
- $O \hookrightarrow O^{p+1}$ with rate $\theta(P_A(t) - p)|O_p|$ for $p < P_A(t)$
- $O \hookrightarrow O^{P_B(t)+1}$ with rate μ
- $O \hookrightarrow O^{P_A(t)-1}$ with rate μ

This chain remains in $\mathcal{O}$ if it starts from there, i.e.

$$P_B(t) \leq P_A(t), \qquad \text{far all } t > 0$$

if it is true at time $t = 0$.

Extensions

With previous models

- Can Compute / Estimate **Probabilities of Conditional Events**
- **Not sufficient for optimal execution strategies**

Need to incorporate other important stylized facts of **order flow**. e.g.

- Limit order arrival rates conditional on distance from e.g. best price on same side
- Existing limit orders cancelled and immediately resubmitted
- Aggressiveness of orders depends on depth
- Fewer market orders when the spread is large
- More limit orders inside spread when depth at best is large
- (Long-range) autocorrelation of signs of consecutive market orders

Optimization Problems

Goal of a LOB model is to

- **Understand the costs of transactions**
- **Develop efficient** (if not **optimal**) trading procedures

Typical challenge

- **Sell** x_0 units of an asset and maximize the sales revenues, using a limited number of **market orders** **only**

$$\sup_{\tau_1 < \cdots < \tau_n < T} \mathbb{E}\left[U\left(\sum_{i=1}^{n} P_B(\tau_i)\right)\right]$$

where U is a utility function and $\mathbb{E}$ is the expectation over a model for the dynamics of the LOB O_t

Searching for optimal ***strategies*** / ***market timing rules*** is a **stochastic control problem** in **prohibitively** **high** dimension

SCREAMING for FORENSIC ANALYSIS

Optimal Execution

We already saw that:

- **splitting**
- and **spreading** over time

large orders can produce better effective prices. So

- How can we capture market **price impact** in a model?
- What are the desirable properties of a *Price Impact* model?
- How can we compute **optimal execution** trading strategies?
- What happens when several execution strategies **interact**?

"Amlgren-Chriss Price Impact" Model

- **Unaffected (fair) price** given by a semi-martingale (Wiener process)
- **Mid-price** affected by trading
  - **Permanent price impact** given by a function g of trading speed $v(t)$
  $$dP_t^{mid} = g(v(t))dt + \sigma dW_t$$
  - **Temporary price impact** given by function h of trading speed
  $$P_t^{trans} = P_t^{mid} + h(v(t))$$
- **Problem:** find deterministic continuous *transaction path* to maximize **mean-variance** reward.
  - Closed form solution when **permanent** and **instantaneous price impact** functions g and h are **linear**
  - **Efficient frontier**: Speed of trading and hence risk/return controlled by risk aversion parameter

Widely used in industry

Criticisms

- Mid-price P_t^{mid} **arithmetic Brownian motion** + drift
  - Can become negative
  - Reasonable only for short times
- Possible issues with **rate of trading** in continuous time?
- **Price impact** more complex than instantaneous + permanent
- What is the link between **Price Impact** and **LOB** dynamics?
  - e.g. can we combine elegant description of risk-return trade-off in Almgren / Chriss with detail of Smith-Farmer type models?
- **Empirical evidence** that instantaneous price impact is stochastic in many markets

Optimal Execution

An execution algorithm has **three** layers:

- At the **highest level** one decides how to **slice** the order, when to trade, in what size and for how long.
- At the **mid level**, given a slice, one decides whether to place **market** or **limit** orders and at what price level(s).
- At the **lowest level**, given a limit or market order, one decides to **which venue** should this order be routed?

We shall not discuss the last bullet point here.

Optimal Execution Set-Up

Goal: sell $x_0 > 0$ shares by time $T > 0$

- $\underline{X} = (X_t)_{0 \leq t \leq T}$ execution strategy
- X_t position (nb of shares held) at time t. $X_0 = x_0$, $X_T = 0$
- Assume X_t absolutely continuous (differentiable)
- $\tilde{P}_t$ **mid-price** (unaffected price), P_t **transaction price**, I_t **price impact**

$$P_t = \tilde{P}_t + I_t$$

  e.g. Linear Impact A-C model:

$$I_t = \gamma[X_t - X_0] + \lambda \dot{X}_t$$

- **Objective:** Maximize *some form of revenue* at time T

Revenue $\mathcal{R}(\underline{X})$ from the execution strategy $\underline{X}$

$$\mathcal{R}(\underline{X}) = \int_0^T (-\dot{X}_t) P_t dt$$

Specific Challenges

- **First generation**: Price impact models (e.g. Almgren - Chriss)
  - Risk Neutral framework (maximize $\mathbb{E}\mathcal{R}(\underline{X})$) versus utility criteria
  - More complex portfolios (including options)
  - Robustness and performance constraints (e.g. slippage or tracking market VWAP)
- **Second generation**: Simplified LOB models
  - Simple liquidation problem
  - performance constraints (e.g. slippage or tracking market VWAP) and using **both** market and limit orders

Optimal Execution Problem in A-C Model

$$
\begin{aligned}
\mathcal{R}(\underline{X}) &= \int_0^T (-\dot{X}_t) P_t dt \\
&= -\int_0^T \dot{X}_t \tilde{P}_t dt - \int_0^T \dot{X}_t I_t dt \\
&= x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - \mathcal{C}(\underline{X})
\end{aligned}
$$

with $\mathcal{C}(\underline{X}) = \int_0^T \dot{X}_t I_t dt$.

Interpretation

- $x_0 \tilde{P}_0$ (initial) **face value** of the portfolio to liquidate
- $\int_0^T X_t d\tilde{P}_t$ **volatility risk** for selling according to $\underline{X}$ instead of immediately!
- $\mathcal{C}(\underline{X})$ **execution costs** due to market impact

Special Case: the Linear A-C Model

$$\mathcal{R}(\underline{X}) = x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - \lambda \int_0^T \dot{X}_t^2 dt - \frac{\gamma}{2} x_0^2$$

Easy Case: Maximizing $\mathbb{E}[\mathcal{R}(X)]$

$$\mathbb{E}[\mathcal{R}(\underline{X})] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \lambda \mathbb{E} \int_0^T \dot{X}_t^2 dt$$

Jensen's inequality & constraints $X_0 = x_0$ and $X_T = 0$ imply

$$\dot{X}_t^* = -\frac{x_0}{T}$$

trade at a constant rate **independent of volatility** !
Bertsimas - Lo (1998)

More Realistic Problem

Almgren - Chriss propose to maximize

$$\mathbb{E}[\mathcal{R}(\underline{X})] - \alpha \mathrm{var}[\mathcal{R}(X)]$$

(α risk aversion parameter – late trades carry volatility risk)

For **DETERMINISTIC** trading strategies $\underline{X}$

$$\mathbb{E}[\mathcal{R}(\underline{X})] - \alpha \mathrm{var}[\mathcal{R}(X)] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \int_0^T \left(\frac{\alpha \sigma^2}{2} X_t^2 + \lambda \dot{X}_t^2 \right) dt$$

maximized by (standard variational calculus with constraints)

$$\dot{X}_t^* = x_0 \frac{\sinh \kappa (T - t)}{\sinh \kappa T} \qquad \text{for} \qquad \kappa = \sqrt{\frac{\alpha \sigma^2}{2\lambda}}$$

For **RANDOM** (adapted) trading strategies $\underline{X}$, **more difficult** as
Mean-Variance not amenable to dynamic programming

Maximizing Expected Utility

Choose $U : \mathbb{R} \to \mathbb{R}$ increasing concave and

$$\text{maximize} \qquad \mathbb{E}[U(\mathcal{R}(\underline{X}_T)]$$

Stochastic control formulation over a state process $(X_t, R_t)_{0 \le t \le T}$.

$$v(t,x,r) = \sup_{\underline{\xi} \in \Xi(t,x)} \mathbb{E}[u(R_T) | X_t = x, R_T = r]$$

value function, where $\Xi(t,x)$ is the set of **admissible controls**

$$\left\{ \underline{\xi} = (\xi_s)_{t \le s \le T} ; \text{progressively measurable}, \int_t^T \xi_s^2 ds < \infty, \int_t^T \xi_s ds = x \right\}$$

$$X_s = X_s^{\underline{\xi}} = x - \int_t^s \xi_u du, \qquad \dot{X}_s = -\xi_s,\ X_t = x$$

and (choosing $\tilde{P}_t = \sigma W_t$)

$$R_s = R_s^{\underline{\xi}} = R + \sigma \int_t^s X_u dW_u - \lambda \int_t^s \xi_u^2 du, \quad dR_s = \sigma X_s dW_s - \lambda \xi_s^2 ds,\ R_t = r$$

Finite Fuel Problem

Non Standard Stochastic Control problem because of the constraints

$$\int_0^T \xi_s ds = x_0.$$

Still, one expects

- For any admissible $\underline{\xi}$, $[v(t, X_t^{\xi}, R_t^{\xi})]_{0 \le t \le T}$ is a super-martingale
- For some admissible $\underline{\xi}^*$, $[v(t, X_t^{\xi^*}, R_t^{\xi^*})]_{0 \le t \le T}$ is a **true** martingale

If v is smooth, and we set $V_t = v(t, X_t^{\xi}, R_t^{\xi})$, Itô's formula gives

$$\begin{aligned} dV_t \quad &= \quad \Big(\partial_t v(t, X_t, R_t) + \frac{\sigma^2}{2} \partial_{rr}^2 v(t, X_t, R_t) \\ &\qquad -\lambda \xi_t^2 \partial_r v(t, X_t, R_t) - \xi_t \partial_x v(t, X_t, R_t) \Big) dt \\ &\qquad + \sigma \partial_x v(t, X_t, R_t) dW_t \end{aligned}$$

Hamilton-Jabobi-Bellman Equation

One expects that v solves the **HJB** equation (nonlinear PDE)

$$\partial_t v + \frac{\sigma^2}{2}\partial^2_{xx} v - \inf_{\xi\in\mathbb{R}}[\xi^2\lambda\partial_r v + \xi\partial_x v] = 0$$

in some sense, with the (non-standard) terminal condition

$$v(T,x,r) = \begin{cases} U(r) & \text{if } x = X_0 \\ -\infty & \text{otherwise} \end{cases}$$

Solution for CARA Exponential Utility

For $u(x) = -e^{-\alpha x}$ and κ as before

$$v(t, x, r) = e^{-\alpha r + x_0^2 \alpha \lambda \kappa \coth \kappa (T-t)}$$

solves the HJB equation and the unique maximizer is given by the **DETERMINISTIC**

$$\xi_t^* = x_0 \kappa \frac{\cosh \kappa (T-t)}{\sinh \kappa T}$$

Schied-Schöneborn-Tehranchi (2010)

- Optimal solution **same** as in Mean - Variance case
- **Schied-Schöneborn-Tehranchi**'s trick shows that optimal trading strategy is **generically** deterministic for **exponential utility**
- **Open problem** for general utility function
- Partial results in **infinite horizon** versions

Shortcomings

- Optimal strategies
  - are **DETERMINISTIC**
  - do not react to **price changes**
  - are **time inconsistent**
  - are **counter-intuitive** in some cases
- Computations require
  - solving **nonlinear PDEs**
  - with **singular** terminal conditions

Recent Developments

Gatheral - Schied (2011), Schied (2012)

- In the spirit of Almgren-Chriss mean-variance criterion, maximize

$$\mathbb{E}\left[\mathcal{R}(\underline{X}) - \tilde{\lambda}\int_0^T X_t P_t dt\right]$$

- The solution happens to be **ROBUST**
  - $\tilde{P}_t$ can be a semi-martingale, optimal solution does not change

Recent Developments

Almgren - Li (2012), Hedging a large option position

- $g(t, \tilde{P}_t)$ price at time t of the option (from Black-Scholes theory)
- Revenue

$$\mathcal{R}(\underline{X}) = g(T, \tilde{P}_T) + X_T \tilde{P}_T - \int_0^T \tilde{P}_t \dot{X}_t dt - \lambda \int_0^T \dot{X}_t^2 dt$$

- Using Itô's formula and the fact that g solves a PDE,

$$\mathcal{R}(\underline{X}) = R_0 + \int_0^T [X_t + \partial_x g(t, \tilde{P}_t)] dt - \lambda \int_0^T \dot{X}_t^2 dt \qquad R_0 = x_0 \tilde{P}_0 + g(0, \tilde{P}_0)$$

- **Introduce** $Y_t = X_t + \partial_x g(t, \tilde{P}_t)$ for **hedging correction**

$$\begin{cases} d\tilde{P}_t = \gamma \dot{X}_t dt + \sigma dW_t \\ dY_t = [1 + \gamma \partial^2_{xx} g(t, \tilde{P}_t)] dt + \sigma \partial^2_{xx} g(t, \tilde{P}_t) dW_t \end{cases}$$

- **Minimize**

$$\mathbb{E}\left[G(\tilde{P}_T, Y_T) + \int_0^T \left(\frac{\sigma^2}{2} Y_t^2 - \gamma \dot{X}_t Y_t + \lambda \dot{X}_t^2 \right) dt \right]$$

Explicit solution in some cases (e.g. $\partial^2_{xx} g(t, x) = c$, G quadratic)

Transient Price Impact

Flexible price impact model

- **Resilience function** $G : (0, \infty) \to (0, \infty)$ measurable bounded
- Admissible $\underline{X} = (X_t)_{0 \le t \le T}$ cadlag, adapted, **bounded variation**
- Transaction price

$$P_t = \tilde{P}_t + \int_0^t G(t-s)\, dX_s$$

- Expected cost of strategy $\underline{X}$ given by

$$-x_0 P_0 + \mathbb{E}[\mathcal{C}(\underline{X})]$$

where

$$\mathcal{C}(\underline{X}) = \int \int G(|t-s|) dX_s dX_t$$

Transient Price Impact: Some Results

- No **Price Manipulation** in the sense of **Huberman - Stanzl (2004)** if $G(|\cdot|)$ positive definite
- Optimal strategies (if any) are **deterministic**
- Existence of an optimal $\underline{X}^*$ $\Leftrightarrow$ solvability of a Fredholm equation
- Exponential Resilience $G(t) = e^{-\rho t}$

$$dX^*_t = -\frac{x_0}{\rho T + 2}\Big(\delta_0(dt) + \rho dt + \delta_T(dt)\Big)$$

- $\underline{X}^*$ purely discrete measure on $[0, T]$ when $G(t) = (1 - \rho t)^+$ with $\rho > 0$
  - $dX^*_t = -\frac{x_0}{2}[\delta_0(dt) + \delta_T(dt)]$ if $\rho < 1/T$
  - $dX^*_t = -\frac{x_0}{n+1}\sum_{i=0}^{n} \delta_{iT/n}(dt)$ if $\rho < n/T$ for some integer $n \geq 1$

Obizhaeva - Wang (2005), Gatheral - Schied (2011)

Optimal Execution in a LOB Model

- Unaffected price $\tilde{P}_t$ (e.g. $\tilde{P}_t = P_0 + \sigma W_t$)
- Trader places only **market sell orders**
  - Placing buy orders is not optimal
- Bid side of LOB given by a function $f : \mathbb{R} \to (0, \infty)$ s.t. $\int_0^\infty f(x)dx = \infty$. At any time t

$$\int_a^b f(x)dx = \text{bids available in the price range } [\tilde{P}_t + a, \tilde{P}_t + b]$$

- The **shape function** f does not depend upon t or $\tilde{P}_t$

Obizhaeva - Wang (2006), Alfonsi - Fruth - Schied (2010), Alfonsi - Schied - Schulz (2011), Predoiu - Shaikhet - Shreve (2011)

Optimal Execution Tracking a Benchmark

R.C. - M. Li

Goal: sell $v > 0$ shares by time $T > 0$ (finite horizon)

- P_t **mid-price** (unaffected price),

$$P_t = P_0 + \int_0^t \sigma(u) dW_u, \qquad 0 \le t \le T,$$

- $V(t)$ volume traded in the market up to (and including) time t
- Market **VWAP**$= \frac{1}{V} \int_0^T P_t dV(t)$
- Fraction of shares still to be executed in the market

$$X(t) = \frac{V - V(t)}{V} = \frac{T - t}{T}$$

(deterministic $V(t)$ used to **change clock**). **Convenient simplification** !

Broker Problem

v_t volume executed by the broker up to time t

$$x_t = \frac{v - v_t}{v}$$

fraction of shares left to be executed by the broker at time t

$$x_t = 1 - \ell_t - m_t$$

Where

- ℓ_t **cumulative** volume executed through **limit orders**
- m_t **cumulative** volume executed through **market orders**
- Broker average liquidation price
  vwap$= \frac{1}{v} \int_0^T \left(P_t - \frac{S}{2}\right) dm_t + \left(P_t + \frac{S}{2}\right) d\ell_t$
- **Objective:** Minimize discrepancy between **vwap** and **VWAP**

Naive Model for the Dynamics of the Order Book

Controls of the broker:

- $(m_t)_{0\leq t\leq T}$ **non-decreasing** adapted process
- $(L_t)_{0\leq t\leq T}$ **predictable** process

$$\ell_t = \int_0^t \int_{[0,1]} y \wedge L_u \; \mu(du, dy) = \sum_{i=1}^{N_t} Y_i \wedge L_{\tau_i}$$

where

$$\mu(du, dy)$$

point measure (Poisson) compensator $\nu_t(du)\nu(t)dt$.

$$x_t = 1 - \int_0^t \int_{[0,1]} y \wedge L_u \; \mu(du, dy) - m_t = 1 - \sum_{i=1}^{N_t} Y_i \wedge L_{\tau_i} - m_t$$

So the dynamics of x_t are given by

$$dx_t = -\int_{[0,1]} y \wedge L_t \; \mu(dt, dy) - dm_t,$$

with initial condition $x_{0-} = 1$.

Optimization Problem

Goal of the broker

$$\sup_{(\underline{L},\underline{m})\in\mathcal{A}} \mathbb{E}\Big[U(\text{vwap} - \text{VWAP})\Big],$$

For the CARA exponential utility, approximately

$$\inf_{(\underline{L},\underline{m})\in\mathcal{A}} \mathbb{E}\left[\exp\left(-\gamma\left(\frac{S}{2} + \int_0^T [x_u^{L,m} - X(u)]dP_u - S\,dm_u\right)\right)\right],$$

We will work with a **Mean - Variance** criterion

$$\inf_{(\underline{L},\underline{m})\in\mathcal{A}} \mathbb{E}\left[\int_0^T \gamma\frac{\sigma(u)^2}{2}[x_u^{L,m} - X(u)]^2 du + S\,m_T\right],$$

- S spread
- $X(u) = (T-u)/T$ fraction of shares left to be executed in the market.

Stochastic Control Problem

Singular control problem of a pure jump process

Value function
$$J(t,x) = \inf_{(\underline{L},\underline{m}) \in \mathcal{A}(t,x)} J(t,x,\underline{L},\underline{m})$$

where

$$J(t,x,\underline{L},\underline{m}) = \mathbb{E}\left[\int_t^T \gamma \frac{\sigma(u)^2}{2} [x_u^{L,m} - X(u)]^2 du + S m_T \right].$$

$J(t,x)$ **is non-decreasing in** t **for** $x \in [0,1]$ **fixed.** ($\mathcal{A}(t_2,x) \subset \mathcal{A}(t_1,x)$ whenever $t_1 \le t_2$)

Tough Luck: Problem is NOT Convex

The set $\mathcal{A}$ of admissible controls is not convex.

For any number $\ell \in (0,1)$, the two controls $(\underline{L}^1, \underline{m}^1)$ and $(\underline{L}^2, \underline{m}^2)$ by:

$$L_t^1 = \mathbf{1}_{\{t \le \tau_1\}} + \sum_{k=2}^{\infty} x_{\tau_{k-1}} \mathbf{1}_{\{\tau_{k-1} < t \le \tau_k\}}, \qquad \text{and} \qquad m_t^1 = x_{T-} \mathbf{1}_{\{T \le t\}},$$

and:

$$L_t^2 = \frac{\ell}{2} \mathbf{1}_{\{t \le \tau_1\}} + \sum_{k=2}^{\infty} x_{\tau_{k-1}} \mathbf{1}_{\{\tau_{k-1} < t \le \tau_k\}}, \qquad \text{and} \qquad m_t^2 = x_{T-} \mathbf{1}_{\{T \le t\}},$$

are admissible, but the pair $(\underline{L}, \underline{m})$ defined by

$$L_t = \frac{1}{2}(L_t^1 + L_t^2), \qquad \text{and} \qquad m_t = \frac{1}{2}(m_t^1 + m_t^2),$$

IS NOT

Closest Related Works

- Poisson random measure $\mu(dt, dy)$ for claim sizes Y_t
- **insurer** pays $Y_t \wedge \alpha_t$ up to a **retention level** α_t
- **re-insurer** covers the excess $(Y_t - \alpha_t)^+$

Wealth process of the Insurance Company

$$X_t = x + \int_0^t p(\alpha_s)ds - \int_0^t y \wedge \alpha_s \, \mu(ds, dy) - \int_0^t dD_s$$

- $p(\alpha)$ insurer net premium (after paying the reinsurance company)
- D_t cumulative dividends paid up to (and including) time t

$$\sup_{(\alpha_t)_t, (D_t)_t} \mathbb{E}\left[\int_0^\tau e^{-ru} dD_u\right]$$

- time of bankruptcy $\tau = \inf\{t \geq 0;\ X_t \leq 0\}$

Jeanblanc-Shyryaev (1995) optimal dividend distribution for Wiener process, Asmunssen- Hjgaard-Taksar (1998) optimal dividend distribution for diffusion, Mnif-Sulem (2005) prove existence and uniqueness of a viscosity solution,
Goreac (2008) multiple contracts

Similarities & Differences

Similarities

- $\alpha_t \leftrightarrow$ standing limit orders L_t
- $D_t \leftrightarrow$ cumulative market orders m_t

Differences

- We work in a **finite horizon** (PDEs instead of ODEs)
- We use a **Mean - Variance** criterion
- We exhibit a **classical** solution (as opposed to a viscosity solution)
- We derive a **system of ODEs** identifying
  - the value function
  - the optimal stratagy

Technical Assumptions

$\nu_t(dy)\nu(t)dt$ intensity of Poisson measure $\mu(dt, dy)$ with $\nu_t([0,1]) = 1$.

- $\int_0^T \sigma(t)^2 dt < \infty$
- $\sup_{0 \leq t \leq T} \nu(t) < \infty$
- $t \hookrightarrow \frac{\sigma(t)^2}{\nu(t)}(X(t) - x)$ is increasing for each $x \in [0,1]$
- $t \hookrightarrow \frac{1}{\nu(t)}\nu_t(\,\cdot\,)$ is decreasing (in the sense of *stochastic dominance*)

Hamilton-Jabobi-Bellman Equation (QVI)

$$\boxed{\min\left[[A\phi](t,x), \partial_t\phi(t,x) + [B\phi](t,x)\right] = 0.}$$

where

$$[A\phi](t,x) = S - \partial_x\phi(t,x)$$

and

$$[B\phi](t,x) = \gamma\frac{\sigma(t)^2}{2}[X(t)-x]^2+\nu(t)\inf_{0\leq L\leq x}\int_{[0,1]}[\phi(t,x-y\wedge L)-\phi(t,x)]\nu_t(dy)$$

with terminal condition

$$\phi(T-,x) = Sx, \qquad \text{(notice that } \phi(T,x)=0\text{)}$$

and boundary condition:

$$\phi(t,0) = \int_t^T \frac{\gamma\sigma(u)^2}{2}X(u)du.$$

Classical Solution

Theorem

The value function is the ***unique*** *solution of*

$$-\dot{J}(t,x) = \min\Bigg[\inf_{0\le y\le x} -\dot{J}(t,x),$$
$$\gamma\frac{\sigma(t)^2}{2}[X(t)-x]^2 + \nu(t)\int_{[0,1]}[J(t,(x-y)\vee\tilde{L}(t,y)) - J(t,x)]\nu_t(dy)\Bigg]$$

with

$$J(t,0) = \gamma\int_0^t \frac{\sigma(u)^2}{2}X(u)^2du, \qquad \textit{and} \qquad J(T,x) = Sx$$

where

$$\tilde{L}(t,x) = \arg\min_{0\le y\le x} J(t,y)$$

- J is $C^{1,1}$
- $x \hookrightarrow J(t,x)$ convex for t fixed
- $t \hookrightarrow J(t,x)$ non-decreasing for x fixed

Free Boundary (No-Trade Region)

$$[0,T]\times[0,1] = A \cup B \cup C$$

with

- $A = \{(t,x); \partial_x J(t,x) < 0\} = \{(t,x);\ 0 \le t < \tau_\ell(x)\}$
- $B = \{(t,x); 0 \le \partial_x J(t,x) \le S\} = \{(t,x);\ \tau_\ell(x) \le t \le \tau_m(x)\}$
- $C = \{(t,x); \partial_x J(t,x) = S\} = \{(t,x);\ \tau_m(x) \le t\}$

where

- $\tau_\ell(x) = \inf\{t > 0;\ \partial_x J(t,x) \ge 0\}$
- $\tau_m(x) = \inf\{t > 0;\ \partial_x J(t,x) \ge S\}$

$$\boxed{\tau_\ell(x) \le T(1-x) \le \tau_m(x)}$$

Optimal Trading Strategy

- If $t > \tau_m(x_t)$ i.e. $(t, x_t) \in C$ (never happens)
  - **place market orders**

  $\Delta m_t > 0$ (just enough to get into B)
- If $t = \tau_m(x_t)$ i.e. $(t, x_t) \in \partial C$
  - **place market orders at a rate** $dm_t = -\dot{\tau}_m(x_t)dt$

  (just enough so not to exit B)
- If $\tau_\ell(x_t) \le t < \tau_m(x_t)$ i.e. $(t, x_t) \in B \cup \partial A$
  - **place** $L_t = x_t - \tilde{L}(t)$ **limit orders**

  (as much as possible without getting ahead too much)
- If $t < \tau_\ell(x_t)$ i.e. $(t, x_t) \in A$ (never happens)
  - **no trade**

Premises for Predatory Trading

- Large Trader facing a Forced Liquidation
- Especially if the need to liquidate is known by other traders
  - hedge funds with (nearing) margin call
  - traders who use portfolio insurance, stop loss orders, . . .
  - some institutions / funds cannot hold on to downgraded instruments
  - Index-replication funds (at re-balancing dates) e.g. Russell 3000

Forced liquidation can be **very costly** because of **price impact**

Business Week

> *...if lenders know that a hedge fund needs to sell something quickly, they will sell the same asset, driving the price down even faster. Goldman Sachs and other counter-parties to LTCM did exactly that in 1998.*

> *When you smell blood in the water, you become a shark . . . . when you know that one of your number is in trouble . . . you try to figure out what he owns and you start shorting those stocks . . .*

Cramer (2002)

Typical Predatory Trading Scenario

- Distressed trader (**prey**) needs to unload a large position
  - Size will have impact on price
- **Predator** initially **trades in the same direction** as the prey
  - Effect is to withdraw liquidity
  - Market impact of the liquidation becomes greater
  - Price fall is exaggerated (**over-shooting**)
- Predator **reverses direction**, profiting from the price over-shoot
- Predator **closes position** for a profit.

Brunnermeier - Pedersen (2005)
Carlin - Lobo - Viswanathan (2005)
Schied - Schöneborn (2008)

Multi-Player Game Model

- One risk free asset and one risky asset
- Trading in continuous time, interest rate $r = 0$
- $n + 1$ strategic players and a number of noise traders
- $X_0(t), X_1(t), \cdots, X_n(t)$ risky asset positions of the strategic players
- Trades at time t are executed at the price (**Chriss-Almgren price impact model**)

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

where $\tilde{P}(t)$ is a **mean zero** martingale (say a Wiener process).

Goal of the Mathematical Analysis

- Understand predation
- Illustrate benefits of
  - Stealth trading
  - Sunshine trading

Modeling extreme markets

- **Elastic** markets:
  - temporary impact $\lambda >>$ permanent impact γ
- **Plastic** markets:
  - permanent impact $\gamma >>$ temporary impact λ

Assumptions of the One Period Game

- Each strategic player $i \in \{0, 1, \cdots, n\}$ knows
  - all other strategic players initial asset positions $X_j(0)$ for $j \neq i$
  - Their target $X_j(T)$ at some fixed time point $T > 0$ in the future
- Objective (all players are **risk neutral**)
  - Players maximize their expected return by choosing an optimal trading strategy $X_i(t)$ satisfying their constraints $X_i(0)$ and $X_i(T)$

One **distressed trader** / **prey** (e.g **seller**), player 0

$$X_0(0) = x_0 > 0, \qquad X_0(T) = 0$$

n **predators** players $1, 2, \cdots, n$

$$X_i(0) = X_i(T) = 0, \qquad i = 1, \cdots, n$$

Optimization Problem

A strategy $X_i = (X_i(t))_{0 \leq t \leq T}$ is **admissible** (for player i) if it is an a

- adapted process
- with continuously differentiable sample paths

Given a set $\underline{X} = (X_0, X_1, \cdots, X_n)$ of admissible strategies

- Each player $i \in \{0, 1, \cdots, n\}$ tries to maximize his expected return

  $$J^i(\underline{X}) = \mathbb{E}[\int_0^T (-\dot{X}_i(t))P(t)dt]$$

  under the constraint

  $$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

- Search for **Nash Equilibrium**

Deterministic Strategies

If we restrict the admissible strategies $\underline{X} = (X_0, X_1, \cdots, X_n)$ to be **DETERMINISTIC**

$$J^i(\underline{X}) = \mathbb{E}[\int_0^T (-\dot{X}_i(t))P(t)dt] = \int_0^T (-\dot{X}_i(t))\overline{P}(t)dt$$

where

$$\overline{P}(t) = P(0) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

THE SOURCE OF RANDOMNESS IS GONE !

Carlin - Lobo - Viswanathan (2005) Schied - Schoenborn (2008)

Solution in the Deterministic Case

Unique Optimal Strategies

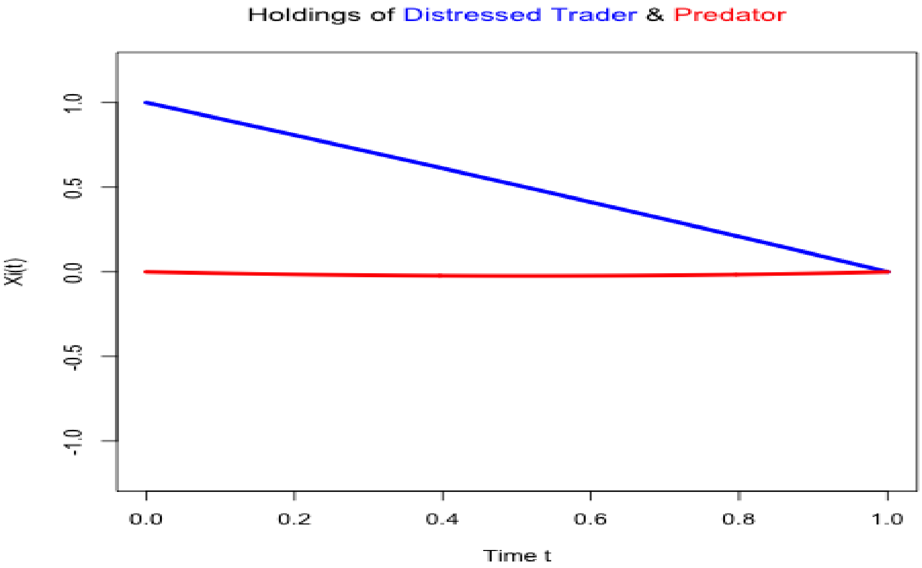
$$X_i(t) = ae^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t}$$

where

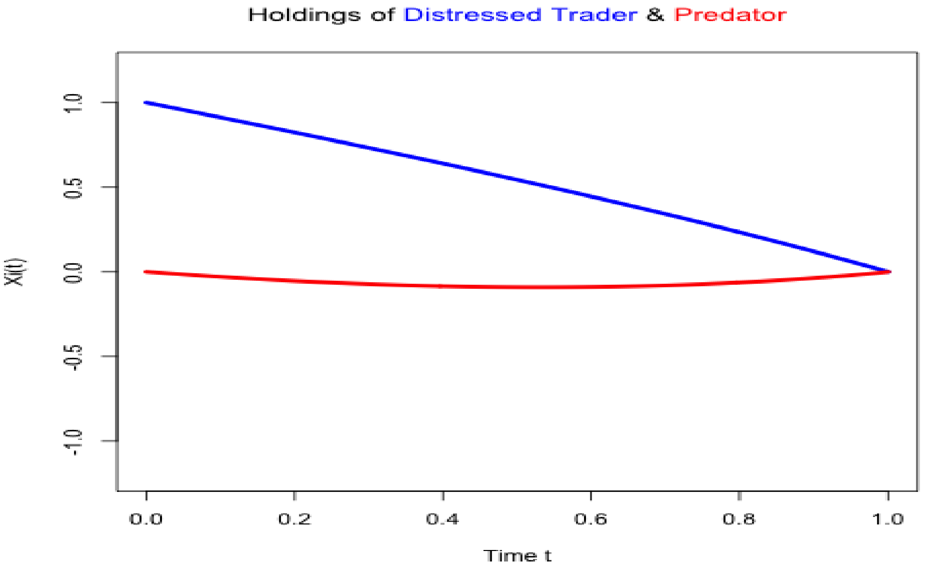
$$\begin{aligned} a &= \frac{n}{n+2}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T}\right)^{-1}\frac{1}{n+1}\sum_{i=0}^{n}[X_i(T) - X_i(0)] \\ b_i &= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\left(X_i(T) - X_i(0) - \frac{1}{n+1}\sum_{i=0}^{n}[X_i(T) - X_i(0)]\right) \end{aligned}$$

Carlin - Lobo - Viswanathan (2005)

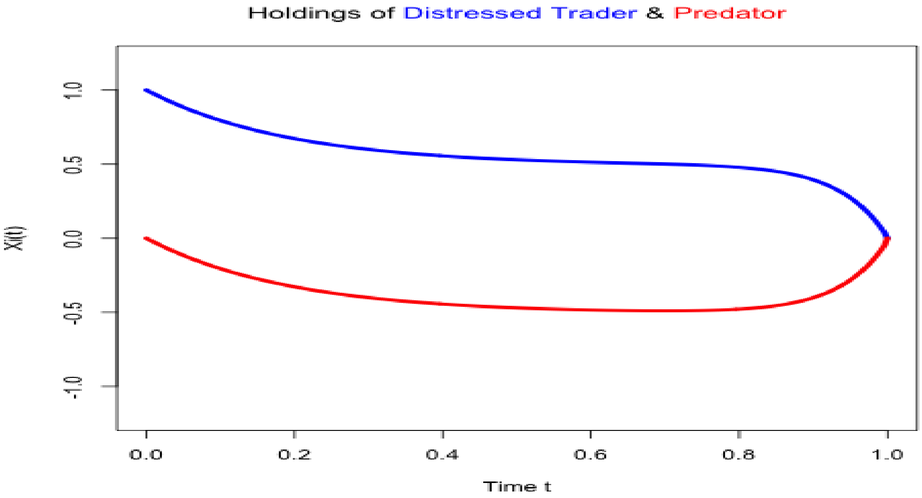
$n = 1$ **predator,** $\gamma/\lambda = 0.3$



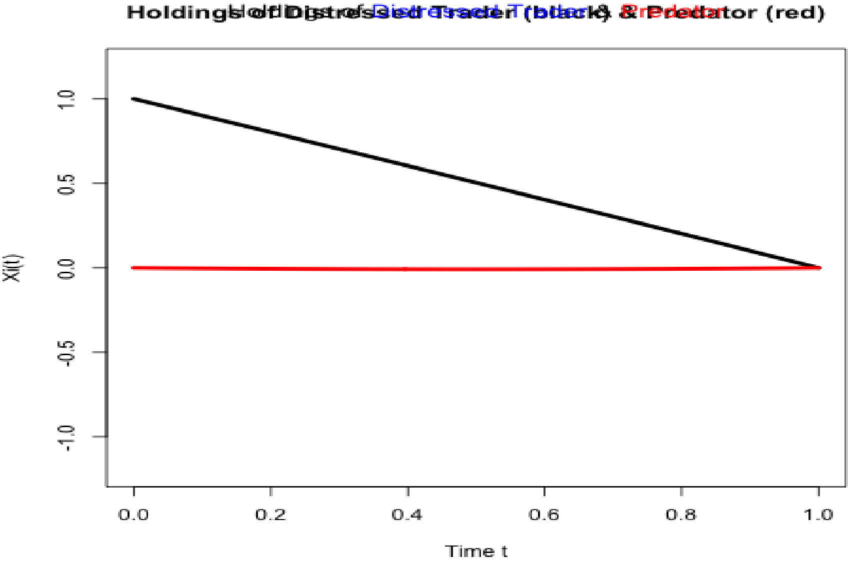
$n = 1$ predator, $\gamma = \lambda$



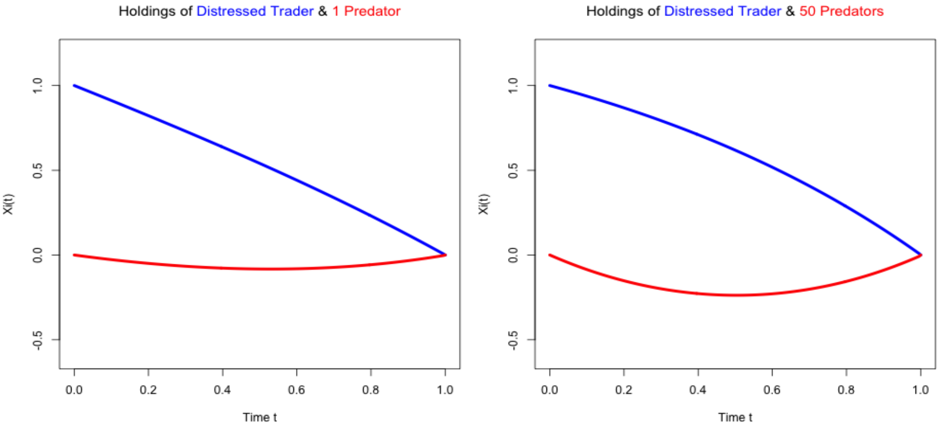
$n = 1$ **predator,** $\gamma = 15.5\lambda$



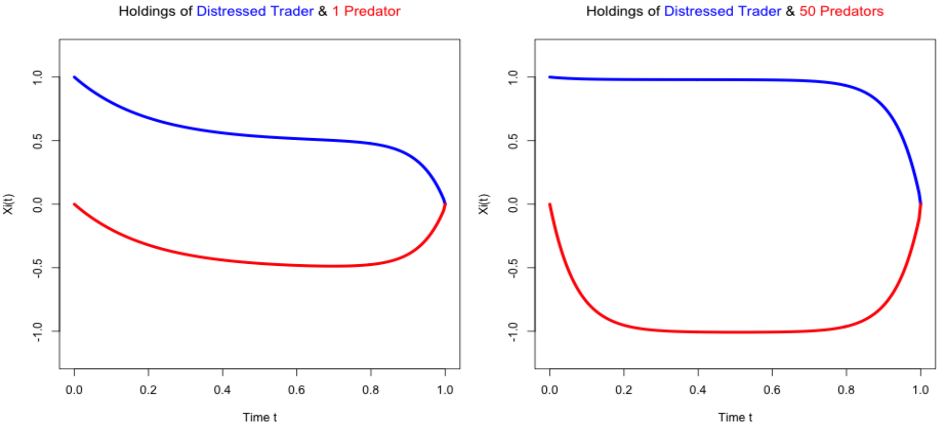
Fancy Plots of the Holdings of the Distressed Trader & Predator



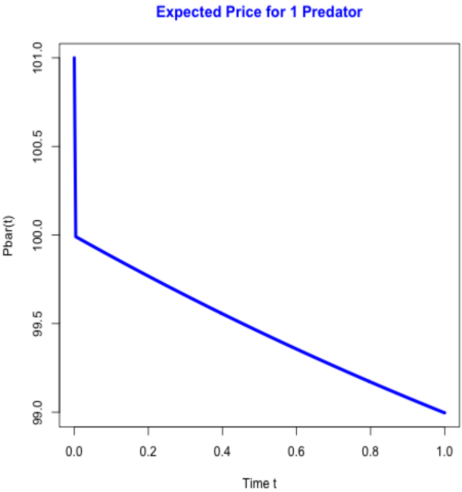
Impact of the Number of Predators: $\gamma = \lambda$

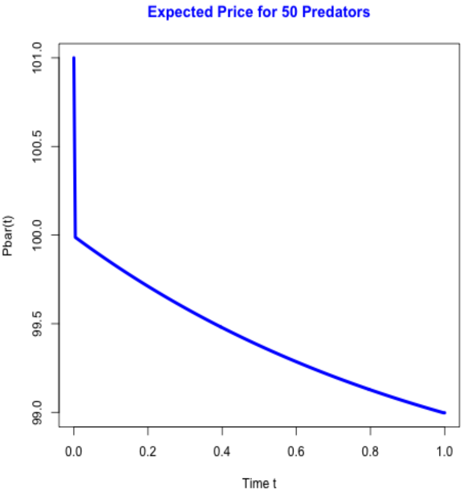


Impact of the Number of Predators: $\gamma = 15.5\lambda$

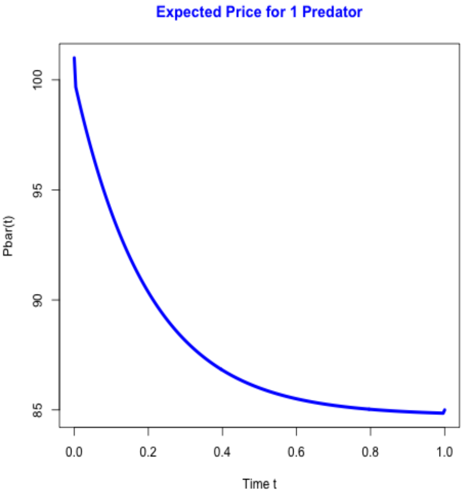


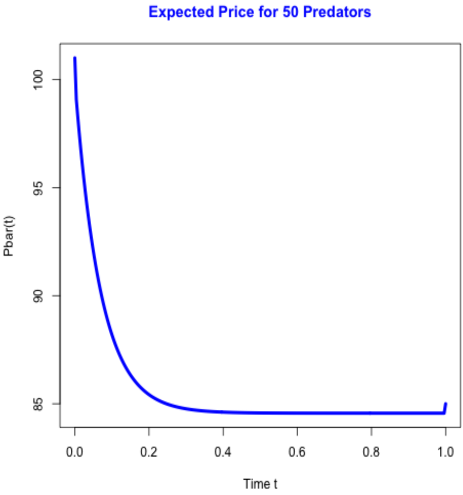
Expected Price: $\gamma = \lambda$



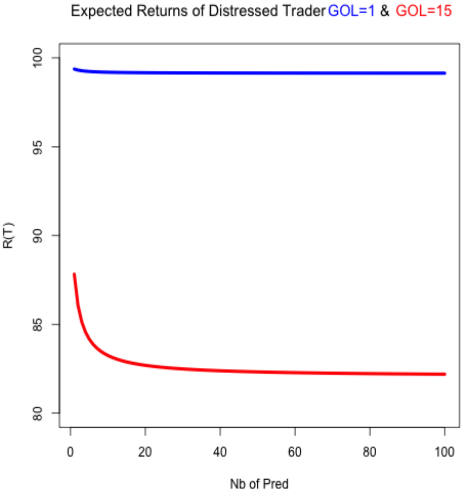


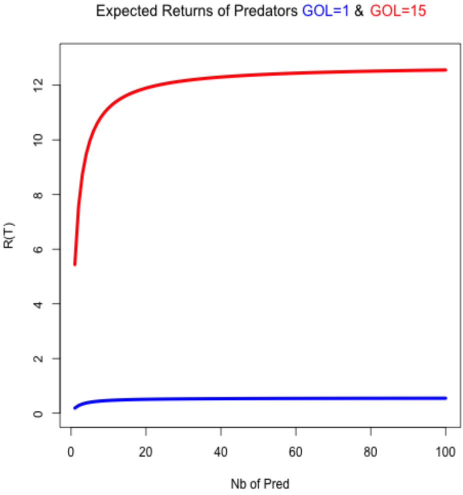
Expected Price: $\gamma = 15\lambda$





Impact of Nb of Predators on Expected Returns





Two Period Model

- Prey has to liquidate $X_0 > 0$ by time T_1, i.e. $X_0(T_1) = 0$
- Predators can stay in the game longer $X_i(0) = X_i(T_2) = 0$ for some $T_2 > T_1$ for $i = 1, \cdots, n$
- Prey does not trade in second period $[T_1, T_2]$, i.e. $X_0(t) = 0$ for $T_1 \leq t \leq T_2$.

Markovian Structure $\Longrightarrow$

Solution determined by predators' positions at time T_1

Nash Equilibrium for Deterministic Strategies

UNIQUE Nash Equilibrium

- **ALL** Predators have the same position at time T_1

$$X_i(T_1) = \frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} X_0, \qquad i = 1, \cdots, n$$

- Coefficients depend upon n but converge as $n \to \infty$
- Asymptotic formulas for expected returns
- Asymptotic comparison of **Stealth** versus Sunshine trading for some regimes of γ/λ

Schöneborn - Schied (2008)