Mathematical Behavioural Finance

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May 2015 @ IPAM

Overview of This Lecture Series

- Part 1: Introduction and Background

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- Part 2: Portfolio Choice and Quantile Formulation

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- Part 3: Market Equilibrium and Asset Pricing

Part I: Introduction and Background

Section 1

Expected Utility Theory and Challenges

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- Axiomatic approach: completeness, transivity, continuity and independence
- Behaviour of a **rational** agent necessarily coincides with that of an agent who values uncertain payoffs using expected **concave** utility

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- *Neoclassical economics*

Paradoxes/Puzzles with EUT

EUT is systematically violated via experimental work, and challenged by many paradoxes and puzzles

- Allais paradox: Allais (1953)
- Ellesberg paradox: Ellesberg (1961)
- Friedman and Savage puzzle: Friedman and Savage (1948)
- Equity premium puzzle: Mehra and Prescott (1985)
- Risk-free rate puzzle: Weil (1989)

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- Merton Miller: "If you transfer a dollar from your right pocket to your left pocket, you are no wealthier. Franco (Modigliani) and I proved that rigorously"

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- **Behaviour does depend on frame**

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- **Reference point: what matters is *deviation* of wealth from certain benchmark, not wealth itself**

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- **Risk averse on gains, risk seeking on losses**

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- **Loss aversion: pain from a loss is more than joy from a gain of the same magnitude**

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- **Exaggeration of extremely small probabilities of both winning big and losing big**

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- *Risk-free rate puzzle* (Weil 1989): observed risk-free rate is too low to be explainable by classical CCAPM

Economic Data 1889–1978 (Mehra and Prescott 1985)

	Consumption growth		riskless return		equity premium		S&P 500 return	
Periods	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1889–1978	**1.83**	**3.57**	**0.80**	**5.67**	**6.18**	**16.67**	**6.98**	**16.54**
1889–1898	2.30	4.90	5.80	3.23	1.78	11.57	7.58	10.02
1899–1908	2.55	5.31	2.62	2.59	5.08	16.86	7.71	17.21
1909–1918	0.44	3.07	-1.63	9.02	1.49	9.18	-0.14	12.81
1919–1928	3.00	3.97	4.30	6.61	14.64	15.94	18.94	16.18
1929–1938	-0.25	5.28	2.39	6.50	0.18	31.63	2.56	27.90
1939–1948	2.19	2.52	-5.82	4.05	8.89	14.23	3.07	14.67
1949–1958	1.48	1.00	-0.81	1.89	18.30	13.20	17.49	13.08
1959–1968	2.37	1.00	1.07	0.64	4.50	10.17	5.58	10.59
1969–1978	2.41	1.40	-0.72	2.06	0.75	11.64	0.03	13.11

EUT Based Theories

- Recall EUT based formulae (single period)

$$\bar{r} - r_f \approx \alpha \mathbf{Cov}(\tilde{g}, \tilde{r}),$$
$$1 + r_f \approx \frac{1+\alpha\bar{g}}{\beta}$$

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  - So $\alpha \geq \frac{6.98\% - 0.80\%}{3.57\% \times 16.54\%} = 10.47$

Puzzles under EUT

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- Under EUT, a puzzle thus arises: the solution simultaneously requires a small relative risk aversion to account for the low risk-free rate and a large relative risk aversion to account for the high equity premium

Section 2

Alternative Theories for Risky Choice

Yaari's Dual Theory

Preference on random payoff $\tilde{X} \geq 0$ represented by (Yaari 1987)

$$V(\tilde{X}) = \int \tilde{X} d(w \circ \mathrm{P}) := \int_0^\infty w\left(\mathrm{P}(\tilde{X} > x)\right) dx$$

where *probability weighting* (or *distortion*) $w : [0,1] \to [0,1]$, $\uparrow$, $w(0) = 0$, $w(1) = 1$

Rank Dependence

Assuming w is differentiable:
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where $F_{\tilde{X}}$ is CDF of $\tilde{X}$

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- For example, ranks of supremium, median, and infimum of $\tilde{X}$: 0, 1/2, and 1 respectively
- $V(\tilde{X})$ depends on ranks of random outcomes
- Consistent with *first-order stochastic dominance*: $V(\tilde{X}) \geq V(\tilde{Y})$ if $F_{\tilde{X}}(x) \leq F_{\tilde{Y}}(x)\ \forall x$

Risk Preference Dictated by Weighting

$V(\tilde{X}) = \int_0^\infty x w'(1 - F_{\tilde{X}}(x)) dF_{\tilde{X}}(x)$

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- Simultaneous risk averse and risk seeking when $w(\cdot)$ is inverse-S shaped

Probability Weighting Functions

- Kahneman and Tversky (1992) weighting

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},$$

- Tversky and Fox (1995) weighting

$$w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma},$$

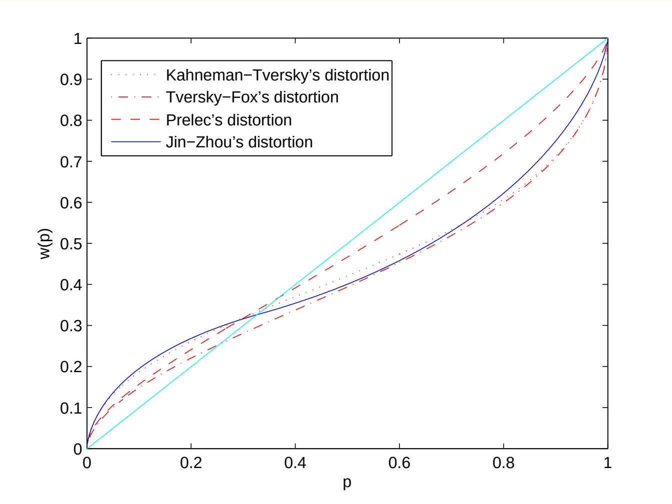
- Prelec (1998) weighting

$$w(p) = e^{-\delta(-\ln p)^\gamma}$$

- Jin and Zhou (2008) weighting

$$w(z) = \begin{cases} y_0^{b-a} k e^{a\mu + \frac{(a\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - a\sigma\right) & z \le 1 - z_0, \\ C + k e^{b\mu + \frac{(b\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - b\sigma\right) & z \ge 1 - z_0 \end{cases}$$

Inverse-S Shaped Functions



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  - A (usually assumed) inverse-S shaped (probability) weighting function: individuals overweight tails

Lopes' SP/A Theory

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  - *Dispositional factor* describes people's natural tendency to achieving security **and** exploiting potential
  - *Situational factor* describes people's responses to specific, immediate needs and opportunities

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- The nonlinear transformation z^{q_s+1} reflects the security and $1-(1-z)^{q_p+1}$ reflects the potential

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- A is the aspiration level, $0 < \alpha < 1$

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  - *Probability weighting*

CPT Preference Function

$$V(\tilde{X}) = \int_0^{\infty} w_+\left(P\left(u_+\left((\tilde{X}-\tilde{B})^+\right) > x\right)\right)dx - \int_0^{\infty} w_-\left(P\left(u_-\left((\tilde{X}-\tilde{B})^-\right) > x\right)\right)dx$$

where

- $\tilde{B}$: reference point in wealth (possibly random)

CPT Preference Function

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- Note: Tversky and Kahneman (1992) used discrete random variables

Section 3

Summary and Further Readings

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- Behavioural theories with new risk preferences have emerged

Further Readings

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