WELLPOSEDNESS OF SECOND ORDER BSDEs

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Outline

1. Introduction
    - Backward stochastic differential equations
    - Probabilistic numerical implications
2. Two intriguing examples
    - Gamma constraints
    - Market illiquidity
3. 2nd order backward SDEs
    - The CSTV framework
    - An alternative formulation of 2BSDEs
    - Main results

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Backward SDEs

Pardoux and Peng (1990, 1992) : W BM on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \mathbb{F}^W$

- For $\xi \in \mathbb{L}^2$, $H_t(y,z)$ Lipschitz in (y,z), $H_.(0,0) \in \mathbb{H}^2$ the BSDE

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

has a unique solution $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$

- For $H \equiv 0$, this is just the martingale representation theorem

- Easy proof by means of a fixed point argument

- Other (important) extensions : obstacle, quadratic in z, multidimensional Y, ...

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Motivation from finance

The BSDE

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

appeared in many contexts :

- Classical hedging problem in finance ($H \equiv 0$)
- Hedging under different lending and borrowing rates, hedging under portfolio constraints (+ nondecreasing process),
- Recursice utility, Risk measures/monetary utility functions
- Portfolio optimization (only in exponential or power expected utility framework)

Connection with PDEs

In the previous context of the BSDE :

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

- Assume further that $H_t(y,z) = h(t, X_t, y, z)$, $\xi = g(X_1)$, and

$$dX_t \;=\; b(t, X_t)dt + \sigma(t, X_t)dW_t$$

Then $Y_t = V(t, X_t)$ for some deterministic measurable function V

- V is a viscosity solution of the semilinear PDE

$$\partial_t V + \frac{1}{2}\sigma^2 D^2 V + bDV + h(t, x, V, \sigma DV) = 0, \quad V(1, x) = g(x)$$

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A Feynman-Kac formula for semilinear PDEs

- Assume that the semilinear Cauchy problem

$$\partial_t v + \frac{1}{2}\sigma^2 D^2 v + bDv + h(t,x,v,\sigma Dv) = 0, \quad v(1,x) = g(x)$$

has a classical solution with appropriate growth

- Then, $v(t,x) = Y_t^{t,x}$, where $(Y^{t,x}, Z^{t,x}) = (Y,Z)$ is the unique solution of the BSDE

$$\begin{aligned} dY_s &= -h(s,X_s,Y_s,Z_s)ds + Z_s dW_s, & Y_1 &= g(X_1) \\ dX_s &= b(s,X_s)ds + \sigma(s,X_s)dW_s & X_t &= x \end{aligned}$$

- One of our objectives is to extend this representation result to fully nonlinear Dirichlet problems

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Typical example of fully nonlinear Cauchy problem : HJB equations

- $\mathcal{U}$ set of control processes with values in U
- For every $\nu \in \mathcal{U}$, the controlled process is defined by :

$$dX_t^\nu = \mu(X_t^\nu, \nu_t)dt + \sigma(X_t^\nu, \nu_t)dW_t$$

- Denote $\beta_s^u := e^{-\int_s^T k(X_u^\nu, \nu_u)du}$, and consider the problem :

$$V(t,x) := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\int_t^T \beta_s^u f(X_s^\nu, \nu_s)ds + \beta_t^u g(X_T^\nu)\right]$$

Then the value function V is characterized by the HJB equation :

$$\begin{aligned} 0 \;=\; & -\frac{\partial V}{\partial t} - \sup_{u \in U}\Big\{\mu(x,u)\cdot\frac{\partial V}{\partial x} + \frac{1}{2}\mathrm{Tr}\left[\frac{\partial^2 V}{\partial x \partial x^{\mathrm{T}}}\sigma\sigma^{\mathrm{T}}(x,u)\right] \\ & -k(x,u)V + f(x,u)\Big\} \end{aligned}$$

Stochastic control and BSDEs

• In general HJB equation is fully nonlinear... no connection to standard BSDEs
• Extension to FBSDEs (Pontryagin Maximum principle)

• If the control only acts on the drift, then HJB equation is semilinear... the solution of the control problem can be represented by means of a BSDE
• In portfolio optimization, both drift and diffusion are controlled

- In general, no connection with standard BSDEs
- However, if utility function is Exponential or Power, then the special structure of the problem leads to a semilinear PDE after a change of variable $\Longrightarrow$ BSDE representation
- Mean-field interacting portfolio managers : conveniently approached by BSDEs (G.-E. Espinosa's PhD thesis)

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Motivation from Probabilistic Numerical Methods

Bally and Pagès (2003), Zhang (2004), Bouchard and T. (2004), Gobet, Lemor and Warin (2005), $\cdots$,

$$\hat{Y}^n_{t_n} \quad = \quad g\left(X^n_{t_n}\right), \quad \text{and for } i = 1, \ldots, n :$$

$$\begin{aligned}
\hat{Y}^n_{t_{i-1}} &= \hat{\mathbb{E}}^n_{i-1}\left[\hat{Y}^n_{t_i}\right] + \Delta t_i\, f\left(t_i, X^n_{t_{i-1}}, \hat{Y}^n_{t_{i-1}}, \hat{Z}^n_{t_{i-1}}\right) \\
\hat{Z}^n_{t_{i-1}} &= \hat{\mathbb{E}}^n_{i-1}\left[\hat{Y}^n_{t_i}\frac{\Delta W_{t_i}}{\Delta t_i}\right]
\end{aligned}$$

Motivation from Probabilistic Numerical Methods

Bally and Pagès (2003), Zhang (2004), Bouchard and T. (2004), Gobet, Lemor and Warin (2005), $\cdots$, Cheridito, Soner, T. Victoir (2006), Fahim, T. and Warin (2009)

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\hat{Z}^n_{t_{i-1}} &= \hat{\mathbb{E}}^n_{i-1}\left[\hat{Y}^n_{t_i} \frac{\Delta W_{t_i}}{\Delta t_i}\right]\\
\hat{\Gamma}^n_{t_{i-1}} &= \hat{\mathbb{E}}^n_{i-1}\left[\hat{Y}^n_{t_i} \frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}$$

- Related work by Kohn and Serfati (2007)...

Convergence of the MC-FD scheme

<Fahim, T. and Warin 09>

Theorem (i) *Suppose that f is Lipschitz uniformly in x and $\varepsilon I \leq \nabla_\gamma f \leq \sigma\sigma^{\mathrm{T}}$. Then*

$$Y_0^n(t,x) \longrightarrow v(t,x) \qquad \textit{uniformly on compacts}$$

where v is the unique viscosity solution of the nonlinear PDE.
(ii) *If f is either convex or concave in (y,z,γ), i.e. HJB operator,*

$$-Ch^{1/10} \leq v - v^h \leq Ch^{1/4}$$

- What about the convergence of the whole process (Y^n, Z^n, Γ^n) ?

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Hedging under Gamma constraints

- nonrisky asset normalized to unity
- Risky asset defined by BS model $dS_t = S_t \sigma dW_t$
- Wealth process : $Y_t^Z = Y_0 + \int_0^t Z_u dS_u$, where Z is the portfolio strategy assumed to be a semimartingale with

$$d\langle Z, S\rangle_t \quad = \quad \Gamma_t d\langle S\rangle_t \ \ \text{(the so-called Gamma)}$$

- Given a payoff $\xi \in \mathbb{L}^0(\mathcal{F}_T)$, superhedging problem :

$$V_0 \quad := \quad \inf\left\{ Y_0 :\ \exists\, Z \in \mathcal{Z},\ \underline{\Gamma} \le \Gamma \le \overline{\Gamma} \text{ and } Y_T^Z \ge \xi \ \mathbb{P}-\text{a.s.}\right\}$$

- Or the corresponding hedging problem : find $Z \in \mathcal{Z}$ and some "minimal" nondecreasing process K, $K_0 = 0$, such that

$$Y_T^Z - K_T = \xi,\ \mathbb{P}-\text{a.s}.$$

(open problem)

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Hedging under market illiquidity

(Çetin, Jarrow and Protter 2004, 2006)
Risky asset price is defined by

- the marginal price S_t, $t \geq 0$
- the supply curve $\nu \longmapsto \mathcal{S}(., \nu)$:

$$\mathcal{S}(S_t, \nu) \;\; \text{price per share of } \nu \text{ risky assets}$$

with $\mathcal{S}(s, 0) = s$

Z^0_t : holdings in cash, Z_t : holdings in risky asset

With $\Delta Z^0_{t_i} = Z^0_{t_{i+1}} - Z^0_{t_i}$, $\Delta Z_{t_i} = Z_{t_{i+1}} - Z_{t_i}$,

$$\text{budget constraint : } \;\; \Delta Z^0_{t_i} + \Delta Z_{t_i} \mathcal{S}\left(S_t, \Delta Z_{t_i}\right) = 0$$

$$\Longrightarrow \;\; Z^0_T = Z^0_0 - \sum \Delta Z_{t_i} \mathcal{S}\left(S_t, \Delta Z_{t_i}\right) = Z^0_0 + \sum Z_{t_{i-1}} \Delta S_{t_i} + \ldots$$

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Continuous-time formulation of Model

Set $Y_t \;:=\; Z^0_t + Z_t S_t$, then :

$$Y_T \;=\; Y_0 + \sum Z_{t_{i-1}} \Delta S_{t_i} - \sum \Delta Z_{t_i} \left[\mathcal{S}\left(S_{t_i}, \Delta Z_{t_i}\right) - \mathcal{S}\left(S_{t_i}, 0\right) \right]$$

Assume $\nu \longmapsto \mathcal{S}(., \nu)$ is smooth (unlike proportional transaction costs models), then :

$$Y_T = Y_0 + \int_0^T Z_t dS_t - \int_0^T \frac{\partial \mathcal{S}}{\partial \nu}\left(S_t, 0\right) d\langle Z^c \rangle_t - \sum_{t \le T} \Delta Z_t \left[\mathcal{S}\left(S_t, \Delta Z_t\right) - S_t \right]$$

- Let $dS_t = S_t \sigma dW_t$, then

$$d\langle Z^c \rangle_t = \Gamma^2_t \; d\langle S \rangle_t \; : \qquad \text{the so-called Gamma...}$$

The Hedging Problem

Option / contingent claim : ξ

Wealth process (jumps in portfolio can be shown to be sub-optimal) :

$$dY_t = Z_t dS_t - \frac{\partial \mathcal{S}}{\partial \nu}(S_t, 0)\Gamma_t^2 \; d\langle S\rangle_t \quad \text{where} \quad d\langle Z, S\rangle_t =: \Gamma_t d\langle S\rangle_t$$

Super-hedging problem

$$V := \inf \left\{ y \;:\; Y_T^{y,Z} \geq \xi \;\mathbb{P}-\text{a.s. for some "admissible" } Z \right\}$$

Here, admissibility is the crucial issue

Main difficulty

Without further restrictions on trading strategies, **the continuous-time problem reduces to Black-Scholes**!

Lemma (Bank-Baum 04) *For predictable $W-$integ. càdlàg process ϕ, and $\varepsilon > 0$, there exists an absolutely continuous predictable process* $\phi^\varepsilon_t = \phi^\varepsilon_0 + \int_0^t \alpha_r dr$ such that

$$\sup_{0\le t\le 1} \left| \int_0^t \phi_r dW_r - \int_0^t \phi^\varepsilon_r dW_r \right| \le \varepsilon$$

$\Longrightarrow$ If the "admissibility" set allows for arbitrary a.c. portfolio $Z_t = Z_0 + \int_0^t \alpha_u du$, then $V = V^{\mathrm{BS}}$ (with $\Gamma = 0$!)

<Cetin, Jarrow, Protter 04>

Asymptotics of the discrete-time solution

BUT Gokay and Soner 09 showed that the discrete-time super-hedging cost (with time step $\frac{1}{n}$)

$$V^n \longrightarrow V^\infty \neq V^{\mathrm{BS}} \;!$$

V^∞ is characterized as the unique viscosity solution of

$$-V_t^\infty(t,s) + \frac{1}{4}s^2\sigma(t,s)^2\ell(s)\left[1-\left(\frac{V_{ss}^\infty(t,s)}{\ell(s)}+1\right)^{+^2}\right] \;=\; 0$$

with $V^\infty(T,.) = g$ and $-C \le V^\infty \le C(1+s)$. Here $\ell := \left(4\frac{\partial \mathcal{S}}{\partial \nu}\right)^{-1}$

In the continuous-time problem, Soner and T. derive directly this nonlinear PDE under appropriate restrictions on the trading strategies... similar to previous work on Gamma consraints with Cheridito

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Second Order Backward SDEs

Cheridito, Soner, T. and Victoir 07 (CSTV) :

- 2BSDE :

$$dY_t \;=\; H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dW_t, \quad \Gamma_t dt := d\langle Z, W\rangle_t, \quad Y_1 = \xi$$

where

$$Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d\langle Z, W\rangle_t = Z_t dW_t + \frac{1}{2}\Gamma_t dt$$

is the Fisk-Stratonovich stochastic integration.

- If $H_t = h(t, W_t, Y_t, Z_t, \Gamma_t)$ and $\xi = g(W_1)$, then $Y_t = V(t, W_t)$, where V is associated with the fully nonlinear PDE :

$$\partial_t V + h(t, x, V, DV, D^2V) = 0 \quad \text{and} \quad V(1, x) = g(x).$$

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The uniqueness result of CSTV

CSTV only deal with the Markov case

- Existence : if corresponding PDE has a smooth solution, then

$$Y_t = V(t, W_t), \quad Z_t = DV(t, W_t), \quad \Gamma_t = D^2 V(t, W_t).$$

- Uniqueness : Second Order Stochastic Target Problem

$$\begin{aligned} V(t,x) &:= \inf\left\{y : \ Y_1^{y,Z} \geq g(W_1) \text{ for some } Z \in \mathcal{Z}\right\} \\ U(t,x) &:= \sup\left\{y : \ Y_1^{y,Z} \leq g(W_1) \text{ for some } Z \in \mathcal{Z}\right\} \end{aligned}$$

V and U are resp. viscosity super and subsolution of the PDE

If the comparison principle for viscosity solutions of PDE holds, then 2BSDE has a unique solution in class $\mathcal{Z}$

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The admissibility set $\mathcal{Z}$ in CSTV

Definition $Z \in \mathcal{Z}$ if it is of the form

$$Z_t = \sum_{n=0}^{N-1} z_n \mathbf{1}_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_s ds + \int_0^t \Gamma_s dW_s$$

- (τ_n) is an ↗ seq. of stop. times, z_n are $\mathcal{F}_{\tau_n}-$measurable, $\|N\|_\infty < \infty$
- Z_t and Γ_t are $\mathbb{L}^\infty-$bounded up to some polynomial of X_t
- $\Gamma_t = \Gamma_0 + \int_0^t a_s ds + \int_0^t \xi_s dW_s$, $0 \le t \le T$, and

$$\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty, \qquad \|\phi\|_{B,b} := \left\| \sup_{0\le t\le T} \frac{|\phi_r|}{1+X_t^B} \right\|_{\mathbb{L}^b}$$

Uniqueness in larger class

Counre-example The following linear 2BSDE with constant coefficients has a nonzero solution in $\mathbb{L}^2$:

$$dY_t = -\frac{1}{2} c \Gamma_t dt + Z_t \circ dW_t, \qquad Y_1 = 0,$$

whenever $c \neq 1$

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General framework

$\Omega := C([0,1],\mathbb{R}^d)$, B : coordinate process, $\mathbb{P}_0$: Wiener measure
$\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq 1}$: filtration generated by B, $\mathbb{F}^+$

$\mathbb{P}$ is a local martingale measure if B local martingale under $\mathbb{P}$

Karandikar 95 : $\int_0^t B_s dB_s$, defined $\omega-$wise, and coincides with the Itô integral, $\mathbb{P}-$a.s. for all local martingale measure $\mathbb{P}$. Then

$$\langle B\rangle_t := B_t B_t^{\mathrm{T}} - 2\int_0^t B_s dB_s^{\mathrm{T}} \quad \text{and} \quad \hat{a}_t := \overline{\lim_{\varepsilon\downarrow 0}} \frac{1}{\varepsilon}\Big(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\Big),$$

defined $\omega-$wise

$\overline{\mathcal{P}}_W$:set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \quad \text{is a. c. in } t \text{ and} \quad \hat{a} \text{ takes values in } \mathbb{S}_d^{>0}(\mathbb{R}),\ \mathbb{P}-\text{a.s.}$$

General framework, continued

For every $\mathbb{F}-$prog. meas. α valued in $\mathbb{S}_d^{>0}(\mathbb{R})$ with $\int_0^1 |\alpha_t| dt < \infty$, $\mathbb{P}_0-$a.s. Define

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0,1], \mathbb{P}_0 - \text{a.s.}$$

$\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$: collection of all such $\mathbb{P}^\alpha$

Then every $\mathbb{P} \in \overline{\mathcal{P}}_S$

- satisfies the Blumenthal zero-one law
- and the martingale representation property

Nonlinear generators

$H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, $D_H \subset \mathbb{R}^{d \times d}$ given, containing 0

- For fixed (y, z, γ), H is $\mathbb{F}-$progressively measurable
- H is uniformly Lipschitz in (y, z), lsc in γ
- H is uniformly continuous in ω under the $\mathbb{L}^\infty-$norm

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}(\mathbb{R});$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0)$$

$\mathcal{P}_H$: set of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \bar{a}_{\mathbb{P}}, \text{for some } \underline{a}_{\mathbb{P}}, \bar{a}_{\mathbb{P}} \text{ and } \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{F}_t^0|^2 dt \right] < \infty.$$

Def $\mathcal{P}_H-$q.s. means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H$ (Denis-Martini 04)

Target problem and relaxations $\mathcal{V}(\xi) \geq \bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi)$

- $Z \in \cap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{S}\mathcal{M}^2(\mathbb{P})$, $d\langle Z, B\rangle_t = \Gamma_t d\langle B\rangle_t$, $\mathcal{P}_H-$q.s. and

$$Y_t^Z = Y_0^Z - \int_0^t H_s(Y_s, Z_s, \Gamma_s)ds + \int_0^t Z_s \circ dB_s$$

Second order target problem :

$$\mathcal{V}(\xi) := \inf\left\{Y_0 : \; Y_T^Z \geq \xi \; \mathcal{P}_H - \text{q.s. } Z \in \cap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{S}\mathcal{M}^2(\mathbb{P})\right\}$$

Relax 1 $\bar{Y}_t^{\mathbb{P},\bar{Z}} = \bar{Y}_0 + \int_0^t \left(\frac{1}{2}\hat{a}_s : \bar{\Gamma}_s - H_s(\bar{Y}_s, \bar{Z}_s, \bar{\Gamma}_s)\right) ds + \int_0^t \bar{Z}_s dB_s$,

$$\bar{\mathcal{V}}(\xi) := \inf\left\{\bar{Y}_0 : \; \exists \; \bar{Z}, \bar{G} \in \cap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{H}^2(\mathbb{P}), \; \bar{Y}_T^{\mathbb{P},\bar{Z}} \geq \xi \; \mathbb{P} - \text{a.s. } \mathbb{P} \in \mathcal{P}_H\right\}$$

Relax 2 $\bar{\bar{Y}}_t^{\mathbb{P},\bar{\bar{Z}}} = \bar{\bar{Y}}_0 + \int_0^t \hat{F}_s\left(\bar{\bar{Y}}_s, \bar{\bar{Z}}_s\right) ds + \int_0^t \bar{\bar{Z}}_s dB_s$,

$$\bar{\bar{\mathcal{V}}}(\xi) := \inf\left\{\bar{\bar{Y}}_0 : \; \exists \; \bar{\bar{Z}} \in \cap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{H}^2(\mathbb{P}), \; \bar{\bar{Y}}_T^{\mathbb{P},\bar{\bar{Z}}} \geq \xi \; \mathbb{P} - \text{a.s. } \mathbb{P} \in \mathcal{P}_H\right\}$$

Definition

For $\mathcal{F}_1-$meas. ξ, consider the 2BSDE :

$$Y_t = \xi - \int_t^1 \hat{F}_s(Y_s, Z_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t,\ 0 \le t \le 1,\ \mathcal{P}_H - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K^{\mathbb{P}}_t := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ t \in [0,1],\ \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ satisfies :

$$K^{\mathbb{P}}_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}{\text{ess inf}}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}'}_t[K^{\mathbb{P}'}_1],\ \ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H, t \le T$$

If the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ can be aggregated into a universal process K, we call (Y, Z, K) a solution of the 2BSDE

Outline

1. Introduction
    - Backward stochastic differential equations
    - Probabilistic numerical implications
2. Two intriguing examples
    - Gamma constraints
    - Market illiquidity
3. 2nd order backward SDEs
    - The CSTV framework
    - An alternative formulation of 2BSDEs
    - Main results

Spaces and norms

- $\mathbb{L}^p_H := \{\xi\ \mathcal{F}_1 - \text{meas.}:\ \ \|\xi\|^p_{\mathbb{L}^p_H} < \infty\}$
- $\mathbb{H}^p_H := \{Z\ \mathbb{F}^+ - \text{prog. meas. in } \mathbb{R}^d:\ \ \|Z\|^p_{\mathbb{H}^p_H} < \infty\}$

$$\|\xi\|^p_{\mathbb{L}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|\xi|^p],\ \ \|Z\|^p_{\mathbb{H}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\Big)^{p/2}\Big]$$

- $\mathbb{D}^p_H := \{Y\ \mathbb{F}^+ - \text{prog. in } \mathbb{R} \text{ càdlàg } \mathcal{P}_H - \text{q.s. } \|Y\|^p_{\mathbb{D}^p_H} < \infty\}$

$$\|Y\|^p_{\mathbb{D}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} |Y_t|^p\Big]$$

- $\mathbb{L}^2_{H,*} := \{\xi \in \mathbb{L}^2_H : \|\xi\|_{\mathbb{L}^2_{H,*}} < \infty\}$

$$\|\xi\|_{\mathbb{L}^2_{H,*}} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} \mathbb{E}^{H,\mathbb{P}}_t[|\xi|^2]\Big],\ \mathbb{E}^{H,\mathbb{P}}_t[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

- $\hat{\mathbb{L}}^2_H :=$ closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^2_{H,*}}$

Representation and uniqueness

Assumption $\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} \mathbb{E}_t^{H,\mathbb{P}}\big[\int_t^1 |\hat{F}_s^0|^2 ds\big]\Big] < \infty$, and there exists C such that for all (y, z_1, z_2) :

$$\big|\hat{F}_t(y,z_1) - \hat{F}_t(y,z_2)\big| \le C|\hat{a}_t^{1/2}(z_1 - z_2)|, \; dt \times d\mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H$$

Theorem Let $\xi \in \mathbb{L}^2_{H,*}$ and suppose $(Y,Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution of the 2BSDE. Then, for any $\mathbb{P} \in \mathcal{P}_H$ and $0 \le t_1 < t_2 \le 1$,

$$Y_{t_1} = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t_1,\mathbb{P})}{}^{\mathbb{P}} \; \mathcal{Y}_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \quad \mathbb{P} - \text{a.s.}$$

Consequently, the 2BSDE has at most one solution in $\mathbb{D}^2_H \times \mathbb{H}^2_H$

Corollary Comparison holds true

A priori estimates

Theorem (i) Let $\xi \in \mathbb{L}^2_{H,*}$ and $(Y,Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ a solution of the 2BSDE. Then

$$\|Y\|^2_{\mathbb{D}^2_H} + \|Z\|^2_{\mathbb{H}^2_H} + \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|K_1^{\mathbb{P}}|^2] \le C\big(\|\xi\|^2_{\mathbb{L}^2_{H,*}} + \|\hat{F}^0\|^2_{\mathbb{H}^2_{H,*}}\big)$$

(ii) Let $\xi^i \in \mathbb{L}^2_{H,*}$ and $(Y^i,Z^i) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ corresponding solutions to the 2BSDE, $i=1,2$. Then, with $\delta\xi := \xi^1 - \xi^2$, $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$, and $\delta K^{\mathbb{P}} := K^{1,\mathbb{P}} - K^{2,\mathbb{P}}$:

$$\begin{aligned} \|\delta Y\|_{\mathbb{D}^2_H} &\le C\|\delta\xi\|_{\mathbb{L}^2_{H,*}} \quad \text{and} \\ \|\delta Z\|^2_{\mathbb{H}^2_H} &+ \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} |\delta K_t^{\mathbb{P}}|^2\Big] \\ &\le C\|\delta\xi\|^2_{\mathbb{L}^2_{H,*}} + C\big(|\xi^1\|_{\mathbb{L}^2_{H,*}} + \|\hat{F}^0\|_{\mathbb{H}^2_{H,*}}\big)\|\delta\xi\|_{\mathbb{L}^2_{H,*}} \end{aligned}$$

Existence

Theorem For any $\xi \in \hat{\mathbb{L}}^2_H$, the 2BSDE admits a unique solution $(Y, Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$.

Recall $\hat{\mathbb{L}}^2_H :=$ closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^2_{H,*}}$, where

$$\|\xi\|_{\mathbb{L}^2_{H,*}} \quad := \quad \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} \mathbb{E}_t^{H,\mathbb{P}}[|\xi|^2]\right]$$

and

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})}{}^{\mathbb{P}} \; \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

Connection with PDEs

Theorem Under "natural conditions", the solution of the 2BSDE satisfies $Y_t = u(t, B_t)$, $t \in [0, T]$, and u is a viscosity solution of

$$\frac{\partial u}{\partial t}(t,x) + \hat{H}\Big(t, x, u(t,x), Du(t,x), D^2u(t,x)\Big) \quad = \quad 0, \ 0 \leq t < 1$$
$$u(1,x) = g(x)$$

where

$$\hat{H}(t,x,y,z,\gamma) \quad = \quad \sup_{a \in \mathbb{S}_d^+(\mathbb{R})} \left\{\frac{1}{2}a : \gamma - F(t,x,y,z,a)\right\}, \ \gamma \in \mathbb{R}^{d\times d}.$$

We also have a Feynman-Kac representation theorem for the Cauchy problem with the latter fully nonlinear PDE