Order book resilience, price manipulation, and the positive portfolio problem

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Workshop on
New Directions in Financial Mathematics
Institute for Pure and Applied Mathematics, UCLA
January 6, 2010

Joint work with Jim Gatheral, Aurélien Alfonsi, and Alla Slynko

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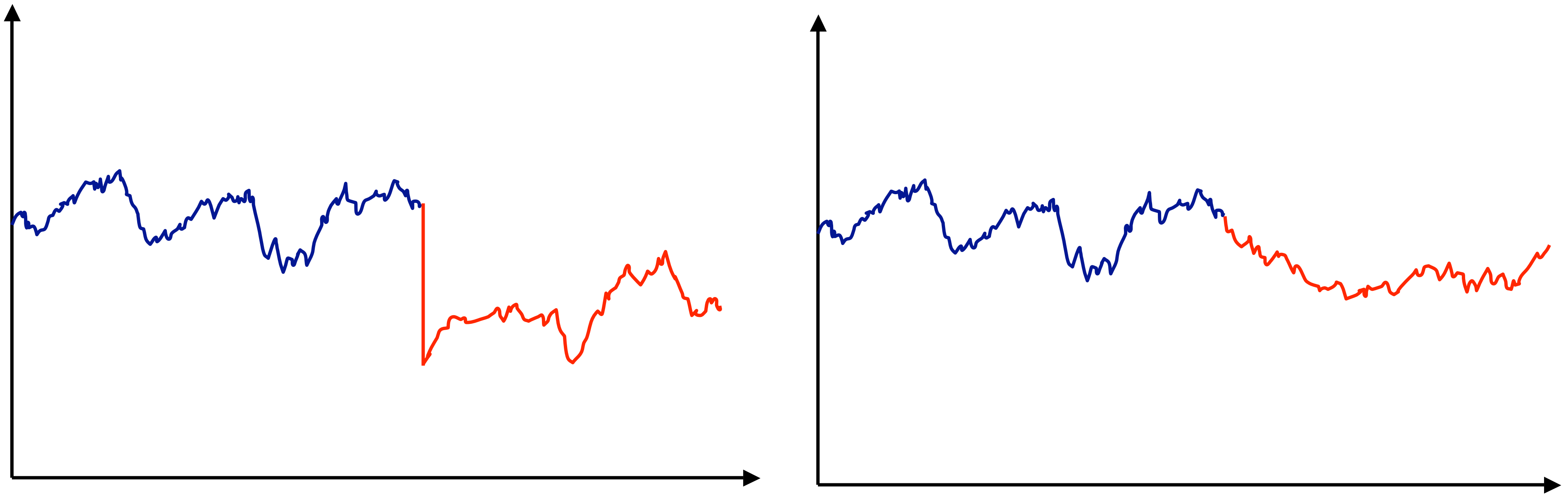
Market impact: adverse feedback effect on the quoted price of a stock caused by one's own trading



Basic observation: liquidity costs of a large trade can be reduced signficantly by splitting the trade into a sequence of smaller trades, which are then spread out over a certain time interval.

Questions:

- What is an appropriate model for market impact?
- When is a model ‘viable’? Can there be undesirable properties?
- What are the optimal trade execution strategies?
- Are strategies and models robust w.r.t. model parameters?

Unaffected price process: martingale S^0

Admissible trategy: adapted process $X = (X_t)$ that describes the number of shares held by the trader

- $t \to X_t$ is rightcontinuous with finite total variation
- the signed measure dX_t has compact support
- w.l.o.g. $X_t = 0$ for large enough t.

For instance, when $X_t = x$ for $t < t_0$ and $X_t = 0$ for $t \geq t_0$, then X describes a single trade of $|x|$ shares, executed at time t_0, which is a sell trade for $x > 0$ and a buy trade for $x < 0$.

Impacted price process:

$$S_t = S_t^0 + \int_{\{s<t\}} G(t-s)\, dX_s,$$

where

$$G : (0, \infty) \to [0, \infty)$$

is the decay kernel. It describes the resilience of price impact between trades; see Bouchaud et al. (2004), Obizhaeva and Wang (2005), Alfonsi et al. (2008, 2007), Gatheral (2008).

We first assume

$$(1) \qquad G \text{ is bounded and } G(0) := \lim_{t \downarrow 0} G(t) \text{ exists.}$$

Costs of a strategy X:

When X is continuous at t, then the infinitesimal order dX_t is executed at price S_t, so $S_t\,dX_t$ is the cost increment.
Thus, the total costs of a continuous strategy are

$$\int S_t\,dX_t = \int S_t^0\,dX_t + \int\int_{\{s<t\}} G(t-s)\,dX_s\,dX_t.$$

When X has a jump ΔX_t, then the price is moved from S_t to

$$S_{t+} = S_t + \Delta X_t G(0)$$

This linear price impact corresponds to a constant supply curve for which $G(0)^{-1}\,dy$ buy or sell orders are available at each price y. The trade ΔX_t is thus carried out at the following cost,

$$\int_{S_t}^{S_{t+}} yG(0)^{-1}\,dy = \frac{1}{2G(0)}\big(S_{t+}^2 - S_t^2\big) = \frac{G(0)}{2}(\Delta X_t)^2 + \Delta X_t S_t.$$

Hence, the total costs of an arbitrary admissible strategy X are given by

$$\begin{aligned}
&\int S_t\,dX_t + \frac{G(0)}{2}\sum(\Delta X_t)^2 \\
&\quad = \int S_t^0\,dX_t + \int\int_{\{s<t\}} G(t-s)\,dX_s\,dX_t + \frac{G(0)}{2}\sum(\Delta X_t)^2 \\
&\quad = \int S_t^0\,dX_t + \frac{1}{2}\int\int G(|t-s|)\,dX_s\,dX_t.
\end{aligned}$$

It therefore follows from the martingale property of S^0 that the **expected costs** of an admissible strategy are

$$S_0^0 X_0 + \frac{1}{2}\mathbb{E}[\,\mathcal{C}(X)\,],$$

where

$$\mathcal{C}(X) := \int\int G(|t-s|)\,dX_s\,dX_t.$$

Remark: Instead of this simple market impact model, one can consider more complicated models for (block-shaped) electronic limit order books. In these models one can then show that

$$\text{Expected costs } \geq S_0^0 X_0 + \frac{1}{2}\mathbb{E}[\,\mathcal{C}(X)\,]$$

with equality for monotone strategies X.

Two questions:

- Can there be model irregularities?
- Existence, uniqueness, and structure of strategies minimizing the expected costs?

Definition 1 (Huberman and Stanzl (2004)). A round trip is an admissible strategy with $X_0 = 0$. A price manipulation strategy is a round trip with strictly negative expected costs.

Clearly, there is no price manipulation when

$$\mathcal{C}(X) \geq 0 \qquad \text{for all strategies } X\,.$$

Proposition 1 (Straightforward extension of Bochner's thm). *$\mathcal{C}(X) \geq 0$ for all strategies $X \iff G(|\cdot|)$ can be represented as the Fourier transform of a positive finite Borel measure μ on $\mathbb{R}$, i.e.,*

$$G(|x|) = \int e^{ixz}\,\mu(dz).$$

If, in addition, the support of μ is not discrete, then $\mathcal{C}(X) > 0$ for every nonzero admissible strategy X.

Optimal trade execution strategies: Minimizing expected costs,

$$S_0^0 x + \frac{1}{2}\mathbb{E}[\,\mathcal{C}(X)\,]$$

for strategies that liquidate a given long or short position of x shares within a given time frame.

Time constraint: compact set $\mathbb{T} \subset [0, \infty)$.

Boils down to minimizing $\mathcal{C}(\cdot)$ over

$$\mathcal{X}(x, \mathbb{T}) := \Big\{ X \,\Big|\, \text{deterministic strategy with } X_0 = x \text{ and support in } \mathbb{T} \Big\}.$$

Simple when $\mathbb{T}$ is discrete. Existence of minimizers not clear when $\mathbb{T}$ is not discrete.

Proposition 2. *When G is strictly positive definite there exists at most one optimal strategy for given x and $\mathbb{T}$.*

Proposition 3. *Suppose that G is positive definite. Then $X^* \in \mathcal{X}(x, \mathbb{T})$ is optimal if and only if there is a constant λ such that X^* solves the generalized Fredholm integral equation*

$$\int G(|t-s|)\, dX_s^* = \lambda \qquad \text{for all } t \in \mathbb{T}. \tag{2}$$

In this case, $\mathcal{C}(X^*) = \lambda x$. In particular, λ must be nonzero as soon as G is strictly positive definite and $x \neq 0$.

Examples

Example 1: Exponential decay

For the exponential decay kernel

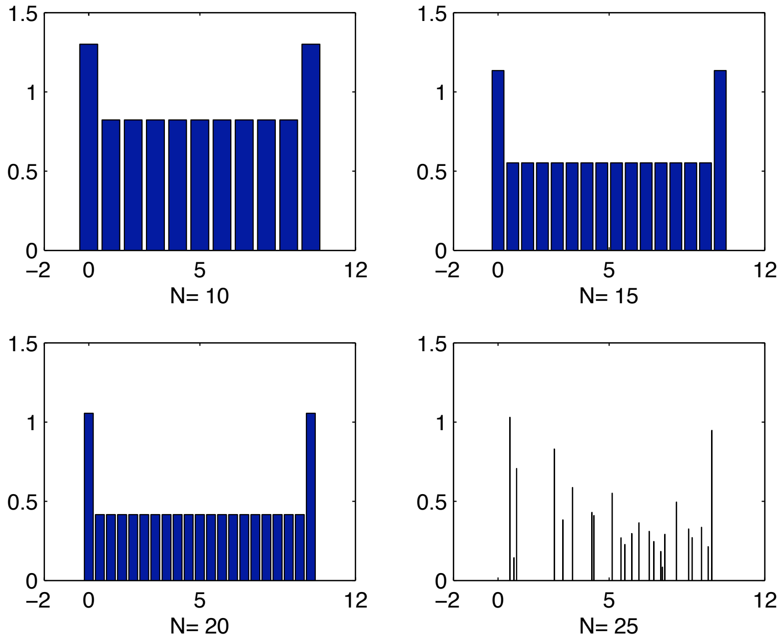
$$G(t) = e^{-\rho t},$$

$G(|\cdot|)$ is the Fourier transform of the positive measure

$$\mu(dt) = \frac{1}{\pi} \frac{\rho}{\rho^2 + t^2}\, dt$$

Hence, G is strictly positive definite.

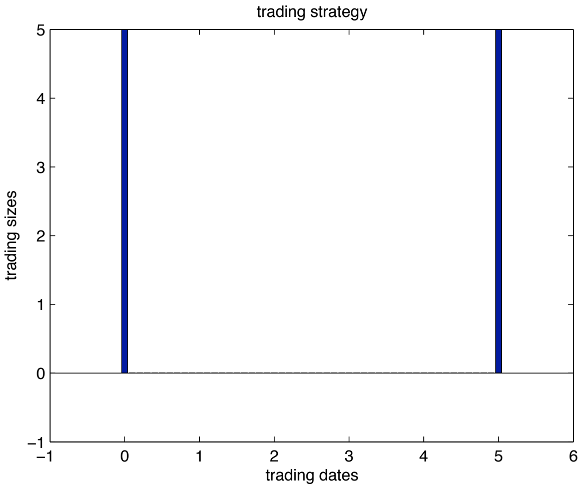
Optimal strategies for $G(t) = e^{-\rho t}$ and discrete $\mathbb{T}$:



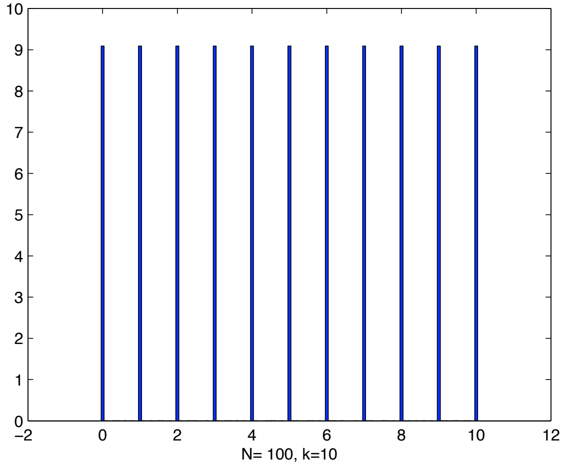
For $\mathbb{T} = [0, T]$:

$$dX_s^* = \frac{x}{\rho T + 2}\Big(\delta_0(ds) + \rho\, ds + \delta_T(ds)\Big).$$

Example 2: Capped linear decay $G(t) = (1 - \rho t)^+$

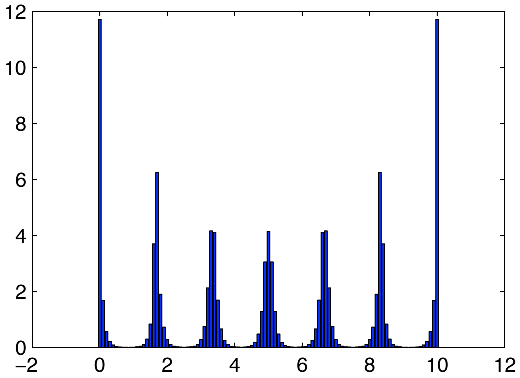


$\rho \leq 1/T$ and arbitrary $\mathbb{T}$

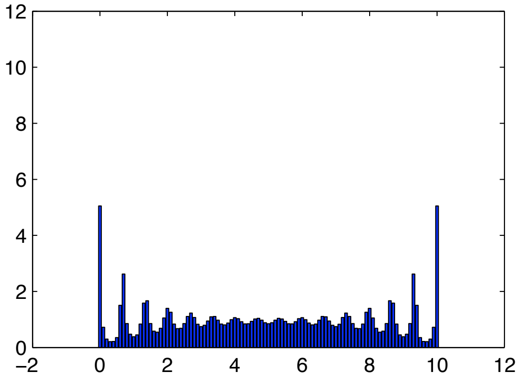


$\rho = N/T$, $\mathbb{T} = [0, T]$ or equisitant

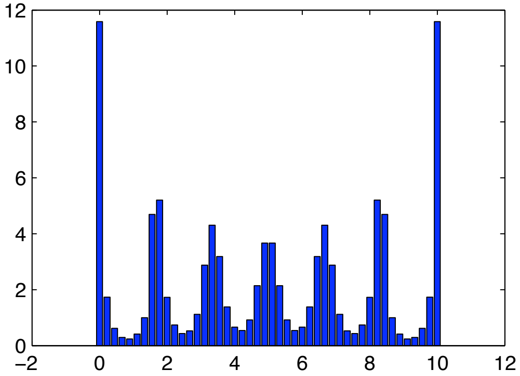
Otherwise, for equistant grid $\mathbb{T}$,



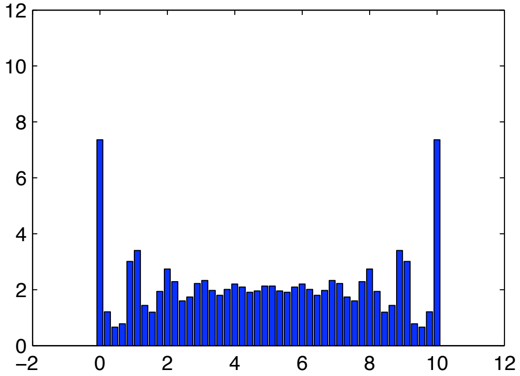
N= 100, k=6



N= 100, k=15



N= 45, k=6



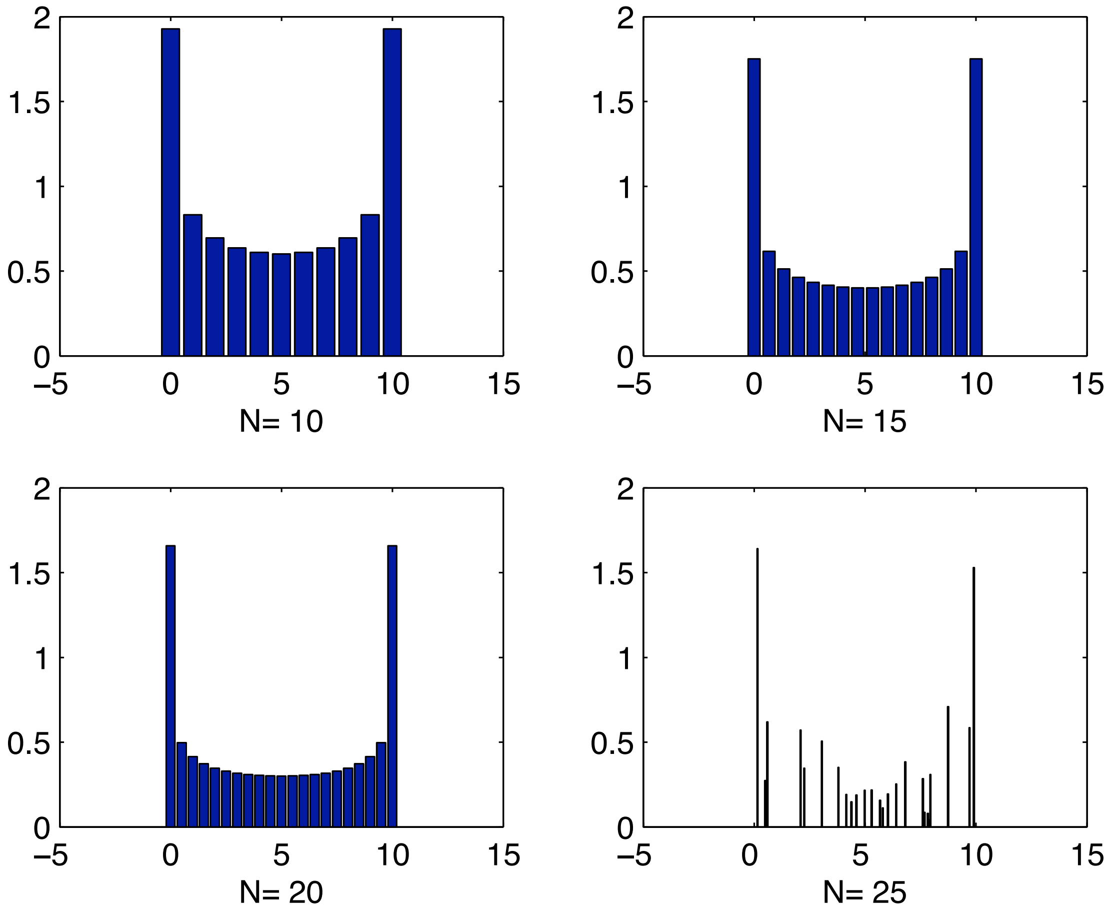
N= 45, k=10

More generally: Convex decay

Theorem [Carathéodory (1907), Toeplitz (1911), Young (1912)]

G is convex, decreasing, nonnegative, and nonconstant $\Longrightarrow$
$G(|\cdot|)$ is strictly positive definite.

Example 3: Power law decay $G(t) = (1+t)^{-\alpha}$ and equidistant grid $\mathbb{T}$,



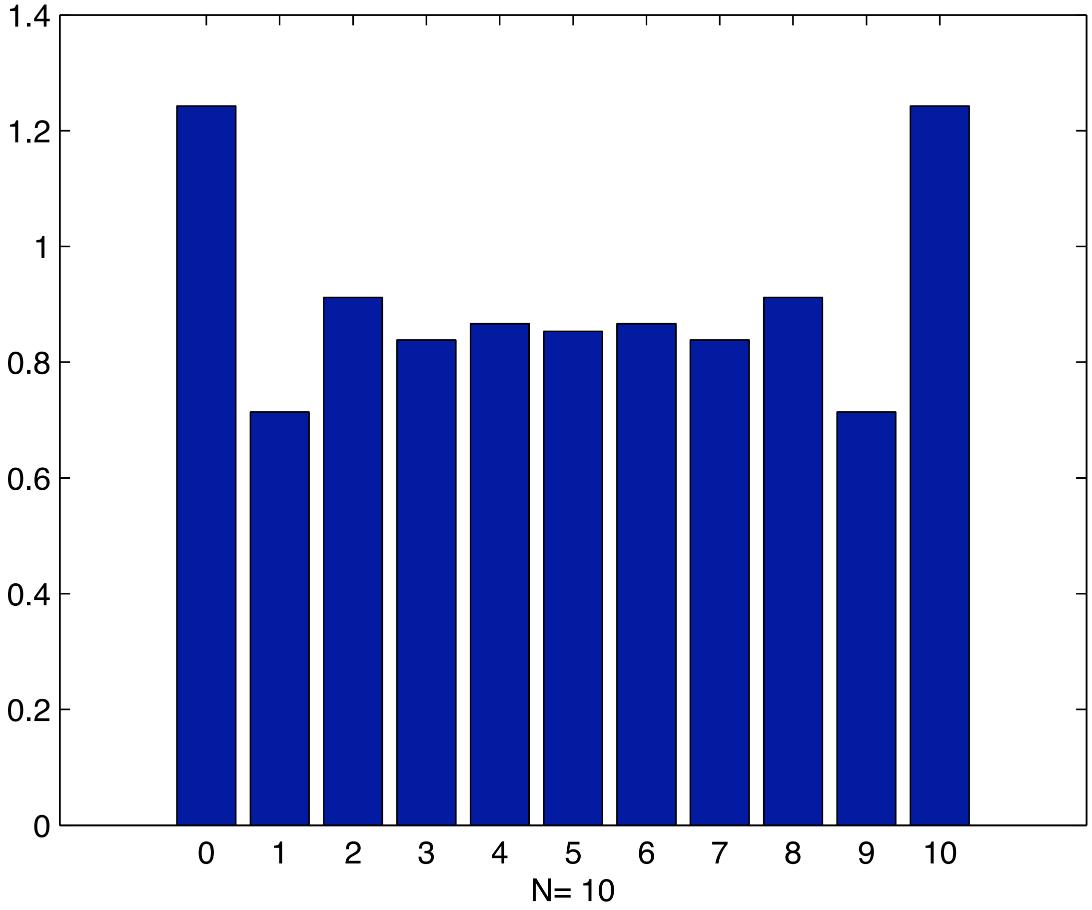
Example 4: Gaussian decay

The Gaussian decay function

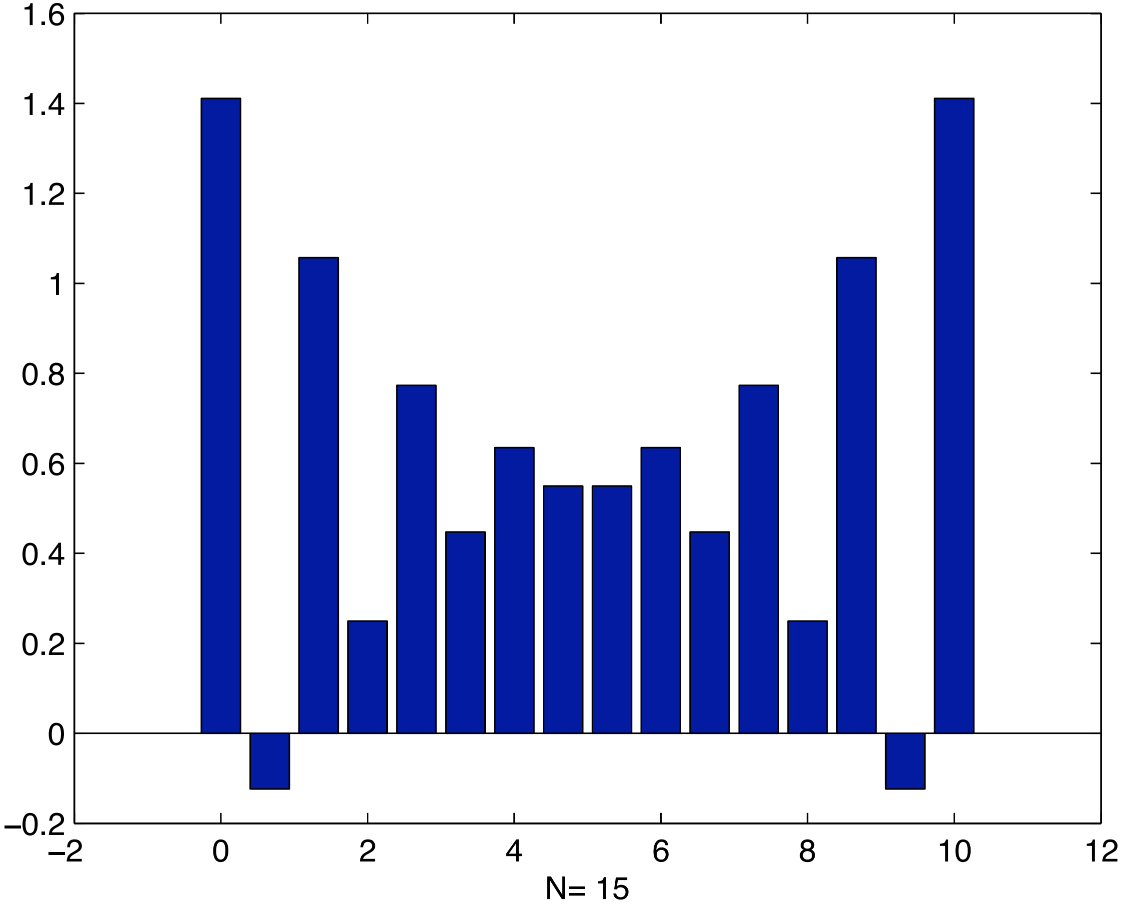
$$G(t) = e^{-t^2}$$

is its own Fourier transform (modulo constants). The corresponding quadratic form is hence positive definite.

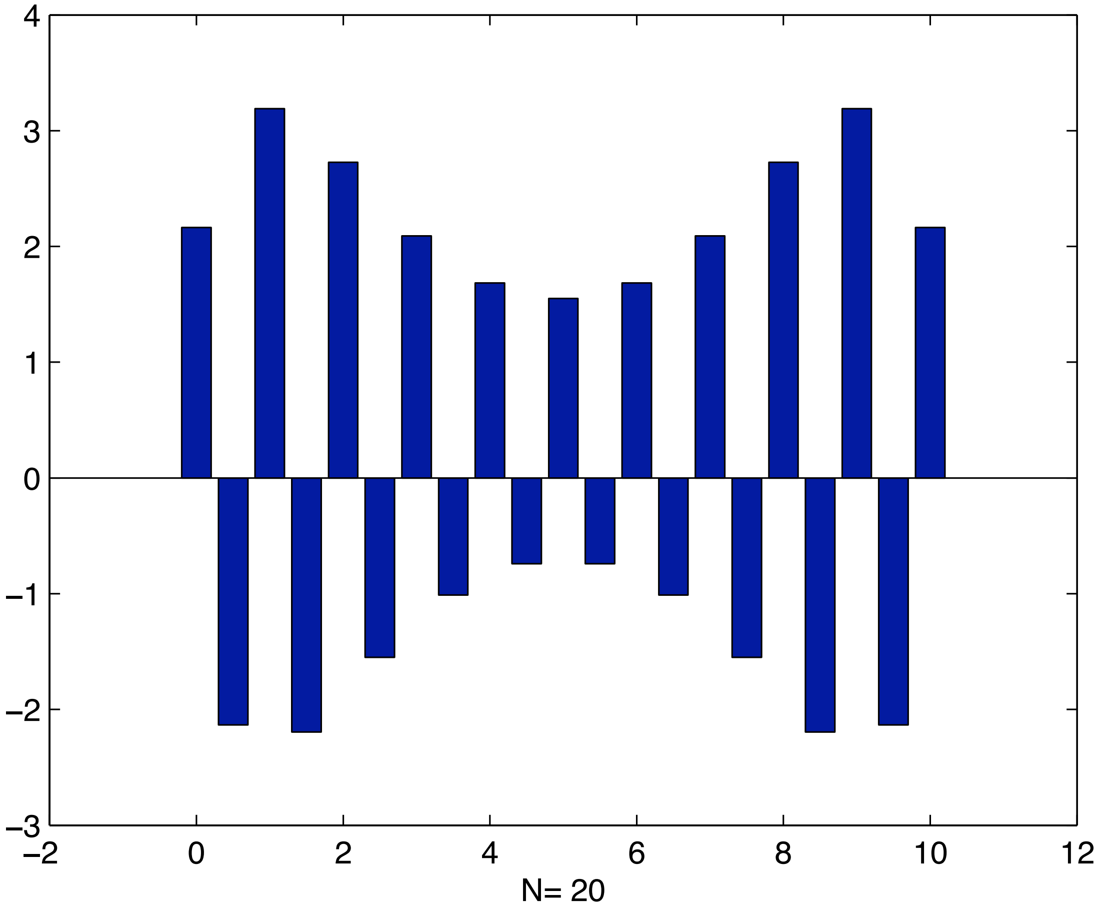
Gaussian decay $G(t) = e^{-t^2}$, $N = 10$



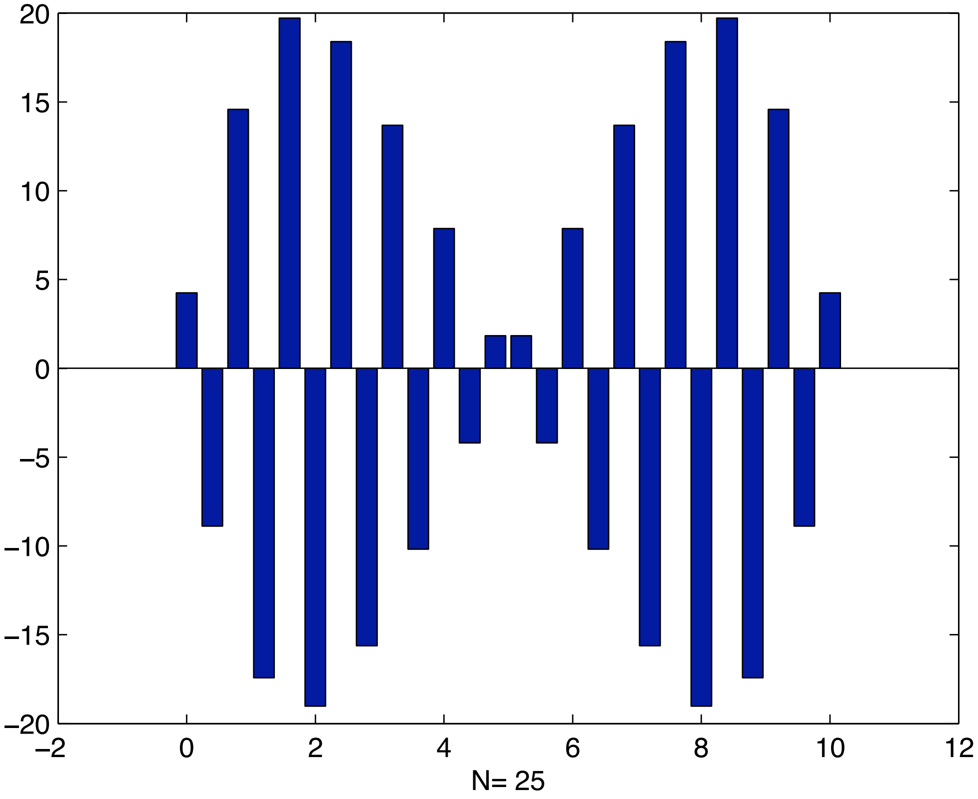
Gaussian decay $G(t) = e^{-t^2}$, $N = 15$



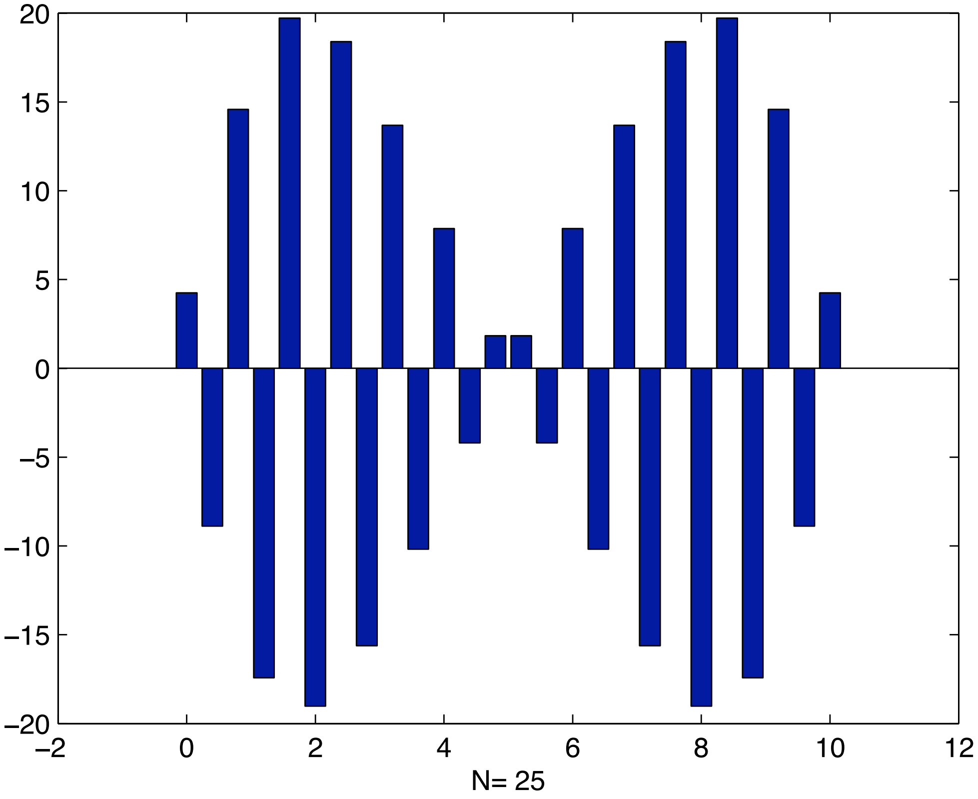
Gaussian decay $G(t) = e^{-t^2}$**,** $N = 20$



Gaussian decay $G(t) = e^{-t^2}$, $N = 25$



Gaussian decay $G(t) = e^{-t^2}$**,** $N = 25$



⇒ absence of price manipulation strategies is not enough

Definition [Hubermann & Stanzl (2004)]
A market impact model admits

price manipulation

if there is a round trip with negative expected liquidation costs.

Definition: Alfonsi, A.S., & Slynko (2009)]
A market impact model admits

transaction-triggered price manipulation

if the expected liquidation costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

Situation for non-discrete $\mathbb{T}$:

Theorem 1. *Suppose that* $G(|\cdot|)$ *is the Fourier transform of a finite Borel measure* μ *for which*

$$\int e^{\varepsilon x}\,\mu(dx) < \infty \qquad \text{for some } \varepsilon > 0. \tag{3}$$

Suppose furthermore that the support of μ *is not discrete. Then there are no optimal strategies in* $\mathcal{X}(x, \mathbb{T})$ *when* $x \neq 0$ *and* $\mathbb{T}$ *is not discrete.*

Examples:

$$G(t) = e^{-\rho t^2} \qquad \text{or} \qquad G(t) := \frac{1}{\rho^2 + t^2},$$

$$\text{or} \qquad G(t) = 2\frac{1 - \cos \rho t}{t^2} \qquad \text{or} \qquad G(t) = 1 + \frac{\sin \rho t}{t},$$

Sketch of proof: Suppose that X^* would be an optimal strategy. Due to the exponential moment condition,

$$h(t) := \int G(|t-s|)\,dX^*_s = \int\int e^{i(s-t)y}\,\mu(dy)\,dX^*_s = \int e^{-ity}\widehat{X}^*(y)\,\mu(dy)$$

admits an holomorphic continuation to the strip

$$S := \big\{z \in \mathbb{C}\,\big|\, -\varepsilon < \Im(z) < \varepsilon\big\}.$$

which is given by

$$h(z) = \int e^{-izy}\widehat{X}^*(y)\,\mu(dy), \qquad z \in S.$$

Next, $h(-t)$ is the Fourier transform of the complex-valued measure $\nu(dy) = \widehat{X}^*(y)\,\mu(dy)$, which is nontrivial. Hence, h is not constant, and so the zero set of $h(t) - \lambda$ must be discrete for any $\lambda \in \mathbb{R}$. $\square$

Our main result solves both problems, existence and monotonicity of strategies:

Theorem 2. *If G is nonconstant, nonincreasing, and convex, then there exists a unique optimal strategy X^* within each class $\mathcal{X}(x, \mathbb{T})$. Moreover, X^*_t is a monotone function of t.*

Our main result solves both problems, existence and monotonicity of strategies:

Theorem 3. *If G is nonconstant, nonincreasing, and convex, then there exists a unique optimal strategy X^* within each class $\mathcal{X}(x, \mathbb{T})$. Moreover, X_t^* is a monotone function of t.*

Remark 1. Everything is fine for

$$G(t) = \frac{1}{(1+t)^2}$$

but there are no optimal strategies for

$$G(t) = \frac{1}{1+t^2}$$

(see above). Hence one must be careful in defining power-law impact.

Our main result solves both problems, existence and monotonicity of strategies:

Theorem 4. *If G is nonconstant, nonincreasing, and convex, then there exists a unique optimal strategy X^* within each class $\mathscr{X}(x, \mathbb{T})$. Moreover, X_t^* is a monotone function of t.*

Proposition 4. *Suppose that there are s, $t > 0$, $s \neq t$, such that*

$$G(0) - G(s) < G(t) - G(t + s). \tag{4}$$

Then there is transaction-triggered price manipulation for the choice $\mathbb{T} := \{0, s, t + s\}$.

Condition (4) is satisfied when $G(t)$ is not convex for small t.

For discrete $\mathbb{T} = \{t_0, \ldots, t_N\}$:

Question: When does the minimizer $\boldsymbol{x}^*$ of

$$\sum_{i,j} x_i x_j G(|t_i - t_j|) \qquad \text{with} \quad \sum_i x_i = X_0$$

have only nonnegative components?

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Related to the **positive portfolio problem** in finance:

When are there no short sales in a Markowitz portfolio?

I.e. when is the solution of the following problem nonnegative

$$\boldsymbol{x}^\top M \boldsymbol{x} - \boldsymbol{m}^\top \boldsymbol{x} \to \min \qquad \text{for } \boldsymbol{x}^\top \mathbf{1} = X_0,$$

where M is a covariance matrix of assets and $\boldsymbol{m}$ is the returns vector?

Partial results, e.g., by Gale (1960), Green (1986), Nielsen (1987)

Theorem 5. [Alfonsi, A.S., Slynko (2009)]

- *If* G *is convex then all components of* $\boldsymbol{x}^*$ *are nonnegative.*

- *If* G *is strictly convex, then all components are strictly positive.*

- *Conversely,* $\boldsymbol{x}^*$ *has negative components as soon as, e.g.,* G *is strictly concave in a neighborhood of* 0*.*

Theorem 5. [Alfonsi, A.S., Slynko (2009)]

- *If G is convex then all components of $\boldsymbol{x}^*$ are nonnegative.*
- *If G is strictly convex, then all components are strictly positive.*
- *Conversely, $\boldsymbol{x}^*$ has negative components as soon as, e.g., G is strictly concave in a neighborhood of 0.*

Proof of first two assertions needs the following duality result:

Lemma 1. [Gale (1960)]
Let M be an symmetric invertible matrix. Then

$$M^{-1}\mathbf{1} \geq \mathbf{0} \qquad \textit{or} \qquad M^{-1}\mathbf{1} \leq \mathbf{0}$$

if and only if there is no vector $\boldsymbol{z}$ such that

$$\boldsymbol{z}^\top \mathbf{1} = 0 \qquad \textit{and} \qquad M\boldsymbol{z} > \mathbf{0}$$

Proof of Theorem 5. Use induction on N to exclude the existence of $\boldsymbol{z} = (z_0, \ldots, z_N)^\top$ such that $\boldsymbol{z}^\top \mathbf{1} = 0$ and $M\boldsymbol{z} > \mathbf{0}$ with $M_{ij} = G(|t_i - t_j|)$. For $N = 0$ the result is evident.

Suppose now that the assertion has already been proved for $N - 1$. Since $\boldsymbol{z}$ must satisfy $\boldsymbol{z}^\top \mathbf{1}_N = 0$ as well as $\boldsymbol{z} \neq 0$, there must be some $k \in \{0, 1, \ldots, N-1\}$ such that $z_k > 0$.

If $k = N$, then the fact that G is decreasing yields

$$G(|t_N - t_m|) z_N \leq G(|t_{N-1} - t_m|) z_N \quad \text{for} \quad m = 0, 1, \ldots, N-1.$$

Hence, the N-dimensional vector

$$\tilde{\boldsymbol{z}} := (z_0, z_1, \ldots, z_{N-2}, z_{N-1} + z_N)^\top$$

satisfies both $\tilde{\boldsymbol{z}}^\top \mathbf{1} = 0$ and $\tilde{M}\tilde{\boldsymbol{z}} > 0$, with $\tilde{M}$ corresponding to the time grid $\{t_0, t_1, \ldots, t_{N-1}\}$. But by induction hypothesis this is impossible.

Next, if $k = 0$, then

$$G(t_m)z_0 \leq G(|t_m - t_1|)z_0 \quad \text{for} \quad m = 1, 2, \ldots, N.$$

Hence,

$$\hat{\boldsymbol{z}} := \big(z_0 + z_1, z_2, \ldots, z_N\big)$$

satisfies both $\hat{\boldsymbol{z}}^\top \mathbf{1} = 0$ and $\hat{M}\hat{\boldsymbol{z}} > 0$, with $\hat{M}$ corresponding to the time grid $\{t_1 - t_1, t_2 - t_1, \ldots, t_N - t_1\}$, which is again impossible due to the induction hypothesis.

Finally, let us suppose that $1 \le k \le N-1$. Let $\alpha \in [0,1]$ be such that $t_k = \alpha t_{k-1} + (1-\alpha) t_{k+1}$. We then have

$$G(|t_k - t_l|) z_k \le \alpha G(|t_{k-1} - t_l|) z_k + (1-\alpha) G(|t_{k+1} - t_l|) z_k \quad \text{for } l \neq k.$$

Hence, the vector

$$\bar{\boldsymbol{z}} := (z_0, z_1, \ldots, z_{k-2}, z_{k-1} + \alpha z_k, z_{k+1} + (1-\alpha) z_k, z_{k+2}, \ldots, z_N)$$

satisfies both $\bar{\boldsymbol{z}}^\top \mathbf{1} = 0$ and $\bar{M} \bar{\boldsymbol{z}} > 0$, with $\bar{M}$ corresponding to the time grid

$$\{t_0, t_1, \ldots, t_{k-1}, t_{k+1}, t_{k+2}, \ldots, t_N\}.$$

This is again impossible due to the induction hypothesis $\square$

Sketch of proof of Theorem 4: $\mathbb{T}$ admits a countable dense subset $\{t_0, t_1, \dots\}$. For $N \in \mathbb{N}$ we define the finite set $\mathbb{T}_N := \{t_0, t_1, \dots, t_N\}$.

It follows from Theorem 5 that for each N there exists a unique optimal strategy X^N within each class $\mathcal{X}(x, \mathbb{T}_N)$, and X_t^N is a nondecreasing or nonincreasing function of $t \in \mathbb{T}_N$, depending on the sign of x. It thus follows that $\frac{1}{x}\, dX^N$ is a Borel probability measure on $\mathbb{T}$. Since the space of all Borel probability measures on $\mathbb{T}$ is compact with respect to the weak topology, there is a subsequence (X^{N_k}) that converges toward a strategy X^* in the sense of weak convergence of the associated probability measures.

Then show $\mathcal{C}(X^{(N_k)}) \to \mathcal{C}(X^*)$ as $k \uparrow \infty$ via continuity arguments.

Finally show that X^* is indeed optimal by proving that it solves the generalized Fredholm integral equation. $\square$

A qualitative property of optimal strategies

Theorem 6. *Let G be nonconstant, nonincreasing, and convex and suppose $x \neq 0$. Then the optimal strategy X^* in $\mathcal{X}(x, \mathbb{T})$ has impulse trades at $t_{\min} := \min \mathbb{T}$ and $t_{\max} := \max \mathbb{T}$, that is*

$$\Delta X^*_{t_{\min}} \neq 0 \textit{ and } \Delta X^*_{t_{\max}} \neq 0.$$

Now we relax the boundedness of G and assume instead

$$G \text{ is nonconstant, nonincreasing, convex, and } \int_0^1 G(t)\,dt < \infty.$$

E.g.,

$$\begin{aligned} G(t) &= t^{-\gamma} \qquad \text{for } 0 < \gamma < 1, \text{ or} \\ G(t) &= \log^-(t). \end{aligned}$$

Let

$$\mathcal{X}_G(x,\mathbb{T}) := \left\{ X \in \mathcal{X}(x,\mathbb{T}) \;\Big|\; \int\!\!\int G(|t-s|)\,d|X|_s\,d|X|_t < \infty \right\}$$

Note: $\mathcal{X}_G(x,\mathbb{T})$ can be empty, e.g., for discrete $\mathbb{T}$.

Theorem 7. *When $\mathcal{X}_G(x,\mathbb{T}) \neq \emptyset$, there exists a unique optimal strategy X^* in $\mathcal{X}_G(x,\mathbb{T})$. Moreover, X^*_t is a monotone function of t.*

A set $A \subset \mathbb{R}$ will be called exceptional when there exists a G_δ-set $G \supset A$ that is a nullset for every finite Borel measure ν on $\mathbb{R}$ for which $\int\int G(|t-s|)\,\nu(ds)\,\nu(dt) < \infty$.

Clearly: $\mathcal{X}_G(x, \mathbb{T})$ is empty for $x \neq 0$ iff $\mathbb{T}$ is exceptional.

Theorem 8. *A strategy $X^* \in \mathcal{X}_G(x, \mathbb{T})$ is optimal if and only if there is a constant λ such that X^* solves the generalized Fredholm integral equation*

$$\int G(|t-s|)\,dX_s^* = \lambda \qquad \textit{for quasi every } t \in \mathbb{T}. \tag{5}$$

Moreover, λ must be nonzero as soon as $x \neq 0$.

Example: (Power-law decay kernel) $G(t) = t^{-\gamma}$ with $0 < \gamma < 1$

$$\int_0^1 \frac{u(s)}{|t-s|^\gamma}\, ds = 1 \qquad \text{for } 0 \le t \le 1,$$

is solved by

$$u^*(s) = \frac{c}{(s(1-s))^{\frac{1-\gamma}{2}}},$$

where c is a suitable constant. Thus, the unique optimal strategy in $\mathcal{X}_G(x,[0,1])$ is

$$X_t^* = x\left(1 - \frac{\Gamma(3-\gamma)}{\Gamma\bigl(\frac{3-\gamma}{2}\bigr)^2} \int_0^t \frac{1}{(s(1-s))^{\frac{1-\gamma}{2}}}\, ds\right).$$

Example: (Logarithmic decay kernel) $G(t) = \log^{-}(t)$

$$\int_0^1 u(s)G(|t-s|)\,ds = -\int_0^1 u(s)\log|t-s|\,ds = 1 \qquad \text{for } 0 \le t \le 1$$

solved by

$$u^*(s) = \frac{ds}{2\pi \log 2\sqrt{s(1-s)}}.$$

This fact was discovered by Carleman (1922). The unique optimal strategy in $\mathcal{X}_G(x,[0,1])$ is thus given by

$$X_t^* = x\Big(1 - \frac{1}{\pi}\int_0^t \frac{1}{\sqrt{s(1-s)}}\,ds\Big) = \frac{2x}{\pi}\arccos\sqrt{t}.$$

What about optimal strategies with risk aversion?

Consider return $R(X) = -\text{costs}$ instead of costs

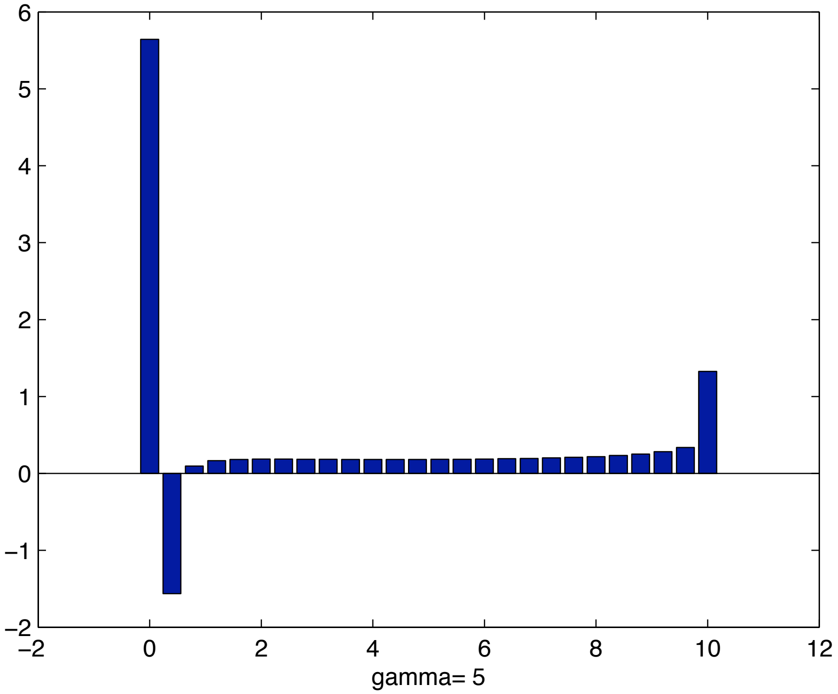
Theorem 9 (A.S., Schöneborn, & Tehranchi (2009)). *Suppose that G is strictly positive definite and that the unaffected price process S^0 satisfies $dS^0_t = \sigma_t\, dW_t$ for a Brownian motion W and a bounded and deterministic volatility function σ_s. Then the following conditions are equivalent for any strategy X^*.*

(a) *X^* maximizes the expected utility $\mathbb{E}[\,-e^{-\gamma R(X)}\,]$ in the class of all strategies X.*

(b) *X^* is deterministic and maximizes*

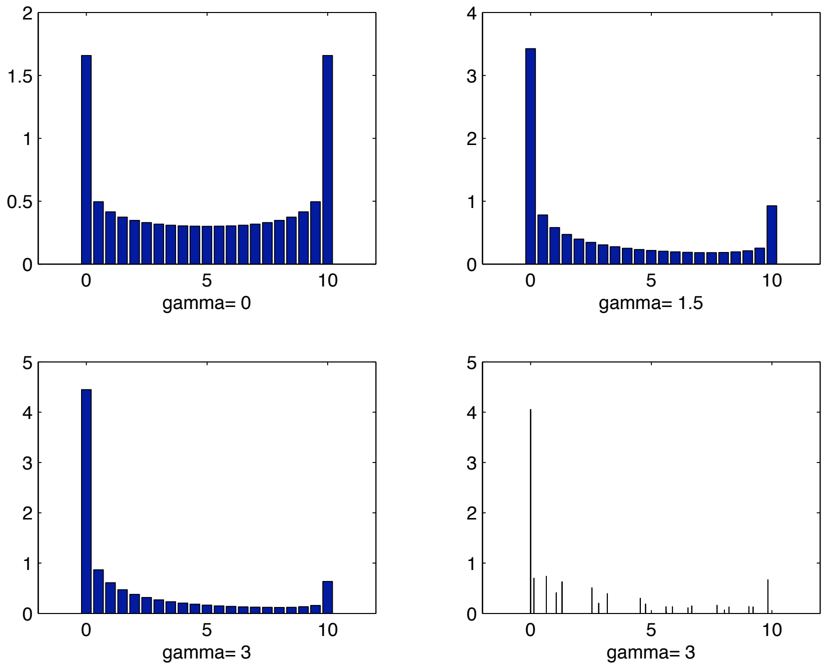
$$\mathbb{E}[\,R(X)\,] - \frac{\gamma}{2}\mathrm{var}(R(X)),$$

in the class of deterministic strategies X.



Mean-variance optimal strategy for power-law decay $G(t) = (1+t)^{-0.4}$, covariance function $\varphi(t) = \sigma^2 t^{1/5}$ with volatility $\sigma = 0.3$, risk aversion $\gamma = 5$, and $N = 25$.

Theorem 10. *Suppose that $G(t)$ is convex, $\mathbb{T}$ is discrete, and the variance of S_t^0 increases as a convex function of t. Then any mean-variance optimal deterministic strategy X^* is monotone.*



Mean-variance optimal strategies for power-law decay $G(t) = (1+t)^{-0.4}$, linear covariance $\varphi(t) = \sigma^2 t$ with volatility $\sigma = 0.3$, and various risk aversion parameters γ.

Conclusion:

- Transient market impact can create new types of irregularities: price manipulation, transaction-triggered price manipulation
- The irregularities do not occut for convex decay of price impact
- Non-robustness with respect to G
- Have only considered very simple problem
- Not clear how to best model nonlinear price impact.