When Mathematical Finance and Actuarial Science Cross

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Outline

1. Introduction
2. Credit Risk and Ruin Problems
3. Equity Linked Insurance Pricing
4. Reinsurance and Stochastic Control Problems
5. Concluding Remarks

An Example

Definition (Credit Default Swap (CDS))

A CDS is a contract where

- the "protection buyer" "A" pays rates "R" at times T_{a+1}, ..., T_b (the "premium leg") in exchange for a single protection payment L_{GD} (Loss Given Default, the "protection leg").
- The buyer receives the protection leg by the protection seller "B" at the default time τ of a reference entity "C", provided that $T_a < \tau < T_b$.
- The rates R paid by "A" stop in case of default.

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In terms of "Term Life Insurance":

- Time of death (default) — τ (of the insured "C")
- Death benefit — L_{GD}, payable at the moment of death
- Premium — an annuity (e.g. monthly) at (leveled) rate R
- Coverage period (term) — $[T_a, T_b]$, where $a < b$ are two ages.

Credit Risk vs. Actuarial Problems

	Credit Risk	**Actuarial Science**
τ	Default time	Ruin time, Future life time ($\tau = T(x)$)
$P\{\tau > t\}$	Survival Proba.	Survival Probability (${}_tp_x = P\{T(x) > t\}$)
$\Lambda(t) = -\ln {}_tp_x$	Hazard Process	Hazard Process
$\lambda(t) = \Lambda'(t)$	Default Intensity	"Force of Mortality" ($\mu(x+t) = -({}_tp_x)'/{}_tp_x$)
	Structure	Ruin Problems
	Reduced form	Life Contingencies

An Example in Risk Management

- Recall that the definition of "Value at Risk" of a r.v. Z:

$$\mathrm{VaR}_\alpha(Z) \overset{\triangle}{=} \inf\{x : \mathbb{P}\{x + Z < 0\} \leq \alpha\}.$$

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- Consider the value process $V_t^\pi = x + Q_t^\pi$ ($Q_0^\pi = 0$) for an investment strategy π. Then one can assess the "risk" associated to this strategy by looking at $\mathrm{VaR}_\alpha(\inf_{t\in[0,T]} Q_t^\pi)$.

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- Define

$$\psi(x, T) = \mathbb{P}\{V_t^\pi < 0 : \ \exists\ t \in [0, T]\}. \tag{1}$$

Then

$$\mathrm{VaR}_\alpha(\inf_{t\geq 0} Q_t^\pi) = \inf\{x : \psi(x, T) \leq \alpha\}.$$

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Then

$$\mathrm{VaR}_\alpha(\inf_{t\ge 0} Q_t^\pi) = \inf\{x : \psi(x,T) \le \alpha\}.$$

- Assume now that $\psi(x,T) \sim e^{-r^* x}$ for some $r^* \in \mathbb{R}$, then

$$\mathrm{VaR}_\alpha(\inf_{t\ge 0} Q_t^\pi) \sim -\frac{\log\alpha}{r^*}!$$

Some Remarks

Note

- In Actuarial Sciences, the quantity $\psi(x, T)$ (or $\psi(x) = \mathbb{P}\{V_t^h < 0 : \ \exists\ t > 0\}$) is called "*Ruin Probability*". The estimate $\psi(x, T) \sim e^{-r^* x}$ is called the *Lundberg bound*, with *Lundberg exponent* r^*.

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- Define the "Average VaR" by

$$\rho(Z) \stackrel{\triangle}{=} AVaR_\alpha(Z) \stackrel{\triangle}{=} \frac{1}{\alpha}\int_0^\alpha VaR_u(Z)du.$$

Then ρ is a "*Coherent Risk Measure*" (Cheridito-Delbaen-Kupper, '04).

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  Then ρ is a "*Coherent Risk Measure*" (Cheridito-Delbaen-Kupper, '04).
- The Lundberg bound also implies that

$$\rho(\inf_{t \geq 0} Q_t) \sim (1 - \log \alpha)/r^*.$$

  (The equality can hold if the Lundberg bound is sharp!)

Ruin Probability Revisited

Basic Insurance Models

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- Add interest income: $X_t = x + \int_0^t [r_s X_s + c_s(1 + \rho_s)]ds - S_t$

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- Add interest income: $X_t = x + \int_0^t [r_s X_s + c_s(1+\rho_s)]ds - S_t$
- Generalized Cramér-Lundberg Model

$$X_t = x + \int_0^t [r_s X_s + c_s(1+\rho_s)]ds - \int_0^t \int_{\mathbb{R}_+} f(s,z)\mu(dzds),$$

where $\mu(dtdz)$ is a Poisson random measure on $(0,\infty)\times\mathbb{R}_+$, with Lévy measure $\nu(dz)$. (In the compound Poisson case, $f(t,z)\equiv z$, $S_t = \sum_{k=1}^{N_t}\Delta S_{T_k}$, where N_t is standard Poisson. $\Longrightarrow \nu(dz) = \lambda F_{U_1}(dz)$, and $c_t = \mathbb{E}\{\Delta S_t\} = \lambda\mathbb{E}[U_1]$, $t \geq 0$.

An Exponential Martingale Approach (Gerber, (1973))

- Consider the classical Cremér-Lundberg moded: $X_t = x + Q_t$, and $Q_t = ct - S_t$. Define a process $M_t^x \overset{\triangle}{=} e^{-r(x+Q_t)}/e^{t\theta(r)}$, $t \geq 0$, $x > 0$, and $r > 0$.

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- Choose $\theta(s) \overset{\triangle}{=} \lambda[m(s) - 1] - sc$, where $m(s) = E[e^{sU_1}]$. Then $E[e^{-sQ_t}] = e^{t\theta(s)}$, and $\{M_t^x\}$ is an mg! (called the *Exponential Mg*, and θ is called the *Adjustment coefficient*)

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- Define $\tau_x \overset{\triangle}{=} \inf\{t \geq 0 : X_t < 0\}$. By optional sampling:

$$e^{-rx} = M_0^x = E\left\{M_{\tau_x}^x\right\} \geq E\{e^{-\tau_x\theta(r)}\Big|\tau_x < \infty\}\psi(x). \qquad (2)$$

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- $\psi(x) \leq e^{-rx} \sup_{t\geq 0} e^{t\theta(r)} \leq e^{-r^* t}$ — *Lundberg Bound*

  where $r^* \overset{\triangle}{=} \sup\{r : \theta(r) \leq 0\}$ is the *Lundberg exponent*.

Lundberg Bounds for Reserve with Investments

Consider a general value process of an portfolio $\pi_t = (\pi_t^1, \cdots, \pi_t^k)$ (of an insurance company):

$$X_t = x + \int_0^t \Big\{ \underbrace{X_s[r_s + \langle \pi_s, \mu_s - r_s\mathbf{1} \rangle]}_{b(s,X_s)} + \underbrace{c_s(1+\rho_s)}_{\eta_s} \Big\} ds$$

$$+ \int_0^t \langle \underbrace{X_s\pi_s, \sigma_s}_{\widehat{\sigma}_s} dW_s \rangle - \int_0^t \int_{\mathbb{R}_+} f(s,z)\mu(dsdz), \qquad (3)$$

where W is a Brownian motion, μ is a Poisson random measure, and f is the claim size.

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Question:

Can we find an exponential martingale that leads to the Lundberg bound for the general reserve (3), or even more general Itô-type value processes?

Analysis

Assume $r = 0$, and S is Compound Poisson. Then

$$X_t = x + \int_0^t [b(s, X_s) + \eta_s] ds + \int_0^t \hat{\sigma}_s dW_s - S_t.$$

For each $\delta \in \mathbb{R}$, let $M_t^\delta = \exp\{-\delta X_t - K_t^\delta\}$, and apply Itô.

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For each $\delta \in \mathbb{R}$, let $M_t^\delta = \exp\{-\delta X_t - K_t^\delta\}$, and apply Itô. Then one can see that if $K_t^\delta = -V_t^\delta + \frac{1}{2} Y_t^\delta + Z_t^\delta$, where

- $V_t^\delta = \delta \int_0^t [b(s, X_s) + \eta_s] ds;$
- $Y_t^\delta = \delta^2 \int_0^t |\hat{\sigma}_s|^2 ds;$
- $Z_t^\delta \triangleq \int_0^t m_s^f(\delta) ds, \; m_t^f(\gamma) \triangleq \int_{\mathbb{R}_+} [\exp\{\gamma f(t, z)\} - 1] \nu(dz).$

is a local martingale, and could be a martingale on a good day!

Lundberg Bounds

Theorem

Denote $\mathscr{D} = \{\delta \geq 0 : Z_t^\delta < \infty, \mathbb{P}$*-a.s.,* $\forall t \geq 0\}$*. Then,* $\forall \delta \in \mathscr{D}$*,*

$$\psi(x) \leq e^{-\delta x}\mathbb{E}\Big\{\sup_{t\geq 0} \exp(K_t^\delta)\Big\}.$$

Furthermore, $\delta^* \stackrel{\triangle}{=} \sup\big\{\delta \in \mathscr{D} : \mathbb{E}\big\{\sup_t \exp(K_t^\delta)\big\} < \infty\big\}$ *is the Lundberg exponent.*

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Example

- **Classical Model** $b \equiv 0,\ \rho \equiv 0\ \hat{\sigma}_t \equiv 0.$
- **Discounted Risk Reserve** $\pi_t = \rho_t = \mu_t = \sigma_t \equiv 0,\ r > 0$
  - $K_t^\delta = \int_0^t \{\int_0^\infty [\exp(\delta e^{-rs}x) - 1]\lambda F(dx) - c\delta e^{-r_s}\}ds$

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- **Discounted Risk Reserve** $\pi_t = \rho_t = \mu_t = \sigma_t \equiv 0$, $r > 0$
  - $K_t^\delta = \int_0^t \{\int_0^\infty [\exp(\delta e^{-rs}x) - 1]\lambda F(dx) - c\delta e^{-r_s}\}ds$
- **Perturbed risk reserve** $\pi_t \equiv 1$, $\rho_t = r_t = \mu_t \equiv 0$, $\sigma_t \equiv \varepsilon$,
  - $K_t^\delta = t(-c\delta + \frac{1}{2}\delta^2\varepsilon^2 + \int_0^\infty (e^{\delta x} - 1)\lambda F(dx)) \stackrel{\triangle}{=} k(\delta)t$

A General Exponential Martingale Method

In general, one can try to find $I \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and process K^I so that $M_t^I \overset{\triangle}{=} \exp\{-I(t, X_t) - K_t^I\}$ is a martingale. If so then the Lundberg bound could be

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Question:

How to find a rate function?

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Useful techniques:

- Solving ODEs, PDEs, IPDEs
- "Principle of smooth fit" (differential inequalities)
- BSDEs with jumps, and with quadratic growth
- Large deviations (infinite horizon ruin probabilities) ... ...

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Literature:

- Brennan-Schwartz ('76), Boyle-Schwartz ('77), Delbaen ('86), Aase-Persson ('94), Nielson-Sandmann (1995), Kurz ('96), ...
- Also, Young (with Bayraktar, Jaimungal, Ludkovski, Zariphopoulou, ...), Schweizer, Frittelli, Rouge-El Karoui, ...

Basic elements involved in an UVL insurance

A Life Model

- Single life
- Multiple life

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Benefit Specifications

- Guaranteed benefit/return
- "Multiple decrements" (including death, retirement, long term disability, ...)
- ... ...

The Single Life Case

- $T(x)$ — Future Life-time r.v., where x is the current age, with "force of mortality" $\lambda_x(t)$
- $X_t \in \{0, 1, ..., m\}$ — State Process (Markov, $X_0 = 0$, and the state "1" is cemetery/absorbing, representing "death").
- $dS_t = S_t\{\mu_t dt + \sigma_t dB_t\}$, $S_0 = s$, — tradable
- $dZ_t = Z_t^0\{\mu_t^Z dt + \sigma_t^Z dB_t + \sigma_t d\tilde{B}_t\}$, $Z_0 = z$ —non-tradable

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Dynamics of general reserve

$$d\hat{W}_t^\pi = [r_t \hat{W}_t^\pi + \pi_t(\mu_t - r_t)]dt + \pi_t \sigma_t dB_t - dA_t,$$

where

- $dA_t = \sum_i I_i(t) a^i(t, S_t, Z_t)dt + \sum_{i \neq j} a^{ij}(t, S_t, Z_t) dN_t^{ij}$
- $I_t^i = \mathbf{1}_{\{X_t = i\}},\ N_t^{ij} \overset{\triangle}{=} \#\{\text{jumps of } X \text{ from } i \text{ to } j \text{ during } [0, t]\}$

An Indifference Pricing Problem

- Due to the claim process, the market is always “incomplete”. Thus a common method to determine the price has been “indifference pricing” (Hodegs-Neuberger. ’89)

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- Given a general utility function u, and let $\mathscr{A} \subset L^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ be a set of "admissible strategies". Define

$$V^0(t,w) \overset{\triangle}{=} \sup_{\pi\in\mathscr{A}} E\{u(W^\pi_T)|W_t = w\};$$

$$U^k(t,w,s) \overset{\triangle}{=} \sup_{\pi\in\mathscr{A}} E\{u(\hat{W}^\pi_T)|\hat{W}_t = w, S_t = s, X_t = k\}.$$

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Indifference pricing

Find $p > 0$, payable at the $t = 0$, such that, denote $V = U^0$

$$V^0(0,w) = V(0,w+p,s). \tag{4}$$

$(p = \inf\{z > 0 : V(0,w+z,s) \geq V^0(0,w)\})$

An Easier Problem.

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- $g = g(S_T)$ (My-Yu, '07): Let $c(t, s)$ be the Black-Scholes price of $g(S_T)$. Then the HJB equation for V is:

$$\begin{cases} 0 = V_t + \max_\pi \{(\mu - r)\pi V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww} + s\sigma^2\pi V_{ws}\} + rwV_w \\ \qquad + s\mu V_s + \frac{1}{2}\sigma^2 s^2 V_{ss} + \lambda_x(t)(V^0(w - c, t) - V(w, t, s)), \\ V(T, w, s) = u(w). \end{cases}$$

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- $g \equiv 1$ (Young-Zariphopoulou, '02)
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$$\begin{cases} 0 = V_t + \max_\pi\{(\mu - r)\pi V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww} + s\sigma^2\pi V_{ws}\} + rwV_w \\ \qquad + s\mu V_s + \frac{1}{2}\sigma^2 s^2 V_{ss} + \lambda_x(t)(V^0(w - c, t) - V(w,t,s)), \\ V(T,w,s) = u(w). \end{cases}$$

Note

In the case of exponential utility ($u(w) = -\frac{1}{\alpha}e^{-\alpha w}$) one has

$$V^0(t,w) = -\frac{1}{\alpha}\exp\{-\alpha w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t)\} \qquad (5)$$

Solution for $g = g(S_T)$

- Assume $V(t, w, s) = V^0(t, w)\Phi(t, s)$, then
  $$\Phi_t + rS\Phi_s + \frac{\sigma^2 s^2 \Phi_{ss}}{2} - \frac{s^2\sigma^2\Phi_s^2}{2\Phi} + \lambda_x(e^{\{c\alpha e^{r(T-t)}\}} - \Phi) = 0$$
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- Define $h(t,s) = c(t,s)\alpha e^{r(T-t)} - \ln\Phi$. Then one shows that

$$\begin{cases} h_t + srh_s + \frac{1}{2}\sigma^2 s^2 h_{ss} - \lambda_x(t)(e^h - 1) = 0 \\ h(T,s) = \alpha g(s) \end{cases} \qquad (6)$$

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Remark

The case of $g = g(S_T, Z_T)$ is similar but more complicated, since it no longer has an arbitrage price! (cf. Ma-Yu, SAJ ('07)).

Multiple-decrement case

Theorem (Yu, '07; M.-Yu, '10)

Under suitable conditions, the value function $U = (U^0, U^1, ..., U^m)$ is the *unique* viscosity solution to the system of PDDE's:

$$\begin{cases} U^k_t + F_k(t, w, s, DU^k, D^2U^k) + (\mathscr{H}_k U) = 0, \\ U^k(T, w, s) = u(w), \qquad k = 0, \cdots, m, \end{cases} \tag{7}$$

where

$$\begin{aligned} F_k(\cdots) &= \sup_{\pi \in \Pi} \Big\{ \pi(\mu_t - r_t) U^k_w + \frac{1}{2} |\sigma_t \pi|^2 U^k_{ww} + \pi \sigma^2_t s U^k_{ws} \Big\} \\ &\quad + \mu_t s U^k_s + \frac{1}{2} \sigma^2_t s^2 U^k_{ss} + (r_t w - a^k(t, s)) U^k_w \\ (\mathscr{H}_k U) &= \sum_{j \neq k} \lambda^{kj}_t (U^j(t, w - a^{kj}(t, s), s) - U^k(t, w, s)). \end{aligned}$$

The Case of Bereaved Partner

Main Rationales

- The usual "Multi-Life Contingency" (e.g., pension plans) assumes independent mortality, even for married couples
- Empirical evidence of the bereaved spouse (Hu-Goldman ('90) Mariikainen-Valkonen ('96), and Valkonen et al. ('04)) indicated the possible correlated mortality.

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- $T_{x_1}, T_{x_2}, \cdots, T_{x_n}$ — future life time random variables,
- $T_m = T_{x_1,\cdots,x_n} \overset{\triangle}{=} \min\{T_{x_1},\cdots,T_{x_n}\}$ — (Joint-life)
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- If $n = 2$, one has $T_M + T_m = T_{x_1} + T_{x_2}$, $T_M T_m = T_{x_1} T_{x_2}$.
- $F_M(t) + F_m(t) = F_{T_{x_1}}(t) + F_{T_{x_2}}(t)$, $t \geq 0$ where F_T is the distribution function of T.
- If $T_{x_1} \perp T_{x_2}$, then $F_M(t) = F_{T_{x_1}}(t) F_{T_{x_2}}(t)$...

The Case of Bereaved Partner

Assume $n = 2$, and that the individual force of mortalities take the form:

$$\begin{cases} \mu_{x_1}(t) = \lambda_{x_1}(t) + \mathbf{1}_{\{T_{x_2} \le t\}} \gamma_{x_1}(t - T_{x_2}) \\ \mu_{x_2}(t) = \lambda_{x_2}(t) + \mathbf{1}_{\{T_{x_1} \le t\}} \gamma_{x_2}(t - T_{x_1}), \end{cases} \qquad t \ge 0, \qquad (8)$$

where λ_{x_i}'s are the (marginal) force of mortality and

$$\gamma_{x_i}(t) = \frac{n_i}{r_i e^t + 1}, \quad i = 1, 2, \quad r_1, r_2, n_1, n_2 > 0.$$

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Note:

This essentially becomes a problem of "*Counter-Party Risk*", a well-know topic in "Contagion Models" of correlated default! Existing literature include

- King-Wadhwani, Kodres-Pritsker, Collin-Dufresne, ...
- Jarrow-Yu, Yu (2001, counterparty, two firms)
- ....

Correlated Default vs. Dependent Mortality

- $\mathscr{F}_t \overset{\triangle}{=} \mathscr{F}_t^X \vee \mathscr{F}_t^1 \vee \dots \vee \mathscr{F}_t^I$, where X is some "factor process", and $\mathscr{F}_t^i = \sigma\{\mathbf{1}_{\{\tau^i \le s\}} : 0 \le s \le t\}$, $\forall i$
- $\mathscr{H}_t^i = \mathscr{F}_t^X \vee \mathscr{F}_t^1 \vee \dots \vee \mathscr{F}_t^{i-1} \vee \mathscr{F}_t^{i+1} \vee \dots \vee \mathscr{F}_t^I$, $\Longrightarrow \; \mathscr{F}_t = \mathscr{H}_t^i \vee \mathscr{F}_t^i$.
- The following identity is "fundamental": $\forall Z \in \mathscr{F}$,

$$\mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{Z | \mathscr{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} Z | \mathscr{H}_t^i\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} | \mathscr{H}_t^i\}}.$$

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- Let $\lambda^i \in \mathscr{H}^i$ be the "*conditional intensity*" of τ^i, defined by $\mathbb{P}\{\tau^i > t | \mathscr{H}_t^i\} = e^{-\int_0^t \lambda_s^i ds} \overset{\triangle}{=} (\Gamma_t^i)^{-1}$. Then

$$\mathbb{P}\{\tau^i > T | \mathscr{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} (\Gamma_t^i)^{-1} \mathbb{E}\Big\{\Gamma_T^i \Big| \mathscr{H}_t^i\Big\}$$

- Denote $Z_t^i \overset{\triangle}{=} \mathbf{1}_{\{\tau^i > t\}} \Gamma_t^i$, then for $k = 1, \cdots, I$, $\prod_{i=1}^k Z_t^i$'s are all non-negative $\{\mathscr{F}_t\}$-martingales with mean 1.

Representation of Joint Survival Probability

The idea of "change of measure" (Collin-Dufresne et al. 03-04):

Define $\left.\frac{d\mathbb{P}^{1,\cdots,k}}{d\mathbb{P}}\right|_{\mathscr{F}_T} \overset{\triangle}{=} \tilde{Z}_T^k = \prod_{i=1}^{k} Z_T^i$, $k = 1, \cdots, I$, then

- $\mathbb{E}\{Z_T^1 Z_T^2 ... Z_T^k X | \mathscr{F}_t\} = Z_t^1 Z_t^2 ... Z_t^k \mathbb{E}^{1,\cdots,k}\{X|\mathscr{F}_t\}, \quad \mathbb{P}-a.s.$

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- $\mathbb{E}\{Z^1_T Z^2_T ... Z^k_T X|\mathscr{F}_t\} = Z^1_t Z^2_t ... Z^k_t \mathbb{E}^{1,\cdots,k}\{X|\mathscr{F}_t\}, \quad \mathbb{P}-a.s.$
- If $l = 2$, then for $t_1 \le t_2$,

$$\begin{aligned}\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2\} &= \mathbb{E}\Big\{\mathbf{1}_{\{\tau^1>t_1\}}\mathbb{E}\Big\{Z^2_{t_2}(\Gamma^2_{t_2})^{-1}\Big|\mathscr{F}_{t_1}\Big\}\Big\}\\ &= \mathbb{E}\Big\{Z^1_{t_1}Z^2_{t_1}\mathbb{E}^{\mathbb{P}^2}\Big\{(\Gamma^1_{t_1})^{-1}(\Gamma^2_{t_2})^{-1}\Big|\mathscr{F}_{t_1}\Big\}\Big\}\\ &= \mathbb{E}^{1,2}\Big\{\mathbb{E}^{\mathbb{P}^2}\Big\{(\Gamma^1_{t_1})^{-1}(\Gamma^2_{t_2})^{-1}\Big|\mathscr{F}_{t_1}\Big\}\Big\}.\end{aligned}$$

- In particular, if $t_1 = t_2 = t$, then we have

$$\mathbb{P}\{\tau^1 > t, \tau^2 > t\} = \mathbb{E}^{1,2}\Big\{e^{-\int_0^t(\lambda^1_s+\lambda^2_s)ds}\Big\}(= \mathbb{P}\{\tau_m > t\}!)$$

Bereaved Partner Case (M.-Yun '10)

Assume that the individual T_{x_i}'s follow the Gompertz's law (1825): $\lambda_{x_1}(t) = h_1 e^{g_1(x_1+t)}$, $\lambda_{x_2}(t) = h_2 e^{g_2(x_2+t)}$, $h_i > 0$, $g_i > 0$. Then

$$\mathbb{P}\{T_{x_1} > t_1, T_{x_2} > t_2\}$$

$$= \begin{cases} \frac{c(t_1,t_2)}{(r_2+1)^{n_2}} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{h_1}{g_1} r_2^{n_2-k} B^1 \left(\tilde{\mathbb{D}}_k^1(t_2) - \tilde{\mathbb{D}}_k^1(t_1)\right) & \\ \qquad +c(t_2,t_2) & t_1 \le t_2; \\ \frac{c(t_1,t_2)}{(r_1+1)^{n_1}} \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{h_2}{g_2} r_1^{n_1-k} B^2 \left(\tilde{\mathbb{D}}_k^2(t_1)\right) - \tilde{\mathbb{D}}_k^2(t_2)\Big) & \\ \qquad +c(t_1,t_1) & t_1 > t_2, \end{cases}$$

where

- $\Delta_k^i(t) = \int_0^t y^{\frac{k}{g_i}} e^{-\frac{h_i}{g_i}y} dy$, $\tilde{\mathbb{D}}_k^i(t) = \mathbb{D}_k^i\left(\frac{\lambda_{x_i}(t)}{h_i}\right)$, $i = 1, 2$,
- $B^1 = e^{-k(t_2+x_1)+\frac{h_1}{g_1}e^{g_1(x_1+t_1)}}$, $B^2 = e^{-k(t_1+x_2)+\frac{h_2}{g_2}e^{g_2(x_2+t_2)}}$,
- $c(t_1,t_2) = \exp\left\{-\frac{h_1}{g_1}[e^{g_1(x_1+t_1)} - e^{g_1 x_1}] - \frac{h_2}{g_2}[e^{g_2(x_2+t_2)} - e^{g_2 x_2}]\right\}$.

Back to UVL Insurance Pricing

- Let T_{x_1} and T_{x_2} be two future life time r.v.'s and let K_t be a generic status process, e.g., K could be one of the following:

$$JLI_t = \mathbf{1}_{\{T_{x_1x_2}\le t\}}, \qquad SLI_t = \mathbf{1}_{\{T_{\overline{x_1x_2}}\le t\}}, \qquad t \ge 0,$$

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- If $K_T \equiv 0$, then denote $J^0(t,w;\pi) \overset{\triangle}{=} \mathbb{E}_{t,w}\{u(W_T^\pi)\}, \pi \in \mathscr{A}$.
- $U(t,w) \overset{\triangle}{=} \sup_{\pi\in\mathscr{A}} J(t,w;\pi)$, $V(t,w) \overset{\triangle}{=} \sup_{\pi\in\mathscr{A}} J^0(t,w;\pi)$.

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Recall the “separation of variable”: $U(t, w) = V(t, w)\Phi(t, w)$, where

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- The indifference (selling) price is

$$p_t^* = \frac{1}{\alpha}e^{-r(T-t)}\log\Phi(t,w) = \frac{1}{\alpha}e^{-r(T-t)}\log\mathbb{E}_{t,w}[e^{\alpha K_T}].$$

Reinsurance Problem

An insurance company may choose to "cede" some of its risk to a reinsurer by paying a premium. Thus the reserve may look like

$$X_t = x + \int_0^t c_s^h (1+\rho_s)ds - \int_0^t \int_{\mathbb{R}_+} h(s,x)\mu(dxds),$$

where h is the "**retention function**"

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Common types of retention functions:

- $h(x) = \alpha x$, $0 \le \alpha \le 1$ — Proportional Reinsurance
- $h(x) = \alpha \wedge x$, $\alpha > 0$ — Stop-loss Reinsurance

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- Diffusion approximation: $dX_t = \mu \alpha_t dt + \sigma \alpha_t dW_t, X_0 = x$ (e.g., Asmussen-Hojgaard-Taksar (2000), ...)

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Using the so-called "Profit Margin Principle", one can argue (Liu-M., '09) that a reasonable general reserve dynamics with reinsurance and consumption/investment could look like:

$$\begin{aligned} dX_t &= X_t \Big\{ [r_t + \eta_t^\pi + (1+\rho_t^\alpha) m(t,\alpha)]dt + \langle \pi_t, \sigma_t dW_t \rangle \Big\} \\ &\quad - D_t dt - \int_{\mathbb{R}_+} \alpha(t,z) f(t,z,\cdot) \mu(dtdz), \end{aligned} \tag{10}$$

where $\eta_t^\pi = \langle \pi_t, \mu_t - r_t \mathbf{1} \rangle$, D is the consumption rate, and $m(t,\alpha) = \int_{\mathbb{R}_+} \alpha(t,z) f(t,z) \nu(dz)$ is the reinsurance premium rate.

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Main Difficulty:

For a given $x \geq 0$, find "non-trivial" strategy (π, α, D), s.t.,

- $\pi \in L^2_{\mathbf{F}}([0,T])$, $0 \leq \alpha_t \leq 1$, $D_t \geq 0$, $t \in [0,T]$;
- The corresponding risk reserve $X^{x,\alpha,\pi,D}_\cdot$ satisfies $X_0^{x,\pi,\alpha,D} = x$, $X_t^{x,\pi,\alpha,D} \geq 0$, for all $t \in [0,T]$.

Optimal Reinsurance/Investment/Consumption Problem

Denote the set of all admissible strategies by $\mathscr{A}(x)$. Consider the following

Utility Maximization Problem

- Cost functional: for $(\pi, \alpha, D) \in \mathscr{A}(x)$

$$J(x; \pi, \alpha, D) \stackrel{\triangle}{=} E\Big\{ \int_0^T U_1(t, D_t)dt + U_2\Big(X_T^{x,\alpha,\pi,D}\Big)\Big\};$$

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$$V(x) \stackrel{\triangle}{=} \sup_{(\pi,\alpha,D)\in\mathscr{A}(x)} J(x; \pi, \alpha, D).$$

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Need to do:

- Show that $\mathscr{A}(x) \neq \emptyset$
- Find the optimal $(\pi^*, \alpha^*, D^*) \in \mathscr{A}(x)$.

Some Easier Results

Wider-sense Strategies "$\mathscr{A}^w(x)$":

$$(\pi, \alpha, D) \in \mathscr{A}^w(x) \quad \text{if} \quad \alpha \in L^2_{\mathbf{F}}([0,T] \times \mathbb{R}, dt \otimes d\nu).$$

Theorem

For any consumption process D and $B \in \mathscr{F}_T$ satisfying $B \geq 0$ and $E^Q\Big\{ \int_0^T \beta_s D_s ds + \beta_T B \Big\} = x$ (budget constraint), there exists a (π, α), such that $(\pi, \alpha, D) \in \mathscr{A}^w(x)$ and the corresponding reserve processes $X^{x,\pi,\alpha,D}$ satisfies

$$X_t^{x,\pi,\alpha,D} > 0, \quad 0 \leq t \leq T; \quad \textit{and} \quad X_T^{x,\pi,\alpha,D} = B, \quad P\text{-a.s.}$$

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Problems:

- In general $0 \leq \alpha \leq 1$ does not hold in the above.
- How can one check whether a wider-sense optimal strategy is actually a true optimal strategy?

Some Not-So-Easy Problems

Idea

"Fictitious Market" $\oplus$ "Duality Method" (Karatzas, Shreve, Cvitanic...).

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- Define $\delta(x) \overset{\triangle}{=} -x\mathbf{1}_{(-\infty,0)}(x)$ (*support function* of $[0,1]$) and

$$\mathscr{D} \overset{\triangle}{=} \{v \in L^2_{\mathbf{F}}(dt \otimes d\nu) : \sup_{t\in[0,R]} \int_{\mathbb{R}^+} |v(t,z)|\nu(dz) < C_R, \ \forall R > 0\}.$$

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- For $v \in \mathscr{D}$, $\alpha \in L^2$, consider a market with *fictitious* interest rate, appreciation rate, and expense loading:
$$r_t^{\alpha,v} \stackrel{\triangle}{=} r_t + m(t, \alpha v + \delta(v)), \mu_t^v \stackrel{\triangle}{=} \mu_t + m(t, \delta(v)), \rho_s^v \stackrel{\triangle}{=} \rho_s + vx.$$

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- Then the fictitious reserve equation under EMM Q is

$$dX_t^v = r_t^{\alpha,v} X_t^v dt + X_t^v \langle \pi_t, \sigma_t dW_t^0 \rangle + dN_t^\alpha - D_t dt,\ \ X_0^v = x.$$

Main Results

Theorem

Assume that

- $B \in L^\infty_{\mathscr{F}_T}(\Omega)$, $D \in L^\infty_{\mathbf{F}}([0,T]\times\Omega;\mathbb{R}_+)$, *and*
- $B \geq 0$ *with* $E(B) > 0$.

Suppose that for some $u^* \in \mathscr{D}$ *the associated strategy* $(\pi^*, \alpha^*, D) \in \mathscr{A}^w(x)$ *satisfies, for all* $v \in \mathscr{D}$,

$$E\Big\{H_T^{*,v}B + \int_0^T H_s^{*,v}D_s ds\Big\} \leq E\Big\{H_T^{*,u^*}B + \int_0^T H_s^{*,u^*}D_s ds\Big\} = x,$$

where $H^{*,v}$ *is the "State-Price-Density" process with "fictitious" interest rate* $r^{\alpha^*,v}$. *Then* $(\pi^*, \alpha^*, D) \in \mathscr{A}(x)$. *Further, the corresponding reserve satisfies* $X_T^* = B$, *a.s.*

Main Results

Theorem (Liu-M., '09)

The solvability of the utility optimization problem is equivalent to the solvability of the following "*Forward-Backward SDE*":

$$\begin{cases} H_t = 1 + \int_0^t H_s[r_s + m(s,\delta(v)+\alpha v))]ds - \int_0^t H_s \langle \theta_s, dW_s \rangle \\ \qquad + \int_0^t \int_{\mathbb{R}_+} H_{s-}\rho_s \tilde{N}_p(dsdz); \\ X_t = I_2(yH_T) - \int_t^T \Big\{ X_s[r_s + m(s,\delta(v)+\alpha v) + \langle \pi_s, \sigma_s\theta_s \rangle] \\ \qquad + (1+\rho_s)m(s,\alpha) \Big\} ds + \int_t^T I_1(s, yH_s)ds \\ \qquad - \int_t^T X_s \langle \pi_s, \sigma_s dW_s \rangle + \int_t^T \int_{\mathbb{R}^+} [\alpha f](s,z)N_p(dsdz), \end{cases}$$

where $I_1(s,\cdot) = [U_1'(s,\cdot)]^{-1}$, and $I_2(\cdot) = [U_2'(\cdot)]^{-1}$.

A New Type of Stochastic Control Problem

Recall the general form of a reserve equation with reinsurance:

$$
\begin{aligned}
dX_t &= b(X_t, \alpha_t, \pi_t)dt + \sigma(\pi_t)dB_t - \int_{\mathbb{R}_+} [\alpha f](t,x)\tilde{\mu}(dxdt), \\
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Suppose that the random field α is such that there exists some predictable pair (β, u) so that the martingale

$$M_t^u \stackrel{\triangle}{=} \int_0^t \beta_s dB_s + \int_0^t \int_{\mathbb{R}_+} [\alpha f](s,x)\tilde{\mu}(dxds)$$

satisfies the following property:

$$d[M^u]_t = dt + u_t dM_t^u, \qquad t \geq 0. \tag{11}$$

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Note: Since $\Delta[M^u]_t = (\Delta M_t^u)^2 = u_t \Delta M_t^u$, u exactly controls the jumps of the reserve, that is, the claim size!

Remarks

Such decomposition is possible for all bounded, predictable u, provided

- ν is "atom-free" with $\nu^c([-1,1]) = +\infty$,
- The probability space is "nice" (for example, Wiener-Poisson — Buckdhan-Ma-Rainer, '08)

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Note:

This controlled dynamics is NEW: The driving martingale varies with u in the class of the "*normal martingales*"!

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The Stochastic Control Problem

The set of “admissible controls at time t” — $\mathscr{U}(t)$:

- $(\Omega, \mathscr{F}, P; \mathbf{F}^t = \{\mathscr{F}_s\}_{s \geq t})$ — Wiener-Poisson space
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Basic Elements

- Given $\mu = (\Omega, \mathscr{F}, P, \mathbf{F}^t, \pi, u, X^u) \in \mathscr{U}(t)$, $t \ge 0$, the controlled dynamics is

$$dY_s = b(Y_s, \pi_s, u_s)ds + \sigma(Y_{s-}, \pi_s, u_s)dX_s^u, \;\; Y_t = y, \;\; s \ge t.$$

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where

$$\mathscr{L}_{\pi,u}[\varphi](t,y) = \nabla_y \varphi b(y,\pi,u) + \sum_{i=1}^{d} \Big\{ \mathbf{1}_{\{u^i=0\}} \frac{1}{2}(D^2_{yy}\varphi\sigma^i, \sigma^i) + \mathbf{1}_{\{u^i\neq 0\}} \frac{\varphi(t, y+u^i\sigma^i)) - \varphi(t,y) - u^i\nabla_y\varphi(t,y)\sigma^i}{(u^i)^2} \Big\}.$$

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- Prove that the value function $V(\cdot,\cdot)$ is the **unique viscosity solution** to the HJB eq. (12). (Buckdahn-Ma-Rainer ('08))

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> *"Ask not only what finance can do for insurance. Ask also what insurance can do for finance."*
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> — Hans Bühlmann (1987)

References: http://www-rcf.usc.edu/~jinma/

THANK YOU VERY MUCH!