

# How far can we go with Lieb's concavity theorem and Ando's convexity theorem?

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Based on arXiv:1811.01205, Adv. Math. 2020.

Entropy Inequalities, Quantum Information and Quantum Physics  
IPAM, February 08 -11, 2021

# Stability of concavity

The concavity of trace functionals is stable under

- ▶ linear combinations with positive coefficients;
- ▶ limits;
- ▶ inf;
- ▶ sup with additional conditions;
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Given the concavity/convexity of certain functionals (Lieb, Ando), one can derive the concavity/convexity of more functionals, e.g. those with nice integral formula. We will see that inf/sup can be very useful.

# Warm up exercises

Two easy exercises:

- ▶ **Exercise 1** For which  $p \in \mathbb{R}$ , the function

$$f(x) = x^p, \quad x > 0$$

is convex/concave? (Compute the second derivative of  $f$ .)

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is convex/concave? (Compute the second derivative of  $f$ .)

For a two-variable function, one can ask the joint convexity/concavity:

- ▶ **Exercise 2** For which  $(p, q) \in \mathbb{R}^2$ , the function

$$f(x, y) = x^p y^q, \quad x, y > 0$$

is jointly convex/concave? (Compute the Hessian of  $f$ .)

# Warm up noncommutative exercises

Fix an invertible matrix  $K$ , and suppose that  $A, B$  are any positive definite matrices. The following exercises are noncommutative analogues of **Exercises 1 & 2**:

- ▶ **Exercise 1'** For which  $(p, s) \in \mathbb{R}^2$ , the function

$$\Psi_{p,s}(A) := \text{Tr}(K^* A^p K)^s = \text{Tr}|A^{\frac{p}{2}} K|^{2s}$$

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- ▶ **Exercise 2'** For which  $(p, q, s) \in \mathbb{R}^3$ , the function

$$\Psi_{p,q,s}(A, B) := \operatorname{Tr}(B^{q/2} K^* A^p K B^{q/2})^s = \operatorname{Tr}|A^{\frac{p}{2}} K B^{\frac{q}{2}}|^{2s}$$

is jointly convex/concave for any  $K$ ? (Compute the second derivative? Rather complicated...)

# Aim and some notations

The aim of this talk:

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- ▶ How to solve Exercises 1' & 2'?

For this we will use the following notations:

- ▶  $\mathcal{H}$ : finite-dimensional Hilbert space.
- ▶  $\mathcal{B}(\mathcal{H})$ : all matrices of size  $\dim \mathcal{H} \times \dim \mathcal{H}$ .
- ▶  $\mathcal{B}(\mathcal{H})^\times$ : all invertible matrices . . . .
- ▶  $\mathcal{B}(\mathcal{H})^+$ : all positive definite matrices . . . .
- ▶  $\mathcal{S}(\mathcal{H})$ : all positive definite matrices with unit trace.

# Outline

Wigner-Yanase-Dyson conjecture and Lieb's concavity theorem

Audenaert-Datta conjecture and data processing inequality

Carlen-Frank-Lieb conjecture and solution of **Exercise 2'**

Remarks and questions

# Beginning of the story: WYD conjecture/theorem

## Wigner-Yanase-Dyson conjecture/theorem <sup>1</sup> (63)

For  $0 < p < 1$  and any self-adjoint  $K$ ,

$$S_p(\rho, K) := \frac{1}{2} \text{Tr}([\rho^p, K][\rho^{1-p}, K]) = -\text{Tr}(\rho K^2) + \text{Tr}(\rho^{1-p} K \rho^p K)$$

is concave in  $\rho \in \mathcal{B}(\mathcal{H})^+$ , with  $[A, B] = AB - BA$ .

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<sup>1</sup>E.P. Wigner, M.M. Yanase. Information contents of distributions. Proc. Natl. Acad. Sci. USA, 49 (1963), pp. 910-918.

# Lieb's concavity theorem

Lieb confirmed this conjecture by proving the following celebrated

## Lieb's concavity theorem (73) <sup>2</sup>

For  $0 \leq p, q \leq 1$  such that  $p + q \leq 1$ ,

$$\Psi_{p,q,1}(A, B) = \text{Tr}(K^* A^p K B^q)$$

is jointly concave for all  $K \in \mathcal{B}(\mathcal{H})^\times$ .

After this: plenty of work...

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<sup>2</sup>E. H. Lieb. Convex trace functions and the Wigner-Yanase-Dyson conjecture. *Advances in Math.* 11 (1973), 267–288.

# Ando's convexity theorem

Recall only a convexity result of Ando here, as a complement of Lieb's concavity theorem.

## Ando's convexity theorem (79) <sup>3</sup>

For  $1 \leq p \leq 2$ ,  $-1 \leq q \leq 0$  such that  $p - q \geq 1$ ,

$$\Psi_{p,q,1}(A, B) = \text{Tr}(K^* A^p K B^q)$$

is jointly convex for all  $K \in \mathcal{B}(\mathcal{H})^\times$ .

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<sup>3</sup>T. Ando. Concavity of certain maps on positive definite matrices and applications to Hadamard products. Linear Algebra Appl., 26 (1979), pp. 203-241.

Wigner-Yanase-Dyson conjecture and Lieb's concavity theorem

Audenaert-Datta conjecture and data processing inequality

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# The $\alpha$ - $z$ Rényi relative entropies

The  $\alpha$ - $z$  Rényi relative entropies (Audenaert-Datta <sup>4</sup> 15) are:

$$D_{\alpha,z}(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{Tr}(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}})^z,$$

where  $\alpha \in (-\infty, 1) \cup (1, \infty)$  and  $z > 0$ . They are quantum generalization of classical  $\alpha$ -Rényi relative entropies

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<sup>4</sup>K.M.R. Audenaert, N. Datta.  $\alpha$ - $z$ -Rényi relative entropies. J. Math. Phys., 56 (2) (2015).

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where  $\alpha \in (-\infty, 1) \cup (1, \infty)$  and  $z > 0$ . They are quantum generalization of classical  $\alpha$ -Rényi relative entropies and

- ▶ when  $z = 1$ : quantum  $\alpha$ -Rényi relative entropy

$$D_{\alpha}(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{Tr}(\rho^{\alpha} \sigma^{1-\alpha});$$

- ▶ when  $z = \alpha$ : sandwiched  $\alpha$ -Rényi relative entropy

$$\tilde{D}_{\alpha}(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{Tr}(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha};$$

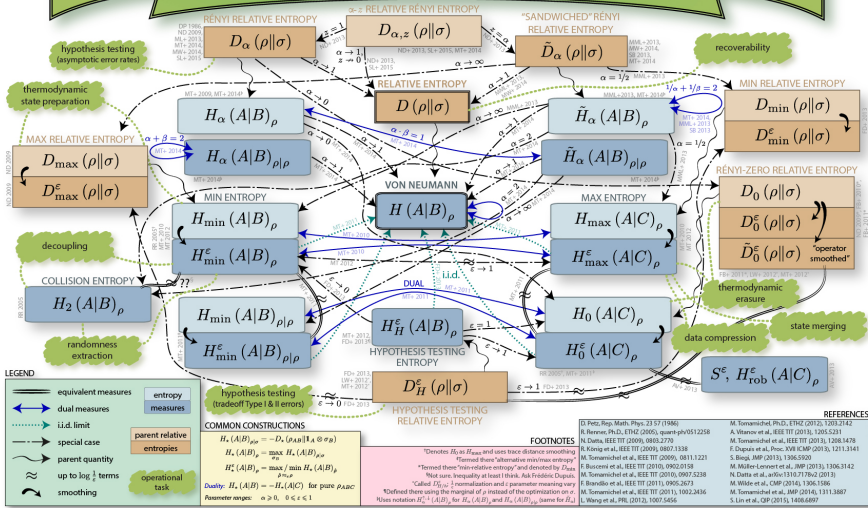
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# WELCOME TO THE ENTROPY ZOO

by Philippe Faist, ETHZ



No guarantee of correctness or completeness. No liability if the proof in your paper is wrong because you believed this diagram. Comments and suggestions are greatly welcome at [pf@isi.phys.ethz.ch](mailto:pf@isi.phys.ethz.ch).

## Monotonicity/data processing inequality for $D_{\alpha,z}$

For which  $(\alpha, z)$  do we have the **data processing inequality** (DPI)

$$D_{\alpha,z}(\mathcal{E}(\rho) \parallel \mathcal{E}(\sigma)) \leq D_{\alpha,z}(\rho \parallel \sigma),$$

for all quantum channel (completely positive trace preserving)  
 $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$  and quantum states  $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ ?

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$\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$  and quantum states  $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ ?

A standard argument shows

### Why study Exercise 2':

Set

$$\Psi(A, B) := \text{Tr}(B^{\frac{q}{2}} A^p B^{\frac{q}{2}})^{\frac{1}{p+q}}, \quad A, B \in \mathcal{B}(\mathcal{H})^+,$$

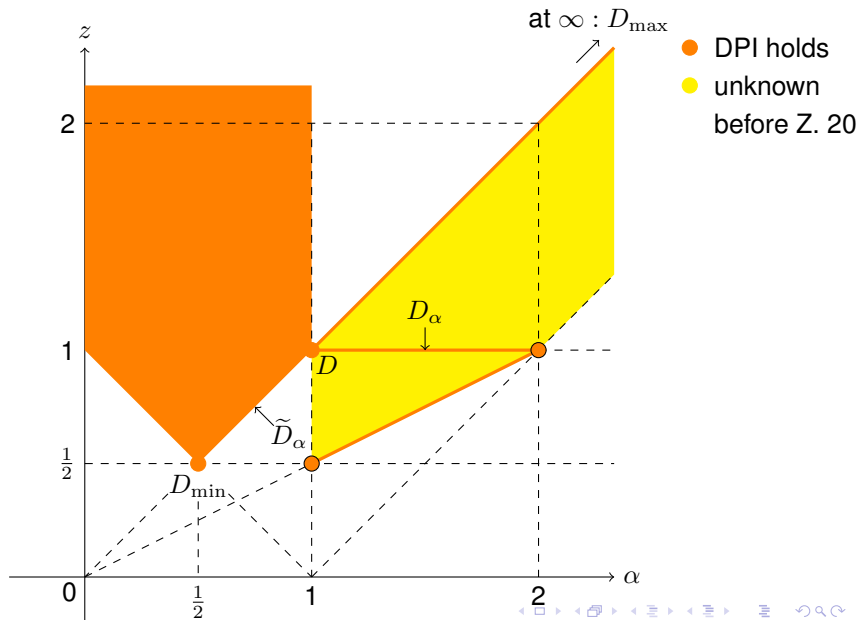
with

$$p := \frac{\alpha}{z}, \quad q := \frac{1-\alpha}{z}.$$

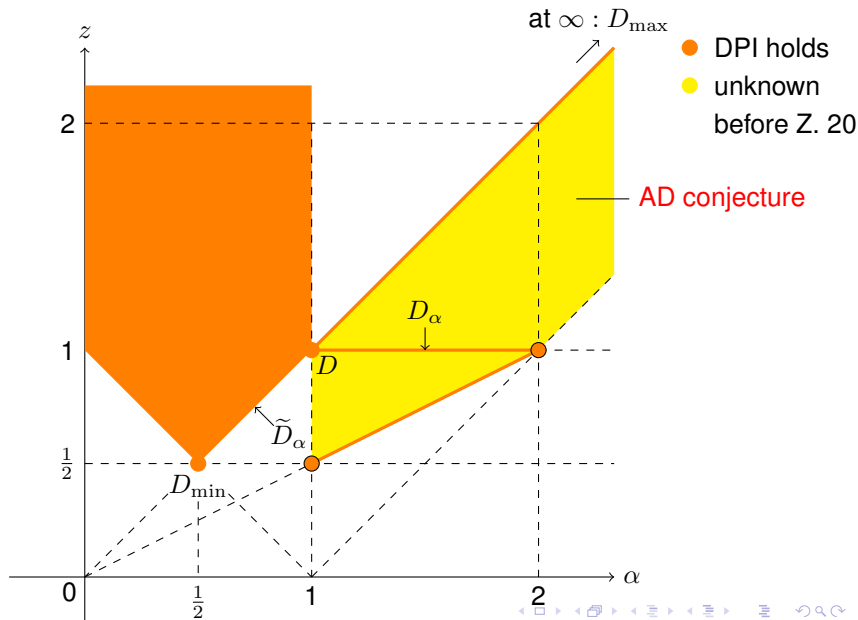
Then DPI holds for  $D_{\alpha,z}$  if and only if one of the following holds

- ▶  $\alpha < 1$  and  $\Psi$  is jointly concave;
- ▶  $\alpha > 1$  and  $\Psi$  is jointly convex.

# Known results of DPI for $D_{\alpha,z}$ before Z. 20



# Known results of DPI for $D_{\alpha,z}$ before Z. 20



# Audenaert-Datta conjecture/theorem

The following is an equivalent form of

## Audenaert-Datta conjecture/theorem (15)

If  $1 \leq p \leq 2$ ,  $-1 \leq q < 0$  and  $(p, q) \neq (1, -1)$ , then for any  $K \in \mathcal{B}(\mathcal{H})^\times$

$$\Psi_{p,q, \frac{1}{p+q}}(A, B) = \text{Tr}(B^{\frac{q}{2}} K^* A^p K B^{\frac{q}{2}})^{\frac{1}{p+q}}, \quad A, B \in \mathcal{B}(\mathcal{H})^+$$

is jointly convex.

Wigner-Yanase-Dyson conjecture and Lieb's concavity theorem

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# Main result: solution of Exercise 2'

Our main result is the solution of Exercise 2' (Z. 20)<sup>5</sup>: the necessary and sufficient conditions for  $(p, q, s) \in \mathbb{R}^3$  s.t.

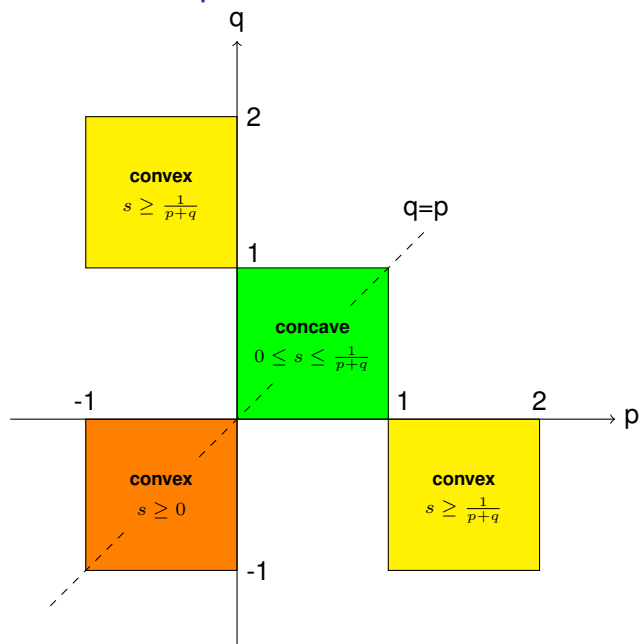
$$\Psi_{p,q,s}(A, B) = \operatorname{Tr}(B^{\frac{q}{2}} K^* A^p K B^{\frac{q}{2}})^s = \operatorname{Tr}|A^{\frac{p}{2}} K B^{\frac{q}{2}}|^{2s},$$

is jointly convex/concave for all  $K \in \mathcal{B}(\mathcal{H})^\times$  and all  $\mathcal{H}$ . For this we may assume that  $s \geq 0$ .

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<sup>5</sup>H. Zhang. From Wigner-Yanase-Dyson conjecture to Carlen-Frank-Lieb conjecture, Adv. Math., 365 (2020).

# Main result: a picture



# Brief history

It takes a long time to complete this picture. Hiai<sup>6</sup> (13) proved that these conditions are **necessary** by testing 4-by-4 matrices. The first significant **sufficient** result is the **Lieb's concavity theorem**. We will complete the whole “sufficient” picture using the following building blocks:

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<sup>6</sup>F. Hiai. Concavity of certain matrix trace and norm functions. *Linear Algebra Appl.*, 439 (5) (2013), pp. 1568-1589.

# Building blocks

## Building block 1 (Lieb, 73)

For any  $0 < p < 1$ ,

$$\Psi_{p,1-p,1}(A, B) = \text{Tr}(K^* A^p K B^{1-p})$$

is **jointly concave** for all  $K \in \mathcal{B}(\mathcal{H})$ .

# Building blocks

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## Building block 2 (Ando, 79)

For any  $-1 < p < 0$ ,  $\Psi_{p,1-p,1}$  is **jointly convex** for all  $K \in \mathcal{B}(\mathcal{H})$ .

# Building blocks

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## Building block 2 (Ando, 79)

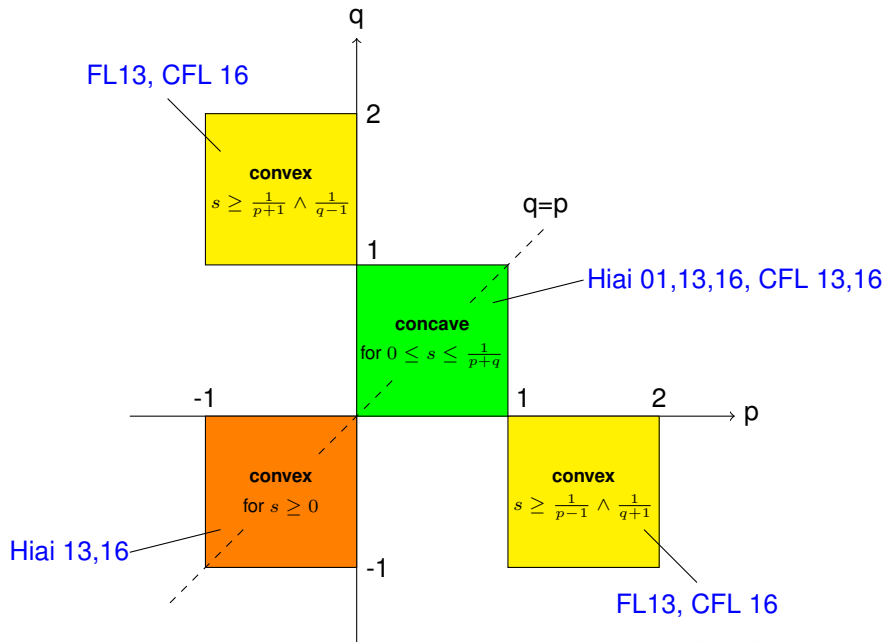
For any  $-1 < p < 0$ ,  $\Psi_{p,1-p,1}$  is **jointly convex** for all  $K \in \mathcal{B}(\mathcal{H})$ .

And the following (well-known) fact:

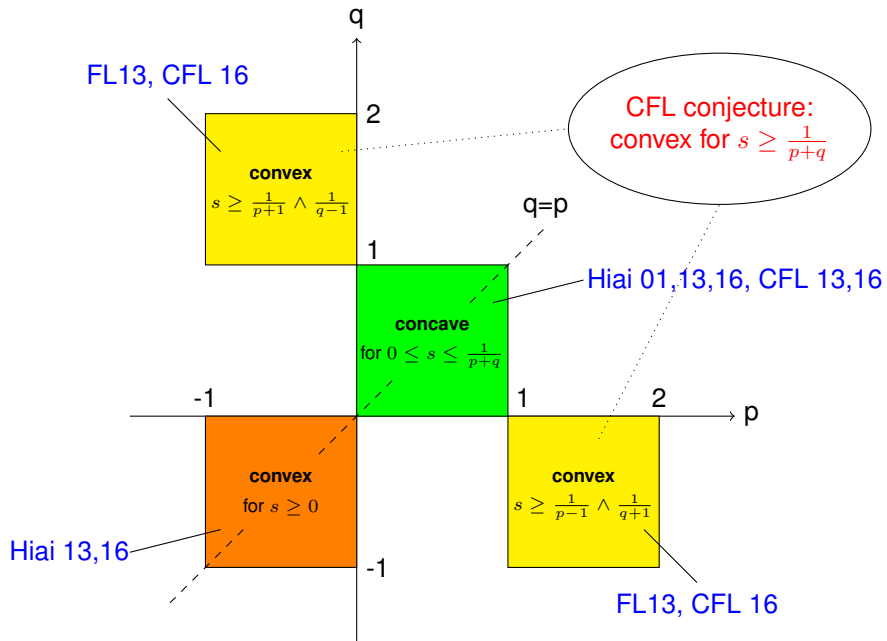
## Building block 3

For any  $t > 0$ ,  $A \mapsto \text{Tr} A^{-t}$  is convex.

# Brief history of sufficient results



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# Carlen-Frank-Lieb conjecture/theorem

## Carlen-Frank-Lieb conjecture/theorem <sup>7</sup>(18)

If  $1 \leq p \leq 2$ ,  $-1 \leq q < 0$ ,  $(p, q) \neq (1, -1)$  and  $s \geq \frac{1}{p+q}$ , then

$$\Psi_{p,q,s}(A, B) = \text{Tr}(B^{\frac{q}{2}} K^* A^p K B^{\frac{q}{2}})^s, \quad A, B \in \mathcal{B}(\mathcal{H})^+$$

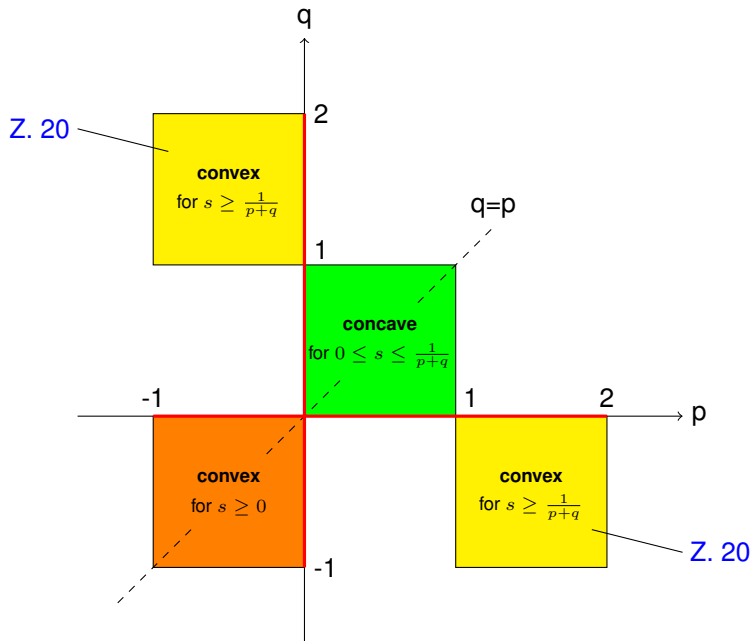
is jointly convex for any  $K \in \mathcal{B}(\mathcal{H})$ .

AD conjecture is a special case of CFL conjecture with  $s = \frac{1}{p+q}$ . We will confirm these two conjectures, thus completing the picture.

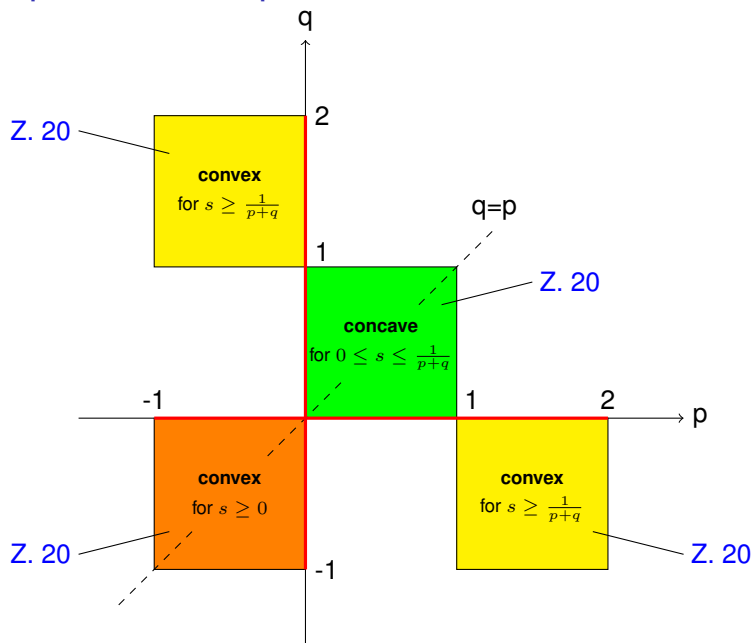
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<sup>7</sup>E.A. Carlen, R.L. Frank, E.H. Lieb. Inequalities for quantum divergences and the Audenaert–Datta conjecture. J. Phys. A, Math. Theor., 51 (48) (2018).

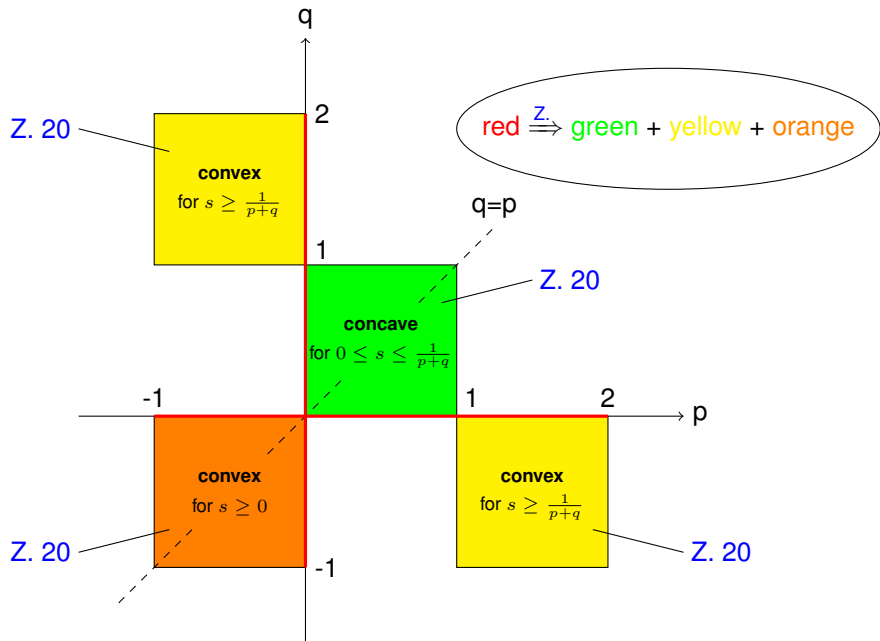
# Completion of the picture



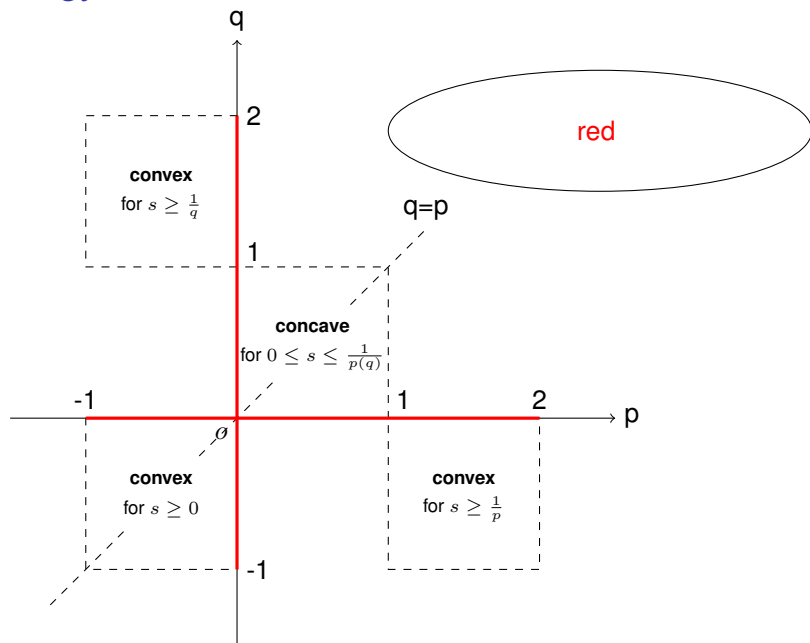
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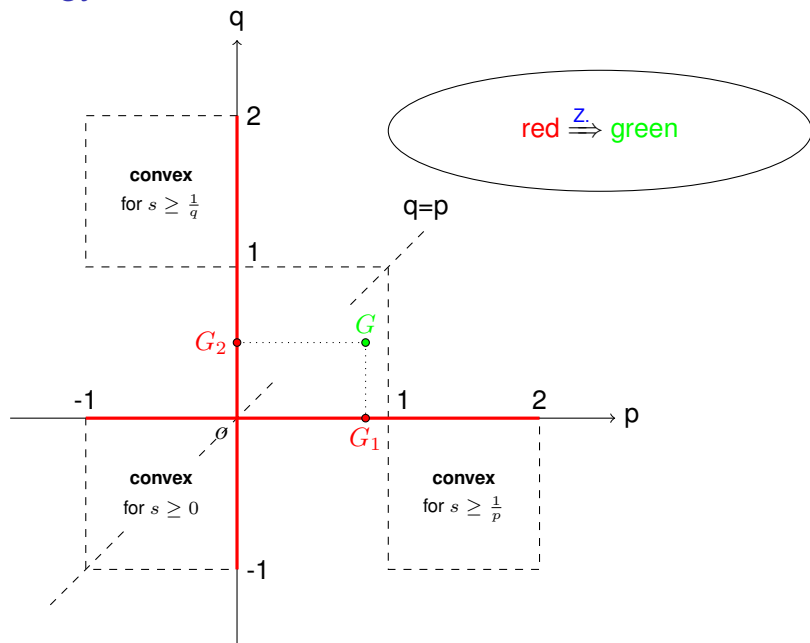
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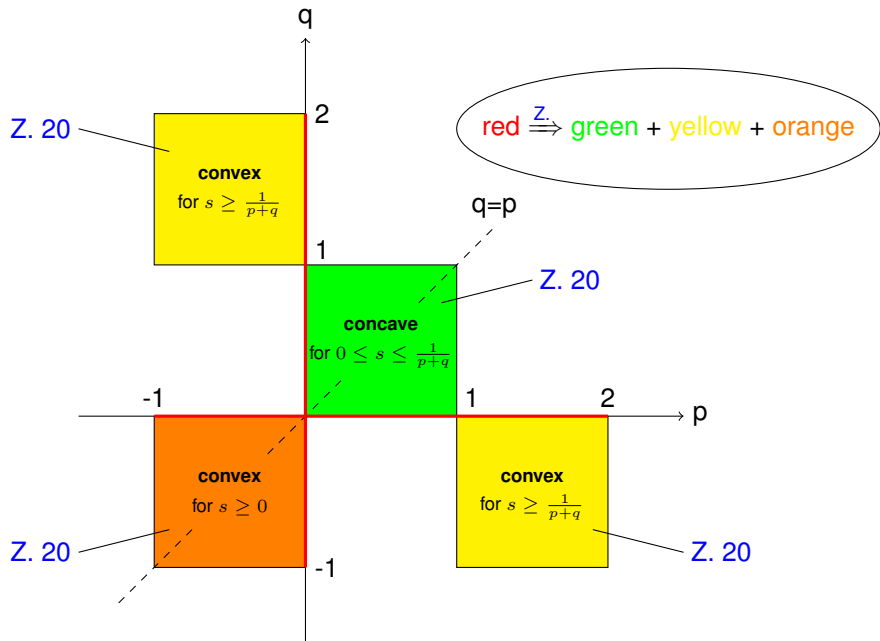
# Strategy: reduction



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## Toy example: another proof of Exercise 2

There is an argument to realize this “reduction”. To explain this let us have a look at a “variational” proof of the joint concavity of

$$\psi_{\alpha,\beta}(x, y) = x^\alpha y^\beta$$

for  $0 < \alpha, \beta < 1$  such that  $\alpha + \beta \leq 1$ .

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### Idea I (concave-min)

If  $f(\cdot, y)$  is concave for each  $y$ , then  $\min_y f(\cdot, y)$  is concave.

Then the desired joint concavity follows from the variational formula

$$\psi_{\alpha,\beta}(x, y) = x^\alpha y^\beta = \min_{z>0} \left\{ \frac{\alpha}{\alpha + \beta} (xz^{\frac{1}{\alpha}})^{\alpha+\beta} + \frac{\beta}{\alpha + \beta} (yz^{-\frac{1}{\beta}})^{\alpha+\beta} \right\},$$

and the concavity of  $t \mapsto t^{\alpha+\beta}$ .

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and the concavity of  $t \mapsto t^{\alpha+\beta}$ . The proof of this variational formula is easy. But we present a proof which will suggest what to prove in noncommutative setting.

## Where is the variational formula from?

Recall that for  $\frac{1}{p} + \frac{1}{q} = 1$ , we have by Young's (Hölder's) inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

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Replace  $(a, b)$  with  $(ac, c^{-1}b)$ , then for any  $c > 0$ :

$$ab = ac \cdot c^{-1}b \leq \frac{1}{p}(ac)^p + \frac{1}{q}(c^{-1}b)^q.$$

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It is easy to find  $c$  to capture the equality, so that we have

$$ab = \min_{c>0} \left\{ \frac{1}{p}(ac)^p + \frac{1}{q}(c^{-1}b)^q \right\}.$$

This gives

$$x^\alpha y^\beta = \min_{z>0} \left\{ \frac{\alpha}{\alpha + \beta} (xz^{\frac{1}{\alpha}})^{\alpha + \beta} + \frac{\beta}{\alpha + \beta} (yz^{-\frac{1}{\beta}})^{\alpha + \beta} \right\},$$

for  $0 < \alpha, \beta < 1$  such that  $\alpha + \beta \leq 1$ .

# The variational method

This “variational method” is quite applicable for **Exercise 2'**:

## The key variational formulas (Z.20)

For  $X, Y \in \mathcal{B}(\mathcal{H})^\times$  and  $r_i > 0, i = 0, 1, 2$  such that  $\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}$ :

$$\mathrm{Tr}|XY|^{r_0} = \min_{Z \in \mathcal{B}(\mathcal{H})^\times} \left\{ \frac{r_0}{r_1} \mathrm{Tr}|XZ|^{r_1} + \frac{r_0}{r_2} \mathrm{Tr}|Z^{-1}Y|^{r_2} \right\},$$

$$\mathrm{Tr}|XY|^{r_1} = \max_{Z \in \mathcal{B}(\mathcal{H})^\times} \left\{ \frac{r_1}{r_0} \mathrm{Tr}|XZ|^{r_0} - \frac{r_1}{r_2} \mathrm{Tr}|Y^{-1}Z|^{r_2} \right\}.$$

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## Remarks

Special cases were known and used (Carlen-Lieb 08), as a noncommutative counterpart of Legendre transformation. Here  $Z \in \mathcal{B}(\mathcal{H})^\times$  is **not unique**, but  $ZZ^* \in \mathcal{B}(\mathcal{H})^+$  is.

## Solution to Exercise 1': red line

On the red line  $q = 0$  (or  $p = 0$ )

$$\Psi_{p,s}(A) := \text{Tr}(K^* A^p K)^s, \quad A \in \mathcal{B}(\mathcal{H})^+,$$

where  $K \in \mathcal{B}(\mathcal{H})^\times$  and  $s > 0$ .

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<sup>8</sup>E.A. Carlen, E.H. Lieb. A Minkowski type trace inequality and strong subadditivity of quantum entropy. II. Convexity and concavity. Lett. Math. Phys., 83 (2) (2008), pp. 107-126.

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where  $K \in \mathcal{B}(\mathcal{H})^\times$  and  $s > 0$ . Solution to Exercise 1' was known:

### Red line (Epstein 73, Carlen-Lieb<sup>8</sup> 08, Hiai 13)

Fix  $K \in \mathcal{B}(\mathcal{H})^\times$ , then  $\Psi_{p,s}$  is

- (1) concave if  $0 < p \leq 1$  and  $0 < s \leq \frac{1}{p}$ ;
- (2) convex if  $-1 \leq p \leq 0$  and  $s > 0$ ;
- (3) convex if  $1 \leq p \leq 2$  and  $s \geq \frac{1}{p}$ .

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<sup>8</sup>E.A. Carlen, E.H. Lieb. A Minkowski type trace inequality and strong subadditivity of quantum entropy. II. Convexity and concavity. Lett. Math. Phys., 83 (2) (2008), pp. 107-126.

## Solution to Exercise 2': red $\Rightarrow$ green

Note that

$$\Psi_{p,q,s}(A, B) = \text{Tr}(B^{\frac{q}{2}} K^* A^p K B^{\frac{q}{2}})^s = \text{Tr}|A^{\frac{p}{2}} K B^{\frac{q}{2}}|^{2s}.$$

If  $p = 0$  or  $q = 0$ , then it reduces to **red line** (1).

## Solution to Exercise 2': red $\Rightarrow$ green

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If  $p = 0$  or  $q = 0$ , then it reduces to **red line** (1).

If  $p, q > 0$ , set  $\lambda := s(p + q) \in (0, 1]$  and then by the **key variational formula**:

$$\begin{aligned} \Psi_{p,q,s}(A, B) &\rightsquigarrow G \\ &= \min_{Z \in \mathcal{B}(\mathcal{H})^\times} \left\{ \frac{p}{p+q} \text{Tr}|A^{\frac{p}{2}} K Z|^{\frac{2\lambda}{p}} + \frac{q}{p+q} \text{Tr}|Z^{-1} B^{\frac{q}{2}}|^{\frac{2\lambda}{q}} \right\} \\ &= \min_{Z \in \mathcal{B}(\mathcal{H})^\times} \left\{ \frac{p}{p+q} \underbrace{\text{Tr}(Z^* K^* A^p K Z)^{\frac{\lambda}{p}}}_{G_1} + \frac{q}{p+q} \underbrace{\text{Tr}(Z^{-1} B^q (Z^{-1})^*)^{\frac{\lambda}{q}}}_{G_2} \right\} \end{aligned}$$

Thus from **red line** (1) and **Idea I**,  $\Psi_{p,q,s}$  is jointly concave.

## Further reductions: variational method bis

Using **Idea I**, we have reduced **Exercise 2'** to **red line: Exercise 1'**.  
But the classical proof of **red line** is still not simple, relying on the  
“analytic method” by Epstein (73)<sup>9</sup>.

### Idea II (concave-max)

If  $f(\cdot, \cdot)$  is jointly concave, then  $\max_y f(\cdot, y)$  is concave.

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### Idea II (concave-max)

If  $f(\cdot, \cdot)$  is jointly concave, then  $\max_y f(\cdot, y)$  is concave.

For example, if  $0 < p < 1$ , then by the **key variational formula**:

$$\mathrm{Tr}(K^* A^p K)^{\frac{1}{p}} = \max_{B \in \mathcal{B}(\mathcal{H})^+} \left\{ \frac{1}{p} \mathrm{Tr}(K^* A^p K B^{1-p}) - \frac{1-p}{p} \mathrm{Tr} B \right\}.$$

Hence the concavity of  $A \mapsto \mathrm{Tr}(K^* A^p K)^{\frac{1}{p}}$  will follow from the joint concavity of (Lieb 73, **building block 1**)

$$(A, B) \mapsto \mathrm{Tr}(K^* A^p K B^{1-p}).$$

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## Further reductions: building blocks

Concavity of **green region**:

$$\Psi_{p,q,s} \stackrel{\min}{\longleftarrow} \text{Tr}(K^* A^p K)^s \stackrel{\min}{\longleftarrow} \text{Tr}(K^* A^p K)^{1/p} \stackrel{\max}{\longleftarrow} \text{Tr} K^* A^p K B^{1-p} \text{ (bb1)}$$

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Convexity of **yellow region**:

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Convexity of **orange region**:

$$\Psi_{p,q,s} \stackrel{\leftarrow \max}{\leq} \text{Tr}(K^* A^p K)^s \stackrel{\leftarrow \min}{\leq} \text{Tr} K^* A^p K B^{1-p} \ \& \ \text{Tr} A^{-t} \text{ (bb2+3)}$$

## Conclusion: back to WYD conjecture/theorem

Generally, using **Idea I & II**, to obtain the whole picture, essentially we only need (**building blocks 1+2**)

$$\Psi_{p,1-p,1}(A, B) = \text{Tr}(K^* A^p K B^{1-p})$$

is

- ▶ jointly concave for  $0 < p < 1$  (Lieb, 73);
- ▶ jointly convex for  $-1 < p < 0$  (Ando, 79),

which admit plenty of proofs.

# Conclusion: back to WYD conjecture/theorem

Generally, using **Idea I & II**, to obtain the whole picture, essentially we only need (**building blocks 1+2**)

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which admit plenty of proofs.

To conclude:

WYD conjecture  $\sim$  Lieb-concave & Ando-convex

$\xRightarrow{\text{Idea II}}$  convexity/concavity of  $\Psi_{p,1/p}$  **partial Exercise 1'**

$\xRightarrow{\text{Idea I}}$  convexity/concavity of  $\Psi_{p,q,s}$  **Exercise 2'**

$\Rightarrow$  AD and CFL conjectures/theorems

Wigner-Yanase-Dyson conjecture and Lieb's concavity theorem

Audenaert-Datta conjecture and data processing inequality

Carlen-Frank-Lieb conjecture and solution of **Exercise 2'**

Remarks and questions

## Remark: another concavity result of Lieb

In his famous 1973 paper Lieb also proved the concavity of

$$f(A) = \text{Tr} e^{K + \log A},$$

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## Remark: another concavity result of Lieb

In his famous 1973 paper Lieb also proved the concavity of

$$f(A) = \text{Tr} e^{K + \log A},$$

with  $K = K^*$  fixed. By the Lie product formula

$$e^{K + \log A} = \lim_{n \rightarrow \infty} (e^{\frac{K}{n}} e^{\frac{1}{n} \log A})^n = \lim_{n \rightarrow \infty} (e^{\frac{K}{n}} A^{\frac{1}{n}})^n,$$

where  $n \geq 1$  is an integer. Set  $K_n := e^{\frac{K}{2n}}$ . Then the concavity of

$$\begin{aligned} f(A) &= \lim_{n \rightarrow \infty} \text{Tr}(K_n A^{\frac{1}{n}} K_n)^n \\ &= \lim_{n \rightarrow \infty} \max_{B \in \mathcal{B}(\mathcal{H})^+} \left\{ n \text{Tr}(K_n A^{\frac{1}{n}} K_n B^{1 - \frac{1}{n}}) - (n - 1) \text{Tr} B \right\} \end{aligned}$$

will follow from the joint concavity result of Lieb (bb1). Recall also that

$$f(A) = \max_{B \in \mathcal{B}(\mathcal{H})^+} \{ \text{Tr} B (K + \log A) - \text{Tr} B \log B + \text{Tr} B \}.$$

## Remark: equality conditions

To prove DPI we essentially use: when  $0 < p, q < 1$ :

$$\mathrm{Tr}(B^{\frac{q}{2}} A^p B^{\frac{q}{2}})^{\frac{1}{p+q}} = \min_H \max_{K,L} \left\{ \frac{1}{p+q} \mathrm{Tr} \left( H^{\frac{1}{2}} A^p H^{\frac{1}{2}} K^{1-p} \right) - \frac{1-p}{p+q} \mathrm{Tr} K \right. \\ \left. + \frac{1}{p+q} \mathrm{Tr} \left( H^{-\frac{1}{2}} B^q H^{-\frac{1}{2}} L^{1-q} \right) - \frac{1-q}{p+q} \mathrm{Tr} L \right\},$$

and when  $1 < p \leq 2$  and  $-1 < q < 0$ :

$$\mathrm{Tr}(B^{\frac{q}{2}} A^p B^{\frac{q}{2}})^{\frac{1}{p+q}} = \max_H \min_{K,L} \left\{ \frac{1}{p+q} \mathrm{Tr} \left( H^{\frac{1}{2}} A^p H^{\frac{1}{2}} K^{1-p} \right) + \frac{p-1}{p+q} \mathrm{Tr} K \right. \\ \left. - \frac{1}{p+q} \mathrm{Tr} \left( H^{\frac{1}{2}} B^{-q} H^{\frac{1}{2}} L^{1+q} \right) + \frac{1+q}{p+q} \mathrm{Tr} L \right\}.$$

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<sup>10</sup>H. Zhang. Equality conditions of Data Processing Inequality for  $\alpha$ -z Rényi relative entropies, J. Math. Phys., 61, 102201 (2020)

## Remark: equality conditions

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Moreover, the corresponding minimizers and maximizers are **unique**, which allows us to give equality conditions of DPI <sup>10</sup>.

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# Questions

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# Questions

- ▶ In general, it is not clear if the equality of DPI of  $D_{\alpha,z}$  implies the existence of recovery maps.
- ▶ Now we know well about the convexity/concavity of

$$(A, B) \mapsto \text{Tr}(B^{q/2} K^* A^p K B^{q/2})^s = \text{Tr}|A^{\frac{p}{2}} K B^{\frac{q}{2}}|^{2s},$$

but how much do we know about

$$K \mapsto \text{Tr}(B^{\frac{1}{2}} K^r A K^r B^{\frac{1}{2}})^s = \text{Tr}|A^{\frac{1}{2}} K^r B^{\frac{1}{2}}|^{2s}?$$

It is trivial when  $r = s = 1$  or  $A = B$ . For an application of such concavity/convexity result, see Proposition 3.1.6 in arXiv:2006.07726 (Sarah Chehade).

Thank you!