Spectral methods for entropic contraction coefficients

Cambyse Rouzé, Technische Universität München, MCQST Start Fellow

Joint work with Li Gao (TUM)

February 2021, IPAM

arXiv:2102.04146

Strong data processing inequality

- **Relative entropy**: For any states $\rho \leq \kappa\,\sigma$ on $\mathbb{M}_n$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\ln\rho - \ln\sigma)\right]$$

$\rightarrow$ asymptotic distinguishability between states [Hiai&Petz91,Ogawa&Nagaoka00]

Strong data processing inequality

- **Relative entropy**: For any states $\rho \leq \kappa\,\sigma$ on $\mathbb{M}_n$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\ln\rho - \ln\sigma)\right]$$

$\rightarrow$ asymptotic distinguishability between states [Hiai&Petz91,Ogawa&Nagaoka00]

- **Data processing inequality**: distinguishability decrease under channel $\Phi : A \rightarrow B$

$$\forall \rho, \sigma\,, \quad D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma)$$

Strong data processing inequality

- **Relative entropy**: For any states $\rho \leq \kappa \sigma$ on $\mathbb{M}_n$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\ln\rho - \ln\sigma)\right]$$

$\rightarrow$ asymptotic distinguishability between states [Hiai&Petz91,Ogawa&Nagaoka00]

- **Data processing inequality**: distinguishability decrease under channel $\Phi : A \rightarrow B$

$$\forall \rho, \sigma\,, \quad D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma)$$

- **Question 1: Can we quantify the contraction of information measures?**

Strong data processing inequality

- **Relative entropy**: For any states $\rho \le \kappa\,\sigma$ on $\mathbb{M}_n$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\ln\rho - \ln\sigma)\right]$$

$\to$ asymptotic distinguishability between states [Hiai&Petz91,Ogawa&Nagaoka00]

- **Data processing inequality**: distinguishability decrease under channel $\Phi : A \to B$

$$\forall \rho, \sigma\,, \quad D(\Phi(\rho)\|\Phi(\sigma)) \le D(\rho\|\sigma)$$

- **Question 1: Can we quantify the contraction of information measures?**

- **Strong data processing inequality**: for all ρ, σ

$$D(\Phi(\rho)\|\Phi(\sigma)) \le \eta(\Phi)\, D(\rho\|\sigma)$$

$\eta(\Phi) < 1$ $\Rightarrow$ exponential decay of information through repeated action of Φ

Strong data processing inequality

- **Relative entropy**: For any states $\rho \leq \kappa\,\sigma$ on $\mathbb{M}_n$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\ln\rho - \ln\sigma)\right]$$

$\to$ asymptotic distinguishability between states [Hiai&Petz91,Ogawa&Nagaoka00]

- **Data processing inequality**: distinguishability decrease under channel $\Phi : A \to B$

$$\forall \rho,\sigma\,,\quad D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma)$$

- **Question 1: Can we quantify the contraction of information measures?**

- **Strong data processing inequality**: for all ρ, σ

$$D(\Phi(\rho)\|\Phi(\sigma)) \leq \eta(\Phi)\, D(\rho\|\sigma)$$

$\eta(\Phi) < 1$ $\Rightarrow$ exponential decay of information through repeated action of Φ

- Classical methods: Dobrushin coefficients, hypercontractivity, maximal correlation, log-Sobolev, concentration and information-transportation inequalities... [Raginsky16]

Asymptotic convergence of channels and thermalization

- **Asymptotic convergence**: Given Jordan decomposition $\Phi := \sum_{|\lambda|=1} \lambda P_\lambda + P_{\mathrm{nilp}}$,

$$\|\Phi^m(\rho) - \Phi^m \circ E_{\mathcal{N}}(\rho)\|_{1\to 1} \to 0\,, \quad m \to \infty\,, \qquad E_{\mathcal{N}} = \sum_{|\lambda|=1} P_\lambda$$

- **Question 2: how long does it take for a system to reach equilibrium?**

- (weak) **Answer 1**: spectral gap estimate $m(\epsilon) = \mathcal{O}(\mathrm{poly}\,|\Lambda|)$

Asymptotic convergence of channels and thermalization

- **Asymptotic convergence**: Given Jordan decomposition $\Phi := \sum_{|\lambda|=1} \lambda P_\lambda + P_{\text{nilp}}$,

$$\|\Phi^m(\rho) - \Phi^m \circ E_{\mathcal{N}}(\rho)\|_{1\to 1} \to 0\,, \quad m \to \infty\,, \qquad E_{\mathcal{N}} = \sum_{|\lambda|=1} P_\lambda$$

- **Question 2: how long does it take for a system to reach equilibrium?**

- (weak) **Answer 1**: spectral gap estimate $m(\epsilon) = \mathcal{O}(\text{poly}\,|\Lambda|)$

- (stronger) **Answer 2**: **entropic contraction** $m(\epsilon) = \mathcal{O}(\text{polylog}\,|\Lambda|)$

$$D(\Phi^m(\rho)\|\Phi^m \circ E_{\mathcal{N}}(\rho)) \leq s(\Phi)^m\, D(\rho\|E_{\mathcal{N}}(\rho))$$

Asymptotic convergence of channels and thermalization

- **Asymptotic convergence**: Given Jordan decomposition $\Phi := \sum_{|\lambda|=1} \lambda P_\lambda + P_{\text{nilp}},$

$$\|\Phi^m(\rho) - \Phi^m \circ E_{\mathcal{N}}(\rho)\|_{1\to 1} \to 0\,, \quad m \to \infty\,, \qquad E_{\mathcal{N}} = \sum_{|\lambda|=1} P_\lambda$$

- **Question 2: how long does it take for a system to reach equilibrium?**

- (weak) **Answer 1**: spectral gap estimate $m(\epsilon) = \mathcal{O}(\text{poly}\,|\Lambda|)$

- (stronger) **Answer 2**: **entropic contraction** $m(\epsilon) = \mathcal{O}(\text{polylog}\,|\Lambda|)$

$$D(\Phi^m(\rho)\|\Phi^m \circ E_{\mathcal{N}}(\rho)) \le s(\Phi)^m\, D(\rho\|E_{\mathcal{N}}(\rho))$$

- Classically, **tensor stability** $\Rightarrow$ environment cannot increase contraction coefficient

- Not true quantumly: **complete entropic contraction**: for any ρ on AR, R arbitrary

$$D((\Phi^m \otimes \text{id}_R)(\rho)\|(\Phi^m \circ E_{\mathcal{N}} \otimes \text{id}_R)(\rho)) \le s_c(\Phi)^m\, D(\rho\|(E_{\mathcal{N}} \otimes \text{id}_R)(\rho))$$

Asymptotic convergence of channels and thermalization

- **Asymptotic convergence**: Given Jordan decomposition $\Phi := \sum_{|\lambda|=1} \lambda P_\lambda + P_{\mathrm{nilp}},$

$$\|\Phi^m(\rho) - \Phi^m \circ E_{\mathcal{N}}(\rho)\|_{1\to 1} \to 0\,, \quad m \to \infty\,, \qquad E_{\mathcal{N}} = \sum_{|\lambda|=1} P_\lambda$$

- **Question 2: how long does it take for a system to reach equilibrium?**

- (weak) **Answer 1**: spectral gap estimate $m(\epsilon) = \mathcal{O}(\mathrm{poly}\,|\Lambda|)$

- (stronger) **Answer 2**: **entropic contraction** $m(\epsilon) = \mathcal{O}(\mathrm{polylog}\,|\Lambda|)$

$$D(\Phi^m(\rho)\|\Phi^m \circ E_{\mathcal{N}}(\rho)) \le s(\Phi)^m\, D(\rho\|E_{\mathcal{N}}(\rho))$$

- Classically, **tensor stability** $\Rightarrow$ environment cannot increase contraction coefficient

- Not true quantumly: **complete entropic contraction**: for any ρ on AR, R arbitrary

$$D((\Phi^m \otimes \mathrm{id}_R)(\rho)\|(\Phi^m \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \le s_c(\Phi)^m\, D(\rho\|(E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

- **Question 3: Is $s_c(\Phi) < 1$ for all Φ?**

Markovian dynamics and modified logarithmic Sobolev inequality

- Continuous time dynamics: when $\Phi \equiv \Phi_t := \mathrm{e}^{t\mathcal{L}}$ is a **quantum Markov semigroup**:

$$D((\Phi_t \otimes \mathrm{id_R})(\rho) \| (\mathrm{e}^{t\mathcal{L}} \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq \mathrm{e}^{-\alpha_c(\mathcal{L})t} \, D(\rho \| (E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

$$\Updownarrow$$

- **Complete modified logarithmic Sobolev inequality** [Bardet17,Junge&Gao&Laracuente18]:

$$\alpha_c(\mathcal{L}) \, D(\rho \| (E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq \mathrm{EP}_{\mathcal{L} \otimes \mathrm{id}_R}(\rho) := - \frac{d}{dt}\bigg|_{t=0} D((\mathrm{e}^{t\mathcal{L}} \otimes \mathrm{id}_R)(\rho) \| (\mathrm{e}^{t\mathcal{L}} \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

Markovian dynamics and modified logarithmic Sobolev inequality

- Continuous time dynamics: when $\Phi \equiv \Phi_t := \mathrm{e}^{t\mathcal{L}}$ is a **quantum Markov semigroup**:

$$D((\Phi_t \otimes \mathrm{id}_{\mathrm{R}})(\rho) \| (\mathrm{e}^{t\mathcal{L}} \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq \mathrm{e}^{-\alpha_c(\mathcal{L})t} \, D(\rho \| (E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

$$\Updownarrow$$

- **Complete modified logarithmic Sobolev inequality** [Bardet17,Junge&Gao&Laracuente18]:

$$\alpha_c(\mathcal{L}) \, D(\rho \| (E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq \mathrm{EP}_{\mathcal{L} \otimes \mathrm{id}_R}(\rho) := - \frac{d}{dt}\Big|_{t=0} D((\mathrm{e}^{t\mathcal{L}} \otimes \mathrm{id}_R)(\rho) \| (\mathrm{e}^{t\mathcal{L}} \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

- Almost all finite dimensional symmetric QMS have $\alpha_c(\mathcal{L}) > 0$ [Junge&Gao&Laracuente18]

- Almost all finite dimensional DBC QMS have $\alpha_c(\mathcal{L}) > 0$ [Junge&Laracuente&CR19]

Markovian dynamics and modified logarithmic Sobolev inequality

- Continuous time dynamics: when $\Phi \equiv \Phi_t := \mathrm{e}^{t\mathcal{L}}$ is a **quantum Markov semigroup**:

$$D((\Phi_t \otimes \mathrm{id_R})(\rho)\|(\mathrm{e}^{t\mathcal{L}} \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq \mathrm{e}^{-\alpha_c(\mathcal{L})t}\; D(\rho\|(E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

$$\Updownarrow$$

- **Complete modified logarithmic Sobolev inequality** [Bardet17,Junge&Gao&Laracuente18]:

$$\alpha_c(\mathcal{L})\, D(\rho\|(E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq \mathrm{EP}_{\mathcal{L}\otimes \mathrm{id}_R}(\rho) := -\frac{d}{dt}\Big|_{t=0} D((\mathrm{e}^{t\mathcal{L}} \otimes \mathrm{id}_R)(\rho)\|(\mathrm{e}^{t\mathcal{L}} \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

- Almost all finite dimensional symmetric QMS have $\alpha_c(\mathcal{L}) > 0$ [Junge&Gao&Laracuente18]

- Almost all finite dimensional DBC QMS have $\alpha_c(\mathcal{L}) > 0$ [Junge&Laracuente&CR19]

- **Question 4: is $\alpha_c(\mathcal{L}) > 0$ for all $\mathcal{L}$ in finite dimensions?**

Overview

I A key Lemma

II Complete modified logarithmic Sobolev inequalities (CMLSI)

III Complete strong data processing inequality (CSDPI)

Relative entropy and Fisher information

- **Relative entropy**: For any states $\rho \leq \kappa\,\sigma$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\ln\rho - \ln\sigma)\right]$$

$$\text{“}\sum p\,\ln\frac{p}{q}\text{”}$$

Relative entropy and Fisher information

- **Relative entropy**: For any states $\rho \le \kappa\,\sigma$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\,(\ln\rho - \ln\sigma)\right]$$

"$\sum p \ln \frac{p}{q}$"

- **Fisher information**: For any $\gamma \in \mathbb{M}_{n,\mathbb{R}}$ s.t. $\mathrm{supp}(\gamma) \subseteq \mathrm{supp}(\sigma)$,

$$\|\gamma\|_{F,\sigma}^2 := \int_0^\infty \mathrm{Tr}\left[\gamma\,(\sigma + uI)^{-1}\,\gamma\,(\sigma + uI)^{-1}\right]\,du$$

"$\sum \frac{\gamma^2}{q}$"

Relative entropy and Fisher information

- **Relative entropy**: For any states $\rho \leq \kappa\, \sigma$,

$$D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\left(\ln\rho - \ln\sigma\right)\right]$$

"$\sum p \, \ln \frac{p}{q}$"

- **Fisher information**: For any $\gamma \in \mathbb{M}_{n,\mathbb{R}}$ s.t. $\mathrm{supp}(\gamma) \subseteq \mathrm{supp}(\sigma)$,

$$\|\gamma\|^2_{F,\sigma} := \int_0^\infty \mathrm{Tr}\left[\gamma\left(\sigma + uI\right)^{-1}\gamma\left(\sigma+uI\right)^{-1}\right] du$$

"$\sum \frac{\gamma^2}{q}$"

- The Fisher information is the Hessian of the relative entropy:

$$\|\gamma\|^2_{F,\sigma} = D^2(D(\sigma+\gamma\|\sigma))[\gamma]\,.$$

I – Controlling the relative entropy by the Fisher information

I – Controlling the relative entropy by the Fisher information

For $\rho \leq \kappa\sigma \quad \Rightarrow \quad \boxed{D_{\max}(\rho\|\sigma) := \ln\left(\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\|_\infty\right) \leq \ln(\kappa)}$ [Datta09]

I – Controlling the relative entropy by the Fisher information

For $\rho \leq \kappa\sigma \quad \Rightarrow \quad D_{\max}(\rho\|\sigma) := \ln\left(\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\|_\infty\right) \leq \ln(\kappa)$ [Datta09]

Lemma [Gao&CR 2021] $\forall\gamma \in \mathbb{M}_{n,\mathbb{R}}$ A, B positive definite s.t. $A \leq B$

$$\|\gamma\|_{F,A} \geq \|\gamma\|_{F,B}$$

$\forall\rho, \sigma \in \mathcal{D}(\mathbb{C}^n)$ s.t. $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$

$$D(\rho\|\sigma) \leq \|\rho - \sigma\|^2_{F,\sigma} \leq k(\mathrm{e}^{D_{\max}(\rho\|\sigma)})\, D(\rho\|\sigma)$$

$$k(c) := \frac{(c-1)^2}{c\ln(c) - (c-1)}$$

I – Controlling the relative entropy by the Fisher information

For $\rho \leq \kappa\sigma \quad \Rightarrow \quad D_{\max}(\rho\|\sigma) := \ln\left(\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\|_\infty\right) \leq \ln(\kappa)$ [Datta09]

Lemma [Gao&CR 2021] $\forall \gamma \in \mathbb{M}_{n,\mathbb{R}}$ A, B positive definite s.t. $A \leq B$

$$\|\gamma\|_{F,A} \geq \|\gamma\|_{F,B}$$

$\forall \rho, \sigma \in \mathcal{D}(\mathbb{C}^n)$ s.t. $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$

$$D(\rho\|\sigma) \leq \|\rho - \sigma\|^2_{F,\sigma} \leq k(\mathrm{e}^{D_{\max}(\rho\|\sigma)})\, D(\rho\|\sigma)$$

- **Proof**:

$$\|\gamma\|^2_{F,A} := \int_0^\infty \mathrm{Tr}\left[(\rho - \sigma)\,(A + uI)^{-1}\,(\rho - \sigma)\,(A + uI)^{-1}\right] du$$

$$k(c) := \frac{(c-1)^2}{c\ln(c) - (c-1)}$$

I – Controlling the relative entropy by the Fisher information

For $\rho \leq \kappa\sigma \quad \Rightarrow \quad \boxed{D_{\max}(\rho\|\sigma) := \ln\left(\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\|_\infty\right) \leq \ln(\kappa)}$ [Datta09]

Lemma [Gao&CR 2021] $\forall\gamma \in \mathbb{M}_{n,\mathbb{R}}$ A, B positive definite s.t. $A \leq B$

$$\|\gamma\|_{F,A} \geq \|\gamma\|_{F,B}$$

$\forall \rho, \sigma \in \mathcal{D}(\mathbb{C}^n)$ s.t. $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$

$$\boxed{D(\rho\|\sigma) \;\leq\; \|\rho-\sigma\|^2_{F,\sigma} \;\leq\; k(\mathrm{e}^{D_{\max}(\rho\|\sigma)})\, D(\rho\|\sigma)}$$

- **Proof**: Let $\rho(t) := (1-t)\sigma + t\rho \quad \Rightarrow \quad (1-t)\,\sigma \leq \rho(t) \leq (1-t+t\kappa)\sigma$
- $f(t) := D(\rho(t)\|\sigma)$, $f(0) = f'(0) = 0 \;\Rightarrow\; D(\rho\|\sigma) = f(1) = \int_0^1\int_0^t f''(s)\,ds\,dt$
- $\frac{1}{1-t+t\kappa}\|\rho-\sigma\|^2_{F,\sigma} \leq \boxed{f''(t) \equiv \|\rho-\sigma\|^2_{F,\rho(t)}} \leq \frac{1}{1-t}\,\|\rho-\sigma\|^2_{F,\sigma}$

$$\boxed{\|\gamma\|^2_{F,\sigma} = D^2(D(\sigma+\gamma\|\sigma))[\gamma]}$$

II - Spectral control of CMLSI

II - Spectral control of CMLSI

- Assume Φ_t is **GNS-symmetric** with respect to σ: $\operatorname{tr}(\sigma\Phi_t^*(X)^\dagger Y) = \operatorname{tr}(\sigma X^\dagger \Phi_t^*(Y))$

II - Spectral control of CMLSI

- Assume Φ_t is **GNS-symmetric** with respect to σ: $\mathrm{tr}(\sigma\Phi_t^*(X)^\dagger Y) = \mathrm{tr}(\sigma X^\dagger \Phi_t^*(Y))$
- In the primitive case, MLSI is related to **hypercontractivity**: $\forall t \geq 0$

$$\|\Phi_t^* : L_2(\sigma) \to L_{p(t)}(\sigma)\| \leq 1\,, \quad p(t) = 1 + \mathrm{e}^{2\alpha_2\, t}$$

II - Spectral control of CMLSI

- Assume Φ_t is **GNS-symmetric** with respect to σ: $\operatorname{tr}(\sigma\Phi_t^*(X)^\dagger Y) = \operatorname{tr}(\sigma X^\dagger \Phi_t^*(Y))$
- In the primitive case, MLSI is related to **hypercontractivity**: $\forall t \geq 0$

$$\|\Phi_t^* : L_2(\sigma) \to L_{p(t)}(\sigma)\| \leq 1\,, \quad p(t) = 1 + \mathrm{e}^{2\alpha_2\, t}$$

- **Interpolation** leads to "worst case" bounds for α_2 [Diaconis&Saloffe-Coste96,Olkiewicz&Zegarlinski99]:

$$\frac{\lambda(\mathcal{L})}{\ln(\sigma_{\min}^{-1}) + 2} \leq \alpha_2 \leq \lambda(\mathcal{L})$$

- By discrete **differential calculus** [Bobkov&Tetali06,Kastoryano&Temme13,Carbone14,Bardet17]:

$$\alpha_2 \leq \frac{\alpha(\mathcal{L})}{2} \leq \lambda(\mathcal{L})$$

II - Spectral control of CMLSI

- Assume Φ_t is **GNS-symmetric** with respect to σ: $\operatorname{tr}(\sigma\Phi_t^*(X)^\dagger Y) = \operatorname{tr}(\sigma X^\dagger \Phi_t^*(Y))$
- In the primitive case, MLSI is related to **hypercontractivity**: $\forall t \geq 0$

$$\|\Phi_t^* : L_2(\sigma) \to L_{p(t)}(\sigma)\| \leq 1\,, \quad p(t) = 1 + \mathrm{e}^{2\alpha_2\, t}$$

- **Interpolation** leads to "worst case" bounds for α_2 [Diaconis&Saloffe-Coste96,Olkiewicz&Zegarlinski99]:

$$\frac{\lambda(\mathcal{L})}{\ln(\sigma_{\min}^{-1}) + 2} \leq \alpha_2 \leq \lambda(\mathcal{L})$$

- By discrete **differential calculus** [Bobkov&Tetali06,Kastoryano&Temme13,Carbone14,Bardet17]:

$$\alpha_2 \leq \frac{\alpha(\mathcal{L})}{2} \leq \lambda(\mathcal{L})$$

- **Complete hypercontractivity** [Beigi&King16] fails [Bardet&CR18,Junge&Gao&Laracuente18]:

$$\alpha_{2,\mathrm{c}} = 0$$

II - Spectral control of CMLSI

$$D_{\max}(\mathbb{M}_n \| \mathcal{N}) := \max_{\rho \in \mathcal{D}(\mathbb{C}^n)} D_{\max}(\rho \| E_{\mathcal{N}}(\rho)) \quad \Rightarrow \quad \rho \leq \underbrace{\exp(D_{\max}(\mathbb{M}_n \| \mathcal{N}))}_{C(\mathbb{M}_n : \mathcal{N})} E_{\mathcal{N}}(\rho)$$

$$C(\mathbb{M}_n : \mathcal{N})_{\text{cb}} := \sup_k C(\mathbb{M}_n \otimes \mathbb{M}_k : \mathcal{N} \otimes \mathbb{M}_k) < \infty$$

[Pimsner&Popa86] [Gao&Junge&Laracuente20]

II - Spectral control of CMLSI

$D_{\max}(\mathbb{M}_n\|\mathcal{N}) := \max_{\rho\in\mathcal{D}(\mathbb{C}^n)} D_{\max}(\rho\|E_{\mathcal{N}}(\rho))$ $\Rightarrow$ $\rho \le \underbrace{\exp(D_{\max}(\mathbb{M}_n\|\mathcal{N}))}_{C(\mathbb{M}_n:\mathcal{N})} E_{\mathcal{N}}(\rho)$

$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}} := \sup_k C(\mathbb{M}_n\otimes\mathbb{M}_k : \mathcal{N}\otimes\mathbb{M}_k) < \infty$

[Pimsner&Popa86] [Gao&Junge&Laracuente20]

Spectral gap: $\lambda(\mathcal{L})\|\rho - E_{\mathcal{N}}(\rho)\|^2_{F,E_{\mathcal{N}}(\rho)} \le \mathcal{E}(\rho) := -\langle\rho,\, \mathcal{L}(\rho)\rangle_{F,E_{\mathcal{N}}(\rho)}$

II - Spectral control of CMLSI

$$D_{\max}(\mathbb{M}_n \| \mathcal{N}) := \max_{\rho \in \mathcal{D}(\mathbb{C}^n)} D_{\max}(\rho \| E_{\mathcal{N}}(\rho)) \quad \Rightarrow \quad \rho \leq \underbrace{\exp(D_{\max}(\mathbb{M}_n \| \mathcal{N}))}_{C(\mathbb{M}_n : \mathcal{N})} E_{\mathcal{N}}(\rho)$$

$$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}} := \sup_k C(\mathbb{M}_n \otimes \mathbb{M}_k : \mathcal{N} \otimes \mathbb{M}_k) < \infty$$

[Pimsner&Popa86] [Gao&Junge&Laracuente20]

Spectral gap: $\lambda(\mathcal{L}) \|\rho - E_{\mathcal{N}}(\rho)\|^2_{F, E_{\mathcal{N}}(\rho)} \leq \mathcal{E}(\rho) := -\langle \rho, \mathcal{L}(\rho) \rangle_{F, E_{\mathcal{N}}(\rho)}$

Theorem [Gao&CR21]: For any GNS-symmetric QMS on $\mathbb{M}_n$,

$$\frac{\lambda(\mathcal{L})}{C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}}} \leq \alpha_{\mathrm{c}}(\mathcal{L}) \leq 2\lambda(\mathcal{L}) \, .$$

$$\alpha_c(\mathcal{L}) \, D(\rho \| (E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq \mathrm{EP}_{\mathcal{L} \otimes \mathrm{id}_R}(\rho) = -\operatorname{tr}(\mathcal{L}(\rho)(\ln \rho - \ln E_{\mathcal{N}}(\rho)))$$

II - Spectral control of CMLSI

$D_{\max}(\mathbb{M}_n \| \mathcal{N}) := \max_{\rho \in \mathcal{D}(\mathbb{C}^n)} D_{\max}(\rho \| E_{\mathcal{N}}(\rho))$ $\Rightarrow$ $\rho \leq \underbrace{\exp(D_{\max}(\mathbb{M}_n \| \mathcal{N}))}_{C(\mathbb{M}_n : \mathcal{N})} E_{\mathcal{N}}(\rho)$

$C(\mathbb{M}_n : \mathcal{N})_{\text{cb}} := \sup_k C(\mathbb{M}_n \otimes \mathbb{M}_k : \mathcal{N} \otimes \mathbb{M}_k) < \infty$ [Pimsner&Popa86] [Gao&Junge&Laracuente20]

Spectral gap: $\lambda(\mathcal{L}) \|\rho - E_{\mathcal{N}}(\rho)\|^2_{F, E_{\mathcal{N}}(\rho)} \leq \mathcal{E}(\rho) := -\langle \rho, \mathcal{L}(\rho) \rangle_{F, E_{\mathcal{N}}(\rho)}$

Theorem [Gao&CR21]: For any GNS-symmetric QMS on $\mathbb{M}_n$,

$$\frac{\lambda(\mathcal{L})}{C(\mathbb{M}_n : \mathcal{N})_{\text{cb}}} \leq \alpha_{\text{c}}(\mathcal{L}) \leq 2\lambda(\mathcal{L}) \,.$$

$\alpha_c(\mathcal{L})\, D(\rho \| (E_{\mathcal{N}} \otimes \text{id}_R)(\rho)) \leq \text{EP}_{\mathcal{L} \otimes \text{id}_R}(\rho) = -\operatorname{tr}(\mathcal{L}(\rho)(\ln \rho - \ln E_{\mathcal{N}}(\rho)))$

$\Rightarrow$ the CMLSI constant exists for GNS-symmetric semigroups (see also [Gao&Junge&Li21])

II - Spectral control of CMLSI (proof)

- $\mathcal{L}$ symmetric $\quad\Rightarrow\quad \mathcal{L} = -\sum_i \partial_i^* \partial_i \,, \partial_i(\rho) := [L_i, \rho]$

II - Spectral control of CMLSI (proof)

- $\mathcal{L}$ symmetric $\Rightarrow$ $\mathcal{L} = -\sum_i \partial_i^* \partial_i \, , \partial_i(\rho) := [L_i, \rho]$

- Noncommutative differential calculus:

$$\mathrm{EP}_{\mathcal{L}}(\rho) = -\,\mathrm{tr}\left[\mathcal{L}(\rho)(\ln\rho - \ln E_{\mathcal{N}}(\rho))\right] = \sum_i \mathrm{tr}\left[\partial_i(\rho)^\dagger\, \partial_i(\ln\rho - \ln E_{\mathcal{N}}(\rho))\right] \qquad \text{“}\partial \ln(f)\text{”}$$

$$= \sum_i \int_0^\infty \mathrm{tr}\left[\partial_i\rho^\dagger\,(\rho + uI)^{-1}\,\partial_i\rho\,(\rho+uI)^{-1})\right] du \qquad \text{“}\frac{\partial f}{f}\text{”}$$

$$= \sum_i \|\partial_i\rho\|^2_{F,\rho}$$

$$\mathcal{E}(\rho) = -\langle \rho, \mathcal{L}(\rho)\rangle_{F,E_{\mathcal{N}}(\rho)} = \sum_i \|\partial_i\rho\|^2_{F,E_{\mathcal{N}}(\rho)}$$

II - Spectral control of CMLSI (proof)

- $\mathcal{L}$ symmetric $\Rightarrow$ $\mathcal{L} = -\sum_i \partial_i^* \partial_i$, $\partial_i(\rho) := [L_i, \rho]$

- Noncommutative differential calculus:

$$\mathrm{EP}_{\mathcal{L}}(\rho) = -\operatorname{tr}\left[\mathcal{L}(\rho)(\ln\rho - \ln E_{\mathcal{N}}(\rho))\right] = \sum_i \operatorname{tr}\left[\partial_i(\rho)^\dagger\, \partial_i(\ln\rho - \ln E_{\mathcal{N}}(\rho))\right] \qquad \text{“}\partial \ln(f)\text{”}$$

$$= \sum_i \int_0^\infty \operatorname{tr}\left[\partial_i\rho^\dagger\,(\rho + uI)^{-1}\,\partial_i\rho\,(\rho+uI)^{-1})\right] du \qquad \text{“}\frac{\partial f}{f}\text{”}$$

$$= \sum_i \|\partial_i\rho\|^2_{F,\rho}$$

$$\mathcal{E}(\rho) = -\langle \rho, \mathcal{L}(\rho)\rangle_{F,E_{\mathcal{N}}(\rho)} = \sum_i \|\partial_i\rho\|^2_{F,E_{\mathcal{N}}(\rho)}$$

$$\frac{\mathcal{E}(\rho)}{\mathrm{EP}_{\mathcal{L}}(\rho)} \le \exp(D_{\max}(\rho\|E_{\mathcal{N}}(\rho))) \le C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}}$$

$A \le B \Rightarrow \|\gamma\|_{F,A} \ge \|\gamma\|_{F,B}$ $\qquad$ $\rho \le C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}}\, E_{\mathcal{N}}(\rho)$

II - Spectral control of CMLSI (proof)

- $\mathcal{L}$ symmetric $\Rightarrow$ $\mathcal{L} = -\sum_i \partial_i^* \partial_i \, , \partial_i(\rho) := [L_i, \rho]$

- Noncommutative differential calculus:

$$\mathrm{EP}_{\mathcal{L}}(\rho) = -\operatorname{tr}\left[\mathcal{L}(\rho)(\ln\rho - \ln E_{\mathcal{N}}(\rho))\right] = \sum_i \operatorname{tr}\left[\partial_i(\rho)^\dagger \, \partial_i(\ln\rho - \ln E_{\mathcal{N}}(\rho))\right] \qquad \text{“}\partial \ln(f)\text{”}$$

$$= \sum_i \int_0^\infty \operatorname{tr}\left[\partial_i \rho^\dagger \, (\rho + uI)^{-1} \, \partial_i \rho \, (\rho + uI)^{-1})\right] du \qquad \text{“}\frac{\partial f}{f}\text{”}$$

$$= \sum_i \|\partial_i \rho\|^2_{F,\rho}$$

$$\mathcal{E}(\rho) = -\langle \rho, \mathcal{L}(\rho) \rangle_{F, E_{\mathcal{N}}(\rho)} = \sum_i \|\partial_i \rho\|^2_{F, E_{\mathcal{N}}(\rho)}$$

$$\frac{\mathcal{E}(\rho)}{\mathrm{EP}_{\mathcal{L}}(\rho)} \leq \exp(D_{\max}(\rho \| E_{\mathcal{N}}(\rho))) \leq C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}}$$

- By our key Lemma:

$$D(\rho \| E_{\mathcal{N}}(\rho)) \leq \|\rho - E_{\mathcal{N}}(\rho)\|^2_{F, E_{\mathcal{N}}(\rho)} \leq \frac{1}{\lambda(\mathcal{L})} \underbrace{\sum_i \|\partial_i \rho\|^2_{F, E_{\mathcal{N}}(\rho)}}_{\mathcal{E}(\rho)} \leq \frac{C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}}}{\lambda(\mathcal{L})} \underbrace{\sum_i \|\partial_i \rho\|^2_{F,\rho}}_{\mathrm{EP}_{\mathcal{L}}(\rho)}$$

$\square$

II - Spectral control of CMLSI

- Given algebra $\mathcal{N} := \bigoplus_i \mathbb{M}_{n_i} \otimes \mathrm{Id}_{m_i}$ and unital conditional expectation $E_{\mathrm{tr}} : \mathbb{M}_n \to \mathcal{N}$:

$$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{tr,cb}} = \sum_i m_i^2$$

$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{tr,cb}}$ is the (**complete**) **Pimsner-Popa index** [Pimsner&Popa86,Gao&Junge&Laracuente20]

- For σ-invariant conditional expectation:

$$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}} \leq \sigma_{\min}^{-1}\, C(\mathbb{M}_n : \mathcal{N})_{\mathrm{tr,cb}}$$

II - Spectral control of CMLSI

- Given algebra $\mathcal{N} := \bigoplus_i \mathbb{M}_{n_i} \otimes \mathrm{Id}_{m_i}$ and unital conditional expectation $E_{\mathrm{tr}} : \mathbb{M}_n \to \mathcal{N}$:

$$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{tr,cb}} = \sum_i m_i^2$$

$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{tr,cb}}$ is the (**complete**) **Pimsner-Popa index** [Pimsner&Popa86,Gao&Junge&Laracuente20]

- For σ-invariant conditional expectation:

$$C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}} \leq \sigma_{\mathrm{min}}^{-1}\, C(\mathbb{M}_n : \mathcal{N})_{\mathrm{tr,cb}}$$

- Primitive case: $E_{\mathcal{N}}(\rho) = \sigma$ for all $\rho \Rightarrow C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}} \leq \sigma_{\mathrm{min}}^{-1}\, n^2$

$$\frac{\sigma_{\mathrm{min}}\, \lambda(\mathcal{L})}{n^2} \leq \alpha_{\mathrm{c}}(\mathcal{L}) \leq \alpha(\mathcal{L}) \leq 2\lambda(\mathcal{L})$$

- Can be compared with hypercontractive estimates

$$\frac{2\lambda(\mathcal{L})}{2 + \ln(\sigma_{\mathrm{min}}^{-1})} \leq 2\alpha_2 \leq \alpha(\mathcal{L}) \leq 2\lambda(\mathcal{L})$$

III - Spectral control of CSDPI

III - Spectral control of CSDPI

- $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ σ-symmetric quantum channel $\operatorname{tr}(\sigma\Phi^*(X)^\dagger Y) = \operatorname{tr}(\sigma X^\dagger \Phi^*(Y))$
- **Gap**: $\lambda(\Phi) := \|\Phi^*(\mathrm{id} - E_{\mathcal{N}}^*) : L_2(\sigma) \to L_2(\sigma)\| < 1$

$$D((\Phi \otimes \mathrm{id}_R)(\rho) \| (\Phi \circ E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq s_c(\Phi)\, D(\rho \| (E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho))$$

Theorem [Gao&CR21]: For any GNS-symmetric channel Φ,

$$\lambda(\Phi)^2 \leq s_{\mathrm{c}}(\Phi) \leq c(C(\mathbb{M}_n : \mathcal{N})_{\mathrm{cb}}, \lambda(\Phi)) < 1$$

III - Spectral control of CSDPI (proof)

- Φ symmetric $\quad \rho_t := t\,\rho + (1-t)E_{\mathcal{N}}(\rho),$ $g(t) := D(\rho_t \| E_{\mathcal{N}}(\rho)) - D(\Phi(\rho_t) \| \Phi \circ E_{\mathcal{N}}(\rho))$
- $g(0) = g'(0) = 0,\; g''(t) = \|\rho - E_{\mathcal{N}}(\rho)\|^2_{F,\rho_t} - \|\Phi(\rho) - \Phi \circ E_{\mathcal{N}}(\rho)\|^2_{F,\Phi(\rho)_t}$

III - Spectral control of CSDPI (proof)

- Φ symmetric $\quad \rho_t := t\,\rho + (1-t)E_{\mathcal{N}}(\rho)$, $g(t) := D(\rho_t \| E_{\mathcal{N}}(\rho)) - D(\Phi(\rho_t) \| \Phi \circ E_{\mathcal{N}}(\rho))$
- $g(0) = g'(0) = 0,\ g''(t) = \|\rho - E_{\mathcal{N}}(\rho)\|^2_{F,\rho_t} - \|\Phi(\rho) - \Phi \circ E_{\mathcal{N}}(\rho)\|^2_{F,\Phi(\rho)_t}$
- $g''(t) \geq 0$ for all $t \geq 0$ (data processing for Fisher informations) [Ruskai&Lesniewski99] and

$$\begin{aligned} g''(t) &\geq \frac{1}{1+(C(\mathbb{M}_n:\mathcal{N})-1)t}\|\rho - E_{\mathcal{N}}(\rho)\|^2_{F,E_{\mathcal{N}}(\rho)} - \frac{1}{1-t}\|\Phi(\rho) - \Phi \circ E_{\mathcal{N}}(\rho)\|^2_{F,\Phi\circ E_{\mathcal{N}}(\rho)} \\ &\geq \left(\frac{1}{1+(C(\mathbb{M}_n:\mathcal{N})-1)t} - \frac{\lambda(\Phi)^2}{1-t}\right)\|\rho - E_{\mathcal{N}}(\rho)\|^2_{F,E_{\mathcal{N}}(\rho)} \\ &\geq \left(\frac{1}{1+(C(\mathbb{M}_n:\mathcal{N})-1)t} - \frac{\lambda(\Phi)^2}{1-t}\right) D(\rho \| E_{\mathcal{N}}(\rho)) \end{aligned}$$

- $\lambda(\Phi) < 1 \Rightarrow$ there exists $t_0 > 0$ such that above prefactor is positive for all $t < t_0$
- Reintegrate twice

Discussion

- **Main message**: simple gap estimates can be used to find (untight) relaxations/strengthnings of entropic inequalities
- **Technical tool**: simple two-sided control of the relative entropy distance in terms of its Hessian
- **Result 1**: positivity of the CMLSI constant for reversible QMS in finite dimensions
- **Result 2**: existence of non-trivial CSDPI constant for reversible channels in finite dimensions
- **Result 3**: existence of complete approximate tensorization (CAT) of the relative entropy
- **Open problem 1**: approximation of the CMLSI constant
- **Open problem 2**: find tighter bounds (reducing to SSA) in the CAT (depending on the complexity of fixed states)

Thank you for your attention!

III – Spectral control of CAT

Tensorization and its approximations

- **Tensorization** [Lieb&Ruskai73,Ohya&Petz04]: whenever $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{E_1} & \mathcal{N}_1 \\
{\scriptstyle E_2}\downarrow & \searrow^{E_{\mathcal{N}}} & \downarrow{\scriptstyle E_2} \\
\mathcal{N}_2 & \xrightarrow{E_1} & \mathcal{N}
\end{array}
$$

$$D(\rho \| E_{\mathcal{N}}(\rho)) \leq D(\rho \| E_1(\rho)) + D(\rho \| E_2(\rho))$$

Tensorization and its approximations

- **Tensorization** [Lieb&Ruskai73,Ohya&Petz04]: whenever $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{E_1} & \mathcal{N}_1 \\ {\scriptstyle E_2}\big\downarrow & \searrow^{E_\mathcal{N}} & \big\downarrow{\scriptstyle E_2} \\ \mathcal{N}_2 & \xrightarrow{E_1} & \mathcal{N} \end{array}$$

$$\boxed{D(\rho \| E_\mathcal{N}(\rho)) \leq D(\rho \| E_1(\rho)) + D(\rho \| E_2(\rho))}$$

- **Complete approximate tensorization**: when $E_1 E_2 \neq E_2 E_1$, $\forall R$ and ρ:

$$\boxed{D(\rho \| (E_\mathcal{N} \otimes \mathrm{id}_R)(\rho)) \leq a_c \Big(D(\rho \| (E_1 \otimes \mathrm{id}_R)(\rho)) + D(\rho \| (E_2 \otimes \mathrm{id}_R)(\rho)) \Big)}$$

when $a_c' \equiv f(\| E_1 E_2 - E_\mathcal{N} \|)$.

Tensorization and its approximations

- **Tensorization** [Lieb&Ruskai73,Ohya&Petz04]: whenever $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{E_1} & \mathcal{N}_1 \\ {\scriptstyle E_2}\downarrow & \searrow^{E_{\mathcal{N}}} & \downarrow{\scriptstyle E_2} \\ \mathcal{N}_2 & \xrightarrow{E_1} & \mathcal{N} \end{array}$$

$$D(\rho \| E_{\mathcal{N}}(\rho)) \leq D(\rho \| E_1(\rho)) + D(\rho \| E_2(\rho))$$

- **Complete approximate tensorization**: when $E_1 E_2 \neq E_2 E_1$, $\forall R$ and ρ:

$$D(\rho \| (E_{\mathcal{N}} \otimes \mathrm{id}_R)(\rho)) \leq a_c \Big(D(\rho \| (E_1 \otimes \mathrm{id}_R)(\rho)) + D(\rho \| (E_2 \otimes \mathrm{id}_R)(\rho)) \Big)$$

when $a'_c \equiv f(\| E_1 E_2 - E_{\mathcal{N}} \|)$.

- **Question 5: Does α_c exist?**

- **Question 6: Link to complexity of multipartite quantum states**

IV - Spectral control of CAT

- [Cesi00] Classical ($\mathcal{N} = \mathbb{C}$ Id) case: given $\gamma := \|E_1 \circ E_2 - E_\mathcal{N} : L_1(\sigma) \to L_\infty\|$,

$$D(\rho \| E_\mathcal{N}(\rho)) \leq \frac{1}{1-2\gamma} \Big(D(\rho \| E_1(\rho)) + D(\rho \| E_2(\rho)) \Big)$$

- Extension to quantum ($\mathcal{N} = \mathbb{C}$ Id) case [Bardet&Capel&CR20]

Theorem [Gao&CR21]: Denote

$$\lambda := \|E_1 \circ E_2 - E_\mathcal{N} : L_2(\sigma) \to L_2(\sigma)\|$$

Then, for any $n \in \mathbb{N}$ and all state ρ on $\mathcal{M} \otimes \mathbb{M}_n$,

$$D(\rho \| (E_\mathcal{N} \otimes \mathrm{id}_n)(\rho)) \leq a_\mathrm{c} \Big(D(\rho \| (E_1 \otimes \mathrm{id}_n)(\rho)) + D(\rho \| (E_2 \otimes \mathrm{id}_n)(\rho)) \Big),$$

where $\frac{1}{1-\lambda^2} \leq a_\mathrm{c} \leq \frac{2\, C(\mathcal{M}:\mathcal{N})_{\mathrm{cb}}}{(1-\lambda)^2}$.

- Non-matching bounds when $E_1 \circ E_2 = E_\mathcal{N}$

Thank you for your attention!