DIFFUSIVELY-PERTURBED TRANSPORT IN PSEUDOPERIODIC FLOWS

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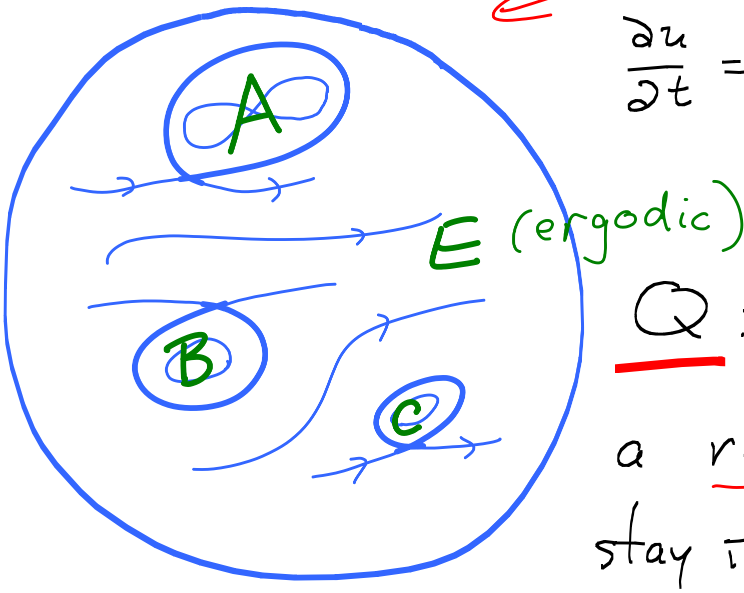
vector field

small diffusivity

$$\frac{\partial u}{\partial t} = (v, \nabla u) + \frac{\varepsilon}{2}\Delta u$$



Q: How long does a representative particle stay in A vs. B vs. C vs. E?

$$dX_t^\varepsilon = v(X_t^\varepsilon)\,dt + \varepsilon\, dW_t$$

2-dim. Brownian motion

Goal: Go from 2-dim PDE (SDE) with small ε to averaged (no ε) 1-dim PDE (SDE)

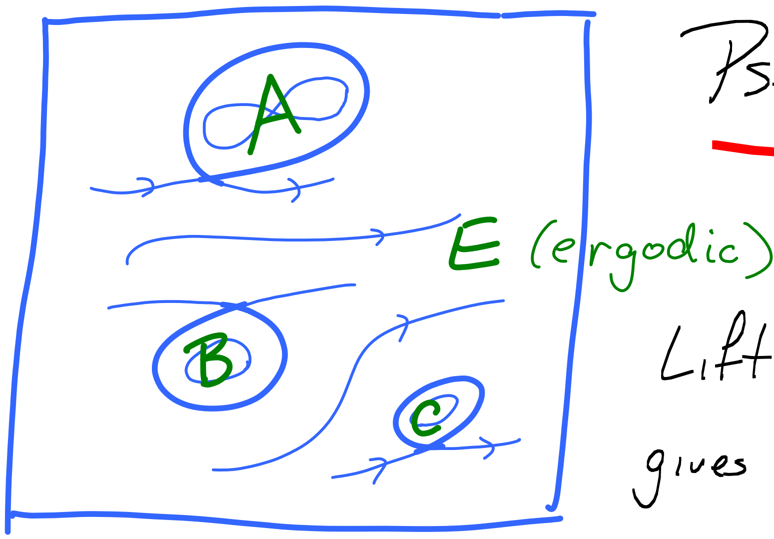
Note: model reduction, not data assimilation

Outline:

1. Modify Topology & Speed of Problem
2. Review of Stochastic Averaging
3. Review of Glueing for fixed points
4. Ergodic Class

① Modify Problem

— Change Topology: Consider Earth-shaped torus $\mathbb{T}^2$



Pseudoperiodic Flow

Arnol'd '91

Lift $\mathbb{T}^2$ to cover $\mathbb{R}^2$ gives Hamiltonian flow on $\mathbb{R}^2$

$$H(x,y) = k_1 x + k_2 y + \varphi(x,y)$$

Assume Morse

$k_1, k_2 \in \mathbb{Z}$ are incommensurable

1-periodic in x & y

vector field on $\mathbb{T}^2$ → $v = \pi_+^* \nabla^{\perp} H$ — skew gradient of H

push forward to $\mathbb{R}^2$

There are (Arnol'd)

a) Traps - disc-like regions where $\exists$ a local Hamiltonian bounded by homoclinic orbits (A, B, C)

b) Ergodic Class (E)

— Change Speed. If $b = \nabla^{\perp}\mathcal{H}$ local Hamiltonian

$$dX_t^{\varepsilon} = v(X_t^{\varepsilon}) + \varepsilon\,\sigma(X_t^{\varepsilon})\circ dW_t$$

2-dim Brownian

σ_1, σ_2

$\frac{1}{2}\{\sigma_1^2 + \sigma_2^2\} = \frac{1}{2}\Delta$

$$\mathcal{H}(X_t^{\varepsilon}) = \mathcal{H}(X_0^{\varepsilon}) + \int_0^t \cancel{(b, \nabla\mathcal{H})}(X_s^{\varepsilon})\,ds + \varepsilon\int_0^t \nabla\mathcal{H}(X_s^{\varepsilon})\circ dW_s$$

orthogonal

$$+ \frac{\varepsilon^2}{2}\int_0^t \Delta\mathcal{H}(X_s^{\varepsilon})\,ds$$

Changes of $\mathcal{H}$ visible on time scale $O(1/\varepsilon^2)$

New SDE: $\underbrace{dX_t^{\varepsilon} = \frac{1}{\varepsilon^2}v(X_t^{\varepsilon})dt + \sigma(X_t^{\varepsilon})\circ dW_t}$

fluctuations of $\mathcal{H}$ visible $t \approx O(1)$

$$\mathcal{H}(X_T^{\varepsilon}) = \mathcal{H}(X_0^{\varepsilon}) + \int_0^t \nabla\mathcal{H}(X_s^{\varepsilon})\circ dW_s + \frac{1}{2}\int_0^t \Delta\mathcal{H}(X_s^{\varepsilon})\,ds$$

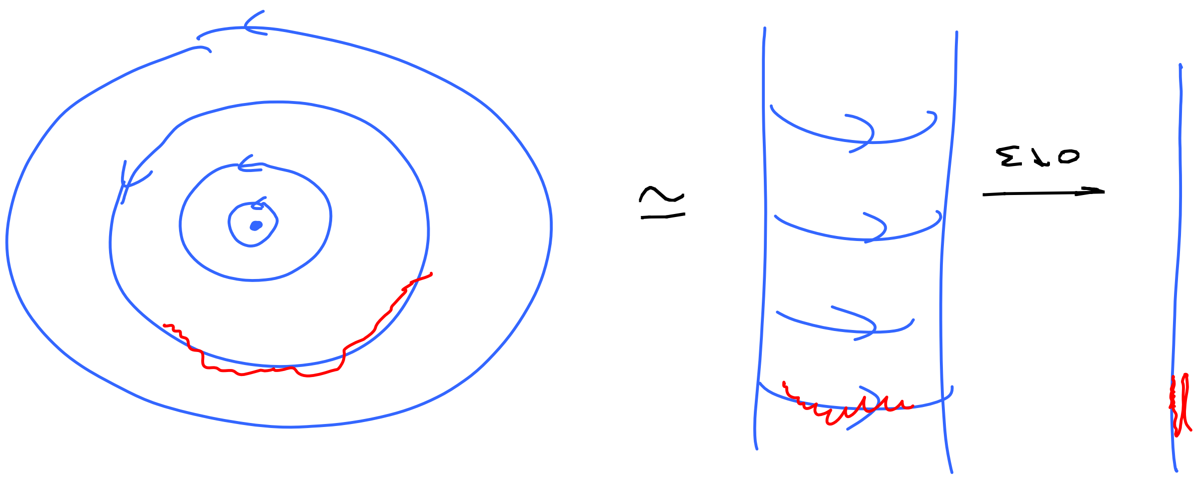
$$dX_t^\varepsilon = \frac{1}{\varepsilon^2} v(X_t^\varepsilon)\,dt + \sigma(X_t^\varepsilon) \circ dW_t$$

$$\mathcal{H}(X_T^\varepsilon) = \mathcal{H}(X_0^\varepsilon) + \int_0^t \nabla\mathcal{H}(X_s^\varepsilon) \circ dW_s + \frac{1}{2}\int_0^t \Delta\mathcal{H}(X_s^\varepsilon)\,ds$$

(2) Review of Stochastic Averaging

Q : What are effective dynamics of $\mathcal{H}(X^\varepsilon)$

(elliptic points) Khasminskii (1960's)



Separation of Scales: Effective Drift & Diffusion coefficients $\mathcal{H}(X^\varepsilon) \to h$

$$dh_t = \hat{b}(h_t)\,dt + \hat{\sigma}(h_t)\,dB_t$$ Brownian motion

$C_h = \{z : \mathcal{H}(z) = h\}$

$$\hat{b}(h) = \oint_{C_h} \frac{1}{2}\Delta\mathcal{H}(z)\,\ell(dz)$$

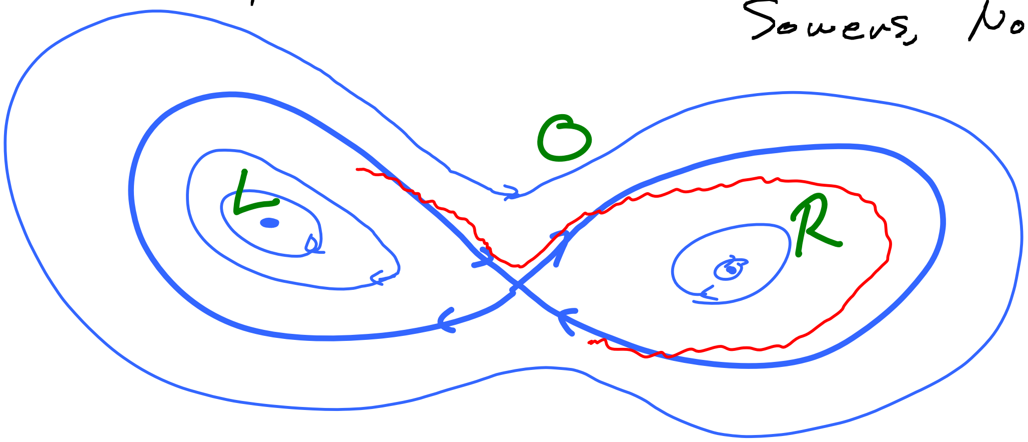
$$\hat{\sigma}^2(h) = \oint_{C_h} |\nabla\mathcal{H}(z)|^2\,\ell(dz)$$

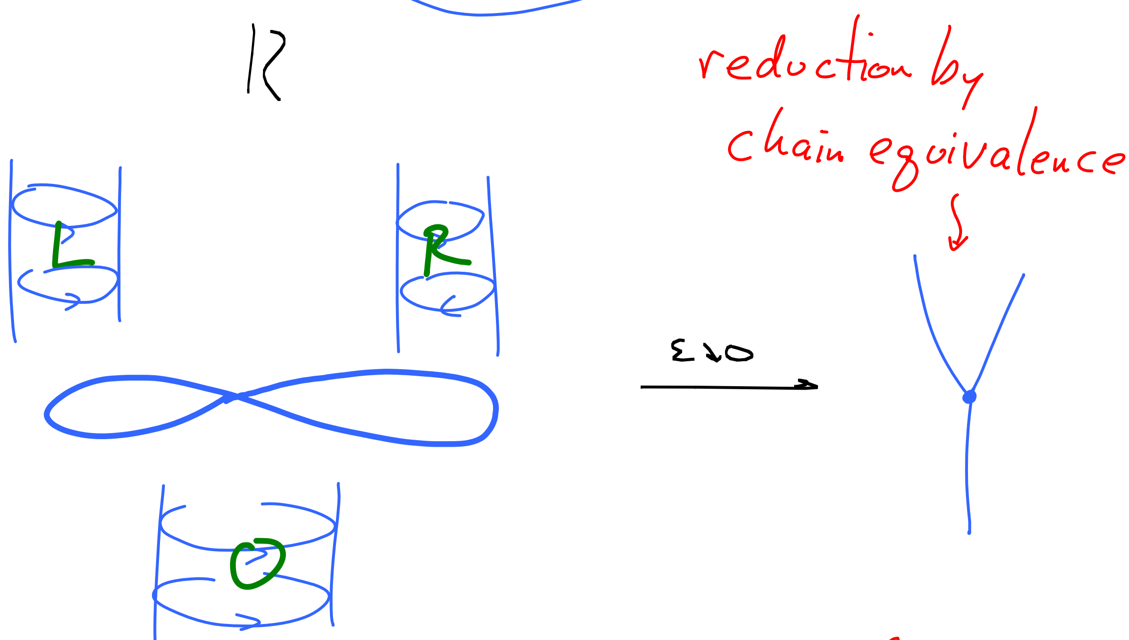
normalized length measure

h has generator $\hat{\mathcal{L}} = \hat{b}(h)\frac{d}{dh} + \frac{1}{2}\hat{\sigma}^2(h)\frac{d^2}{dh^2}$

③ Review of Glueing Conditions

(hyperbolic points) Neishtadt, Freidlin, Sowers, Novikov-Papanicolaou-Ryzhik



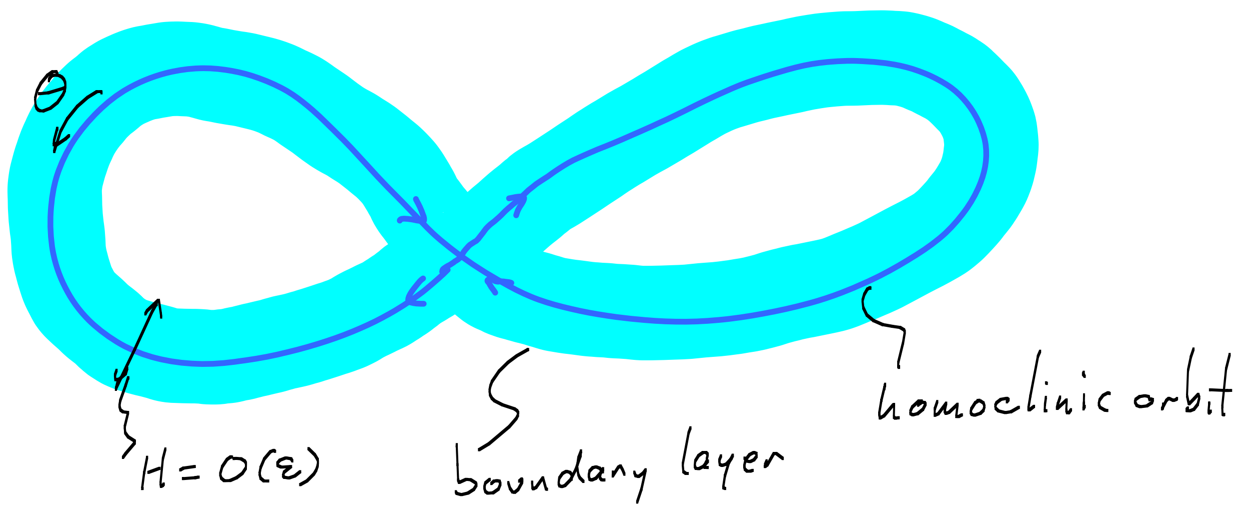


$\ell \in \{L, R, O\}$

Limiting Behavior:
- While on leg ℓ, diffuse according to averaged generator $\hat{\mathcal{L}}_\ell$
- flip biased coin at vertex

(glueing conditions) (Freidlin, 96)

Basic idea of Glueing conditions:



Diffusive generator is $\mathcal{L}^{\varepsilon} = \frac{1}{\varepsilon^2} \nabla^{\perp} H + \frac{1}{2}\Delta$

Boundary coordinates: $\left(\Theta, \frac{H}{\varepsilon}\right)$

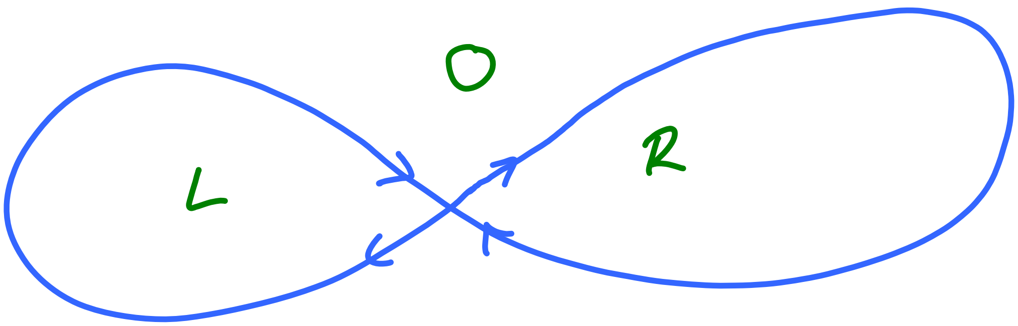
$$\mathcal{L}^{\varepsilon} u \approx \frac{1}{\varepsilon^2}(\nabla^{\perp} H, \nabla \Theta)\frac{\partial u}{\partial \theta} + \frac{1}{\varepsilon^2}|\nabla H|^2 \frac{\partial^2 u}{\partial h^2}$$

equal since boundary layer is H/ε^{1}

choose Θ such that $(\nabla^{\perp} H, \nabla \Theta) = |\nabla H|^2$

Solutions of $\mathcal{L}^{\varepsilon} u^{\varepsilon} \approx 0 \Longleftrightarrow$ How particle leaves boundary layer

CAN SOLVE IN BOUNDARY LAYER WHEN GLUEING CONDITIONS HOLD



Biases are:

$$G_L = \int_{\bigcirc} |\nabla H(z)| \, \ell(dz)$$

A
length measure

$$G_R = \int_{\bigcirc} |\nabla H(z)| \, \ell(dz)$$

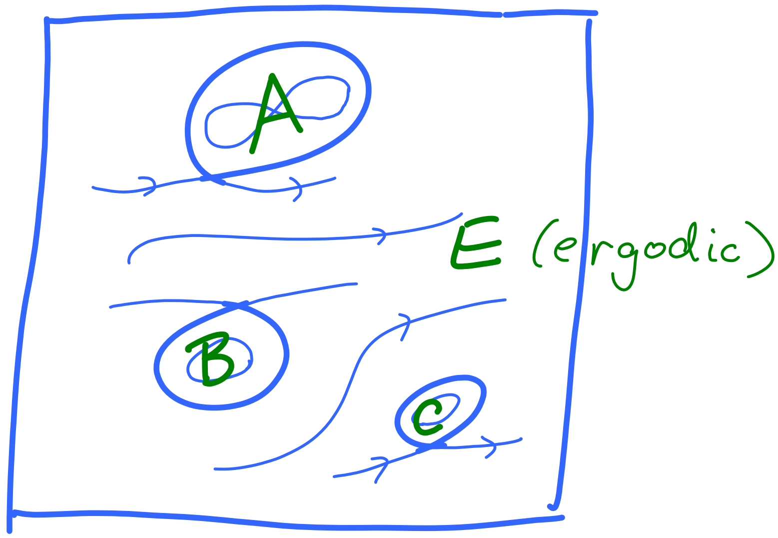
$$G_O = \int_{\infty} |\nabla H(z)| \, \ell(dz)$$

$$|\nabla H(z)| = \frac{|\nabla H(z)|^2}{|\nabla H(z)|}$$

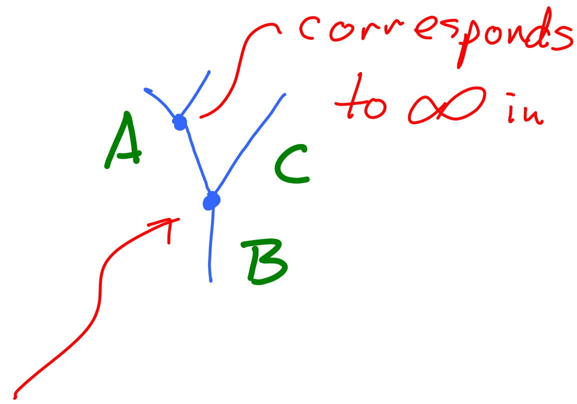
transversal diffusion

fast motion

(4) Ergodic Class (Pseudoperiodic Flow)

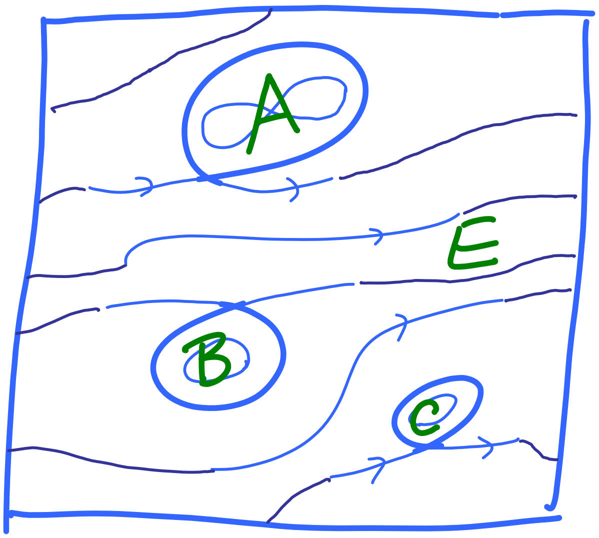


Know what to do in A, B, C



- Chain equivalence collapses all of E to single point

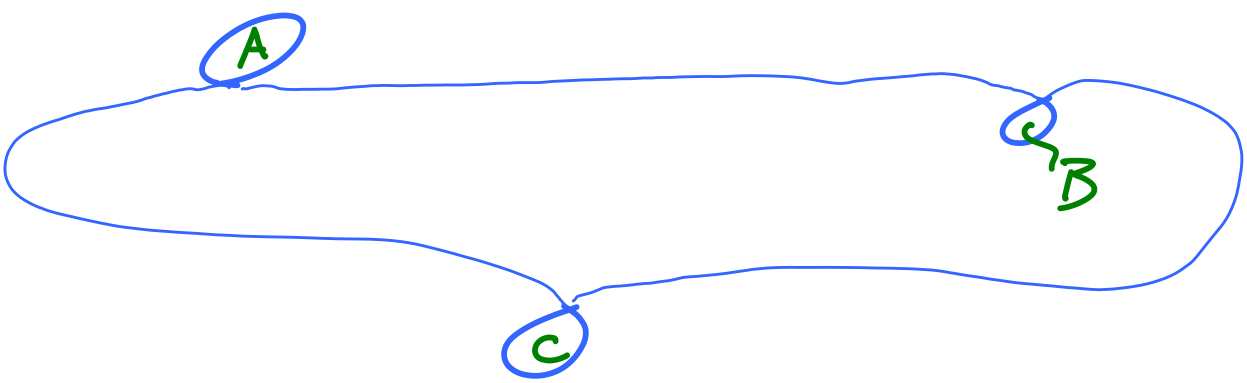
- Interesting new behavior is vertex (ignore behavior within A, B, C



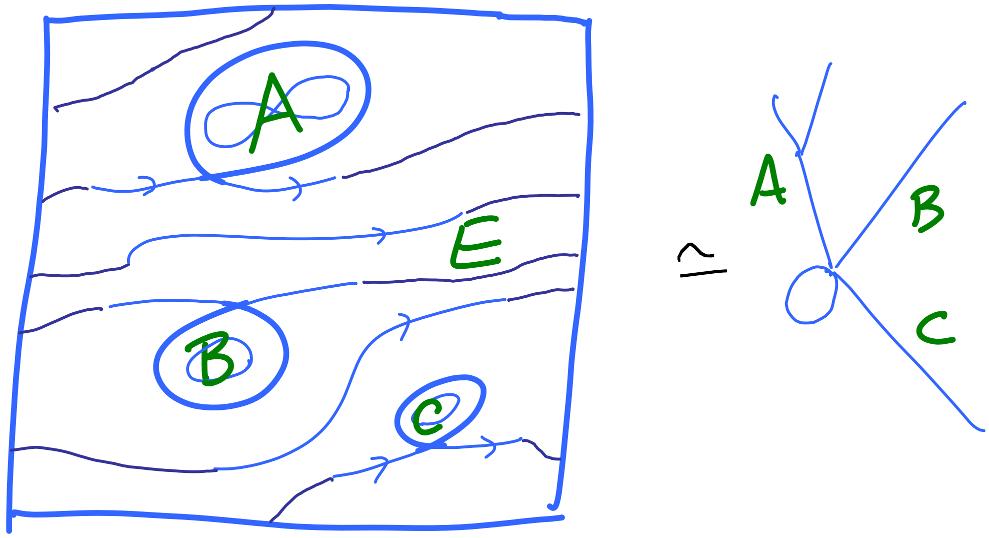
$$H(x,y) = \underbrace{k_1 x + k_2 y}_{\text{incommensurable; } \beta = \frac{k_1}{k_2} \text{ is irrational}} + \varphi(x,y)$$

Approximate β by rationals (continued fraction expansion)

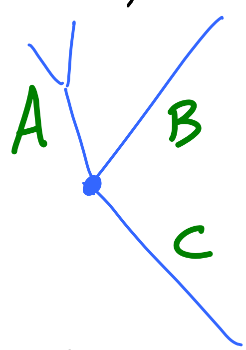
Long Heteroclinic Cycle



Can glue; very likely to go into long part (ie part in E)



Competition: • very likely to go into little loop
• spends very little time in little loop



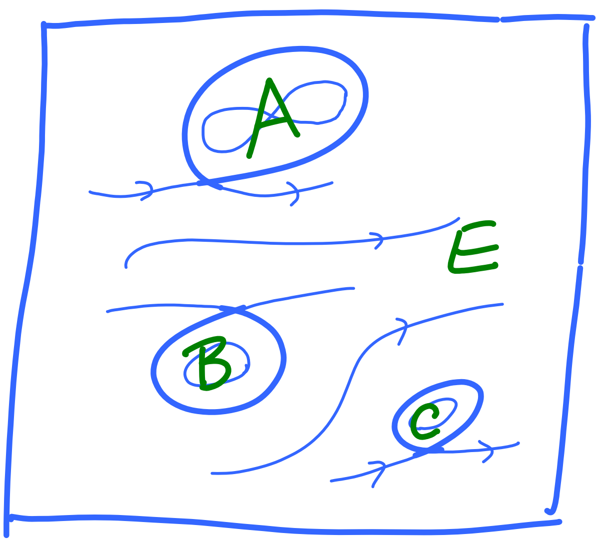
Result: Limit (along a certain subsequence, if partial fraction expansion of ρ converges fast enough) is a stickiness at vertex.

stickiness coefficient: $2|E|$ ← area of E

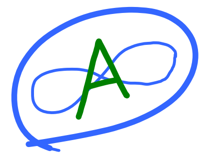
glueing conditions as before:

$$G_A = \int_O \qquad G_B = \int_\gamma \qquad G_C = \int_\ell$$

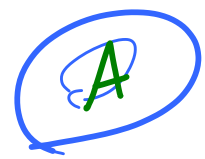
LIMITING PDE for time spent in traps

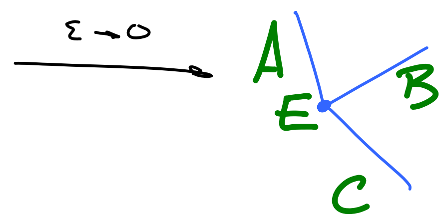


For simplicity, replace



with





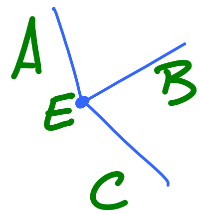
Generator of graph-valued process

domain
$$\left\{ \begin{array}{l} u_A,\ u_B,\ u_C \in C^2 \\ u_A(0) = u_B(0) = u_C(0) \\ 2|E| = G_A \dot{u}_A - G_B \dot{u}_B + G_C \dot{u}_C \end{array} \right.$$

H > 0 on A & C   H < 0 on B

Result Conjectured by Freidlin '96

Invariant measure of graph-valued



(how much time spent in A vs. B vs. C vs. E?)

$$\int \mathcal{L}_A u_A(x)\, v_A(x)\, dx + \int \mathcal{L}_B u_B(x)\, v_B(x)\, dx$$

$$+ \int \mathcal{L}_C u_C(x)\, v_C(x)\, dx + (\mathcal{L}u)(E)\, v(E) = 0$$

all (u_A, u_B, u_C) in domain of generator

$$\mathcal{L}_A^* v_A = \mathcal{L}_B^* v_B = \mathcal{L}^* v_C = 0$$

Conservation of flux at E

$$v_e(0^\pm) = \pm \frac{G_e}{\sigma_e^2(0)\,|E|}\, v(E)$$ continuity

$$\sum_e v_e(x)\, dx = 1$$ almost right

Integrate invariant measure to get residence time