Is Galerkin Projection The Right Scheme?

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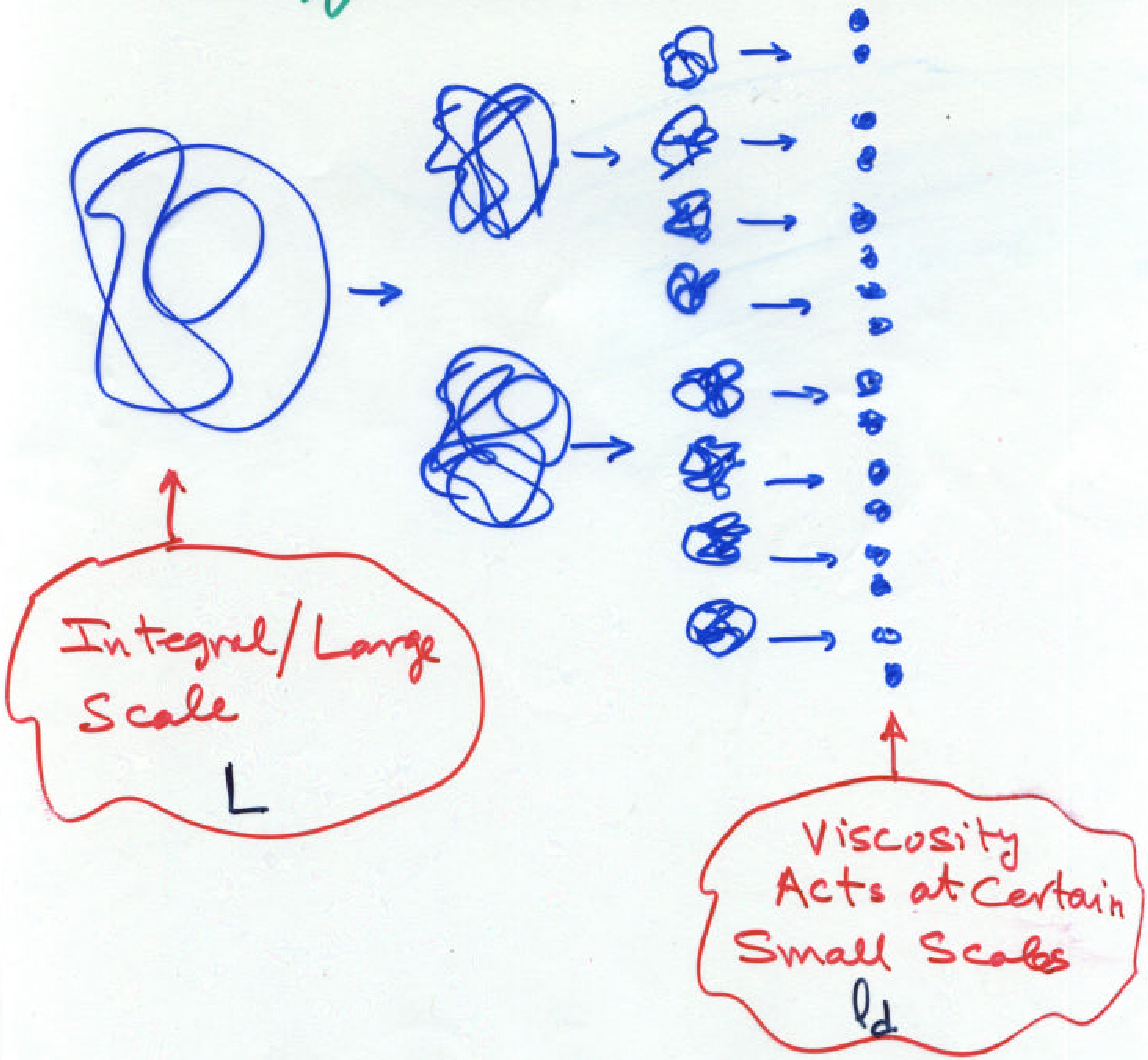
Why Infinite Dimensional Systems Can Be Reduced to Finite Systems?

What is the Underlying Mechanism(s) for Reduction?

Richardson/ Kolmogorov Scenario of Energy Cascade in Turbulence



The Number of Degrees of Freedom in Turbulent Flows is proportional to

$$\left(L/l_d\right)^3.$$

That is Finitely Many Degrees of Freedom.

In Infinite Dimensional Dynamical Systems.

The Navier-Stokes Equations of Viscous Incompressible Fluid:

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u} + \dfrac{1}{\rho_0}\vec{\nabla} p = \vec{f} \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$$

in $\Omega = [0, L]^3$ (Periodic Box)

$$\int_\Omega \vec{\nabla} p \cdot \vec{u} \, dx = 0 \quad \& \quad \int (\vec{u} \cdot \vec{\nabla})\vec{u} \cdot \vec{u} \, dx =$$

"Energy Balance"

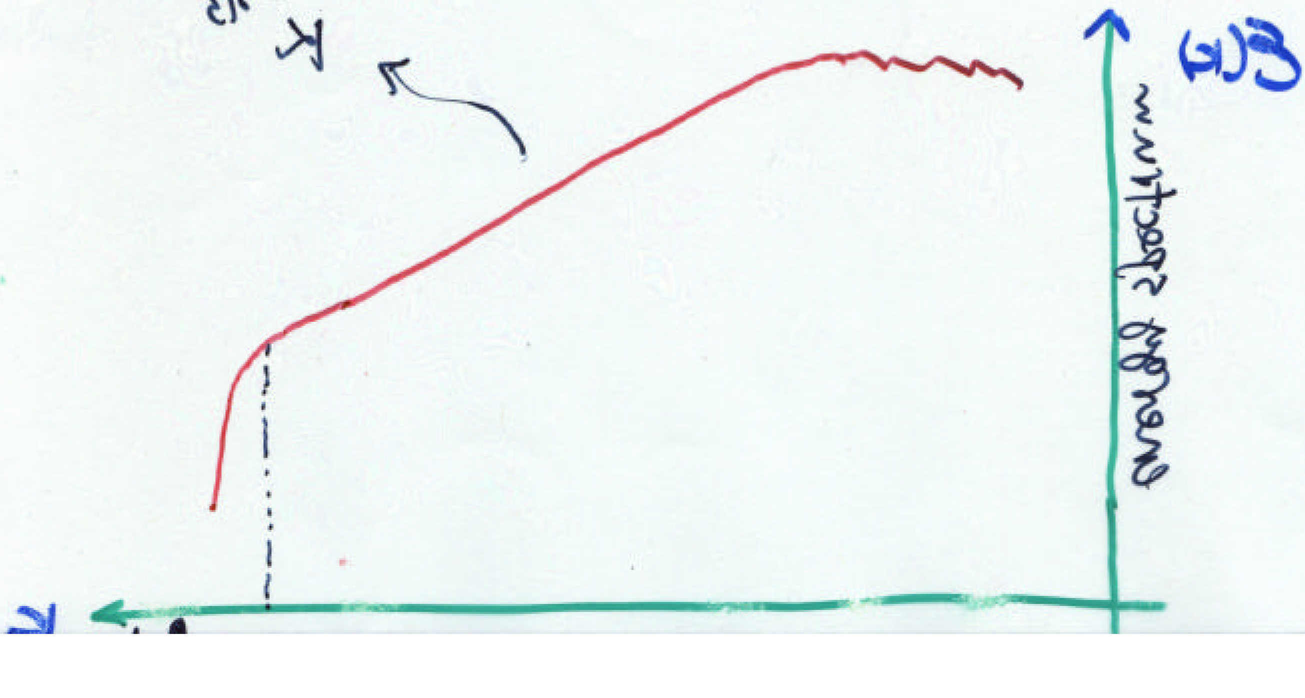
$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\vec{u}(x,t)|^2 dx + \nu \int_\Omega |\vec{\nabla} \vec{u}(x,t)|^2 dx =$$

$$\int \vec{f} \cdot \vec{u}(x,t) dx$$

Mean Rate of Dissipation of Energy:

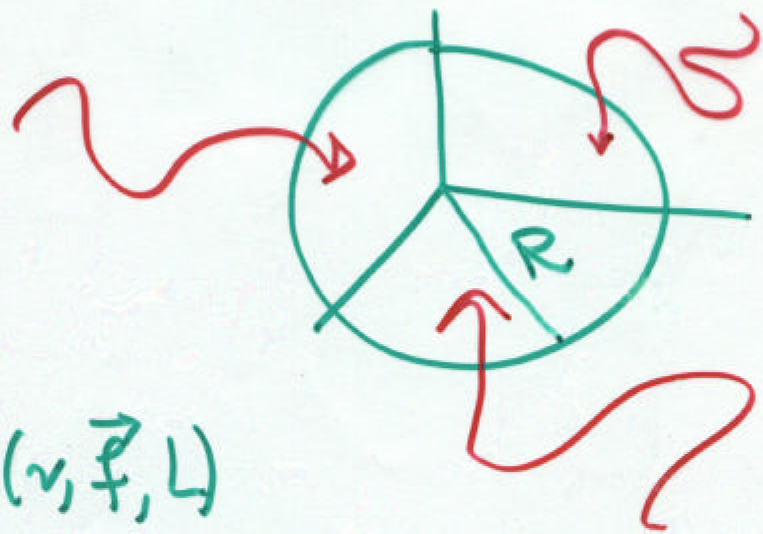
$$\epsilon = \left\langle \frac{\nu}{L^3} \int |\nabla \vec{u}(x,t)|^2 \, dx \right\rangle$$

$$\eta_d = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$



Dynamical System Approach to Turbulence.

$u(t) = S(t)u_0$ Semi-group of Solutions



- $R(\nu, \vec{f}, L)$

- $B(0,R)$ An Absorbing Ball

- $A = \bigcap_{\tau>0} \overline{\bigcup_{t \geq \tau} S(t)B}$

$$d_H(A) \le d_F(A) \le N$$

$N \sim$ # of degrees of freedom.

- In 2-D Navier–Stokes
  [Constantin–Foias–Temam]

- In 3-D Navier–Stokes

???

2-D Navier-Stokes Equations

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p = f \\ \nabla \cdot \vec{u} = 0 \\ + \text{B.C.} \end{cases}$$

1-D Kuramoto-Sivashinsky Equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4}{\partial x^4} u + \alpha\left(\dfrac{\partial^2 u}{\partial x^2} + u \dfrac{\partial u}{\partial x}\right) = 0 \\ \text{periodic B.C. on } [0, 2\pi], \\ \alpha > 0. \end{cases}$$

Complex Ginburg-Landau Eq.

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta u + \gamma |u|^2 u = 0$$

$\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\,\alpha < 0$, $\mathrm{Re}\,\beta < 0$, $\mathrm{Re}\,\gamma > 0$.

Periodic B.C. on $[0, 2\pi]$.

$$\begin{cases} \dfrac{du}{dt} + Au + R(u) = 0 \\ \text{in a Hilbert space } H. \end{cases}$$

- $A > 0$, unbounded operator
- $A w_j = \lambda_j w_j$, $\lambda_j \to \infty$, as $j \to \infty$

  $\{w_j\}$ basis of H

- $u(t) = S(t) u_0$ semigroup of nonlinear compact operator.

- $\exists$ Ball $B \subset H$ s.t.

  for every E bounded set in H, $\exists$ $T^*(E)$ for which

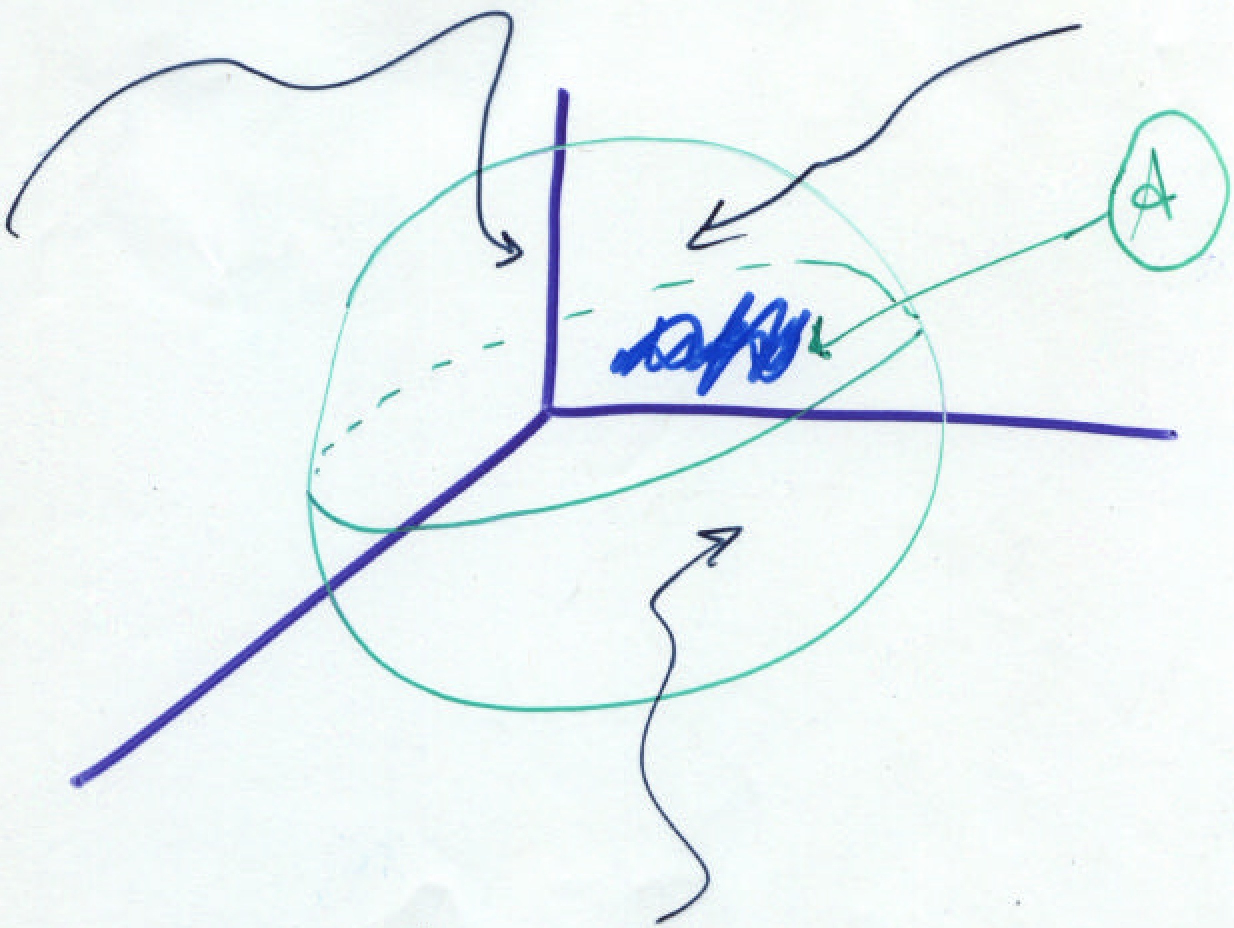
  $$S(t) E \subset B \quad \text{for all } t \geq T^*.$$

Dissipative System

- $\mathcal{A} = \bigcap_{t>0} \overline{\left(\bigcup_{\tau > t} S(\tau) B \right)}$

finite dimensional Global Attractor

Indication of finite-dimension long-term behavior



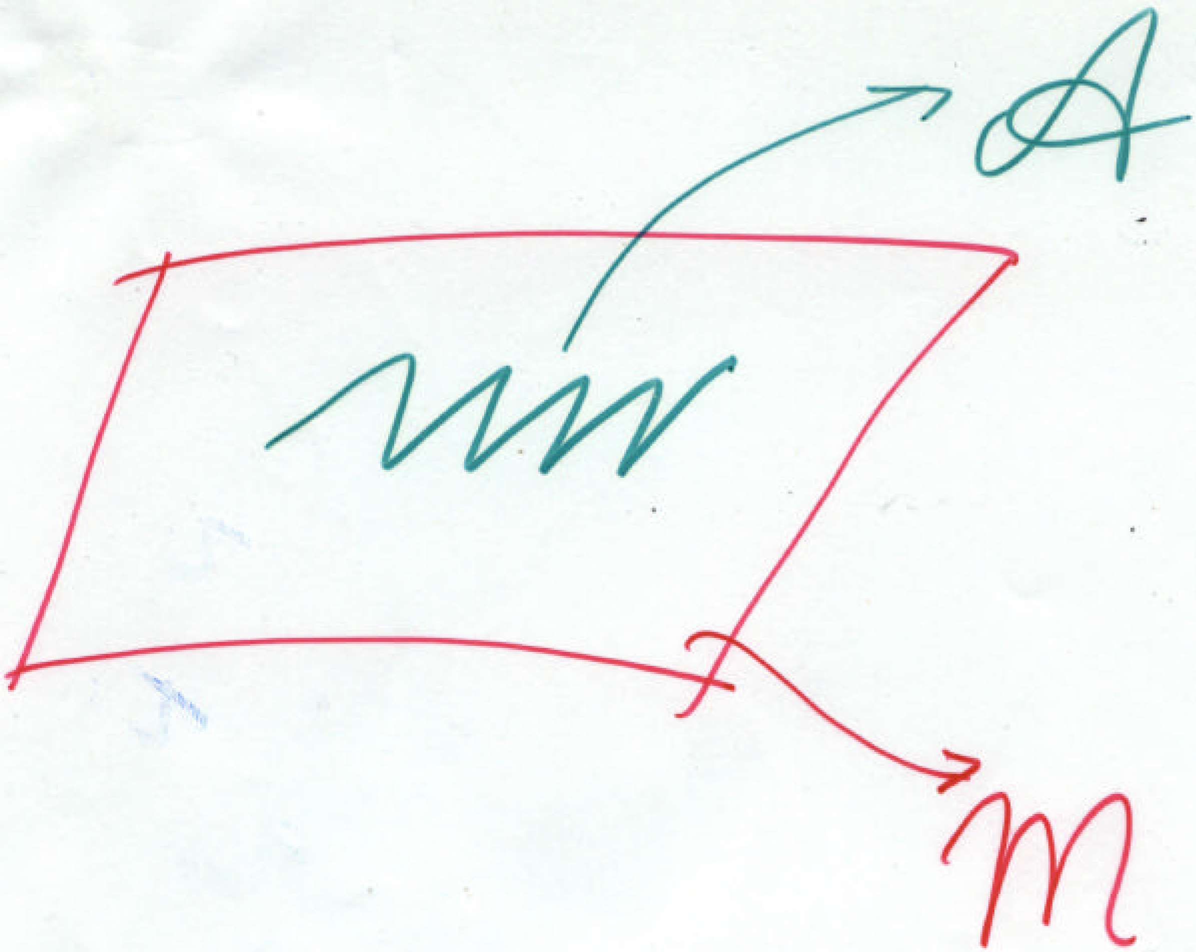
Inertial Manifold

(i) $\mathcal{M} \subset H$ Lipschitz Finite-dimension Manifold.

(ii) $S(t)\mathcal{M} \subset \mathcal{M}$

(iii) $dist(S(t)E, \mathcal{M}) \to 0$, as $t \to \infty$

Exponentially fast, $\forall E \subset H$ bounded.



$$\mathcal{M} = \text{graph } \Phi$$

- $H_m = \text{span}\{w_1, w_2, \ldots, w_m\}$

- P_m Orthogonal projection of H onto H_m.

- $Q_m = I - P_m$.

- $\dfrac{du}{dt} + Au + R(u) = 0$

  Equivalent to

- $$\begin{cases} \dfrac{dp}{dt} + Ap + P_m R(p+q) = 0 \\ \dfrac{dq}{dt} + Aq + Q_m R(p+q) = 0 \\ p = P_m u, \quad q = Q_m u, \quad u = p+q \end{cases}$$

- Search for Inertial Manifolds

$$\mathcal{M} = \text{graph}(\Phi)$$

$$\Phi : H_m \longrightarrow H_m^{\perp}$$

for $m \gg 1$

- That is $q(t) = \Phi(p(t))$ as $t \to \infty$

- That is

- $\begin{cases} \dfrac{dp}{dt} + Ap + P_m R(p + \Phi(p)) = 0 \\ q = \Phi(p) \end{cases}$ [Inertial Form / ODE]

is equivalent to you

- $\dfrac{du}{dt} + Au + R(u) = 0$, $u = p + \Phi(p)$

Sufficient Condition for Existence of An Inertial Manifold Of Dimension m Is That

$$\lambda_{m+1} - \lambda_m >> 1$$

I.e.

<u>Separation</u> <u>In</u> <u>Scales</u>

FOIAS–PRODI 1967, DETERMINING MODES.

THERE EXISTS $M_0(G)$ SUCH THAT IF u AND v ARE TWO EXACT SOLUTIONS TO THE NAVIER–STOKES EQUATIONS SATISFYING

$$\|P_m u - P_m v\| \longrightarrow 0 \quad \text{AS } t \to \infty$$

FOR SOME $m \geq M_0(G)$ THEN

$$\|u - v\| \longrightarrow 0 \quad \text{AS } t \to \infty.$$

i.e.

$$\|(I - P_m)u - (I - P_m)v\| \longrightarrow 0$$

AS $t \to \infty$.

Application of the Determining Modes to Continuous Data Assimilations:

Consider NSE

$$\frac{du}{dt} + \nu Au + B(u,u) = f$$

equivalent to

$$\frac{dp}{dt} + \nu Ap + PB(p+q,p+q) = Pf$$

$$\frac{dq}{dt} + \nu Aq + QB(p+q,p+q) = Qf$$

Suppose $p(t)$ is given from Measurements. Find $q(t)$ (small scales)!

The problem is $q(0)$? (Initialization).

<u>Theorem</u> (Olson-T.)

Let $\tilde{q}$ solves

$$\frac{d\tilde{q}}{dt} + \nu A\tilde{q} + Q\,B(p+\tilde{q}, p+\tilde{q}) = Qf$$

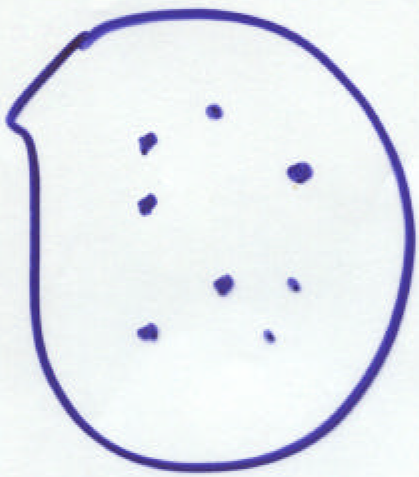
$$\tilde{q}(0) = 0$$

- If $m \gg 1 \Rightarrow \|q(t) - \tilde{q}(t)\| \to 0.$
  $t \to \infty.$

OTHER PARAMETERIZATION OF INERTIAL MANIFOLD

Detrmining Nodes

Foias-Temam



$$\mathcal{E} = \{x_1, \dots, x_N\}$$

Suppose

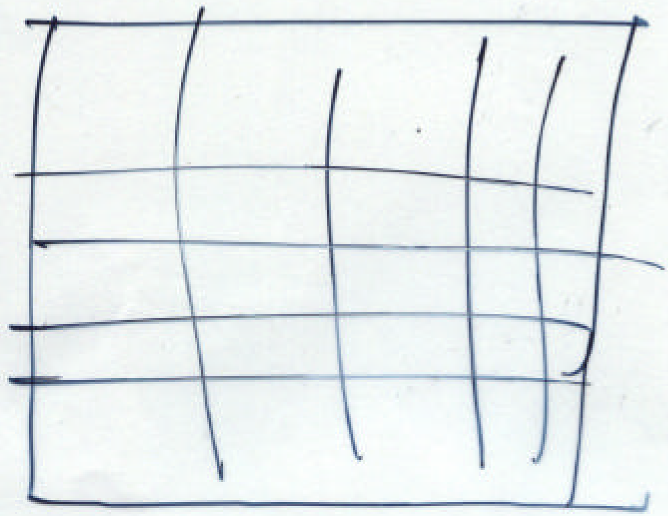
$$|u(x_j,t) - v(x_j,t)| \xrightarrow[t\to\infty]{} 0$$

$$\Longrightarrow \quad \|u(x,t) - v(x,t)\| \longrightarrow 0$$

as $t \to \infty$.

Determining-Volume ELEMENTS

Foias-Titi



$$\Omega = \bigcup_{j=1}^{N} Q_j$$

If $\displaystyle\int_{Q_j} u(x,t) - v(x,t) \xrightarrow[t\to\infty]{} 0$

$$\Rightarrow \|u - v\| \xrightarrow[t\to\infty]{} 0$$

2-D Navier-Stokes best estimates on N are due to Jones-Titi

Questions:

Can one have similar results concerning continuous data assimilations from point value measurements?

Namely, if $u(x_j,t)$, $j=1,\dots,N$ are given can we find an algorithm for computing $u(x,t)$, say, asymptotically in time?

LONG-TERM DYNAMICS & POST-PROCESSING

GALERKIN METHOD

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- LOS ALAMOS NATIONAL LABORATORY

- B. Garcia-Archilla
- J. Novo

- L. Margolin
- S. Wynne

- Devulder
- C. Foias
- I. Kevrekidis
- M. Jolly
- D. Jones
- M. Marion
- G. Sell
- ⋮

- M. Graham
- P. Steen
- ⋮

WORK WITH

B. GARCIA-ARCHILLA

&

J. NOVO

(SPAIN)

Motivated by Inertial Manifolds where

$$q(t) = \Phi(p(t)).$$

Consider an Approximate Inertial Manifolds

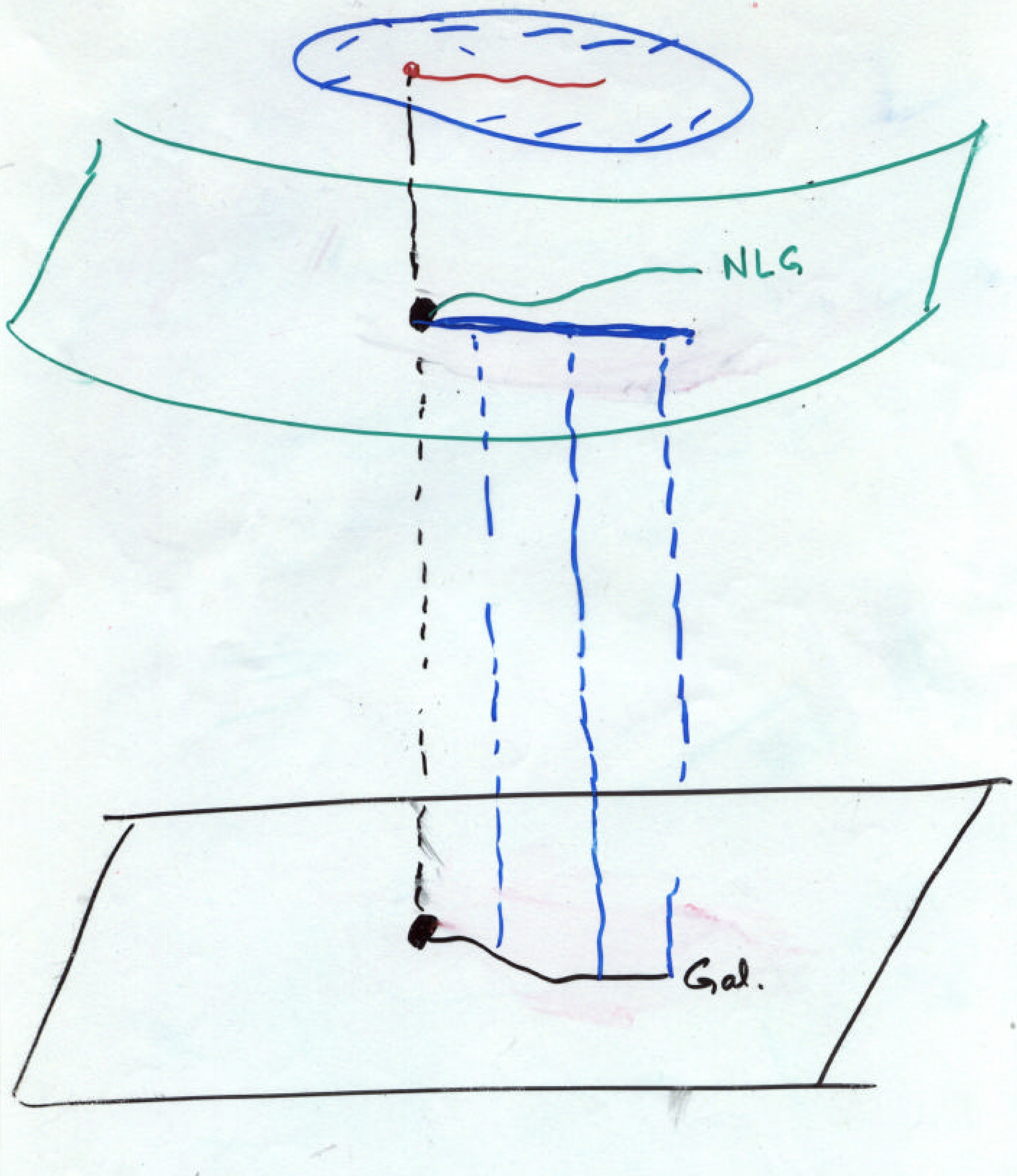
$$q \approx \Psi(p)$$

[Parametrization of Small scales in terms of Large Scales].

- Nonlinear Galerkin:

$$\frac{dp}{dt} + \nu A p + P R(p + \Psi(p)) = 0$$

$$u_{app} = p + \Psi(p).$$


NLG
Gal.

Galerkin:

$$\frac{d y_m}{dt} + A y_m + P_m R(y_m) = 0$$

$$u_{app} = y_m$$

Nonlinear Galerkin

$$\frac{d u_m}{dt} + A u_m + P_m R(u_m + \Psi(u_m)) = 0$$

$$u_{app} = u_m + \Psi(u_m)$$

New Idea

$$\frac{dy_m}{dt} + Ay_m + P_m R(y_m) = 0$$

Galerkin

- $u_{app} = y_m + \Psi(y_m)$

post-processing.

Properties:

- The "Same" CPU Like Galerkin
- ACCURACY: NOT WORSE THAN GALERKIN, BUT POTENTIALY AS GOOD AS THE NONLINEAR GALERKIN.
- CAN BE EASILY IMPLEMENTED FOR FINITE ELEMENTS OR OTHER SPECTRAL METHOD

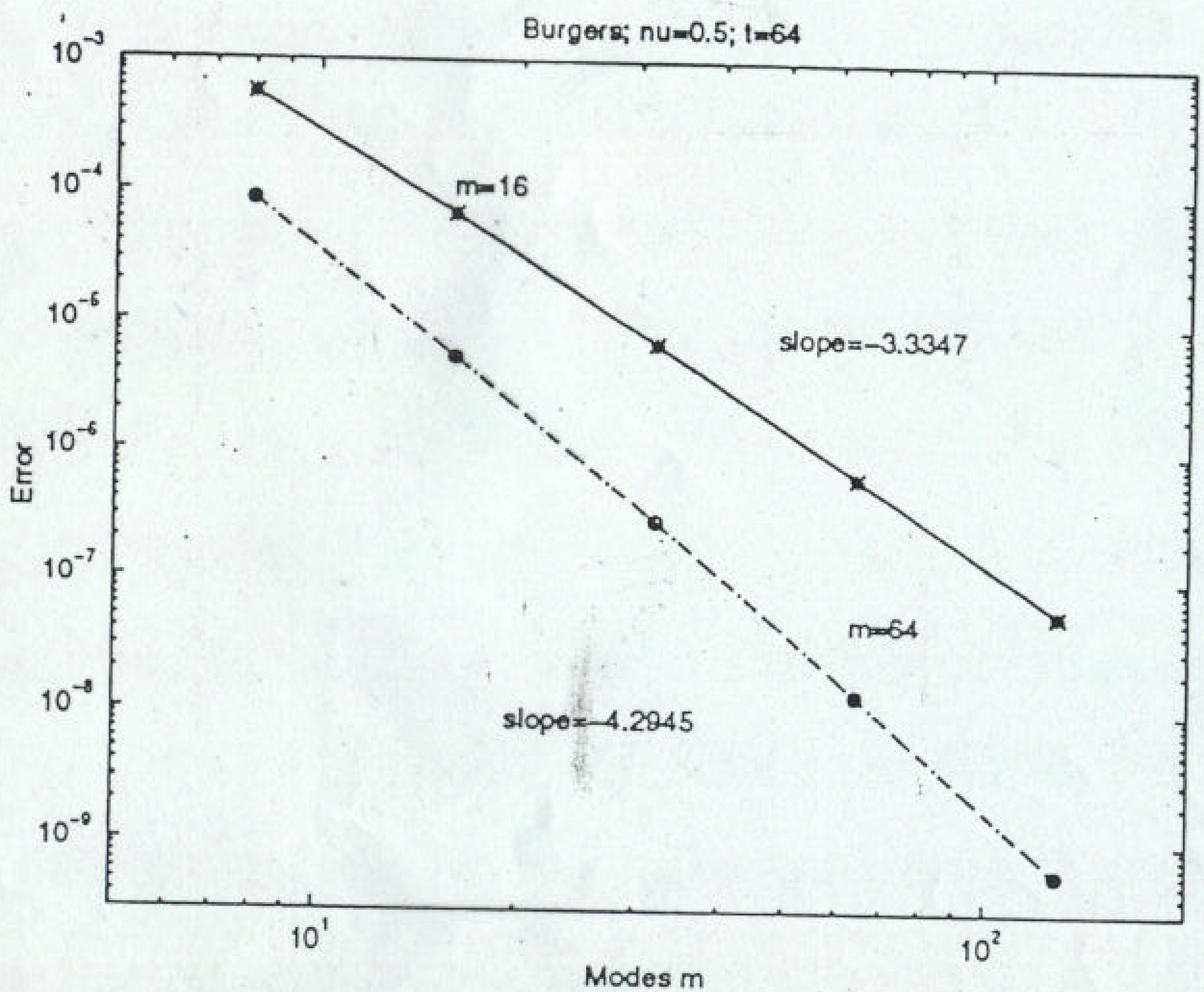


Figure 1: Errors for Burgers' equation with $\nu = 0.5$: $*$ standard Galerkin $\circ$ NLG, + Postprocessed Galerkin

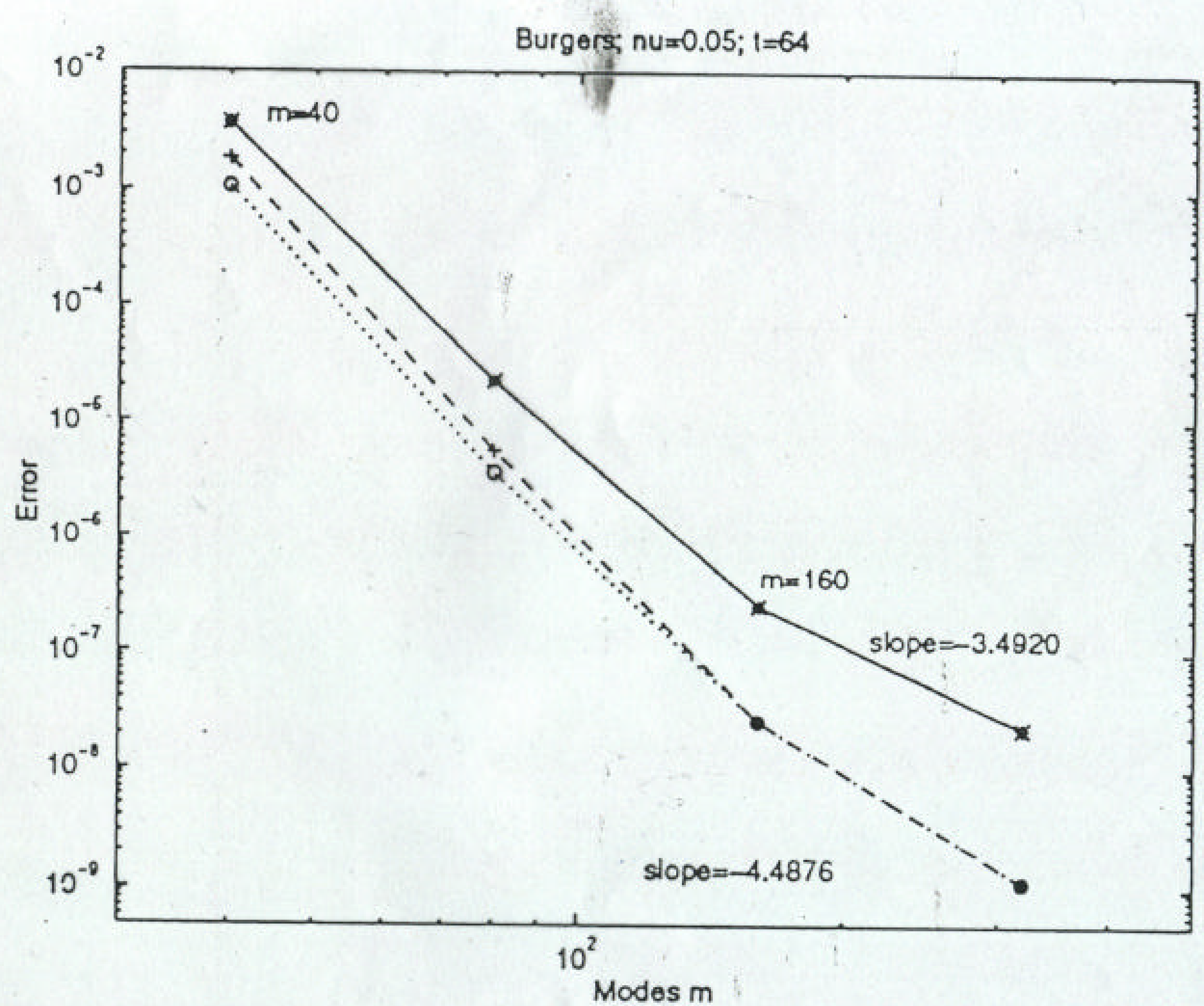
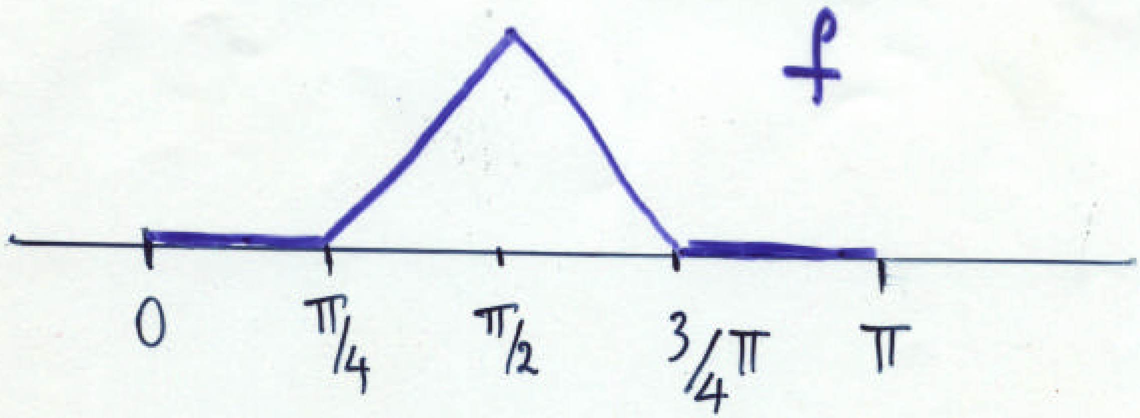
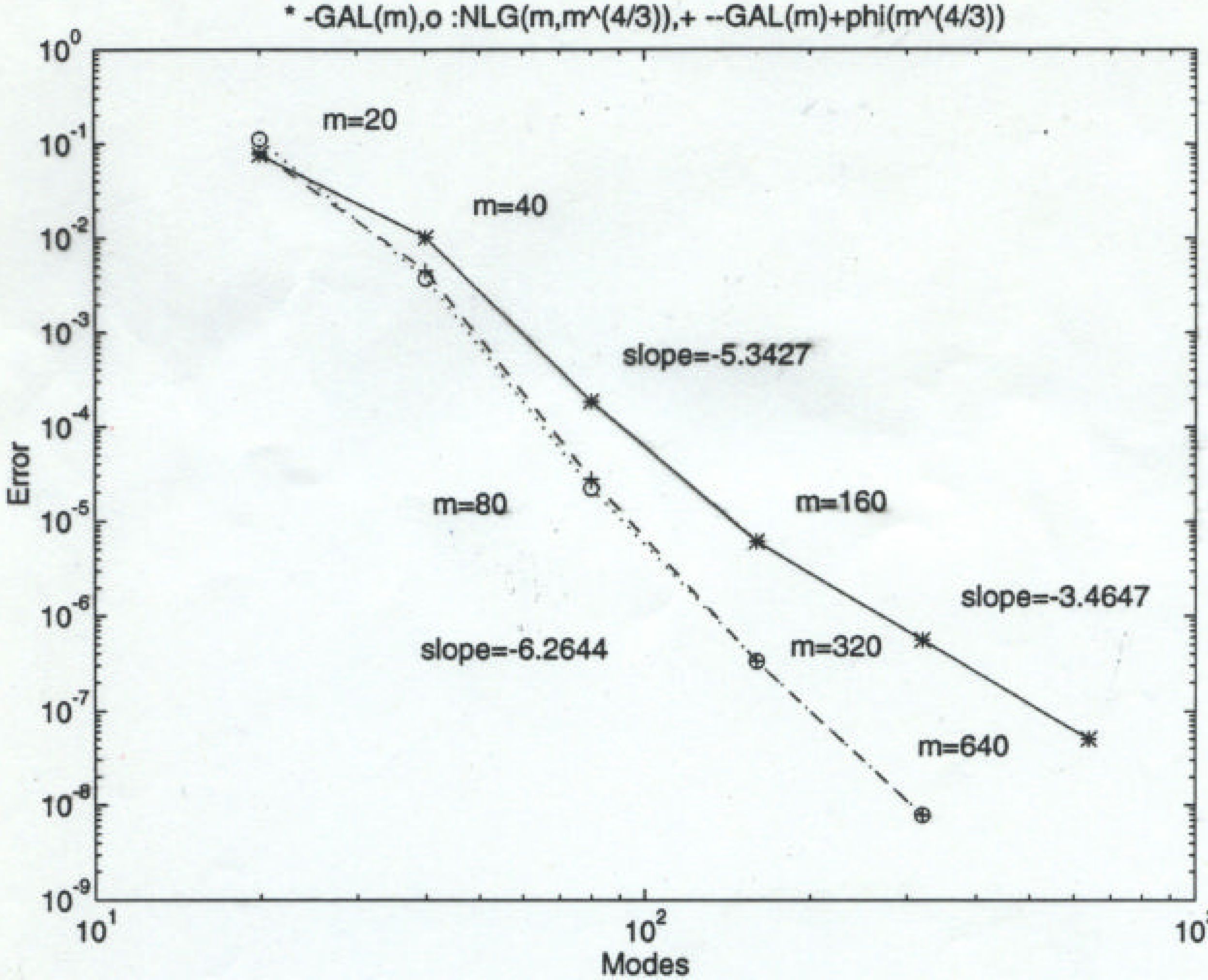


Figure 2: Errors for Burgers' equation with $\nu = 0.05$: $*$ standard Galerkin $\circ$ NLG, $+$ Postprocessed Galerkin

Computations for:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u^3 - u = f$$





* -GAL(m),o :NLG(m,m^(4/3)),+ --GAL(m)+phi(m^(4/3))
Error
Modes
m=20
m=40
m=80
m=160
m=320
m=640
slope=-5.3427
slope=-6.2644
slope=-3.4647

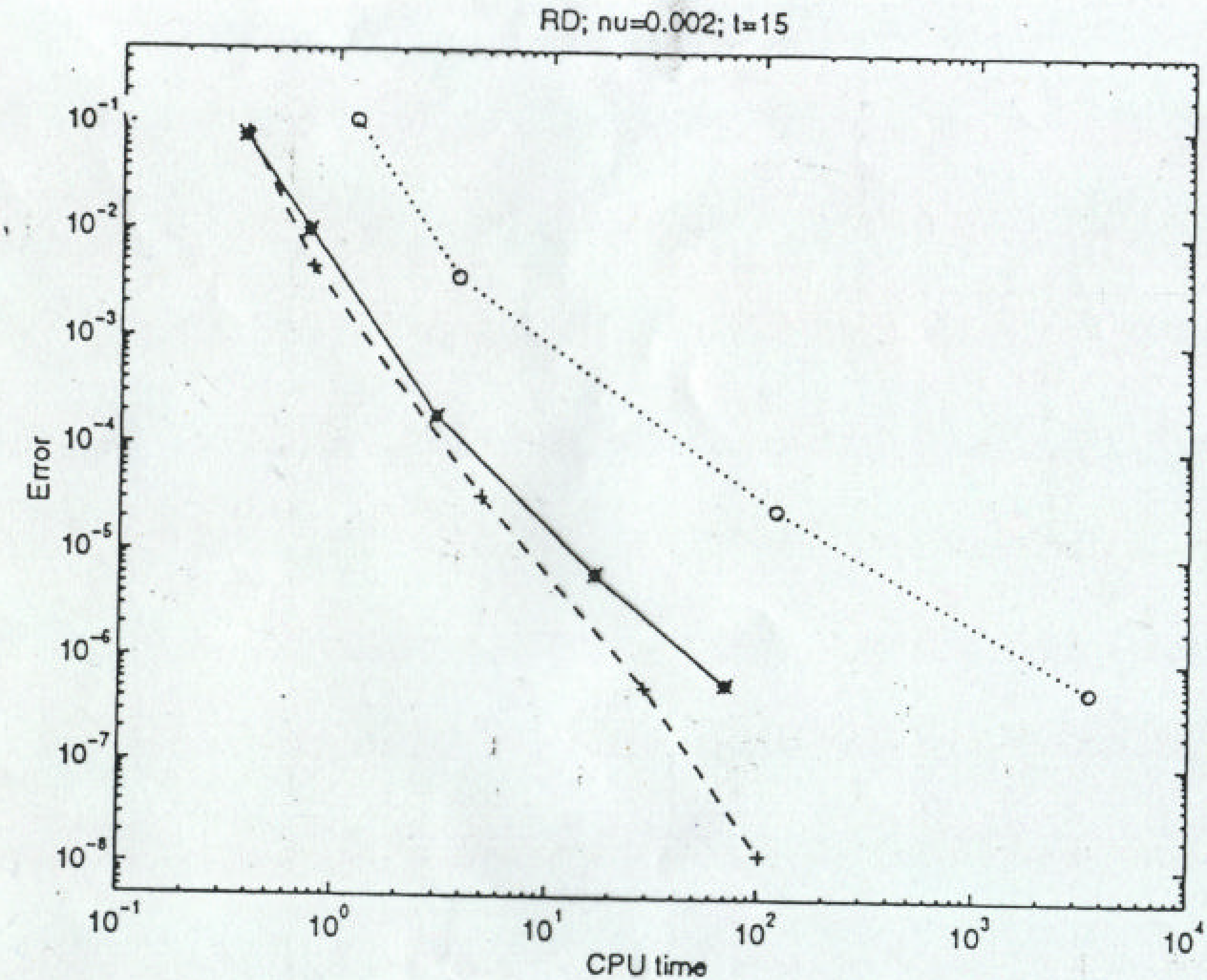


Figure 4: Efficiency diagram for the RD equation with $nu = 0.002$: $*$ standard Galerkin, $\circ$ NLG, $+$ postprocessed Galerkin.

THEOREM (Garcia-Archilla, Novo, T.)

CONSIDER THE EQUATION

$$\frac{du}{dt} + Au + F(u) = 0$$

$$A w_j = \lambda_j w_j$$

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$$

$$\| u - u_{GAL} \|_{L^2} = O\left(\lambda_{m+1}^{-1}\right)$$

$$\| u - u_{NLG} \|_{L^2} = O\left(\lambda_{m+1}^{-3/2}\right)$$

$$\| u - u_{pp} \|_{L^2} = O\left(\lambda_{m+1}^{-3/2}\right).$$

FINITE ELEMENT CASE

(Garcia-Archilla & T.)

$$\frac{du}{dt} + \nu A u + F(u) = 0$$

$\nu > 0.$

$A = -\Delta$ Dirichlet B.C.

$$a(u,v) = \left(A^{1/2} u, A^{1/2} v\right) = (\nabla u, \nabla v)$$

$S_{h,r}$ – Finite Element Space

P_h – L^2 Orthogonal Proj. onto $S_{h,r}$.

- $a_h(\chi, \psi) = (\nabla\chi, \nabla\psi)$

$$\chi, \psi \in S_{h,r}$$

- Inner Product on $S_{h,r}$
- A_h

$$\boxed{a_h(\chi, \psi) = (A_h \chi, \psi) = (\chi, A_h \psi)}$$

- R_h ORTHOGONAL PROJECTION ON $S_{h,r}$ WITH RESPECT TO $a_h(\cdot,\cdot)$.

- GALERKIN:

$$\frac{du_h}{dt} + \nu A_h u_h + P_h F(u_h) = 0$$

$$u_h(0) = R_h u(0)$$

$$\nu A u(T) = -F(u(T)) - \frac{du}{dt}(T)$$

IN THE R.H.S. APPROX. $u(T) = u_h(T)$

$$\nu A u(T) \approx \boxed{\nu A \tilde{u} = -F(u_h(T)) - \frac{du_h}{dt}(T)}$$

- SOLVE AN ELLIPTIC PROBLEM @ FINAL TIME.

THEOREM 1: Let $r \geq 3$. $\exists C > 0$ s.t.

$\forall h$ small enough:

$$\|u - \tilde{u}\|_{L^2(\Omega)} + h \|u - \tilde{u}\|_{H^1(\Omega)} \leq C h^{r+\bar{\mu}} |\log h|^{\bar{r}}$$

$$\bar{\mu} = \begin{cases} 2 & \text{if } r \geq 4 \\ 1 & \text{if } r = 3 \end{cases} \qquad \bar{r} = \begin{cases} 0 & \text{if } r \geq 4 \\ 1 & \text{if } r = 3 \end{cases}$$

$$K_\infty(u) = \max_{0\le t\le T}\left(\|u(\cdot,t)\|_{W^{r,\infty}} + \|u_t(\cdot,t)\|_{W^{r,\infty}}\right)$$

TEOREM2:

$\mathbb{R}^d$, $d=2,3$; $r\geq 3$. $\exists\, C(K_\infty(u))>0$

s.t. $\forall h$ small enough:

(i) $\|u(T)-\tilde{u}\|_{L^\infty} \le C h^{r+2-\frac{d}{2}} |\log h|^{4-\frac{d}{2}}$ if $r=3$

(ii) $\|u(T)-\tilde{u}\|_{L^\infty} \le C h^{r+2} |\log h|^3$ if $r\geq 4$

for some $C = C(\|v\|_{H^r(\Omega)})$. Thus, from (60), (61) and (62), it follows that

$$\|A^{-1}(b(v)\cdot\nabla\epsilon)\|_{L^2(\Omega)} \le C\|\nabla A^{-1}\epsilon\|_{L^2(\Omega)^d}. \tag{63}$$

Finally, the second term on the right-hand side of (58) can be treated similarly as the first-one (i.e. after applying A^{-1} it can be bounded by the right-hand side of (63). This, together with (59) lead to

$$\|A^{-\mu/2}(b(v)\cdot\nabla v - F(\chi)\cdot\nabla\chi)\|_{L^2(\Omega)} \le C(\|A^{(1-\mu)/2}(v-\chi)\|_{L^2(\Omega)} + \|v-\chi\|^2_{H^1(\Omega)}), \tag{64}$$

for $\mu = 2$.

In the case $\mu = 1$, notice first that $\|A^{-1/2}\cdot\|_{L^2(\Omega)} = \|\nabla A^{-1}\cdot\|_{L^2(\Omega)}$. Then, go back to the decomposition (58) and estimate again the first two terms by duality (recall that the third one has already been dealt with). For example, for $\phi \in C_0^\infty(\Omega)^2$, integrating by parts twice one gets $(\nabla A^{-1}(b(v)\cdot\nabla\epsilon),\phi) = (\epsilon, \operatorname{div}((A^{-1}\operatorname{div}(\phi))b(v)))$ from where it is easy to proceed as in the case $\mu = 2$. Finally, the bound (64) in the case $\mu = 0$ is a direct consequence of (58).

The arguments above can be (more easily) applied to the reaction term, yielding

$$\|A^{-\mu/2}(g(v)-g(\chi))\|_{L^2(\Omega)} \le C(\|A^{-\mu/2}(v-\chi)\|_{L^2(\Omega)} + \|v-\chi\|_{L^q(\Omega)}\|v-\chi\|_{L^2(\Omega)}),$$

for $\mu = 0, 1, 2$. This and (64) conclude (57). □

3 Numerical Experiments

We consider the reaction-diffusion system

$$\left.\begin{aligned} \frac{\partial u_1}{\partial t} &= 1 - 4u_1 + u_1^2 u_2 + \nu\Delta u_1 \\ \frac{\partial u_2}{\partial t} &= \quad\;\; 3u_1 - u_1^2 u_2 + \nu\Delta u_2 \end{aligned}\right\} \tag{65}$$

known as the Brusselator (see e.g. [23]), in $\Omega \times [0,T]$, where $\Omega = [0,1]^2$ and $T = 10$, subject to

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0.$$

We set $\nu = 0.002$. We complement this system with the initial condition

$$\begin{aligned} u_1(x,y,0) &= 1 - \tfrac{1}{4}(2y-1)((2y-1)^2-3), \\ u_2(x,y,0) &= 3 - (2x-1)((2x-1)^2-3). \end{aligned}$$

The corresponding solution becomes nearly periodic in time, and develops moderately large gradients. Fig. 1, shows the evolution of $\|u\|_{L^2(\Omega)^2}$ up to

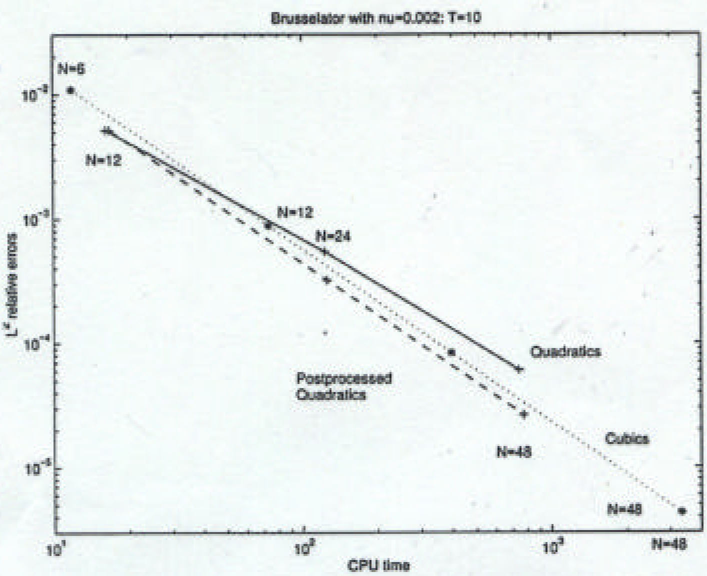


Figure 5: Efficiency diagram

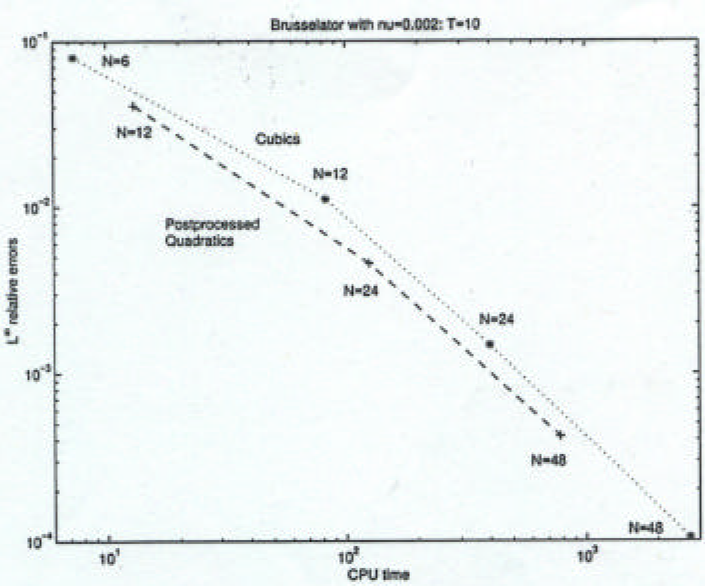


Figure 6: Efficiency diagram in L^∞

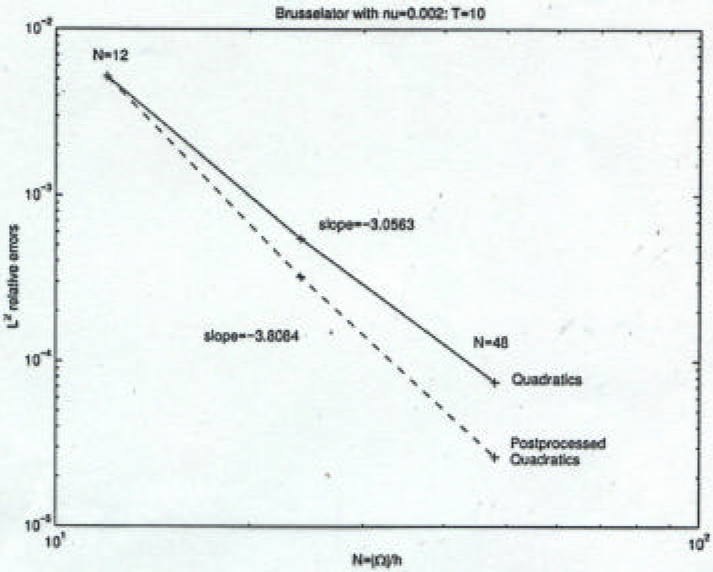


Figure 3: Convergence diagram

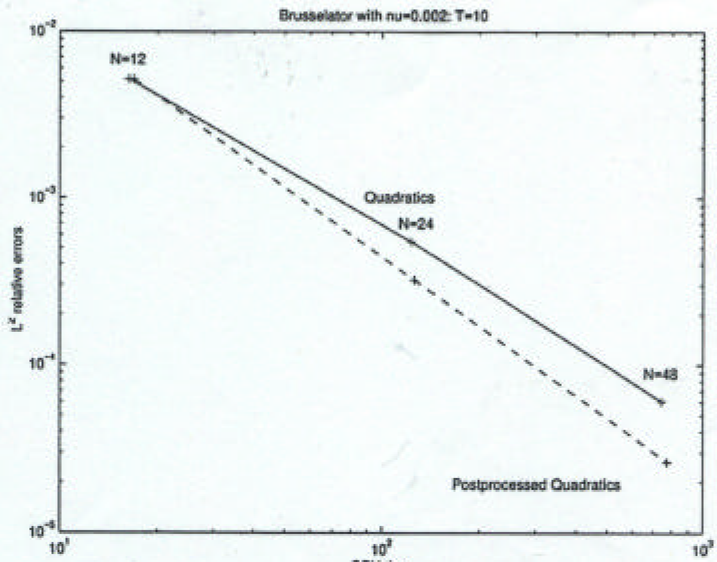


Figure 4: Efficiency diagram

Remark

This is different than any two-level multigrid method.

TRUNCATION ANALYSIS

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- S. Wynne
- E. S. T.

Recall The Navier-Stokes Equations

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} + \vec{u}.\nabla \vec{u} + \nabla p = f \\ \nabla . \vec{u} = 0 \\ + \text{B.C.} \end{cases}$$

is equivelant to:

- $\dfrac{du}{dt} + \nu A u + B(u,u) = f$

- $u = p + q$, $p \in H_m$, $q \in H_m^{\perp}$

$\|q\|_{L^2} = O(\lambda_{m+1}^{-1})$ $\quad \left\|\dfrac{dq}{dt}\right\|_{L^2} = O(\lambda_{m+1}^{-1})$

$\|\nabla q\|_{L^2} = O(\lambda_{m+1}^{-1/2})$ $\quad \|\Delta q\|_{L^2} = O(1)$

- Let $\varepsilon = \lambda_{m+1}^{-1/2}$

Small parameter

- $\|q\|_{L^2} = O(\varepsilon^2)$, $\|\frac{dq}{dt}\|_{L^2} = O(\varepsilon^2)$
- $\|\nabla q\|_{L^2} = O(\varepsilon)$, $\|\Delta q\|_{L^2} = O(1)$

The above estimates are valid on the Global Attractor.

$$\underline{\frac{dp}{dt}} + \underline{\nu A p} + \underline{P_m B(p,p)} + P_m B(p,q) +$$

$$+ P_m B(q,p) + P_m B(q,q) = \underline{P_m f}$$

$$\boxed{\frac{dq}{dt}} + \underline{\nu A q} + \underline{Q_m B(p,p)} + Q_m B(p,q) +$$

$$+ Q_m B(q,p) + Q_m B(q,q) = \underline{Q_m f}$$

Thus to leading Order:

$$\frac{dp}{dt} + \nu A p + P_m B(p,p) = P_m f$$

$$\boxed{\frac{dq}{dt} +} \; \nu A q + Q_m B(p,p) = Q_m f$$

Post-processing &

<u>NOT</u> Galerkin.

General Postprocessing Schemes With Applications to Data Assimilations.

- We have the equation

$$\frac{du}{dt} + \nu A u + R(u) = 0.$$

- We are interested in $u(t=T)$.

$$\nu A u(T) = -\left.\frac{du}{dt}\right|_{t=T} + \left.R(u)\right|_{t=T}$$

- The Right Hand is usually given on a coarse mesh. We use the Coarse data to compute the Left hand by inverting (νA) as accurate as possible.

• Dynamical Post-processing Galerkin

• In the case of transient solutions or time-dependent oscillatory forcing

$\left\| \frac{dq}{dt} \right\|_{L^2}$ might be large

• $$\begin{cases} \frac{dp}{dt} + \nu A p + P_m B(p,p) = P_m f & \leftarrow \text{Galerkin} \\ \frac{dq}{dt} + \nu A q + Q_m B(p,p) = Q_m f & \leftarrow \end{cases}$$

Dynamical Post-processing Galerkin

Dynamical post-processing
Galerkin – Finite Elements
Case:

① Galerkin step

$$\begin{cases} \dfrac{\partial u_h}{\partial t} + A u_h + P_h R(u_h) = 0 \\ u_h(0) = \text{projection of } u_0 \end{cases}$$

② Post-processing step (Dynamical)

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + A \tilde{u} + R(u_h) = 0 \\ \tilde{u}(0) = u_0 \end{cases}$$

$$u_{app}(t) = \tilde{u}(t)$$

$$u_{exact} = \sum_{k=1}^{\infty} \frac{a_k(t)}{k^3} \sin kx$$

$$a_k(t) = \begin{cases} 1 + \gamma \sin k^2 t & 1 \le k \le 100 \\ 1 & 100 < k \end{cases}$$

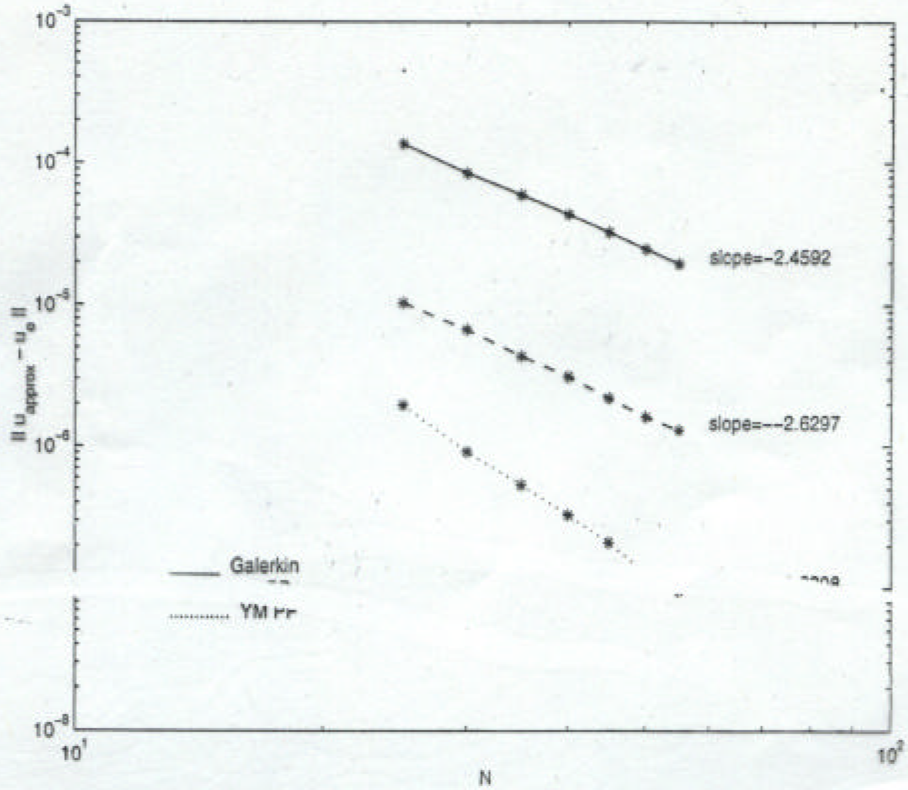

Figure 1: Total errors $\|u_{approx} - u_e\|$

$t = 2.0$

$\gamma = 0.1$

Forced-viscous Burgers