Graph cut, convex relaxation and continuous max-flow problem

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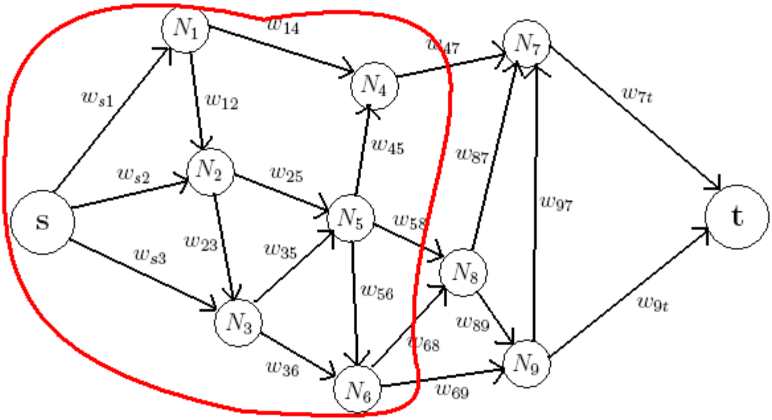
Max-Flow / Min-Cut

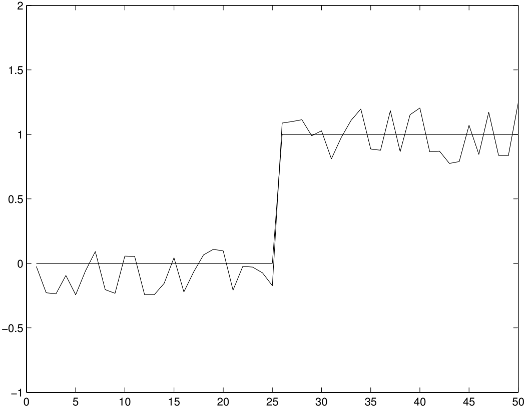


(V_s, V_t) is a cut, w_{ij} = cost of cutting edge(i, j)
cost of cut $c(V_s, V_t) = \sum_{i \in V_s, j \in V_t} w_{ij}$
Min-cut: find cut of minimum cost,
Max-Flow: Find the maximum amount of flow from s to t.

Max-flow = min-cut.

Graph-cut for image segmentation

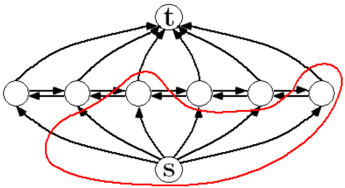
A simple 1d signal $I(x)$:



Graph-cut for images: Boykov-Kolmogorov (2001).

Graph-cut for image segmentation

The graph:



1	1	2	1	1	2

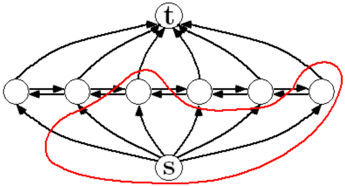
costs: (Chan-Vese)

$$w_{s,p} = |I(p) - c_1|^2, w_{t,p} = |I(p) - c_2|^2, c_1 = 0, c_2 = 1.$$

More generally

$$w_{s,p} = f_1(p),\ \ w_{t,p} = f_2(p),\ \ w(p,q) = \alpha \text{ or } g(p,q) \text{ (edge force)}.$$

Regularized Graph-cut: $\alpha \neq 0$



1	1	2	1	1	2

The "virtual graph and the corresponding label function $u(p), p = 1, 2, \cdots$.

Costs:

$$w_{s,p} = |I(p) - c_1|^2, w_{t,p} = |I(p) - c_2|^2, w_{p,q} = \alpha.$$

The corresponding minimization problem is: ($N(p)$ – neighbors of p)

$$\min_{u(p)\in\{1,2\}} \sum_{p\in\Omega_1} |I(p)-c_1|^2 + \sum_{p\in\Omega_2} |I(p)-c_2|^2 + \alpha \sum_{p} \sum_{q\in N(p)} |u(p)-u(q)|.$$

Discrete vs continuous

Discrete minimization:

$$\min_{u(p)\in\{0,1\}} \sum_{p\in\Omega_1} |I(p)-c_1|^2+\sum_{p\in\Omega_2} |I(p)-c_2|^2+\alpha\sum_{p}\sum_{q\in N(p)} |u(p)-u(q)|.$$

Continuous minimization:

$$\min_{u(x)\in\{0,1\}} \int_{\Omega_1} |I(x)-c_1|^2 + \int_{\Omega_2} |I(x)-c_2|^2 + \alpha\int_{\Omega} |Du|.$$

$$\min_{u(x)\in\{0,1\}} \int_{\Omega} |I(x)-c_1|^2(1-u) + \int_{\Omega} |I(x)-c_2|^2 u + \alpha\int_{\Omega} |Du|.$$

Higher dimensional problems

A graph for 2D images:

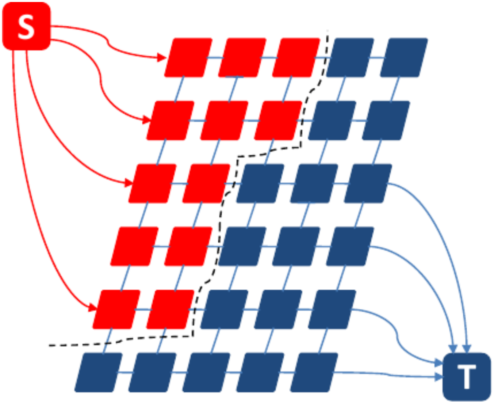


Figure: Graph used for discrete 2D binary labeling

Two-phase Min-cut – Discretized setting

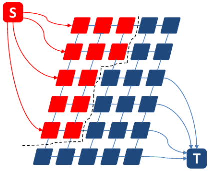


Figure: Graph used for discrete binary labeling

$$\min_{u\in\{0,1\}} \sum_{p\in\mathcal{P}} f_1(p)(1-u(p))+f_2(p)u(p)+\sum_{p\in\mathcal{P}}\sum_{q\in\mathcal{N}_p^k} g(p,q)|u(p)-u(q)|.$$

Costs:

$$w_{s,p} = f_1(p),\ \ w_{t,p} = f_2(p),\ \ w_{p,q} = g(p,a).$$

[1] $\mathcal{N}_p^k$ is the k-neighborhood of $p \in \mathcal{P}$.

Max-Flow / Min-Cut (graph cut)

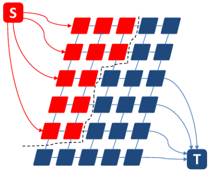


Figure: Graph used for discrete binary labeling

Max-flow formulation

$$\max_{p_s} \sum_{v \in \mathcal{V} \setminus \{s,t\}} p_s(v)$$

subject to

$$\begin{aligned} q(v,u) &\leq g(v,u), & \forall (v,u) \in \mathcal{V} \times \mathcal{V} \\ 0 \leq p_s(v) &\leq f_1(v), & \forall v \in \mathcal{V} \setminus \{s,t\}; \\ 0 \leq p_t(v) &\leq f_2(v), & \forall v \in \mathcal{V} \setminus \{s,t\}; \\ \Big(\sum_{u \in N(v)} q(v,u) \Big) - p_s(v) + p_t(v) &= 0, & \forall v \in \mathcal{V} \setminus \{s,t\};. \end{aligned}$$

Continuous Max-Flow and Min-Cut

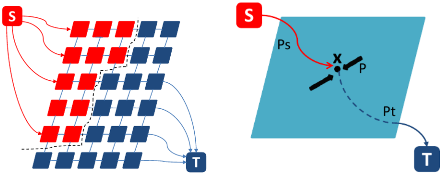


Figure: (left) vs. Continuous (right)

Continuous max-flow formulation (G. Strang (1983))

$$\sup_{p_s, p_t, q} \int_\Omega p_s(x)\, dx$$

subject to

$$
\begin{aligned}
|q(x)| = \sqrt{q_1^2(x) + q_2^2(x)} \; &\leq \; g(x), \quad && \forall x \in \Omega; \\
p_s(x) \; &\leq \; f_1(x), \quad && \forall x \in \Omega; \\
p_t(x) \; &\leq \; f_2(x), \quad && \forall x \in \Omega; \\
\operatorname{div} q(x) - p_s(x) + p_t(x) \; &= \; 0, \quad && \text{a.e. } x \in \Omega.
\end{aligned}
$$

Continuous Max-Flow and Min-Cut

Lagrange multiplier u for flow conservation condition

$$\operatorname{div} q(x) - p_s(x) + p_t(x) \;=\; 0, \qquad \text{a.e. } x \in \Omega.$$

yields primal-dual formulation

$$\sup_{p_s, p_t, q} \inf_{u} \int_\Omega p_s + u\big(\operatorname{div} q - p_s + p_t \big)\, dx$$

$$\text{s.t.} \quad p_s(x) \;\leq\; f_1(x)\,, \quad p_t(x) \leq f_2(x)\,, \quad |q(x)| \;\leq\; g(x)\,.$$

Optimizing for flows p_s, p_t, q results in:

$$\min_{u \in [0,1]} \int_\Omega f_1(x)(1 - u(x)) + f_2(x) u(x)\, dx + g(x) \left| \nabla u(x) \right| \, dx\,.$$

This is exactly the same model as in Chan et at (2006).

Three problems

$$\min_{u(x)\in\{0,1\}} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.$$

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$$\min_{u(x)\in[0,1]} \int_\Omega f_1 u + f_2(1-u) + g(x)|\nabla u| dx.$$

Three problems

$$\min_{u(x)\in\{0,1\}} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.$$

$$\min_{u(x)\in[0,1]} \int_\Omega f_1 u + f_2(1-u) + g(x)|\nabla u| dx.$$

$$\max_{p_s,p_t,q} \int_\Omega p_s dx \text{ subject to:}$$

$$p_s(x) \le f_1(x),\ \ p_t(x) \le f_2(x),\ \ |p(x)| \le g(x),$$

$$\text{div} p(x) - p_s(x) + p_t(x) = 0.$$

Three problems

$$\min_{u(x)\in\{0,1\}} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.$$

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$$p_s(x) \le f_1(x),\ \ p_t(x) \le f_2(x),\ \ |p(x)| \le g(x),$$

$$\mathrm{div} p(x) - p_s(x) + p_t(x) = 0.$$

Three problems

PCLMS or Binary LM (Lie-Lysaker-T.,2005):

$$\min_{u(x)\in\{0,1\}} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.$$

Convex problem (CEN, (Chan-Esdoglu-Nikolova,2006))

$$\min_{u(x)\in[0,1]} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.$$

Graph-cut (Boykov-Kolmogorov,2001)

$$\max_{p_s,p_t,q} \int_\Omega p_s dx \text{ subject to:}$$

$$p_s(x) \le f_1(x), \;\; p_t(x) \le f_2(x), \;\; |p(x)| \le g(x),$$

$$\text{div} p(x) - p_s(x) + p_t(x) = 0.$$

Remarks

The following approached are solving the same problem, but did not know each other:

- max-flow and min-cut (Boykov-Kolmogorv 2001, etc)
- CEN 2006 (convex relaxation approach)
- Binary Level set methods and PCLSM (piecewise constant level set method)

Continuous Max-Flow: Remarks

- Min-cut problem is minimizing an energy functional. Not using the decent (gradient) info of the energy.
- Continuous max-flow/min-cut is a convex minimization problem. A lot of choices, can use decent (gradient) info.

Continuous Max-Flow: How to solve it (Only 2-phase case)?

- Min-cut algorithms: Augmented Path. Push-relabel, etc,
- Split-Bregman, Augmented Lagrangian, Primal-Dual approaches: we can use these approach to solve the convex min-cut problem.

Multiplier-Based Maximal-Flow Algorithm

Augmented lagrangian functional (Glowinski & Le Tallec, 1989)

$$L_c(p_s, p_t, q, \lambda) \ := \int_\Omega p_s \, dx + \lambda\big(\operatorname{div} q - p_s + p_t\big) - \frac{c}{2} |\operatorname{div} q - p_s + p_t|^2 \, dx.$$

minmax subject to:

$p_s(x) \leq f_1(x)\,, \ \ p_t(x) \leq f_2(x)\,, \ \ |q(x)| \leq g(x)$

ADMM algorithm: For k=1,... until convergence, solve

$$q^{k+1} := \arg \max_{\|q\|_\infty \leq \alpha} L_c(p_s^k, p_t^k, q, \lambda^k)$$

$$p_s^{k+1} := \arg \max_{p_s(x) \leq f_1(x)} L_c(p_s, p_t^k, q^{k+1}, \lambda^k)$$

$$p_t^{k+1} := \arg \max_{p_t(x) \leq f_2(x)} L_c(p_s^{k+1}, p_t, q^{k+1}, \lambda^k)$$

$$\lambda^{k+1} = \lambda^k - c\,(\operatorname{div} q^{k+1} - p_s^{k+1} + p_t^{k+1})$$

Other algorithms for solving relaxed problem: $p = \nabla u$

- Bresson et. al.
  - fix μ^k and solve ROF problem

$$\lambda^{k+1} := \arg\min_{\lambda} \left\{ \alpha \int_{\Omega} |\nabla \lambda(x)| \, dx + \frac{1}{2\theta} \|\lambda(x) - \mu^k(x)\|^2 \right\}$$

  - fix λ^{k+1} and solve

$$\mu^{k+1} := \arg\min_{\mu \in [0,1]} \left\{ \frac{1}{2\theta} \|\mu(x) - \lambda^{k+1}\|^2 + \int_{\Omega} \mu(x) \big(f_1(x) - f_2(x) \big) \, dx \right\}$$

- Goldstein-Osher: Split Bregman / augmented lagrangian

Convergence

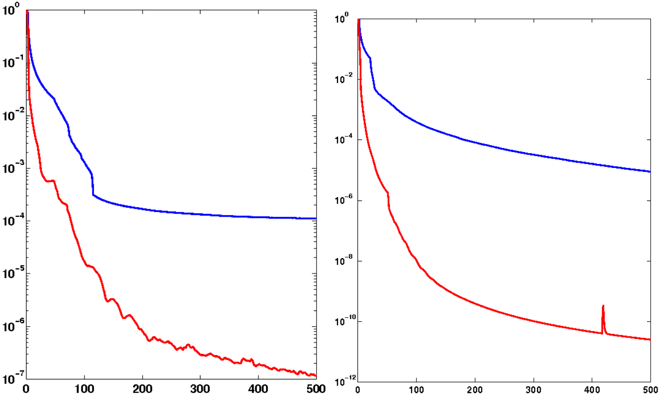


Figure: Red line: max-flow algorithm. Blue line: Splitting algorithm (Bresson et. al. 2007)

Multiphase problems

Multpihase problem

α-expansion and $\alpha - \beta$ swap

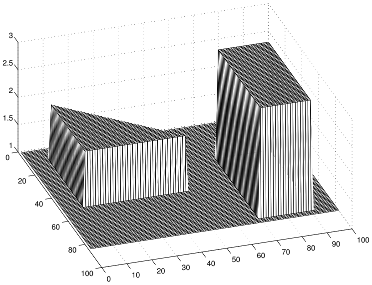
- Related to garph cut, α-expansion and $\alpha - \beta$ swap are mostly popular.
- Approximations are made and upper bounded has been given.
- Boykov-Veksler-Zahib (1999).

Multiphase problems – Approach I

Each point $x \in \Omega$ is labelled by

$$u(x) = i, \quad i = 1, 2, \cdots n.$$

- One label is enough for any n phases.
- More generall $u(x) = \ell_i, \; i = 1, 2, \cdots n.$



Multiphase problems – Approach II

Each point $x \in \Omega$ is labelled by a vector function:

$$u(x) = (u_1(2), u_2(x), \cdots u_d(x)).$$

Multiphase problems – Approach II

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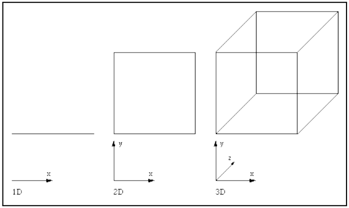
- Multiphase: Total number of phases $n = 2^d$. (Chan-Vse)



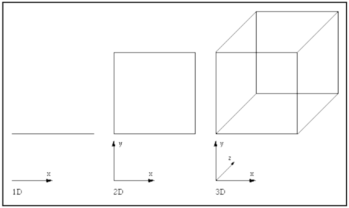
$$u_i(x) \in \{0, 1\}.$$

Multiphase problems – Approach II

Each point $x \in \Omega$ is labelled by a vector function:

$$u(x) = (u_1(2), u_2(x), \cdots u_d(x)).$$

- Multiphase: Total number of phases $n = 2^d$. (Chan-Vse)



$$u_i(x) \in \{0, 1\}.$$

- More than binary labels: Total number of phases $n = B^d$.

$$u_i(x) \in \{0, 1, 2, \cdots B\}.$$

Multiphase problems – Approach III

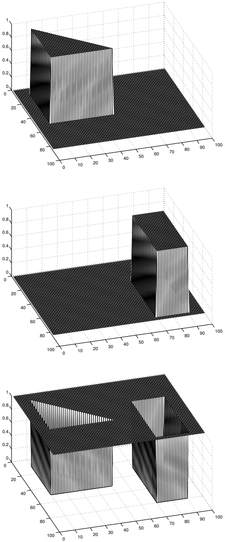
We need to identify n characteristic functions $\psi_i(x),\ i = 1, 2 \cdots n$:

$$\psi_i(x) \in \{0, 1\}, \quad \sum_{i=1}^{n} \psi_i(x) = 1.$$

- Relation between Approach I and III:

$$u(x) = i, i = 1, 2, \cdots n.$$

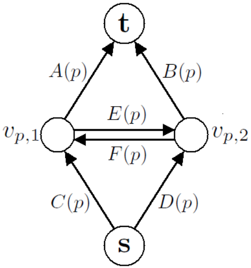
$$u(x) = \sum_{i=1}^{n} i\ \psi_i(x).$$



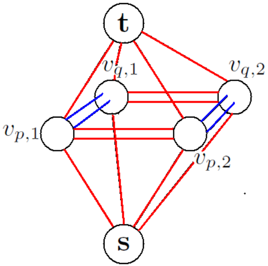
Multpihase problem (I)

Special graph cut for Chan-Vese approach

CV Graph construction (Bae-Tai EMMCVPR2009)



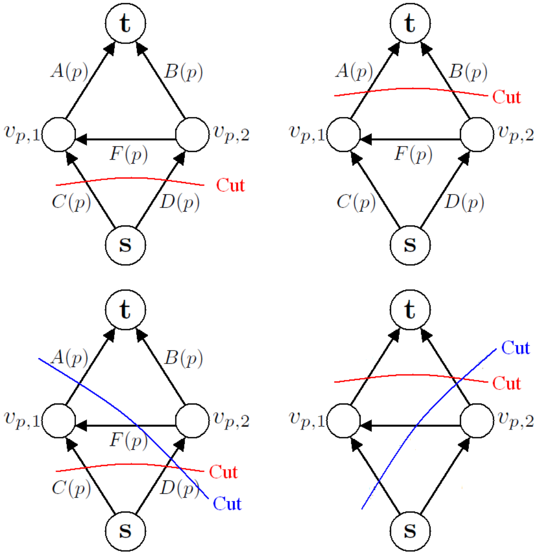
One pixel



two pixels

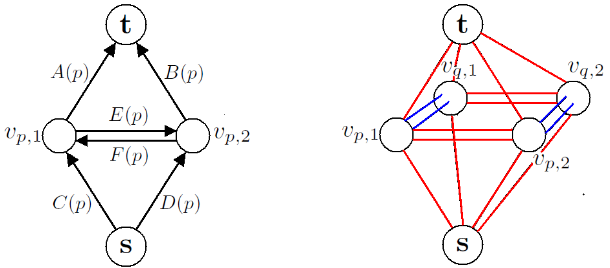
- Associate two vertices to each grid point ($v_{p,1}$ and $v_{p,2}$)
- For any cut (V_s, V_t)
  - If $v_{p,i} \in V_s$ then $\phi^i = 1$ for $i = 1, 2$
  - If $v_{p,i} \in V_t$ then $\phi^i = 0$ for $i = 1, 2$
- Figure left: graph corresponding to one grid point p
- Figure right: graph corresponding to two grid points p and q
  - Red: Data edges, constituting $E^{data}(\phi_1, \phi_2)$
  - Blue: Regularization edges with weight w_{pq}

Cuts for the CV-graph(Bae-Tai, EMMCVPR2009)



Minimization by graph cut

Graph construction



- Linear system for finding edge weights

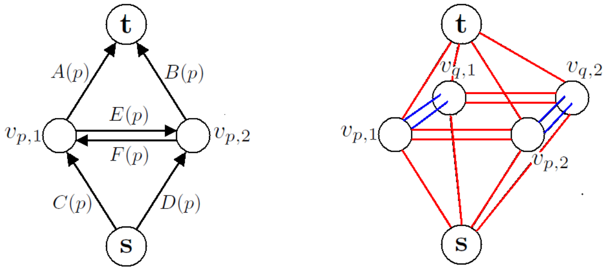
$$\begin{cases} A(p) + B(p) & = |c_2 - u_p^0|^\beta \\ C(p) + D(p) & = |c_3 - u_p^0|^\beta \\ A(p) + E(p) + D(p) & = |c_1 - u_p^0|^\beta \\ B(p) + F(p) + C(p) & = |c_4 - u_p^0|^\beta \end{cases}$$

such that $E(p), F(p) \geq 0$

- For each p, $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ interaction between two binary variables. Linear system has solution iff $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ is submodular.

Global minimizer – conditions

Graph construction



- Restriction $E(p), F(p) \geq 0$ implies

$$|c_1 - u_p^0|^\beta + |c_4 - u_p^0|^\beta = A(p) + B(p) + C(p) + D(p) + E(p) + F(p)$$

$$\geq A(p) + B(p) + C(p) + D(p) = |c_2 - u_p^0|^\beta + |c_3 - u_p^0|^\beta.$$

- Therefore it is sufficient that

$$|c_2 - I|^\beta + |c_3 - I|^\beta \leq |c_1 - I|^\beta + |c_4 - I|^\beta, \qquad \forall\, I \in [0, L],$$

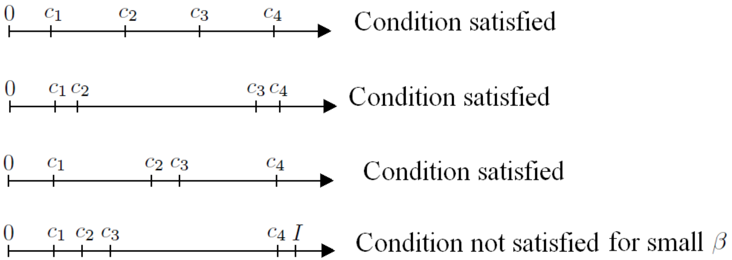
- At most three edges are required for a general submodular function of two binary variables (Kolmogorov et. al.)

Global minimizer – Guarantees

Submodularity condition

$$|c_2 - I|^\beta + |c_3 - I|^\beta \leq |c_1 - I|^\beta + |c_4 - I|^\beta, \qquad \forall\, I \in [0, L],$$

- Proposition 1: Let $0 \leq c_1 < c_2 < c_3 < c_4$. Condition is satisfied for all $I \in [\frac{c_2 - c_1}{2}, \frac{c_4 - c_3}{2}]$.
- Proposition 2: Let $0 \leq c_1 < c_2 < c_3 < c_4$. There exists a $\mathcal{B} \in \mathbb{N}$ such that condition is satisfied for any $\beta \geq \mathcal{B}$.



CV-graph – negative weights

- There are infinite many solution for A, B, C, D, E, F for each pixel.
- We can guarantee $A > 0, B > 0, C > 0, D > 0$. If one of E, F is negative, there is a modified graph.
- Some arts: sort c_i as $c_1 < c_2 < c_3 < c_4$, then choose

$$f_1(p) = |c_2 - u_p^0|^\beta, f_2(p) = |c_3 - u_p^0|^\beta,$$

$$f_3(p) = |c_1 - u_p^0|^\beta, f_4(p) = |c_4 - u_p^0|^\beta.$$

Numerical experiments

Experiment 1

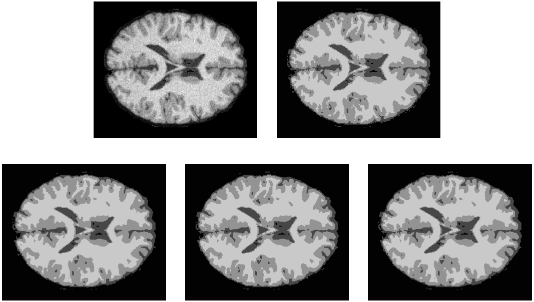


Figure: Experiment 3: (a) Input image, (b) ground truth, (c) gradient descent, (d) our approach, (e) alpha expansion, (f) alpha-beta swap.

Numerical experiments

Experiment 2

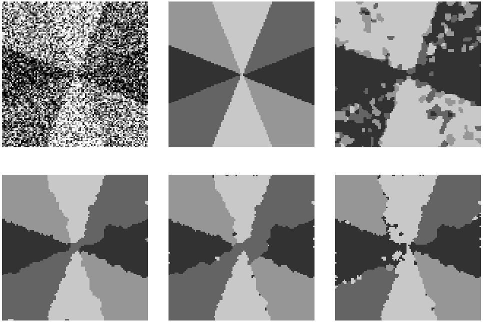


Figure: Experiment 3: (a) Input image, (b) ground truth, (c) gradient descent, (d) our approach, (e) alpha expansion, (f) alpha-beta swap.

Numerical experiments

Experiment 3

- L^2 data term ($\beta = 2$)
- Right: Input image.
- Left: Output.

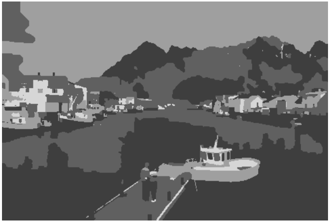
Numerical experiments

Experiment 4, non-submodular minimization

- L^1 data term ($\beta = 1$)
- Right: Input image.
- Left: Set of pixels where residual criterion was not satisfied (empty set).

Numerical experiments

Experiment 4, non-submodular minimization

- L^1 data term $(\beta = 1)$
- Right: Input image.
- Left: Output (global solution).

Exact convex formulation for the Multiphase Chan-Vese model by Continuous max-flow/min-cuts

Multiphase level set representation of CV model

$$\min_{\phi^1,\phi^2,\{c_i\}_{i=1}^4} \alpha \int_\Omega |\nabla H(\phi^1)| + \alpha \int_\Omega |\nabla H(\phi^2)| + E^{data}(\phi^1,\phi^2),$$

where

$$E^{data}(\phi^1,\phi^2) = \int_\Omega \{H(\phi^1)H(\phi^2)|c_2-u^0|^\beta + H(\phi^1)(1-H(\phi^2))|c_1-u^0|^\beta$$
$$+(1-H(\phi^1))H(\phi^2)|c_4-u^0|^\beta + (1-H(\phi^1))(1-H(\phi^2))|c_3-u^0|^\beta\}dx.$$

$$\Omega_1 = \{x \in \Omega \text{ s.t. } \phi^1(x) > 0, \phi^2(x) < 0\}$$
$$\Omega_2 = \{x \in \Omega \text{ s.t. } \phi^1(x) > 0, \phi^2(x) > 0\}$$
$$\Omega_3 = \{x \in \Omega \text{ s.t. } \phi^1(x) < 0, \phi^2(x) < 0\}$$
$$\Omega_4 = \{x \in \Omega \text{ s.t. } \phi^1(x) < 0, \phi^2(x) > 0\}$$

Binary formulation of multiphase Chan-Vese model

Wish to obtain global optimization framework for

$$\min_{\phi^1,\phi^2\in\{0,1\}} \alpha \int_\Omega |\nabla \phi^1| dx + \alpha \int_\Omega |\nabla \phi^2| dx + E^{data}(\phi^1, \phi^2),$$

with

$$E^{data}(\phi^1,\phi^2) = \int_\Omega \{\phi^1\phi^2|c_2 - u^0|^\beta + \phi^1(1-\phi^2)|c_1 - u^0|^\beta$$

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Binary formulation of multiphase Chan-Vese model

Wish to obtain global optimization framework for

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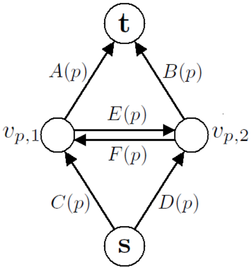

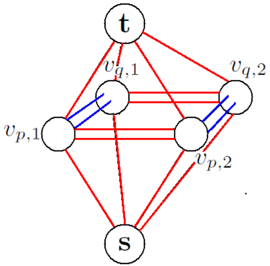

Continuous max-flow formulation

$$\sup_{p_s^i, p_t^i, p^{12}, q^i;\, i=1,2} \int_\Omega p_s^1(x) + p_s^2(x)\, dx$$

subject to

$$p_s^1(x) \leq C(x), \quad p_s^2(x) \leq D(x), \quad p_t^1(x) \leq A(x), \quad p_t^2 \leq B(x), \quad |q^i(x)| \leq \alpha$$

$$-F(x) \leq p^{12}(x) \leq E(x),$$

$$\operatorname{div} q^1(x) - p_s^1(x) + p_t^1(x) + p^{12}(x) = 0$$

$$\operatorname{div} q^2(x) - p_s^2(x) + p_t^2(x) - p^{12}(x) = 0$$

Lagrange multipliers λ_1 and λ_2 for flow conservation constraints.
Lagrangian functional:

$$\max_{p_s^i, p_t^i, p^{12}, q^i; i=1,2} \inf_{\lambda_1, \lambda_2} \int_\Omega (1 - \lambda_1(x)) p_s^1(x) + (1 - \lambda_2(x)) p_s^2(x) \, dx$$

$$+ \int_\Omega \lambda_1(x) p_t^1(x) + \lambda_2(x) p_t^2(x) + (\lambda_1(x) - \lambda_2(x)) p^{12}(x) \, dx$$

$$+ \int_\Omega \lambda_1(x) \operatorname{div} q^1(x) + \int_\Omega \lambda_2(x) \operatorname{div} q^2(x).$$

subject to

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Maximizing Lagrangian for all flows results in

$$\min_{\lambda_1,\lambda_2} \int_\Omega (1-\lambda_1(x))C(x)+(1-\lambda_2(x))D(x)+\lambda_1(x)A(x)+\lambda_2(x)B(x)\,dx$$

$$+\int_\Omega \max\{\lambda_1(x)-\lambda_2(x),0\}E(x)\,dx-\min\{\lambda_1(x)-\lambda_2(x),0\}F(x)\,dx$$

$$+\alpha\int_\Omega |\nabla\lambda_1(x)|\,dx+\alpha\int_\Omega |\nabla\lambda_2(x)|\,dx.$$

subject to $\lambda_1(x),\lambda_2(x)\in[0,1],\ \forall x\in\Omega$.

$$\begin{cases} A(x)+B(x) & = |c_2-u^0(x)|^\beta \\ C(x)+D(x) & = |c_3-u^0(x)|^\beta \\ A(x)+E(x)+D(x) & = |c_1-u^0(x)|^\beta \\ B(x)+F(x)+C(x) & = |c_4-u^0(x)|^\beta \end{cases}$$

- Convex, iff $E(x),F(x)\geq 0$
- Theorem: Thresholding optimal $\lambda_1(x)$ and $\lambda_2(x)$ will give a binary global solution to multiphase Chan-Vese model

Corollaries

- No approximation: the global minimizer of the max-flow (convex CV) is the global minimizer of the original non-convex CV model.

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$$R(u) = \int_\Omega |\nabla u_1| + |\nabla u_2|.$$

- We can also regularize the length of the interface, then Thresholded solution is not guaranteed to be exact.

Multiphase problems

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  More than four-phase, cannot guarantee global binary solution.
- Other multiphase relaxations:
  J. Lellmann-Kappes-Yuan-Becker-Schnörr (2008),
  Lellmann-et-al(2009, 2010),
  Brown-Chan-Bresson (2011),
  Goldstein-Bresson-Osher (2009),
  Chambolle-Cremers-Pock (2009, 2012).

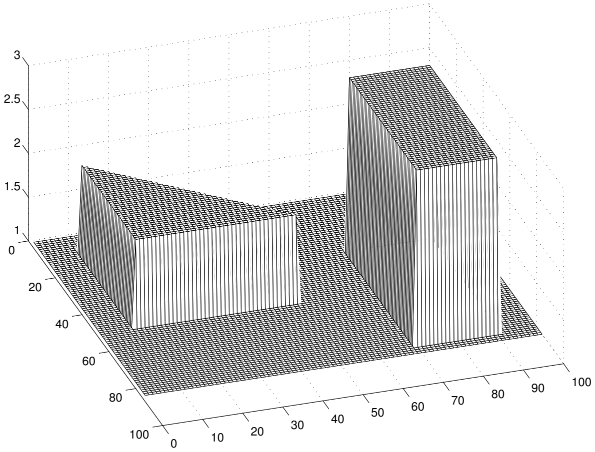
Multiphase problem (II)

Layered Graph[1]

[1]Boykov-Kolmogorov (PAMI 2001), Ishikawa (PAMI 2003), Darbon-Segle(JMIV, 2006), Bae-Tai (SSVM 2009)

Multiphase problems

To identify n phases, we need one label function, but n labels.



Multiphase problem

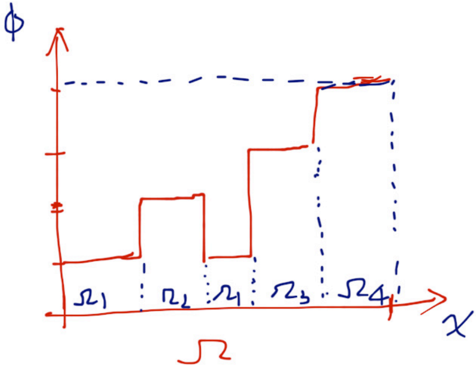


Figure: Need multi-labels $\phi(x) = i$ in Ω_i, $i = 1, 2, 3, 4$.

Increase dimension – only need two phases

$$|\nabla \phi| = |\nabla u|.$$

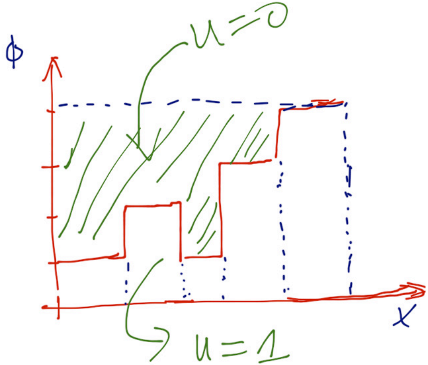
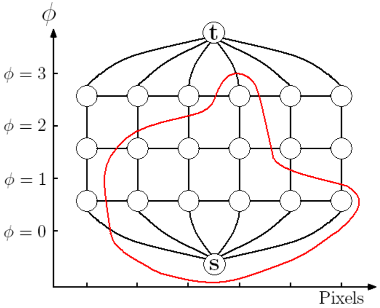


Figure: Just need one label: Increase the dimension, we just need $u(x, \phi) = 0$ or 1.

1D signal and multiphase



0	2	2	3	1	1

Figure: Left: Example cut on the graph $\mathcal{G}$ corresponding to a 1d image of 6 grid points. Right: Values of ϕ corresponding to the cut

Historical review

- This graph was proposed in Ishikaka (PAMI 2003).

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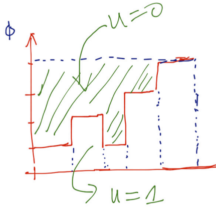
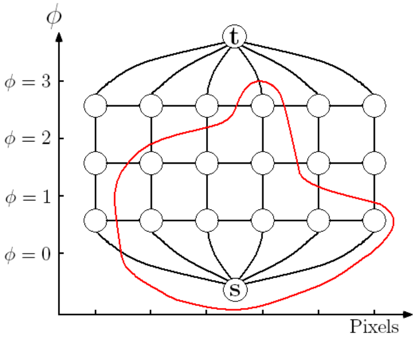


T. Pock and D. Cremers and H. Bischof and A. Chambolle (2010): gives a convex relaxation in case both image domain and the labels are continuous.

Continuous max-flow and cut

This part is based on: Bae-Yuan-T.-Boykov: CAM-10-62 (2010): a fast continuous max-flow approach to non-convex multilabeling problems.

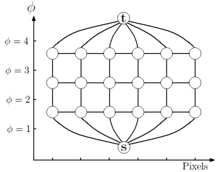
Multiphases



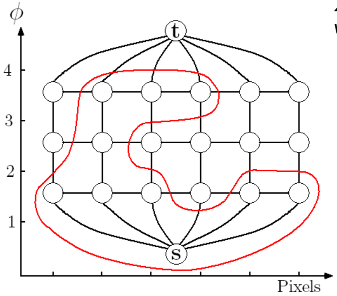
0	2	2	3	1	1

Costs: $\rho(u(p), p)$, $C(p, q)$, $i = 1, 2, 3$.

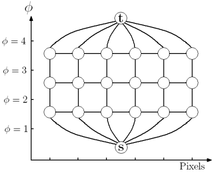
Discrete min-cut

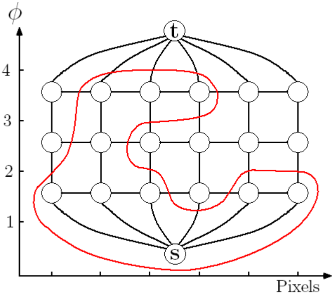


$$\min \sum_{v \in \mathbf{P}} \rho(u_v, v) + \sum_{(u,v) \in \mathcal{N}} C(u,v) |u_v - u_w|.$$



Discrete max-flow





$$\max \sum_{v \in \mathbf{P}} p_1(v)$$

$$p_i(v) \leq \rho(\ell_i, v), \;\; i = 1, 2, \cdots n,$$

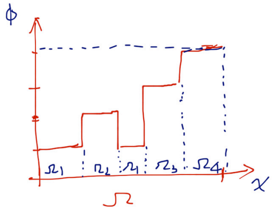
$$|q_i(v, w)| \leq C(v, w).$$

Continuous min-cut and max-flow

Continuous min-cut:

$$\min_{u \in U} \int_{\Omega} \rho(u(x), x) dx + \int_{\Omega} C(x) |\nabla u| dx.$$

$$U = \{ u : \Omega \mapsto \{ \ell_1, \ell_2, \cdots \ell_n \} \}.$$



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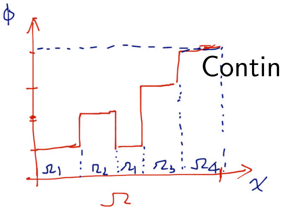

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$$\max \int_\Omega p_1(x)dx$$

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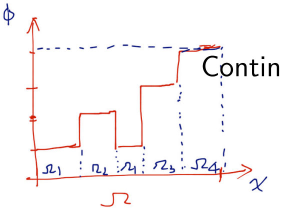

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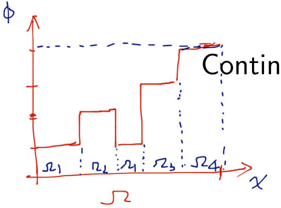

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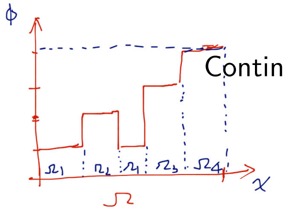

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$$(\text{div} q_i - p_i + p_{i+1})(x) = 0, \;\; q_i \cdot \mathbf{n} = 0.$$

Equivalence

Theorem: The continuous min-cut and max-flow problems are dual to each other. A "threshold" of any solutions of the "convex" min-cut problem is a global minimizer for the "non-convex" min-cut problem.

Algorithm

Algorithm: Primal-dual algorithm is tested and is fast.
Primal variables: The flow variables.
Dual variables: The cut u which turn out to the Lagrangian of the "flow conservation" constraints.

Infinite number of labels

For the number of labels, instead of:

$$U = \{u : \Omega \mapsto \{\ell_1, \ell_2, \cdots \ell_n\}\}.$$

we use "infinite number of labels":

$$U = \{u : \Omega \mapsto [\ell_{min}, \ell_{max}]\}.$$

This is exactly the same problem considered in:
T. Pock and D. Cremers and H. Bischof and A. Chambolle (2010).

Continuous labels

As the number of labels goes to the limit of infinity, the max-flow problem with the flow constraints turns into:

$$\sup_{p,q} \quad \int_\Omega p(\ell_{\min}, x)\, dx$$

$$\text{s.t.} \quad p(\ell, x) \leq \rho(\ell, x)\,, \quad |q(\ell, x)| \leq \alpha, \qquad \forall x \in \Omega,\ \forall \ell \in [\ell_{\min}, \ell_{\max}]$$

$$\mathrm{div}_x\, q(\ell, x) + \partial_\ell\, p(\ell, x) = 0\,, \qquad \text{a.e. } x \in \Omega, \quad \ell \in [\ell_{\min}, \ell_{\max}].$$

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The convex min-cut problem (the dual problem to the max-flow) is:

$$\min_{\lambda(\ell,x)\in[0,1]} \int_{\ell_{\min}}^{\ell_{\max}} \int_\Omega \left\{ \alpha\, |\nabla_x \lambda| - \rho(\ell, x) \partial_\ell\, \lambda(\ell, x) \right\} dx d\ell$$

$$+ \int_\Omega (1 - \lambda(\ell_{\min}, x)) \rho(\ell_{\min}, x) + \lambda(\ell_{\max}, x) \rho(\ell_{\max}, x)\, dx$$

subject to

$$\partial_\ell\, \lambda(\ell, x) \le 0, \quad \lambda(\ell_{\min}, x) \le 1, \quad \lambda(\ell_{\max}, x) \ge 0, \quad \forall x \in \Omega, \quad \forall \ell \in [\ell_{\min}, \ell_{\max}] \tag{3}$$

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$$+ \int_{\Omega} (1 - \lambda(\ell_{\min},x))\rho(\ell_{\min},x) + \lambda(\ell_{\max},x)\rho(\ell_{\max},x) \, dx$$

subject to

$$\partial_\ell \, \lambda(\ell,x) \, \leq \, 0 \, , \quad \lambda(\ell_{\min},x) \leq 1 \, , \quad \lambda(\ell_{\max},x) \geq 0 \, ,$$

The following is the model from Poct et al (2010): (Note the difference)

$$\min_{\lambda(\ell,x)\in\{0,1\}} \int_{\ell_{\min}}^{\ell_{\max}} \int_{\Omega} \left\{ \alpha \, |\nabla_x \lambda| + \rho(\ell,x) \, |\partial_\ell \lambda(\ell,x)| \, \right\} dx d\ell \, .$$

subject to

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Multiphase problem (III)

Graph for characteristic functions[1]

[1] Yuan-Bae-T.-Boykov (ECCV'10)

Multi-partitioning problem (Pott's model)

$$\min_{\{\Omega_i\}} \sum_{i=1}^{n} \int_{\Omega_i} f_i dx + \sum_{i=1}^{n} \int_{\partial\Omega_i} g(x) ds,$$

such that $\cup_{i=1}^{n} \Omega_i = \Omega, \quad \cap_{i=1}^{n} \Omega_i = \emptyset$

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Pott's model in terms of characteristic functions

$$\min_{u_i(x)\in\{0,1\}} \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x)\, dx + \sum_{i=1}^{n} \int_{\Omega} g(x)\, |\nabla u_i|\, dx\,, \text{ s.t. } \sum_{i=1}^{n} u_i(x) = 1$$

Multi-partitioning problem

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$$\text{such that} \quad \cup_{i=1}^{n} \Omega_i = \Omega, \quad \cap_{i=1}^{n} \Omega_i = \emptyset$$

Pott's model in terms of characteristic functions

$$\min_{u_i(x)\in\{0,1\}} \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x)\, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) \left|\nabla u_i\right| \, dx \,, \ \text{s.t.} \ \sum_{i=1}^{n} u_i(x) \,=\, 1$$

$$u_i(x) \,=\, \chi_{\Omega_i}(x) \,:=\, \begin{cases} 1, & x \in \Omega_i \\ 0, & x \notin \Omega_i \end{cases} , \quad i = 1, \ldots, n$$

A convex relaxation approach

Relaxed Pott's model in terms of characteristic functions (primal model)

$$\min_{u} \; E^P(u) = \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x)\, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) \left| \nabla u_i \right| \, dx \,,$$

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$$\min_u \; E^P(u) = \sum_{i=1}^n \int_\Omega u_i(x) f_i(x)\, dx + \sum_{i=1}^n \int_\Omega g(x)\, |\nabla u_i| \, dx\,,$$

$$\text{s.t. } \; u \in \triangle_+ = \{(u_1(x), \ldots, u_n(x)) \,|\, \sum_{i=1}^n u_i(x) \,=\, 1\,; \quad u_i(x) \geq 0\,\}$$

- Convex optimization problem
- Optimization techniques: Zach et. al. alternating TV minimization. Lellmann et. al: Douglas Rachford splitting and special thresholding, Bae-Yuan-T. (2010), Chambolle-Crmer-Pock (2012).

Dual formulation of relaxation: Bae-Yuan-T. (IJCV, 2010)

Dual model: $C_\lambda := \{p : \Omega \mapsto \mathbb{R}^2 \,|\, |p(x)|_2 \leq g(x), \; p_n|_{\partial\Omega} = 0\}$,

- Hence the primal-dual model can be optimized pointwise for u

$$\min_{u \in \triangle_+} \sum_{i=1}^{n} \int_\Omega u_i(x) f_i(x) \, dx + \sum_{i=1}^{n} \int_\Omega g(x) \, |\nabla u_i| \; dx \,,$$

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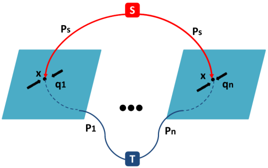
$$\begin{aligned}
\max_{p_i \in C_\lambda} \min_{u \in \triangle_+} E(u,p) &= \int_\Omega \sum_{i=1}^n u_i (f_i + \operatorname{div} p_i)\, dx \\
&= \max_{p_i \in C_\lambda} \int_\Omega \min_{u(x) \in \triangle_+} \sum_{i=1}^n u_i(x)(f_i(x) + \operatorname{div} p_i(x))\, dx \\
&= \max_{p_i \in C_\lambda} \int_\Omega \left\{ \min(f_1 + \operatorname{div} p_1, \ldots, f_n + \operatorname{div} p_n) \right\} dx \\
&= \max_{p_i \in C_\lambda} E^D(p)
\end{aligned}$$

Multiple Phases: Convex Relaxed Potts Model (CR-PM) –Yuan-Bae-T.-Boykov (ECCV'10)

Continuous Max-Flow Model (CMF-PM)



1. n copies Ω_i, $i = 1, \ldots, n$, of Ω;
2. For $\forall x \in \Omega$, the same source flow $p_s(x)$ from the source s to x at Ω_i, $i = 1, \ldots, n$, simultaneously;
3. For $\forall x \in \Omega$, the sink flow $p_i(x)$ from x at Ω_i, $i = 1, \ldots, n$, of Ω to the sink t. $p_i(x)$, $i = 1, \ldots, n$, may be different one by one;
4. The spatial flow $q_i(x)$, $i = 1, \ldots, n$ defined within each Ω_i.

Max-flow on this graph

Max-Flow:

$$\max_{p_s,p,q}\{P(p_s,p,q)=\int_\Omega p_s dx\}$$
$$|q_i(x)|\le g(x),\quad p_i(x)\le f_i(x),$$
$$(\mathrm{div} q_i - p_s + p_i)(x)=0,\ \ i=1,2,\cdots n.$$

Note that

$$p_s(x)=\mathrm{div} q_i(x)+p_i(x), i=1,2\cdots n.$$

Thus

$$p_s(x)=\min(f_1+\mathrm{div}\, p_1,\dots,f_n+\mathrm{div}\, p_n).$$

Therefore, the maximum of $\int_\Omega p_s(x)$ is:

$$\max_{|q_i(x)|\le g(x)}\int_\Omega \min(f_1+\mathrm{div}\, p_1,\dots,f_n+\mathrm{div}\, p_n)dx$$

(Convex) min-cut on this graph

$$\max_{p_s,p,q} \min_u \{E(p_s,p,q,u) = \int_\Omega p_s dx + \sum_{i=1}^m u_i(\text{div} q_i - p_s + p_i)dx\}$$

$$\text{s.t. } p_i(x) \leq f_i(x), \quad |q_i(x)| \leq g(x).$$

Rearranging the energy functional $E(\cdot)$, we that

$$E(p_s,p,q,u) = \int_\Omega (1 - \sum_{i=1}^m u_i)p_s + \sum_{i=1}^m u_i p_i + \sum_{i=1}^m u_i \text{div} q_i . dx.$$

The following constraint are automatically satisfied from the optimization:

$$u_i(x) \leq 0, \sum_{i=1}^m u_i = 1.$$

Summary

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- If we replace the isotropic TV by antisotropic TV, then all the models we have investigated has a discrete "graph". However, some recent (ongoing) work show that some continuous max-flow models do not have a discrete graph.
- The CV (Chan-Vese) model has special properties in term of global minimization through max-flow and min-cut approach. Two-phase and four-phase problems have global binary minimizers, but not 2^n-phases ($n \geq 3$).