

Convex Relaxations for the Mumford-Shah Functional



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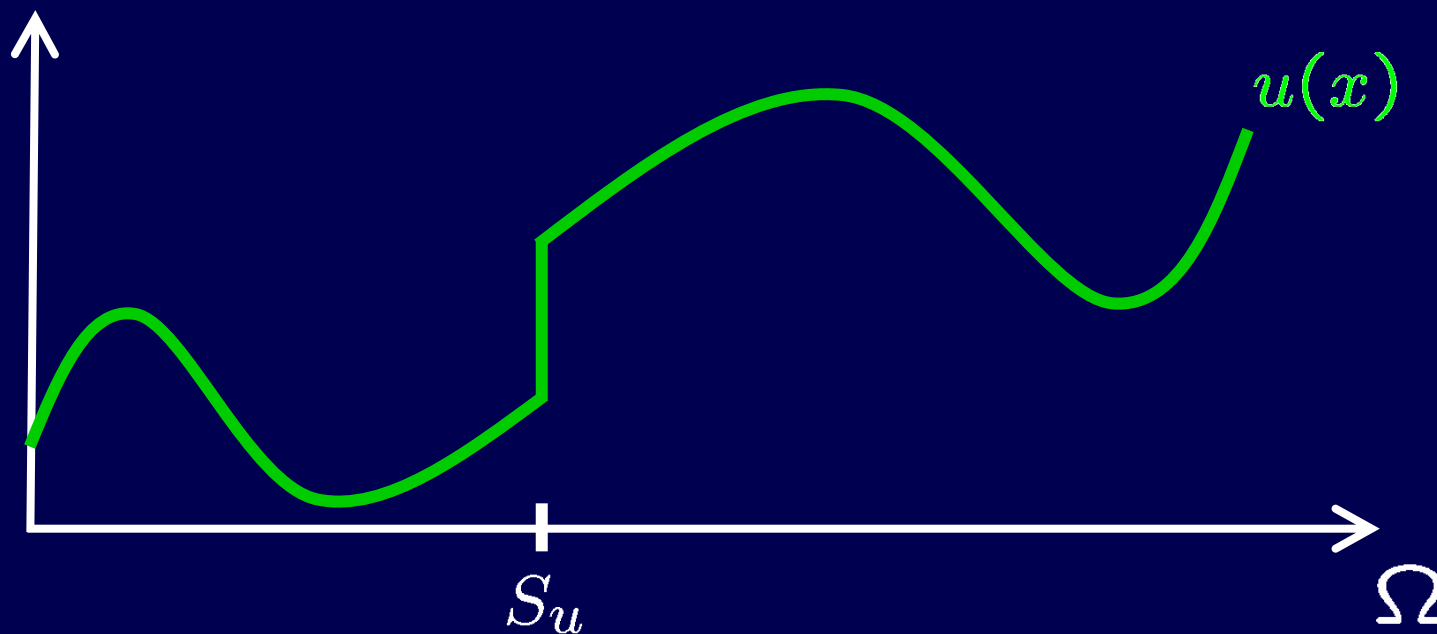
with Evgeny Strelakovski, Mohamed Souiai, Claudia Nieuwenhuis, Tom Pock and Antonin Chambolle

The Mumford-Shah Functional

Let $\Omega \subset \mathbb{R}^d$ and $f, u : \Omega \rightarrow \mathbb{R}^k$.

$$E(u) = \int_{\Omega} |f - u|^2 dx + \lambda \int_{\Omega \setminus S_u} |\nabla u|^2 dx + \nu \mathcal{H}^1(S_u)$$

*Mumford, Shah '89, Blake, Zisserman '87
Ambrosio, Tortorelli '90, Vese, Chan '02*



The Mumford-Shah Functional

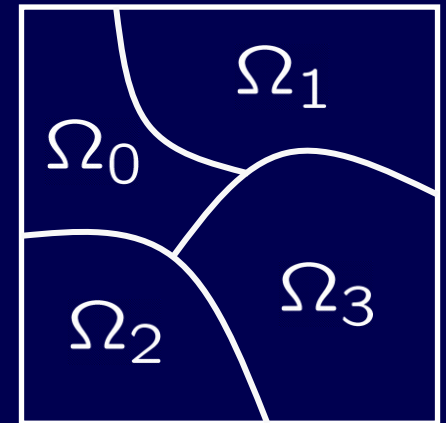
Let $\Omega \subset \mathbb{R}^d$ and $f, u : \Omega \rightarrow \mathbb{R}^k$.

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Ambrosio, Tortorelli '90, Vese, Chan '02*

Piecewise constant approximation for $\lambda \rightarrow \infty$:

$$E(\{\Omega_i, \mu_i\}_i) = \sum_i \int_{\Omega_i} |f(x) - \mu_i|^2 dx + \frac{\nu_0}{2} |\partial\Omega_i|$$



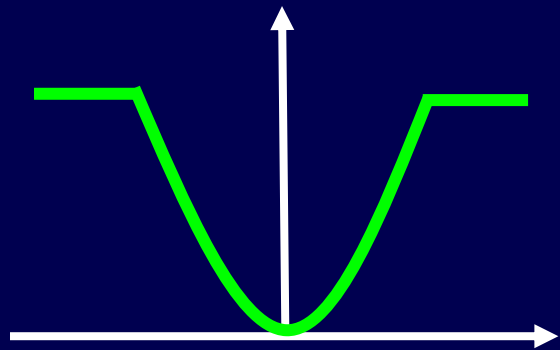
s.t. $\bigcup_i \Omega_i = \Omega$, and $\Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j$

Mumford, Shah '89, Chan, Vese '01, Potts '52, Ising '25

Comparison of Regularizers

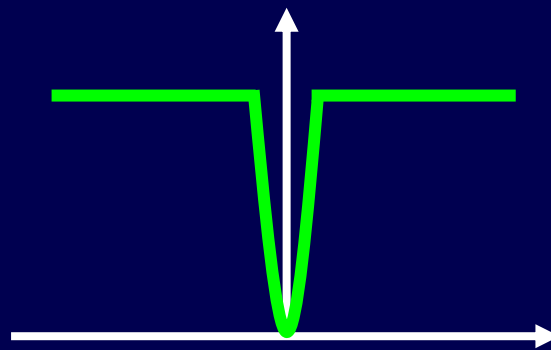
$$E(u) = \int_{\Omega} |f - u|^2 dx + \int_{\Omega} \psi(\nabla u) dx$$

$$\psi(s) = \min(|s|^2, \nu)$$



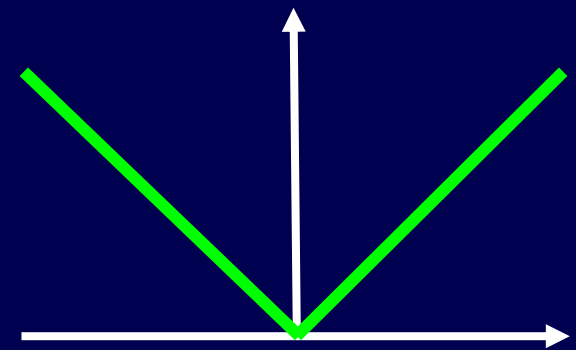
Truncated quadratic

$$\psi(s) = \begin{cases} 0, & \text{if } s=0 \\ \nu, & \text{else} \end{cases}$$



Potts model

$$\psi(s) = |s|$$



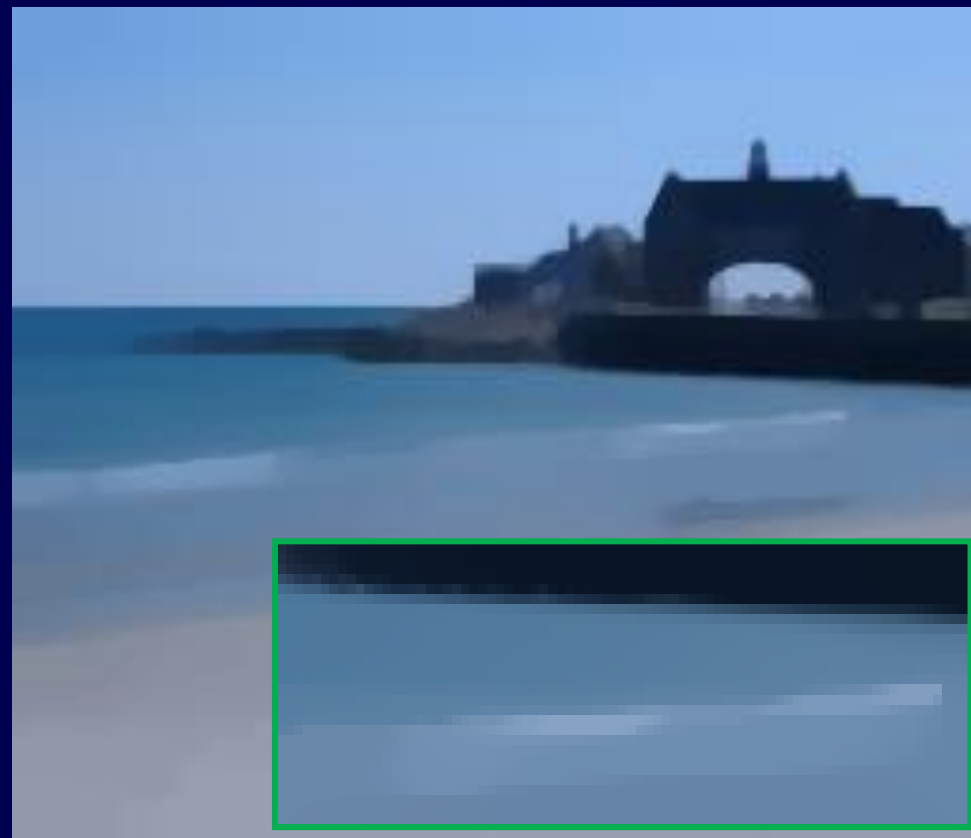
Linear / TV



Total Variation Denoising



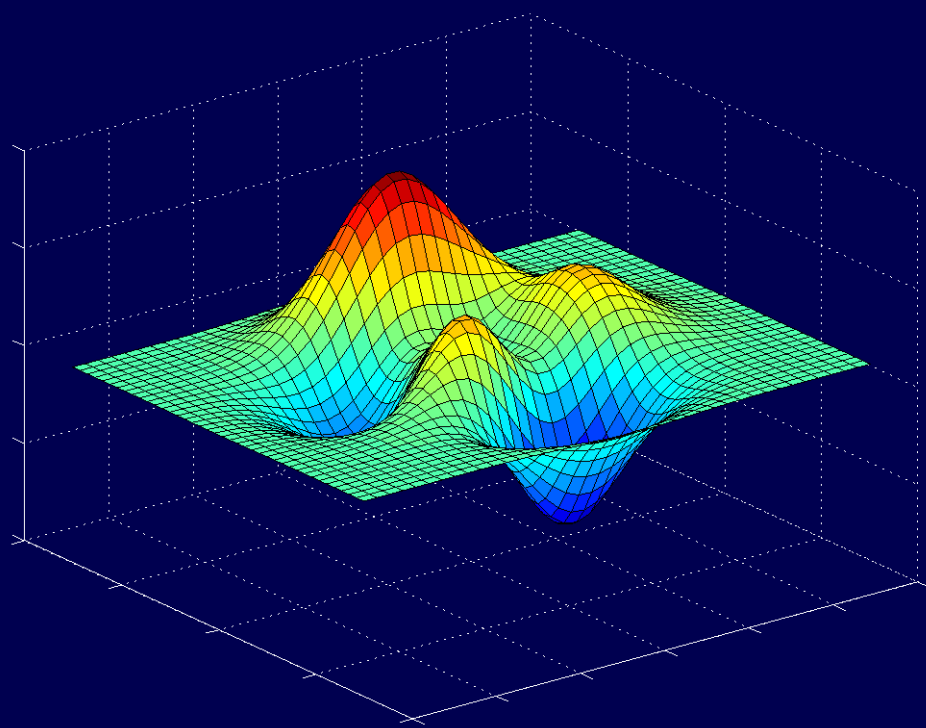
Input image



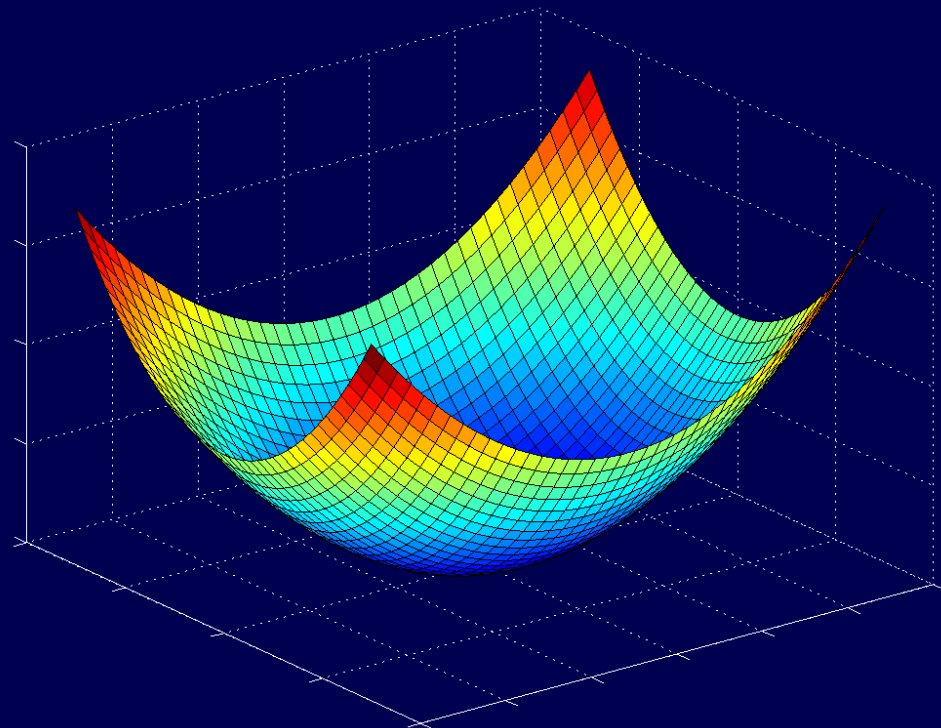
TV-denoised

- + Convex & fast to minimize
- Oversmoothing in flat regions (staircasing)
- Reduces contrast at edges

Non-convex versus Convex Energies



Non-convex energy

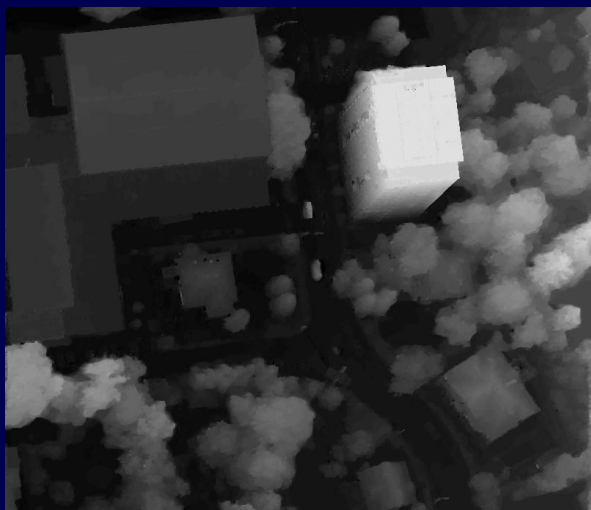


Convex energy

Some related work: *Brakke '95, Alberti et al. '01, Ishikawa '01, Chambolle '01, Attouch et al. '06, Nikolova et al. '06, Bresson et al. '07, Zach et al. '08, Lellmann et al. '08, Zach et al. '09, Brown et al. '10, Bae et al. '10, Yuan et al. '10,...*



Overview



Convex multilabel optimization



Minimal partitions



Semantic segmentation



Mumford-Shah



Overview



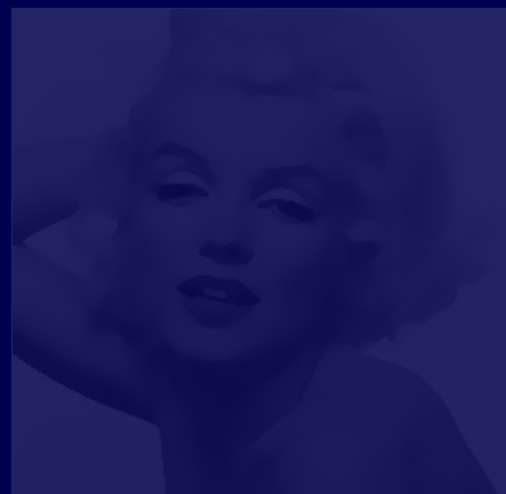
Convex multilabel optimization



Minimal partitions



Semantic segmentation

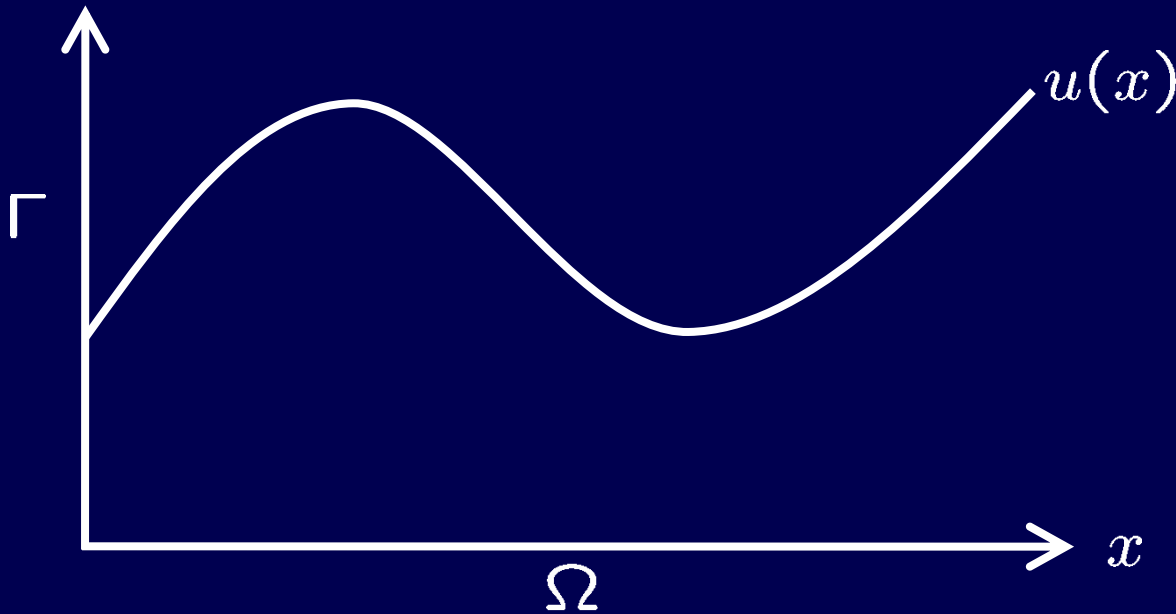


Mumford-Shah

Cartesian Currents and Relaxation

$$u : \Omega \rightarrow \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \underbrace{\int_{\Omega} \rho(x, u(x)) dx}_{\text{nonconvex data term}} + \underbrace{\int_{\Omega} |\nabla u(x)| dx}_{\text{label regularity}} \quad (*)$$



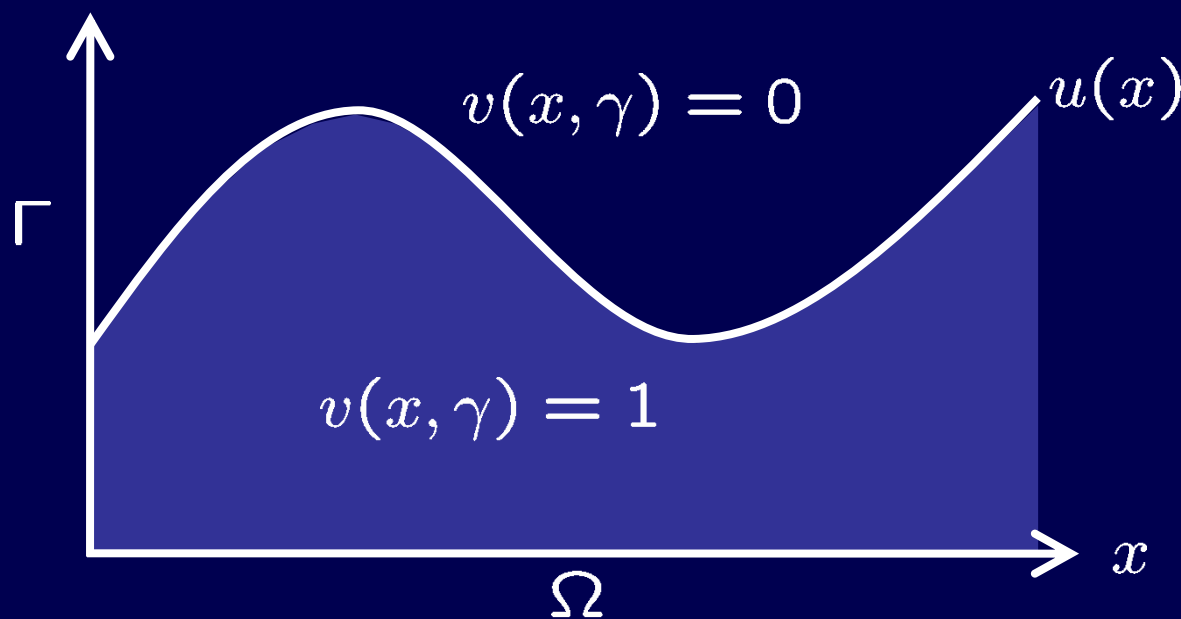
Pock, Schoenemann, Graber, Bischof, Cremers ECCV '08

Cartesian Currents and Relaxation

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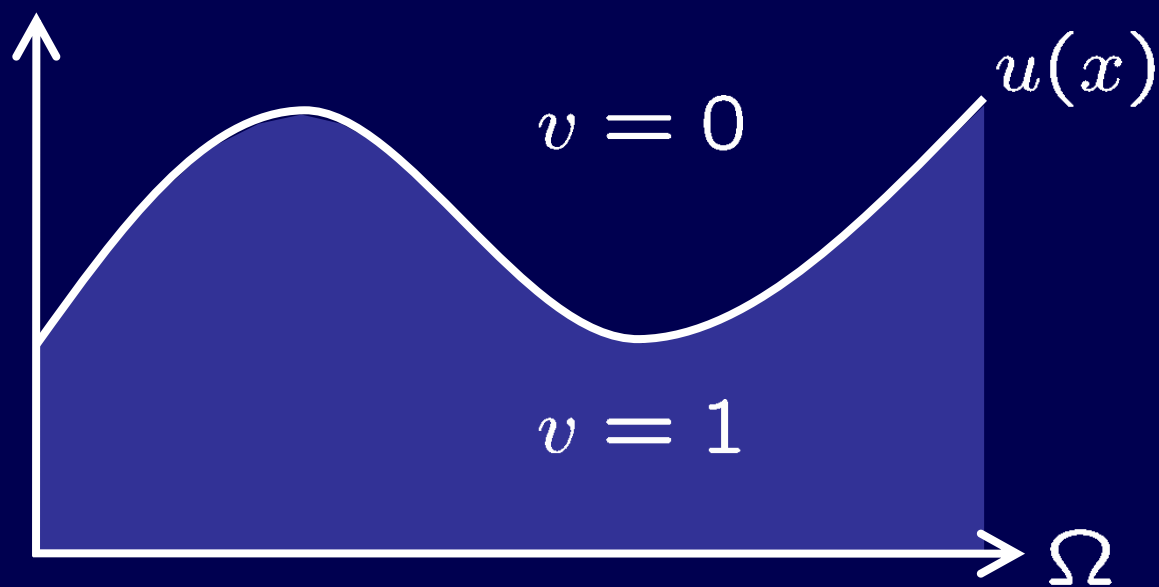
$$E(u) = \int_{\Omega} \rho(x, u(x)) dx + \int_{\Omega} |\nabla u(x)| dx \quad (*)$$

Let $v : (\Sigma = \Omega \times \Gamma) \rightarrow \{0, 1\}$ $v(x, \gamma) = \mathbf{1}_{u \geq \gamma}(x)$



Ishikawa, PAMI '03

Pock, Schoenemann, Graber, Bischof, Cremers ECCV '08



Cartesian Currents and Relaxation

$$u : \Omega \rightarrow \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \int_{\Omega} \rho(x, u(x)) dx + \int_{\Omega} |\nabla u(x)| dx \quad (*)$$

nonconvex functional

$$\text{Let } v : (\Sigma = \Omega \times \Gamma) \rightarrow \{0, 1\} \quad v(x, \gamma) = \mathbf{1}_{u \geq \gamma}(x)$$

Theorem: Minimizing (*) is equivalent to minimizing

$$E(v) = \int_{\Sigma} \rho(x, \gamma) |\partial_{\gamma} v(x, \gamma)| + |\nabla v(x, \gamma)| dx d\gamma \quad (**)$$

convex functional

Solve (**) in relaxed space ($v : \Sigma \rightarrow [0, 1]$) and threshold to obtain a globally optimal solution.

Pock, Schoenemann, Graber, Bischof, Cremers ECCV '08



Evolution to Global Minimum





Reconstruction from Aerial Images

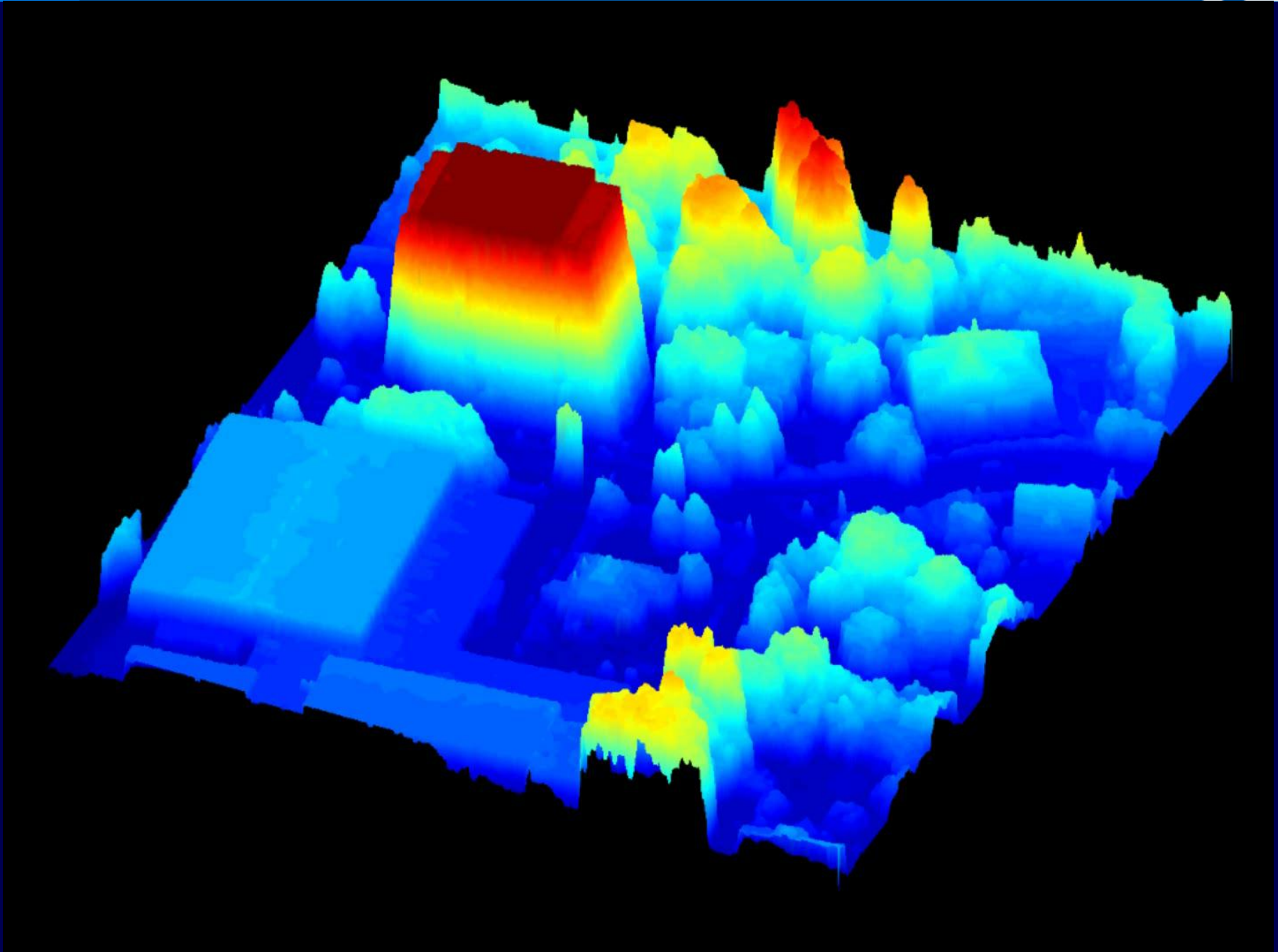


One of two input images
Courtesy of Microsoft

Depth reconstruction



Reconstruction from Aerial Images



Let

$$E(u) = \int_{\Omega} f(x, u, \nabla u) dx$$

be continuous in $x \in \mathbb{R}^d$ and u , and convex in ∇u .

Theorem:

For any function $u \in W^{1,1}(\Omega; \mathbb{R})$ we have:

$$E(u) = F(\mathbf{1}_u) := \sup_{\phi \in \mathcal{K}} \int_{\Omega \times \mathbb{R}} \phi \cdot D\mathbf{1}_u,$$

where ϕ is constrained to the convex set

$$\mathcal{K} = \left\{ \phi = (\phi^x, \phi^t) \in C_0(\Omega \times \mathbb{R}; \mathbb{R}^d \times \mathbb{R}) : \right. \\ \left. \phi^t(x, t) \geq f^*(x, t, \phi^x(x, t)), \forall x, t \in \Omega \times \mathbb{R} \right\}.$$

Pock, Cremers, Bischof, Chambolle, SIAM J. on Imaging Sciences '10

The functional $E(u)$ can be minimized by solving the relaxed saddle point problem

$$\min_v F(v) = \min_v \sup_{\phi \in \mathcal{K}} \int_{\Omega \times \mathbb{R}} \phi \cdot Dv,$$

Theorem:

The functional F fulfills a generalized coarea formula:

$$F(v) = \int_{-\infty}^{\infty} F(\mathbf{1}_{v \geq s}) ds.$$

As a consequence, we have a thresholding theorem assuring that we can globally minimize the functional $E(u)$.

Pock, Cremers, Bischof, Chambolle, SIAM J. on Imaging Sciences '10

An Efficient Saddle Point Solver

Given the saddle point problem

$$\min_{x \in C} \max_{y \in K} \langle Ax, y \rangle + \langle g, x \rangle - \langle h, y \rangle$$

with closed convex sets C and K and linear operator A of norm L .

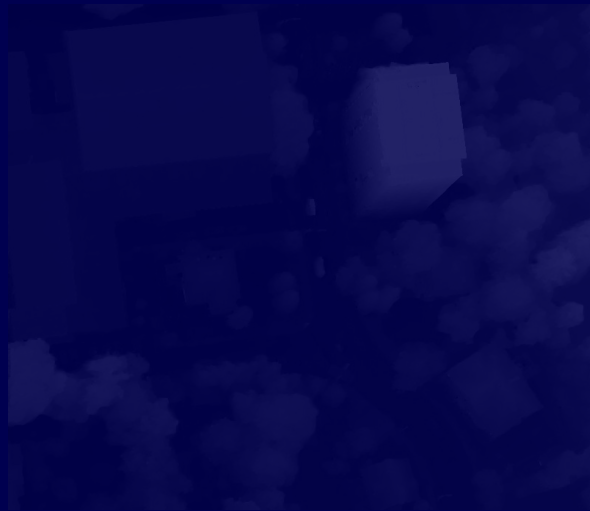
The iterative algorithm

$$\begin{cases} y^{n+1} = \Pi_K(y^n + \sigma(A\bar{x}^n - h)) \\ x^{n+1} = \Pi_C(x^n - \tau(A^*y^{n+1} + g)) \\ \bar{x}^{n+1} = 2x^{n+1} - x^n \end{cases}$$

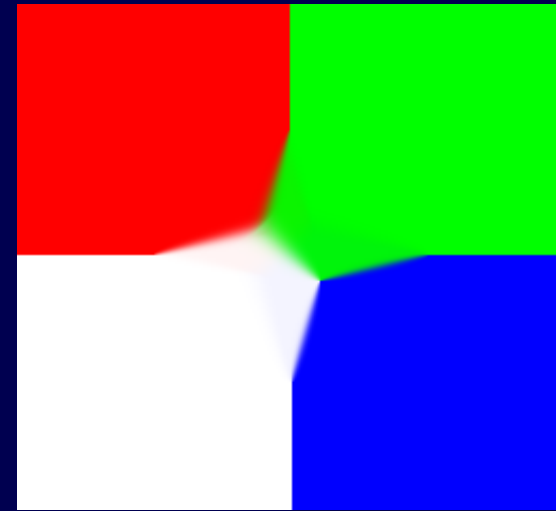
converges with rate $O(1/n)$ to a saddle point for $\sigma \tau L^2 \leq 1$.

Pock, Cremers, Bischof, Chambolle, ICCV '09, Chambolle, Pock '10

Overview



Convex multilabel optimization



Minimal partitions



Semantic segmentation

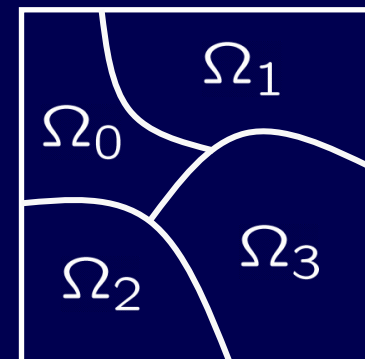


Mumford-Shah

The Minimal Partition Problem

$$\min_{\Omega_0, \dots, \Omega_n} \frac{1}{2} \sum_i |\partial \Omega_i| + \sum_i \int_{\Omega_i} f_i(x) dx$$

$$\text{s.t. } \bigcup_i \Omega_i = \Omega \subset \mathbb{R}^d, \text{ and } \Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j$$



Potts '52, Blake, Zisserman '87, Mumford-Shah '89, Vese, Chan '02

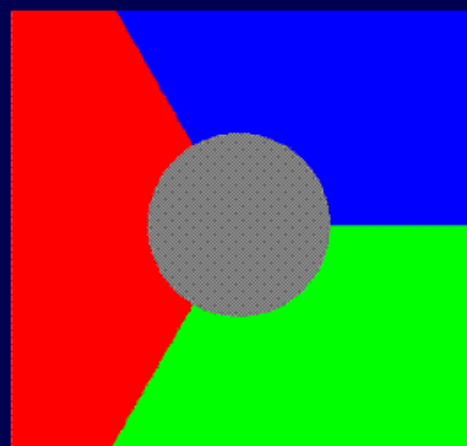
Proposition: With $v_i = 1_{\Omega_i}$, this is equivalent to

$$\min_{v \in \mathcal{B}} \frac{1}{2} \sum_i \int_{\Omega} |Dv_i| + \int_{\Omega} v_i f_i dx = \min_{v \in \mathcal{B}} \sup_{p \in \mathcal{K}} \sum_i \int_{\Omega} v_i \operatorname{div} p_i dx + \int_{\Omega} v_i f_i dx$$

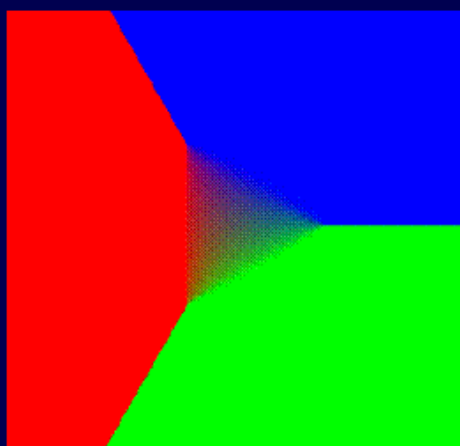
$$\text{where } \mathcal{K} = \left\{ p = (p_1, \dots, p_n)^{\top} \in \mathbb{R}^{n \times d} : |p_i - p_j| \leq 1, \forall i < j \right\}$$

Chambolle, Cremers, Pock '08, SIIMS '12

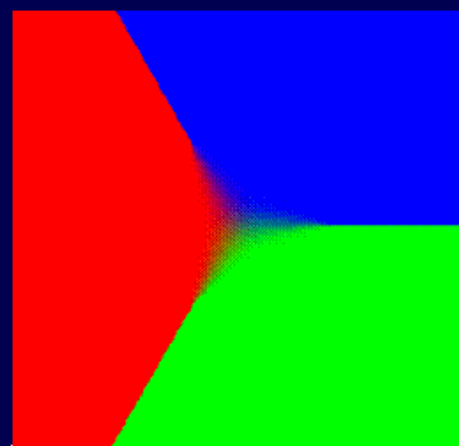
Test Case: The Triple Junction



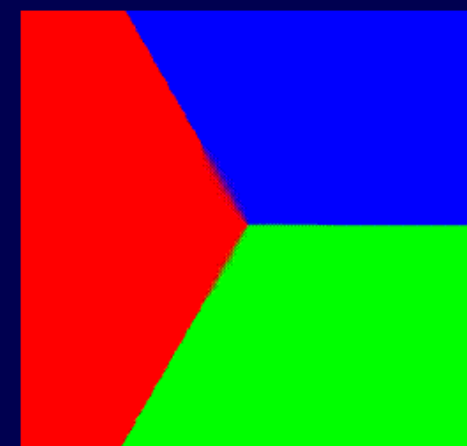
Input image



Lellmann et al. '08



Zach et al. '08

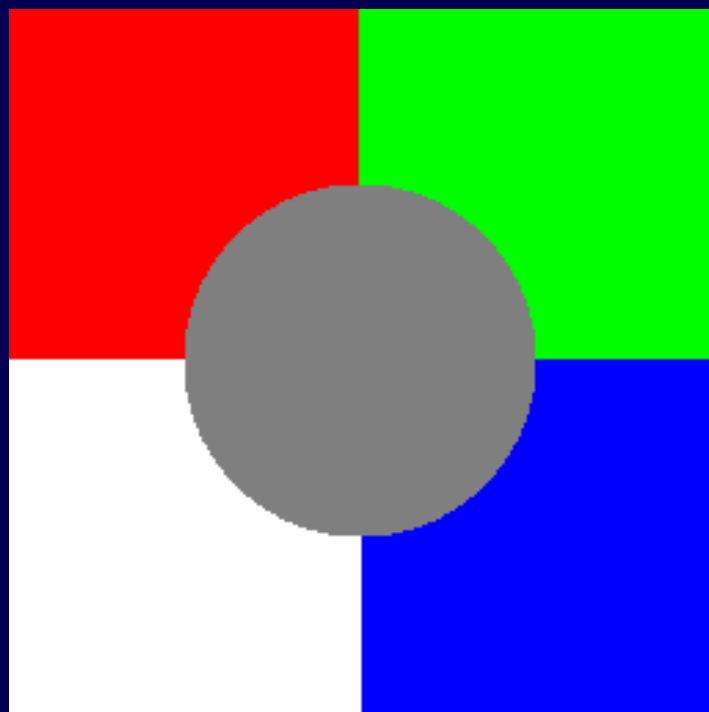


our approach

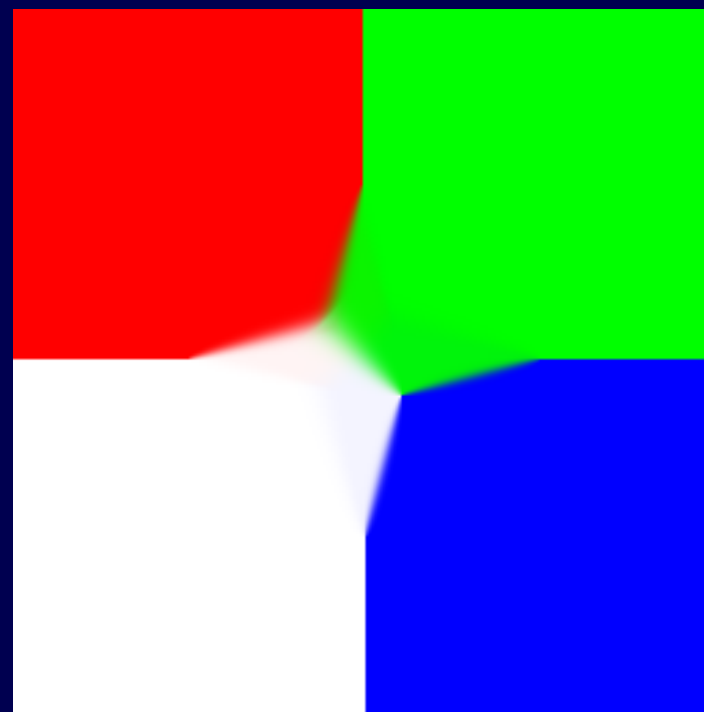
Proposition: The proposed relaxation strictly dominates alternative relaxations.

Chambolle, Cremers, Pock '08, SIIMS '12

Four-Region Case



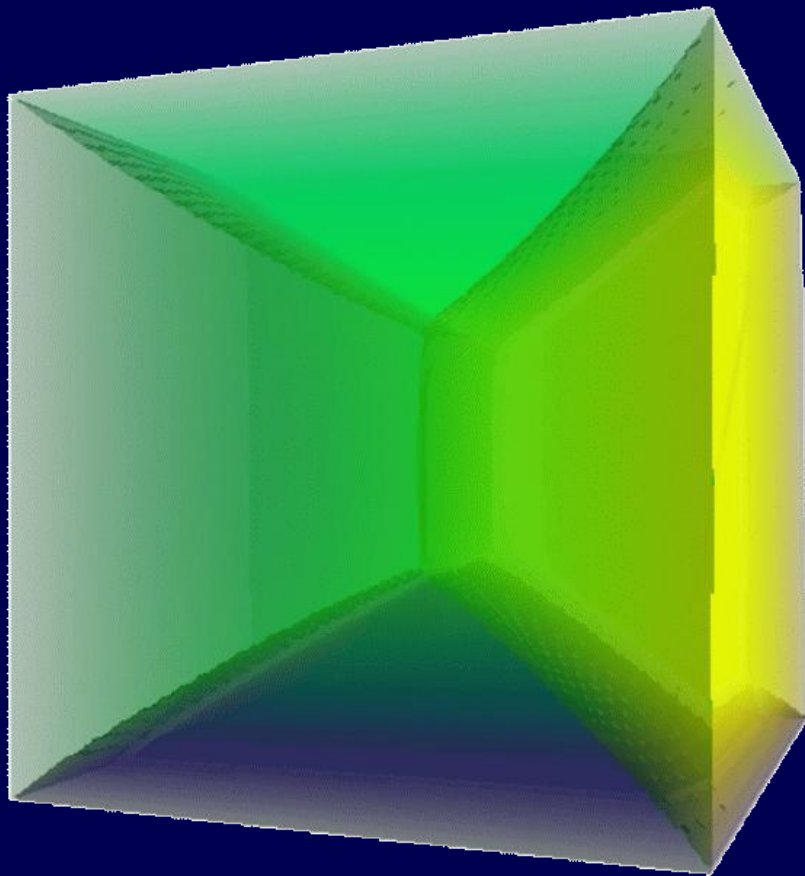
Input image



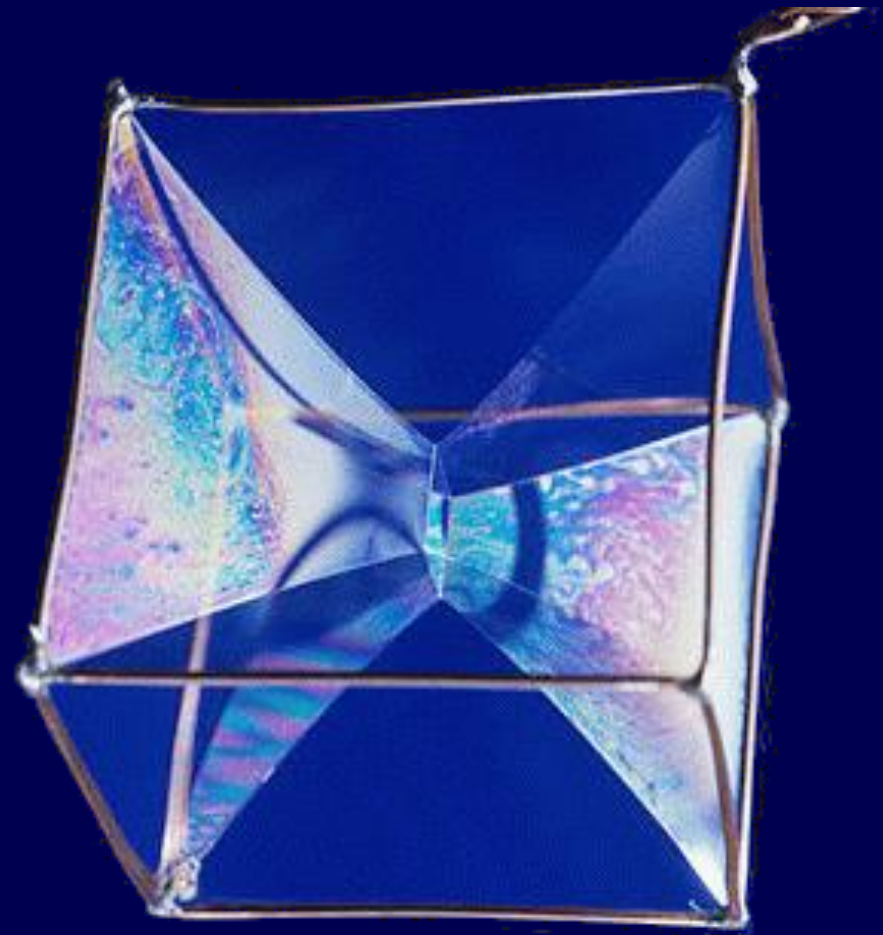
Inpainted

Chambolle, Cremers, Pock '08, SIIMS '12

Minimal Surfaces in 3D



3D min partition inpainting



Photograph of a soap film

Chambolle, Cremers, Pock '08, SIIMS '12

The Minimal Partition Problem



Input color image



10 label segmentation

Chambolle, Cremers, Pock '08, SIIMS '12

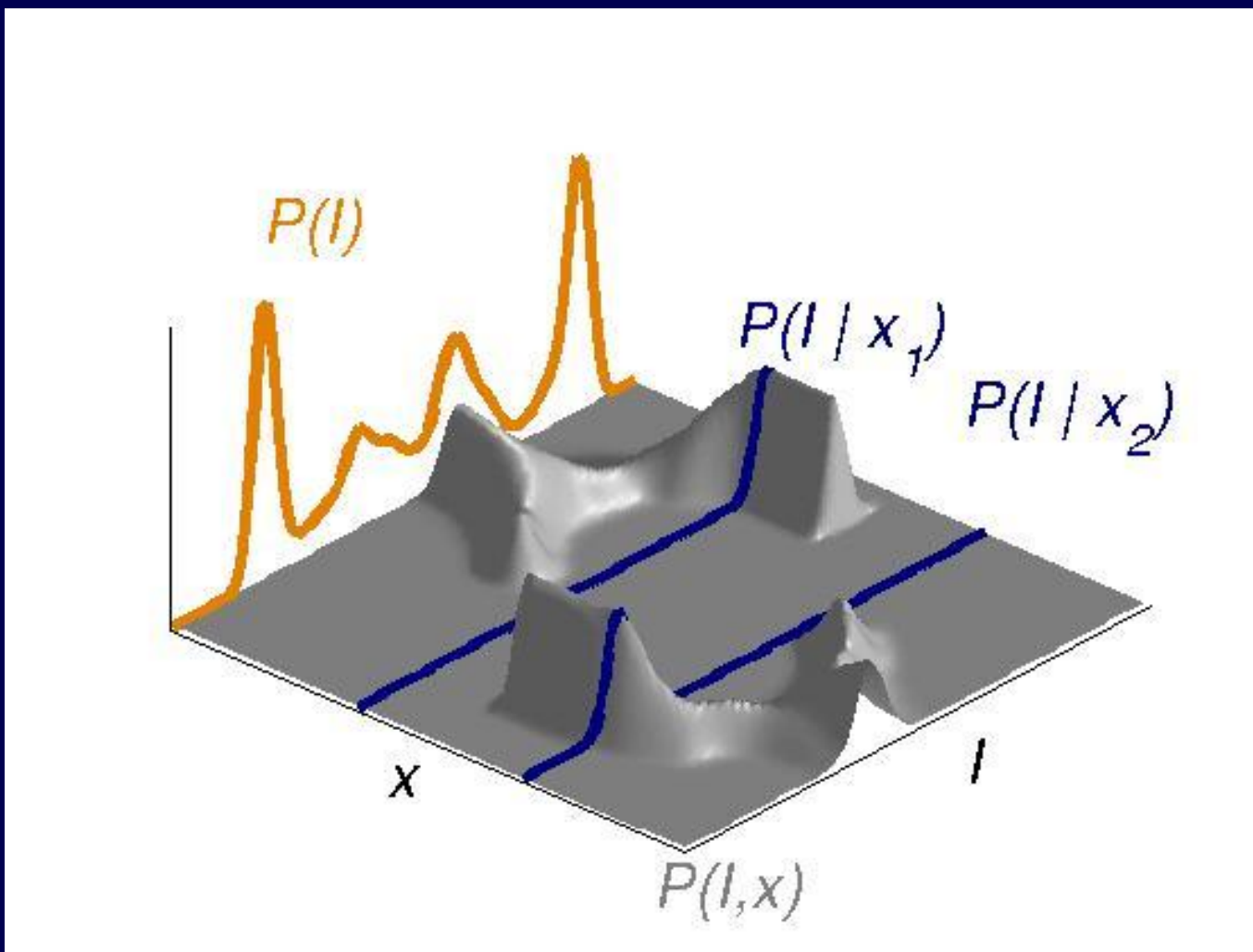


Interactive Segmentation



Nieuwenhuis, Cremers, PAMI '12

Space-dependent Color Likelihoods



Nieuwenhuis, Cremers, PAMI '12



Interactive Segmentation



Nieuwenhuis, Cremers, PAMI '12



Interactive Segmentation



Nieuwenhuis, Cremers, PAMI '12



Overview



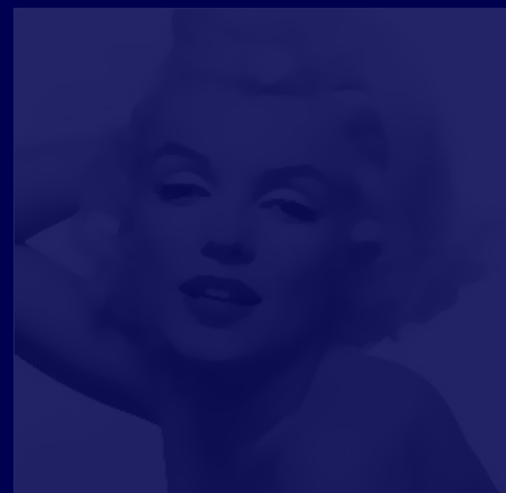
Convex multilabel optimization



Minimal partitions



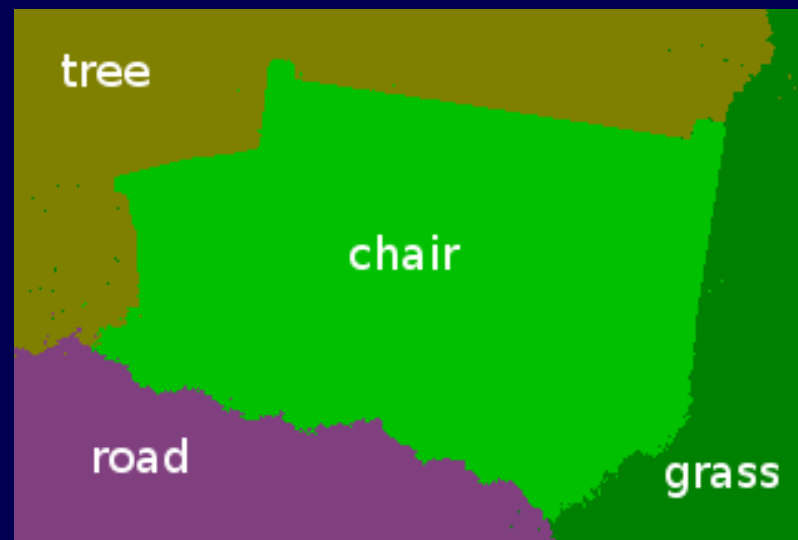
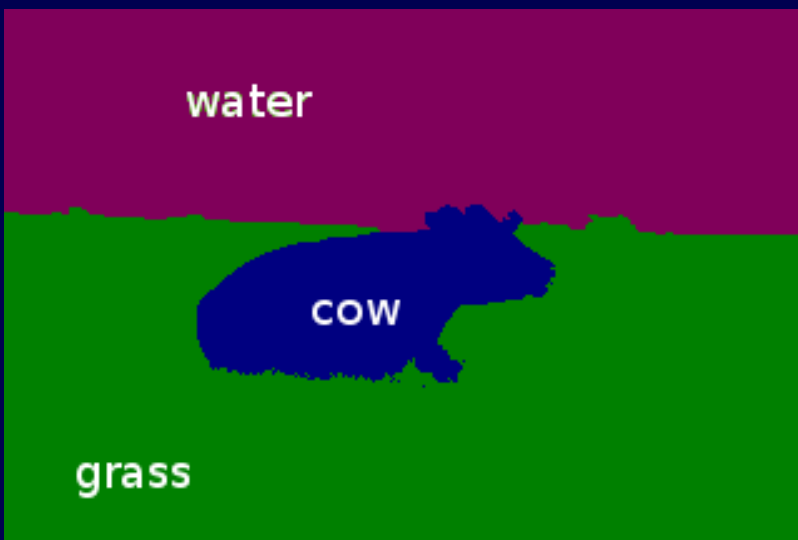
Semantic segmentation



Mumford-Shah



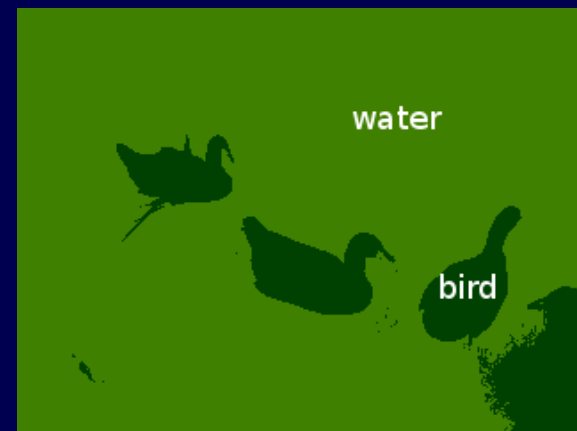
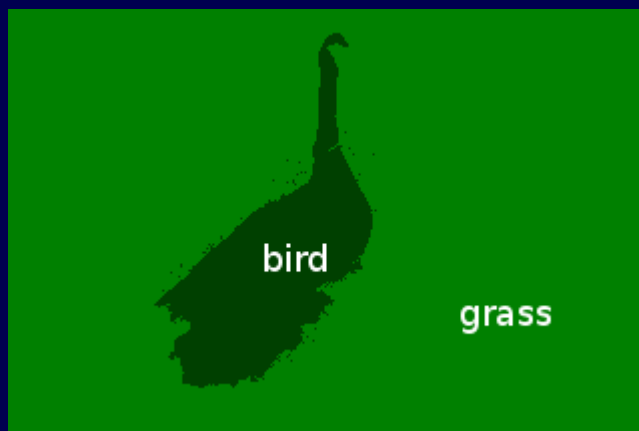
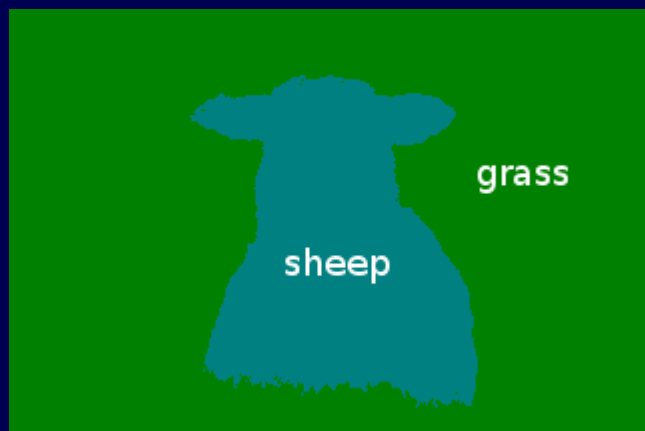
Semantic Image Segmentation



Souiai et al. '13



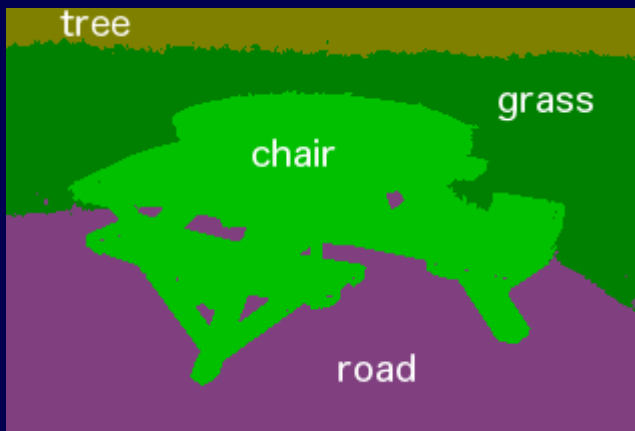
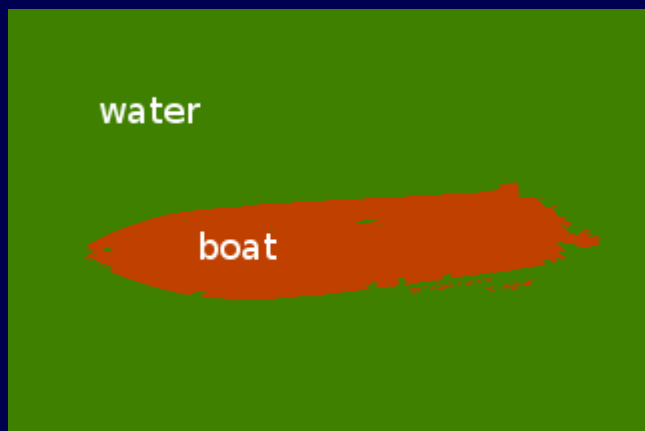
Semantic Image Segmentation



Souiai et al. '13



Semantic Image Segmentation



Souiai et al. '13

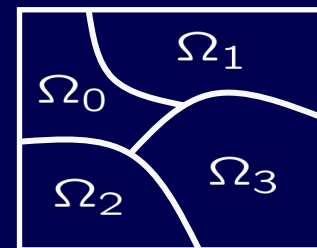
General Ordering Constraints



Strekalovskiy, Cremers, ICCV 2011
Related discrete approach: *Liu et al. PAMI '10*

General Ordering Constraints

Reminder: With $v_i = 1_{\Omega_i}$, the minimal partition problem is:



$$\min_{v \in \mathcal{B}} \frac{1}{2} \sum_i \int_{\Omega} |Dv_i| + \int_{\Omega} v_i f_i dx = \min_{v \in \mathcal{B}} \sup_{p \in \mathcal{K}} \sum_i \int_{\Omega} v_i \operatorname{div} p_i dx + \int_{\Omega} v_i f_i dx$$

where $\mathcal{K} = \{p = (p_1, \dots, p_n)^{\top} \in \mathbb{R}^{n \times m} : |p_i - p_j| \leq 1, \forall i < j\}$

Consider instead the more general convex set:

$$\mathcal{K}_d = \{p \in \mathbb{R}^{n \times m} : \langle p_i - p_j, \nu \rangle \leq d(i, j, \nu), \forall i < j, \nu \in \mathbb{S}^{m-1}\}$$

Penalize transitions depending on label values i, j and orientation ν .

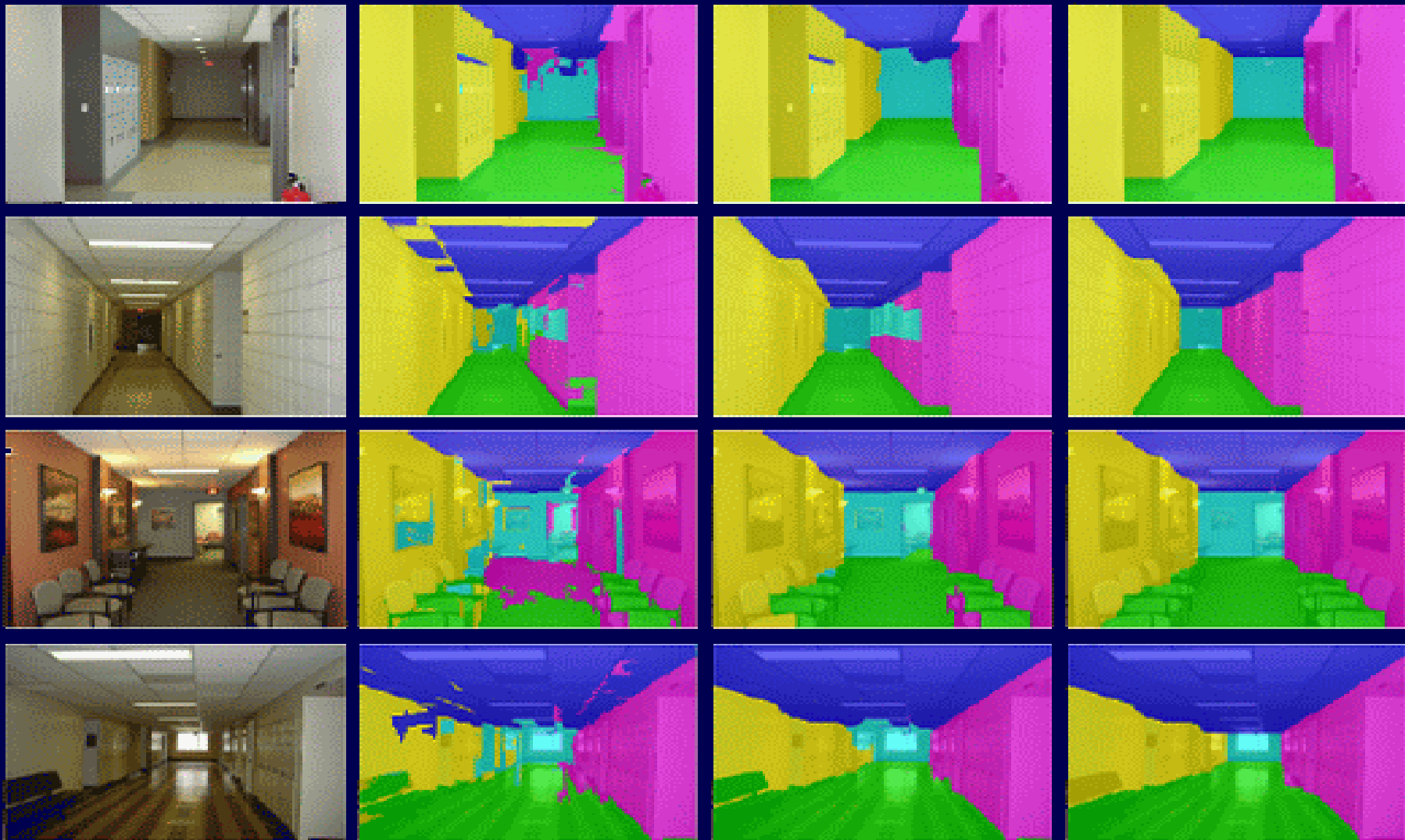
Stekalovskiy, Cremers, ICCV 2011

General Ordering Constraints



Stekalovskiy, Cremers, ICCV 2011

General Ordering Constraints



Input

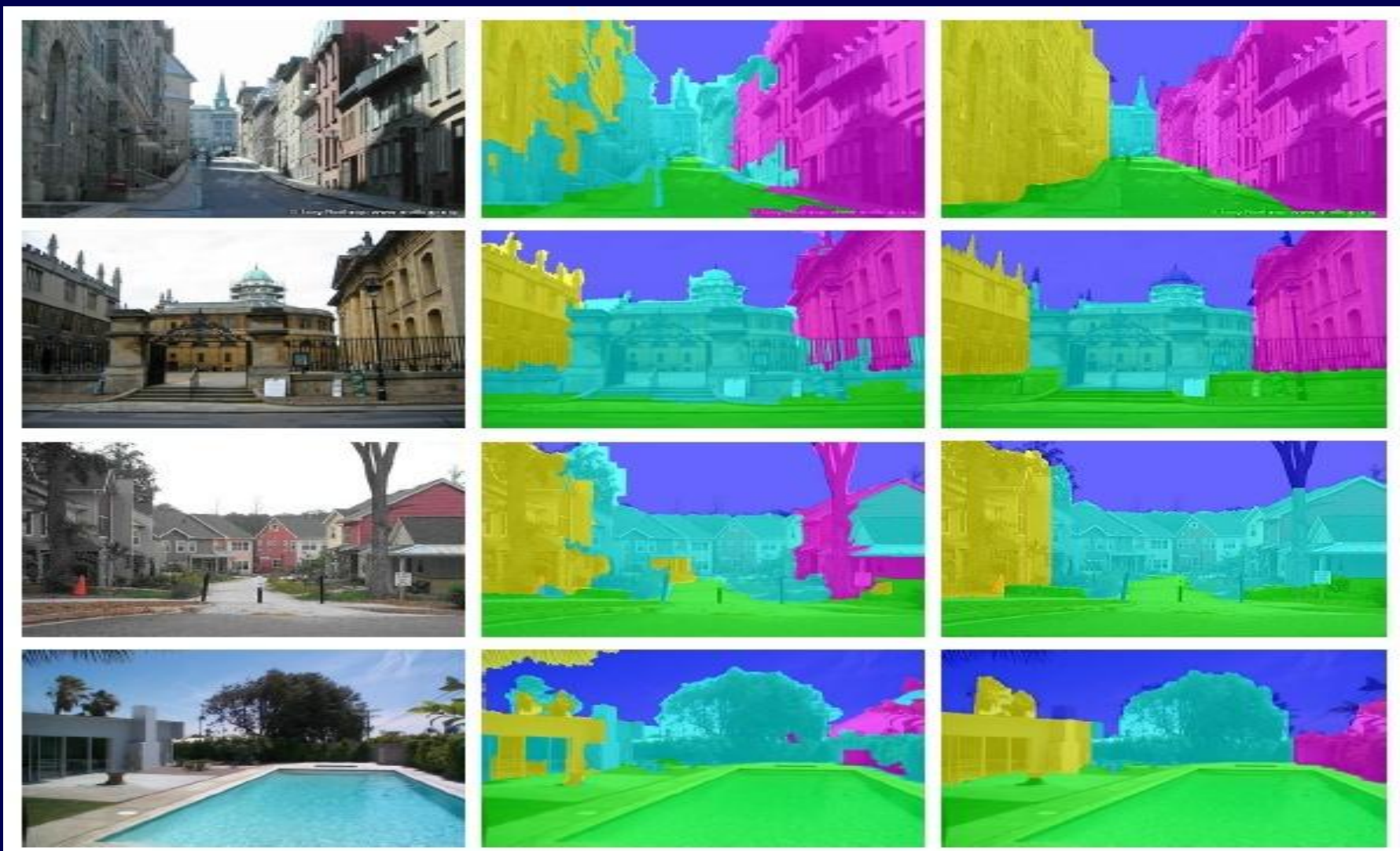
Data term

Min. partition

Ordering

Stekalovskiy, Cremers, ICCV 2011

General Ordering Constraints



Input

Min. partition

Ordering

Strekalovskiy, Cremers, ICCV 2011



Overview



Convex multilabel optimization



Minimal partitions



Semantic segmentation



Mumford-Shah



Piecewise Smooth: Scalar Case



$$E(u) = \lambda \int_{\Omega} (f-u)^2 dx + \int_{\Omega \setminus S_u} |\nabla u|^2 dx + \nu \mathcal{H}^1(S_u) \quad (*)$$

Mumford, Shah '89

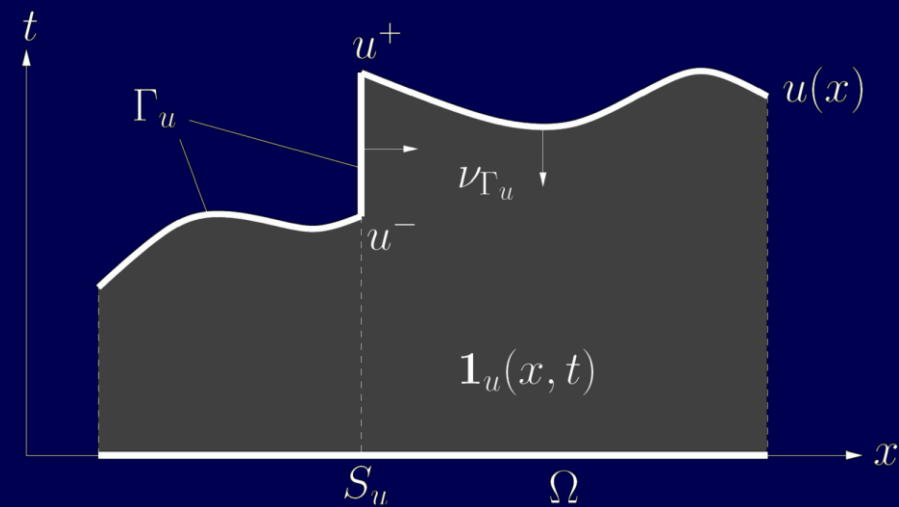
For $u \in SBV(\Omega, \mathbb{R})$, $\Omega \subset \mathbb{R}^n$, (*) can be written as

$$E(u) = \sup_{\varphi \in K} \int_{\Omega \times \mathbb{R}} \varphi D\mathbf{1}_u,$$

with a convex set

$$K = \left\{ \varphi \in C_0(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}) : \right.$$

$$\left. \varphi^t(x, t) \geq \frac{\varphi^x(x, t)^2}{4} - \lambda(t - f(x))^2, \left| \int_{t_1}^{t_2} \varphi^x(x, s) ds \right| \leq \nu \right\},$$



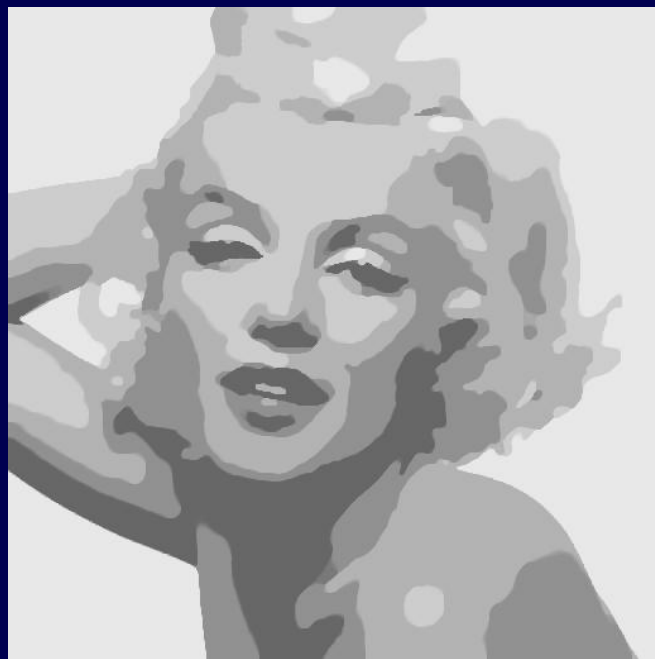
Alberti, Bouchitte, Dal Maso '04



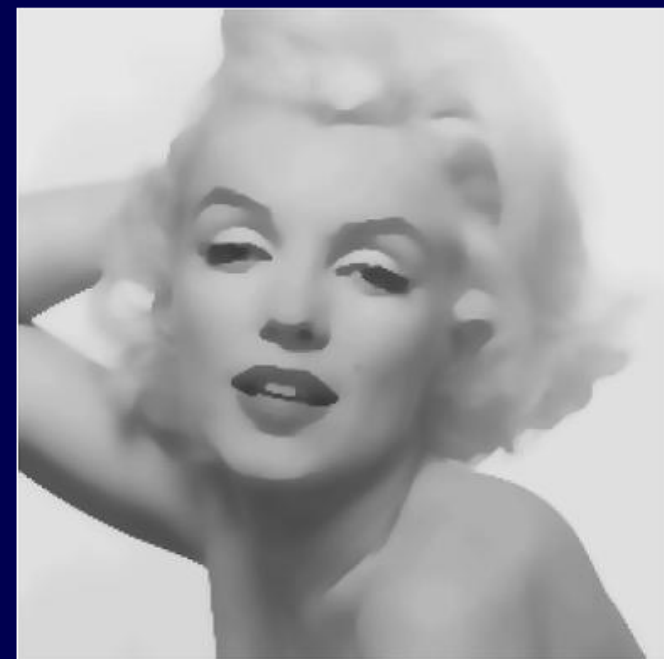
Piecewise Smooth: Scalar Case



Input image



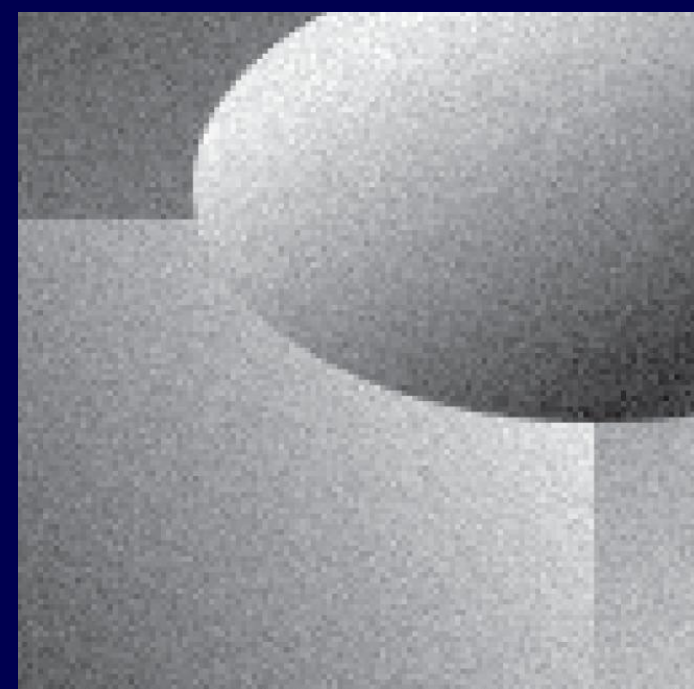
piecewise constant



piecewise smooth

Pock, Cremers, Bischof, Chambolle ICCV '09

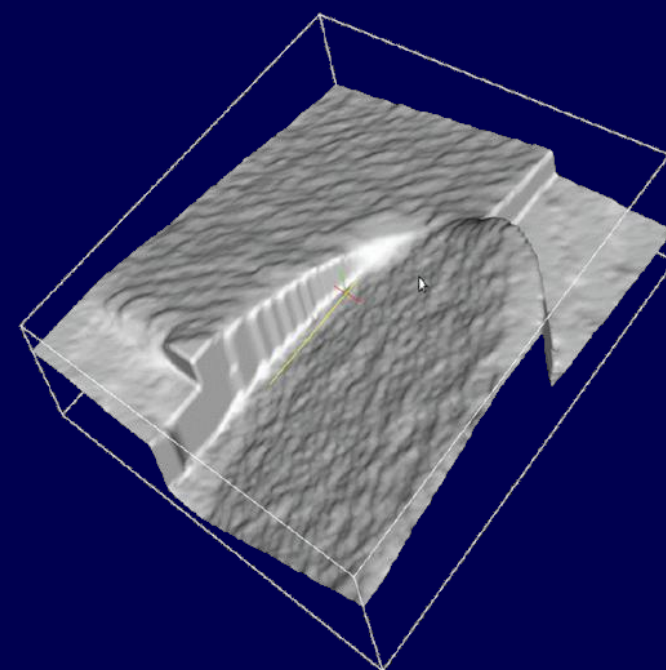
Piecewise Smooth: Scalar Case



noisy input



restoration



surface plot

Pock, Cremers, Bischof, Chambolle ICCV '09

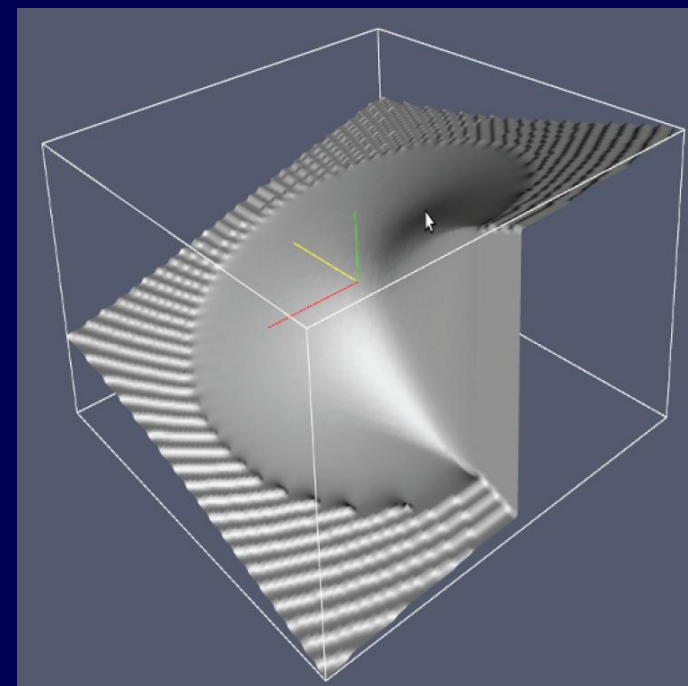
The Crack Tip & Open Boundaries



fixed boundary values



inpainted crack tip



surface plot

Pock, Cremers, Bischof, Chambolle ICCV '09

The Vectorial Mumford-Shah Problem

For $u \in SBV(\Omega, \mathbb{R}^k)$, we consider the functional

$$E(u) = \int_{\Omega} |f - u|^2 dx + \lambda \int_{\Omega \setminus S_u} \sum_{i=1}^k |\nabla u_i|^2 dx + \nu \mathcal{H}^1(S_u).$$

Proposition: For $v = \mathbf{1}_u = (1_{u_1}, \dots, 1_{u_k})$, we have:

$$E(u) = \mathcal{F}(v) := \sup_{\sigma \in \mathcal{K}} \sum_{i=1}^k \int_{\Omega \times \mathbb{R}} \sigma_i(x, t) \cdot Dv_i(x, t)$$

with the convex set:

$$\mathcal{K} = \left\{ \sigma \mid (\sigma_i^x, \sigma_i^t) \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}), \right.$$

$\mathcal{O}(n_1^2 \cdots n_k^2)$ constraints!

$$\left. \begin{aligned} &\sigma_i^t(x, t_i) \geq \frac{1}{4\lambda} |\sigma_i^x(x, t_i)|^2 - (t_i - f_i(x))^2, \\ &\sum_{j=1}^k \left| \int_{t_j}^{t'_j} \sigma_j^x(x, s) ds \right| \leq \nu, \quad \forall 1 \leq i \leq k, x \in \Omega, t_j < t'_j \end{aligned} \right\}.$$

Stekalovskiy, Chambolle, Cremers, CVPR '12

An Efficient Reformulation

Proposition: The constraint set

$\mathcal{O}(n_1^2 \cdots n_k^2)$ constraints

$$\mathcal{K} = \left\{ \sigma \mid \begin{aligned} &(\sigma_i^x, \sigma_i^t) \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}), \\ &\sigma_i^t(x, t_i) \geq \frac{1}{4\lambda} |\sigma_i^x(x, t_i)|^2 - (t_i - f_i(x))^2, \\ &\sum_{j=1}^k \left| \int_{t_j}^{t'_j} \sigma_j^x(x, s) ds \right| \leq \nu, \quad \forall 1 \leq i \leq k, x \in \Omega, t_j < t'_j \end{aligned} \right\}.$$

is equivalent to the constraint set

$\mathcal{O}(n_1^2 + \dots + n_k^2)$ constraints

$$\mathcal{K}' := \left\{ (\sigma, \mathbf{m}) \mid \begin{aligned} &(\sigma_i^x, \sigma_i^t) \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}), \\ &\sigma_i^t(x, t_i) \geq \frac{1}{4\alpha} |\sigma_i^x(x, t_i)|^2 - (t_i - f_i(x))^2, \\ &\left| \int_{t_i}^{t'_i} \sigma_i^x(x, s) ds \right| \leq m_i(x), \quad \sum_{j=1}^k m_j(x) \leq \nu \quad \forall i, x \in \Omega, t_i < t'_i \end{aligned} \right\}.$$



Same complexity as channel-wise processing.

Stekalovskiy, Chambolle, Cremers, CVPR '12

Generalizations and Variants

For $u \in SBV(\Omega, \mathbb{R}^k)$, φ_i convex and superlinear, we consider

$$E(u) = \sum_{i=1}^k \int_{\Omega} \rho_i(x, u_i(x)) dx + \int_{\Omega \setminus S_u} \varphi_i(\nabla u_i) dx + \nu \mathcal{H}^1(S_u).$$

Proposition: For $v = \mathbf{1}_u = (1_{u_1}, \dots, 1_{u_k})$, we have:

$$E(u) = \mathcal{F}(v) := \sup_{(\sigma, m) \in \mathcal{K}} \sum_{i=1}^k \int_{\Omega \times \mathbb{R}} \sigma_i(x, t) \cdot Dv_i(x, t)$$

with the convex set:

$$\mathcal{K} := \left\{ (\sigma, m) \mid (\sigma_i^x, \sigma_i^t) \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}), \right.$$

$$\sigma_i^t(x, t) \geq \varphi_i^*(\sigma_i^x(x, t)) - \rho_i(x, t),$$

$$\left. \left| \int_{t_i}^{t'_i} \sigma_i^x(x, s) ds \right| \leq m_i(x), \sum_{j=1}^k m_j(x) \leq \nu \quad \forall i, x \in \Omega, t_i < t'_i \right\}.$$

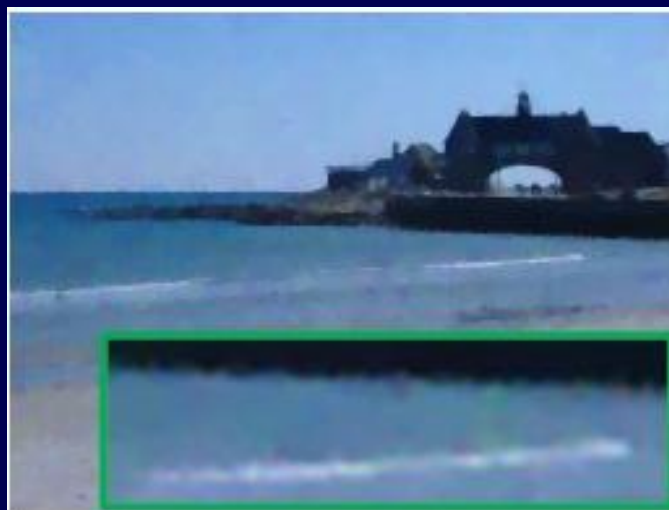
Strekalovskiy, Chambolle, Cremers, CVPR '12



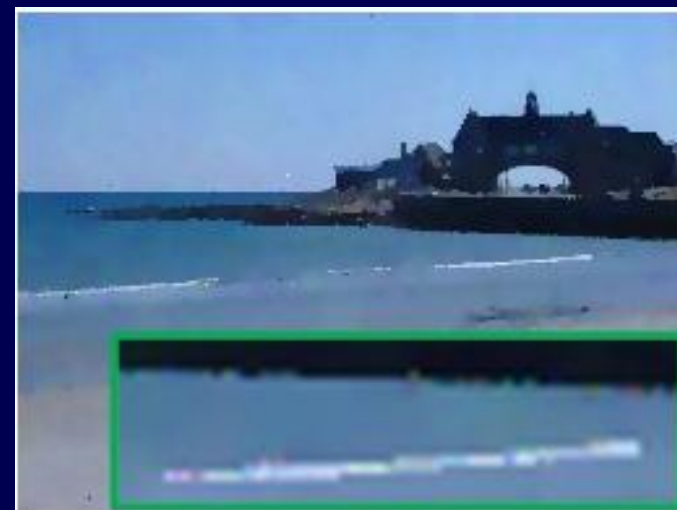
The Vectorial Mumford-Shah Problem



Input image



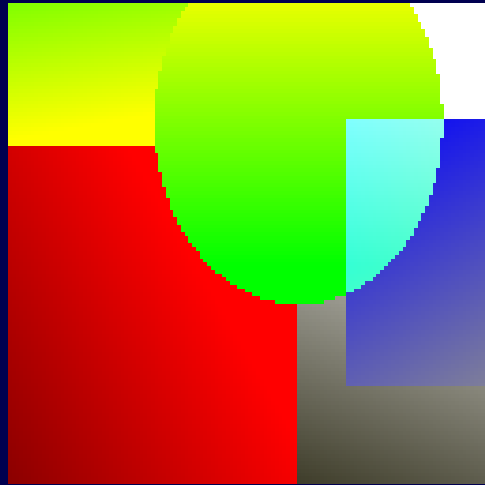
TV denoised



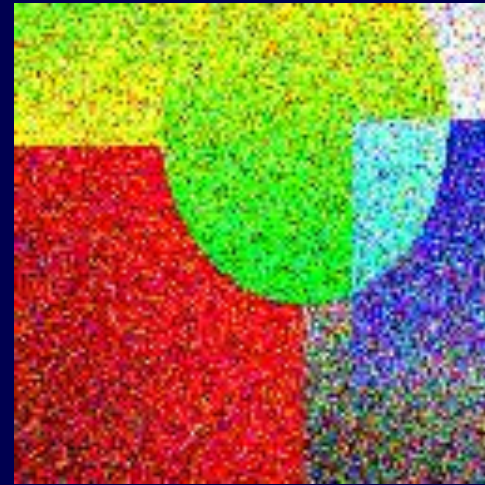
Vectorial Mumford-Shah

Strekalovskiy, Chambolle, Cremers, CVPR '12

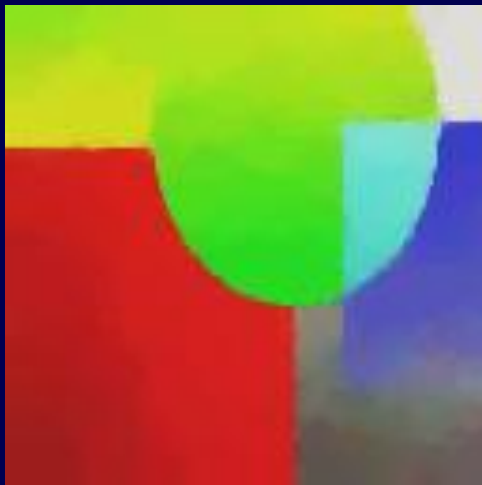
Experimental Comparison



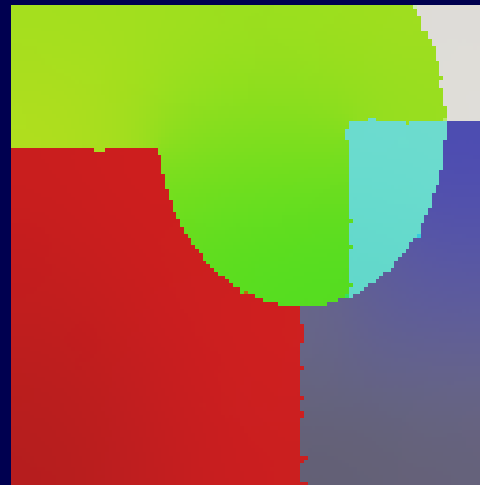
Input image



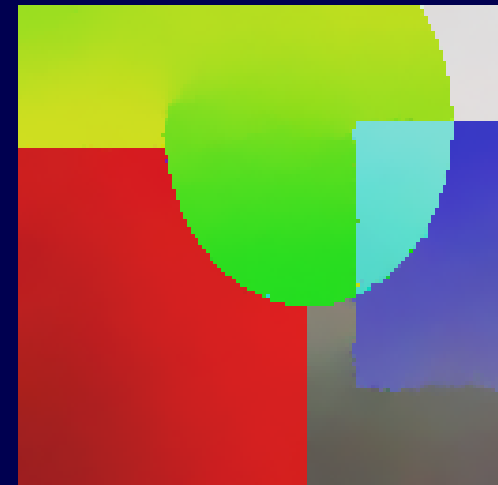
with 30% noise



TV denoised



Ambrosio-Tortorelli

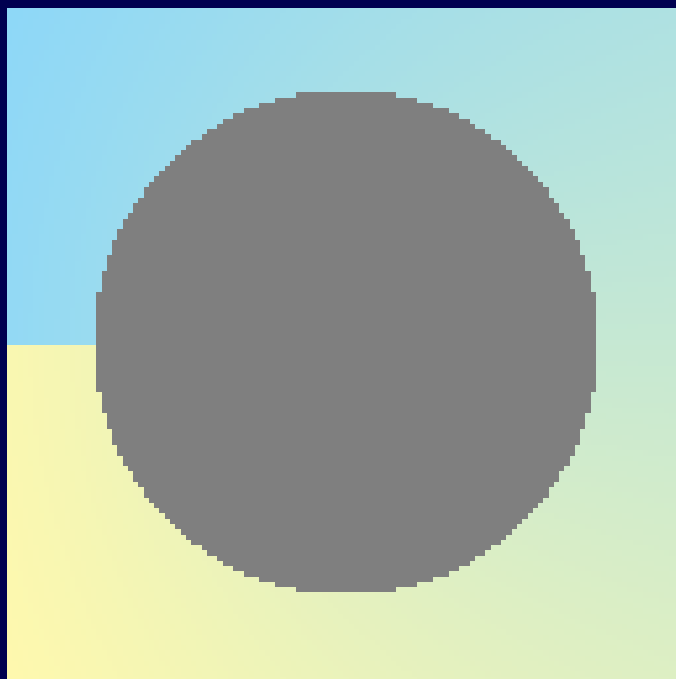


Vectorial MS

Strekalovskiy, Chambolle, Cremers, CVPR '12



The Color Crack Tip



Input image



Ambrosio-Tortorelli



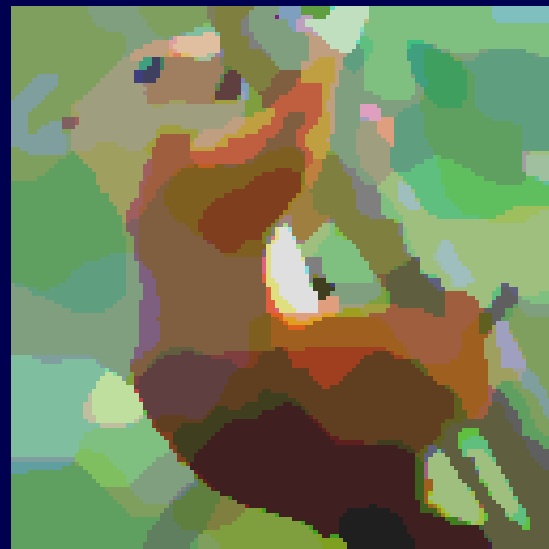
Vectorial MS

Stekalovskiy, Chambolle, Cremers, CVPR '12

Channelwise versus Vectorial



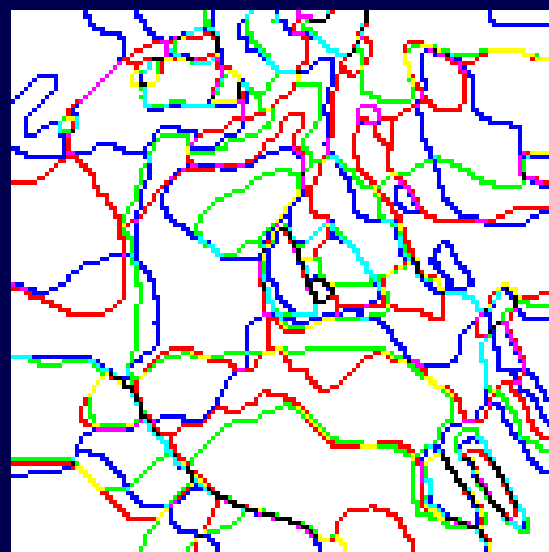
Input image



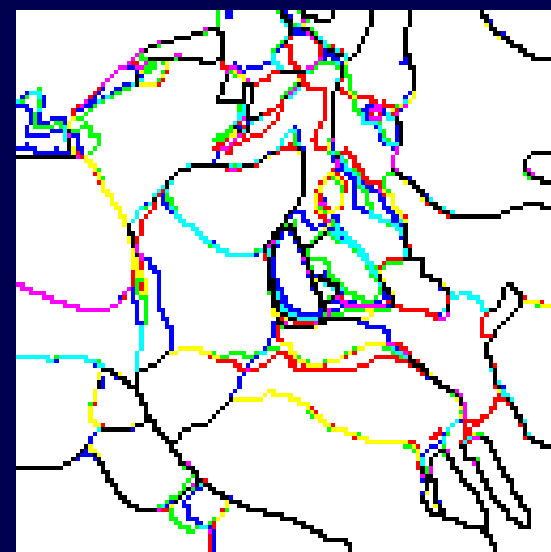
Channelwise MS



Vectorial MS



Jump set S_u



Jump set S_u



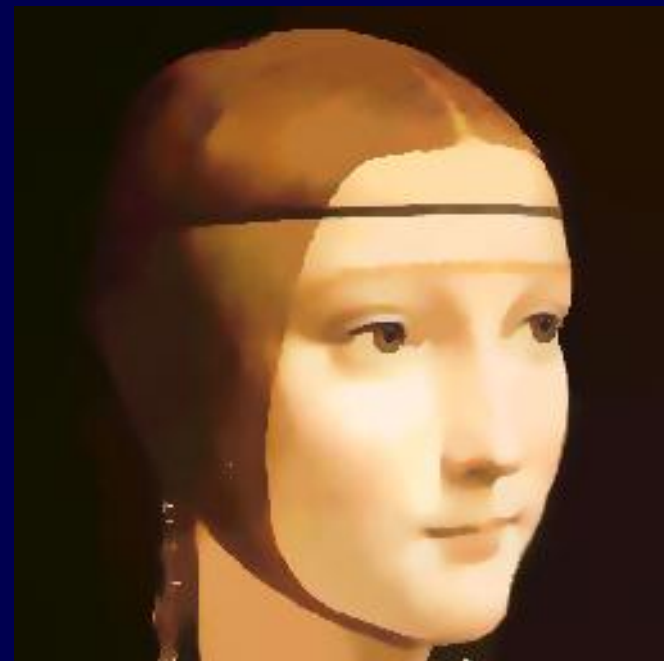
Channelwise versus Vectorial



Input image



Channelwise MS



Vectorial MS

Stekalovskiy, Chambolle, Cremers, CVPR '12



Piecewise Constant Color Segmentation



Input image



$\lambda \approx \infty, \nu = 0.05$



$\lambda \approx \infty, \nu = 0.1$



$\lambda \approx \infty, \nu = 0.2$

Strekalovskiy, Chambolle, Cremers, CVPR '12