Filtering a statistically exactly solvable test model for turbulent tracers from partial observations

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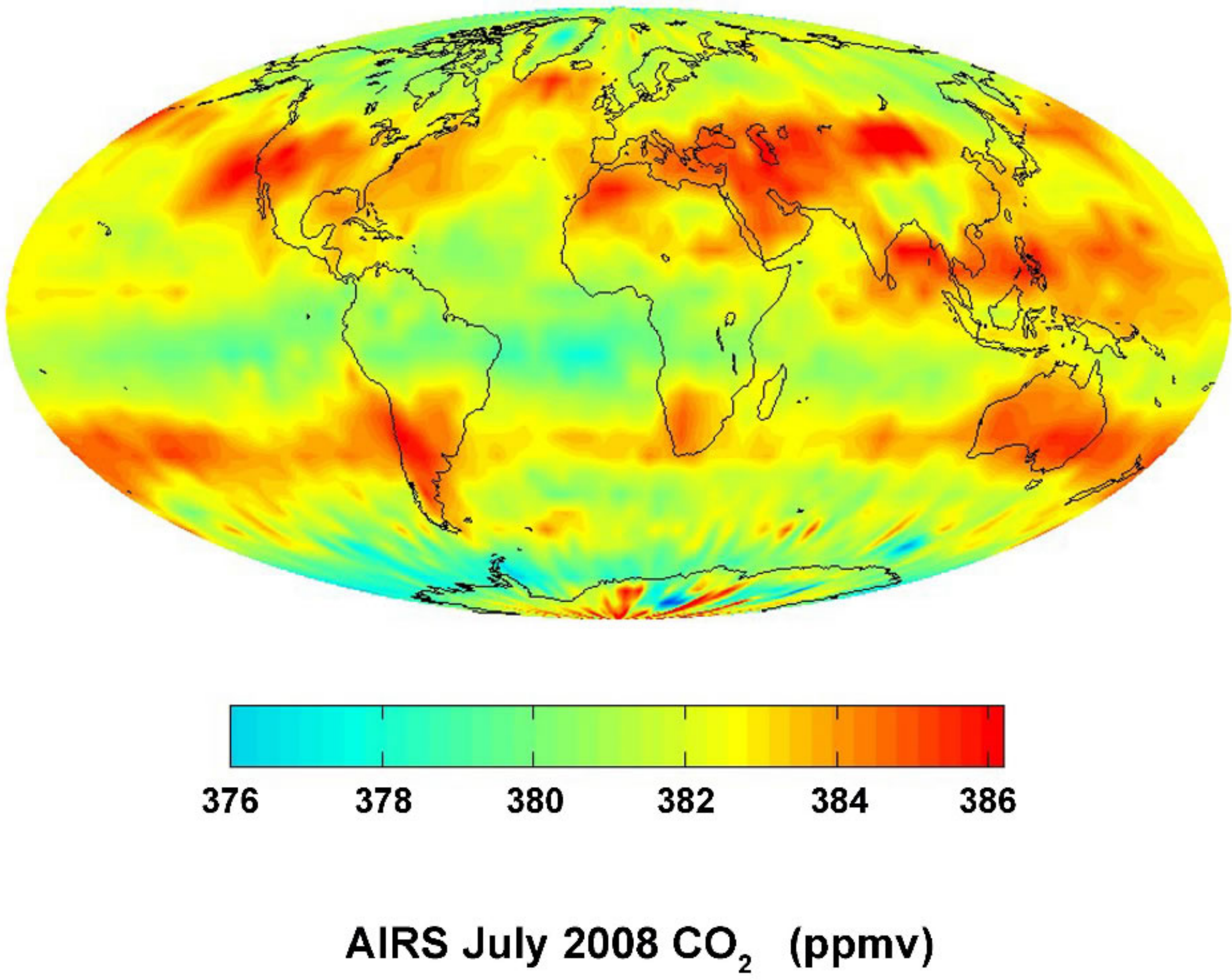
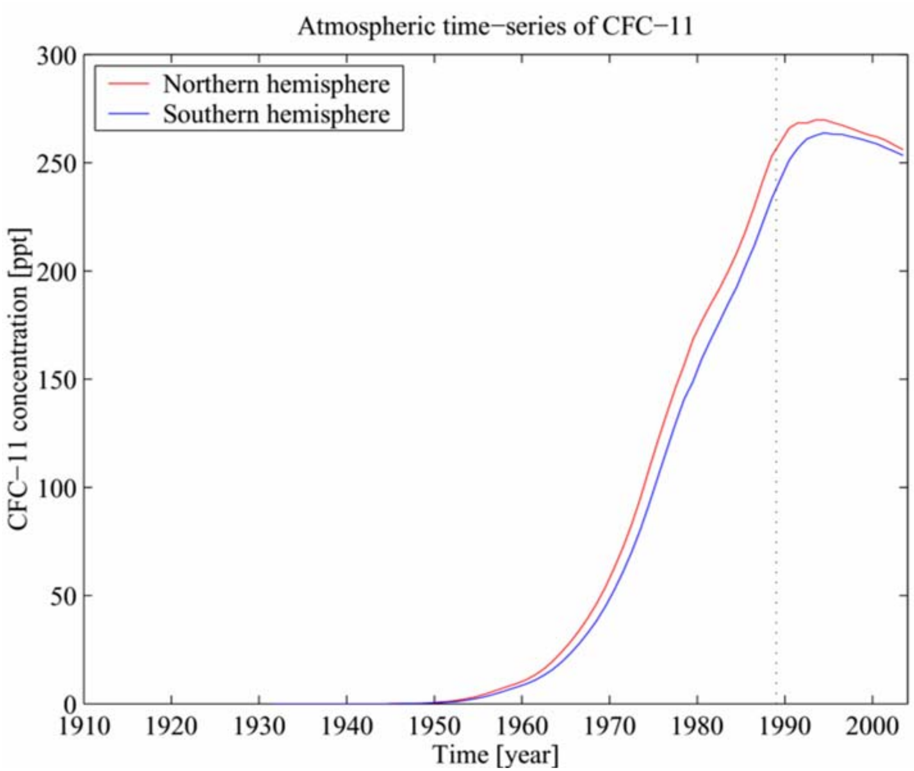
Tracers in the atmosphere

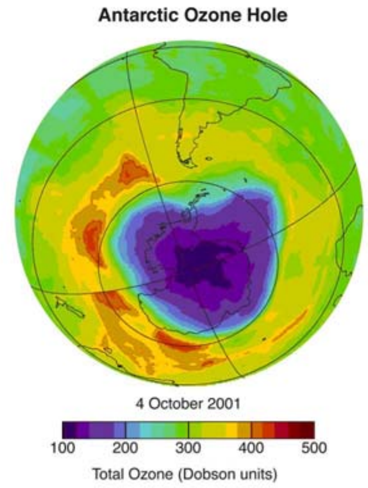


$$CO_2$$

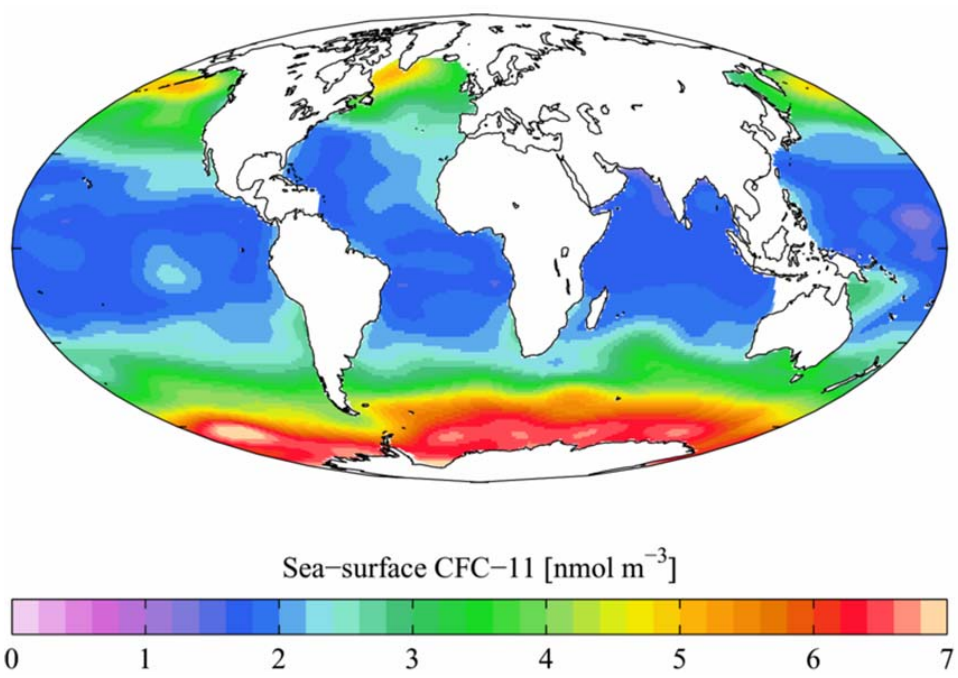
Tracers in the atmosphere and ocean

CFC-11



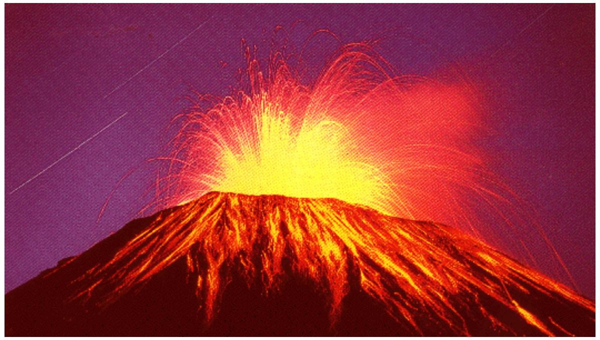


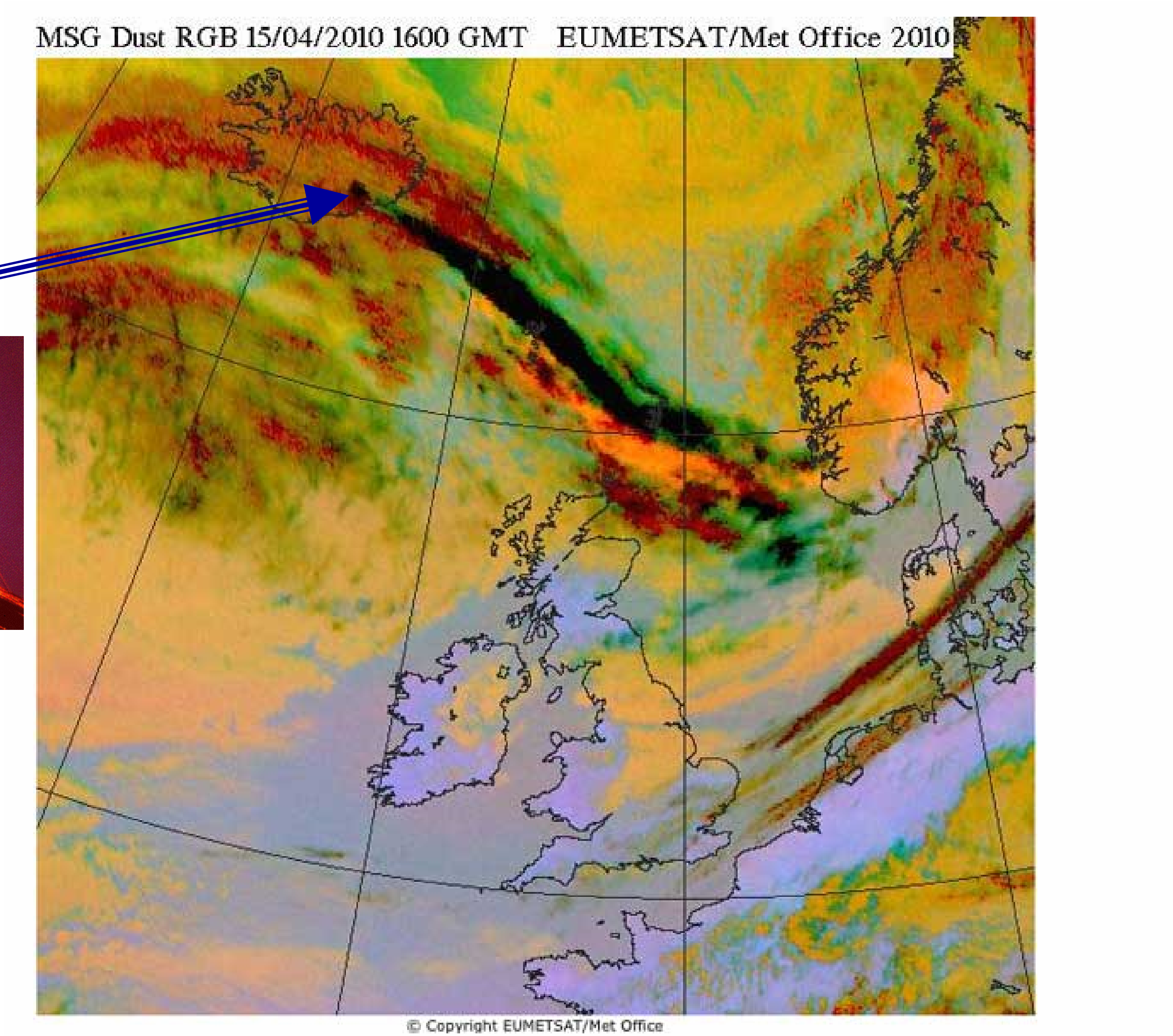
O_3



Tracers in the atmosphere: filtering

Source



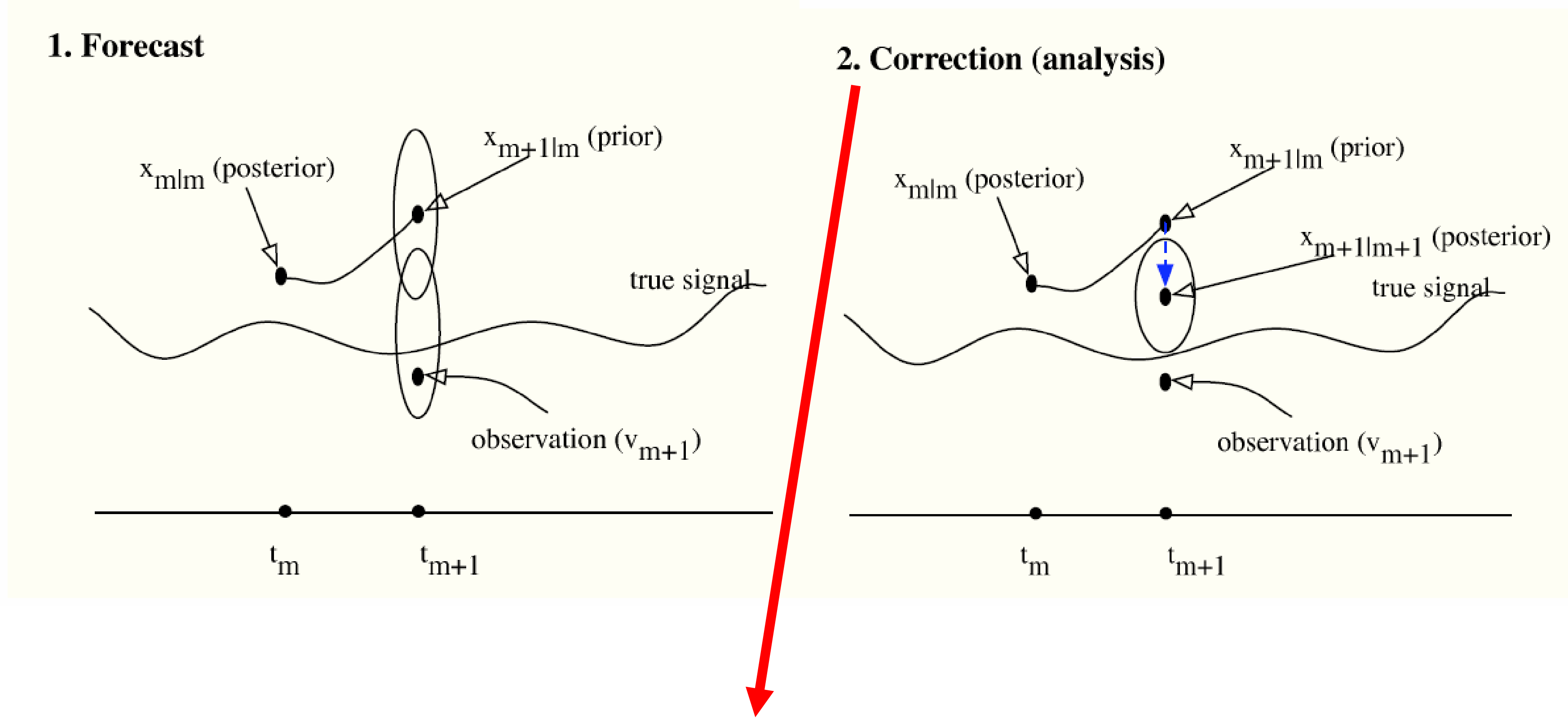


Tracers in the ocean: filtering

Why is filtering of turbulent tracers important?

- Knowing short and long term dynamics and statistics of a certain tracer can be extremely important
- By studying the tracers one can trace back the physics of the flow
- Applications: climate science, weather prediction, pollution of the air or water due to natural or anthropogenic disasters
- Optimization of measurements: get maximum information from minimum measurements
- Model error: study how well approximate models perform

Kalman filter



$$x_{m+1|m+1} = x_{m+1|m} + K_{m+1}(v_{m+1} - G x_{m+1|m})$$
$$r_{m+1|m+1} = (\mathcal{I} - K_{m+1} G) r_{m+1|m}$$
$$K_{m+1} = r_{m+1|m} G^* (G r_{m+1|m} G^* + r^o)^{-1}$$

Nonlinear Extended Kalman filter

Dynamics is NONLINEAR but
the exact mean and covariance
are used for the prior estimate
(no linearization applied)

Use the linear Kalman filter
formalism for constructing posterior

Advection-diffusion equation

diffusion source

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{x}} T = \kappa \Delta_{\vec{x}} T - \gamma T + S(\vec{x}, t)$$

advection damping

Incompressible flow: $\nabla_{\vec{x}} \vec{v} = 0$

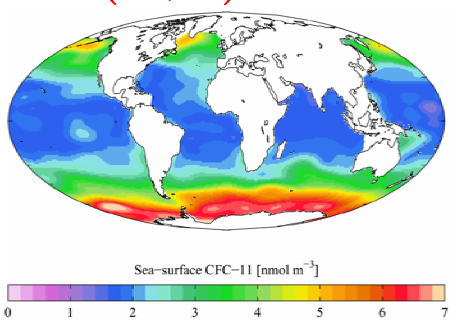
Tracer with the mean gradient

$$T(x,y,t) = \alpha y + T'(x,t)$$

given mean gradient
in the north-south direction

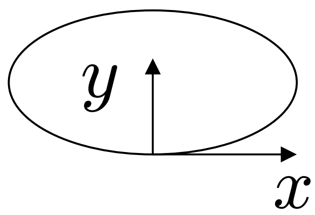
$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{x}} T = \kappa \Delta_{\vec{x}} T - \gamma T + S(\vec{x},t)$$

$$S(\vec{x},t) = 0$$



$$\vec{v}(x,t) = (U(t), v(x,t))^T$$

incompressible Gaussian
velocity field



$$\frac{\partial T'(x,t)}{\partial t} + U(t)\frac{\partial T'(x,t)}{\partial x} = -\alpha v(x,t) + \kappa \frac{\partial^2 T'(x,t)}{\partial x^2} - \gamma T'(x,t)$$

"T"$=T'$

in Fourier space:

$$\frac{\partial T_k(t)}{\partial t} + ikU(t)T_k(t) = -\kappa k^2 T_k(t) - \gamma T_k(t) - \alpha v_k(t)$$

Tracer path-wise solution

$$\frac{\partial T_k(t)}{\partial t} + ikU(t)T_k(t) = -(\kappa k^2 + \gamma)T_k(t) - \alpha v_k(t)$$

$$\begin{aligned} T_k(t) &= D_k(t_0, t)T_k(t_0)e^{-ikJ_U(t_0,t)} \\ &\quad -\alpha \int_{t_0}^{t} D_k(s,t)v_k(s)e^{-ikJ_U(s,t)}ds \end{aligned}$$

Gaussian

$$\begin{aligned} D_k(s,t) &= e^{-(\gamma+\kappa k^2)(t-s)} \\ J_U(s,t) &= \int_s^t U(s')ds' \end{aligned}$$

Tracer statistics

Tracer is not Gaussian!

$$\langle T_k(t)\rangle = D_k(t_0,t)\underline{\langle T_k(t_0)e^{-ikJ_U(t_0,t)}}\rangle - \alpha\int_{t_0}^{t} D_k(s,t)\langle v_k(s)e^{-ikJ_U(s,t)}\rangle ds$$

For Gaussian z and x:

$$\langle ze^{ix}\rangle = \Big(\langle z\rangle + iCov(z,x)\Big)e^{i\langle x\rangle - \frac{1}{2}Var(x)}$$

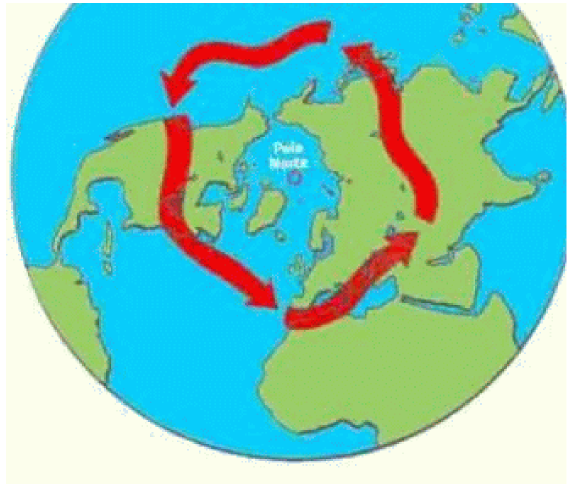
$$\underline{\langle T_k(t_0)e^{-ikJ_U(t_0,t)}\rangle} = \Big(\langle T_k(t_0)\rangle - ikCov(T_k(t_0), J_U(t_0,t))\Big) \times e^{-ik\langle J_U(t_0,t)\rangle - \frac{k^2}{2}Var(J(t_0,t))}$$

First and second order statistics for the tracer

Statistically exactly solvable

Gershgorin, Majda ('08,'10), Bourlioux, Majda ('02)

Velocity field



$$\vec{v}(x,t) = (U(t), v(x,t))^T$$

$U(t)$: cross-sweep (east-west zonal jet)

$$U(t) = Re[V(t)]$$

$$\frac{dV(t)}{dt} = \lambda_U V(t) + f_U(t) + \sigma_U \dot{W}_U(t)$$

$$\lambda_U = -\gamma_U + i\omega_U \qquad f_U(t) = A_f e^{i(\eta t + \phi_f)} + B_f$$

$v(x,t)$: transverse waves (north-south Rossby waves)

$$\frac{dv(x,t)}{dt} = P\left(\frac{\partial}{\partial x}\right) v(x,t) + f_v(x,t) + \sigma_v(x)\dot{W}_v$$

Waves in Fourier space

$$\frac{dv_k(t)}{dt} = \lambda_k v_k(t) + f_k(t) + \sigma_k \dot{W}_k(t)$$
$$\lambda_k = -\gamma_k + i\omega_k$$

Choice of parameters:

nondispersive waves: $\omega_k = -ck, \quad \gamma_k = d_v + \mu k^2$

Rossby waves: $\omega_k = \dfrac{\beta k}{k^2 + F_s}, \quad \gamma_k = \nu(k^2 + F_s)$

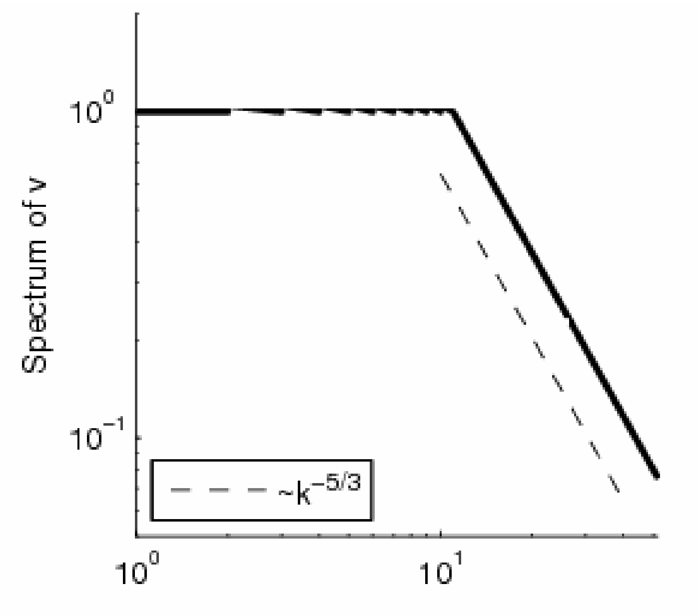
Equilibrium spectrum:

$$E_k \equiv Var_{eq}(v_k) = \frac{\sigma_k^2}{2\gamma_k}, \quad \sigma_k = \sqrt{2\gamma_k E_k}$$

external forcing: $f_k(t) = A_k e^{i(\xi_k t + \phi_k)}$

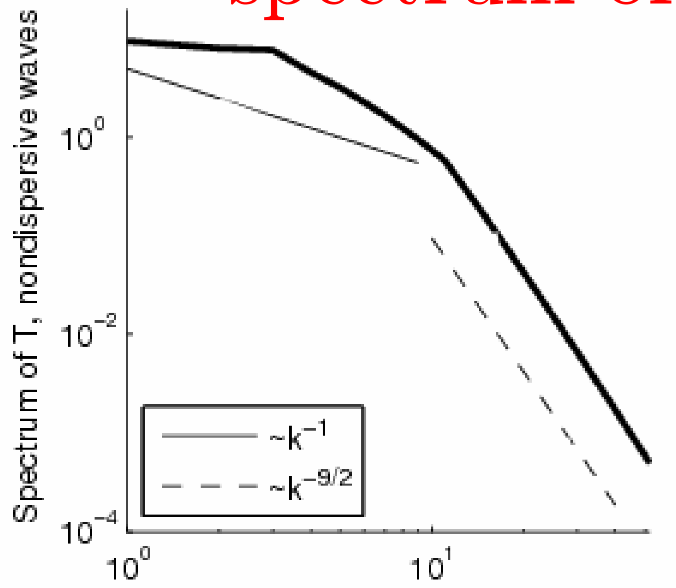
Spectra

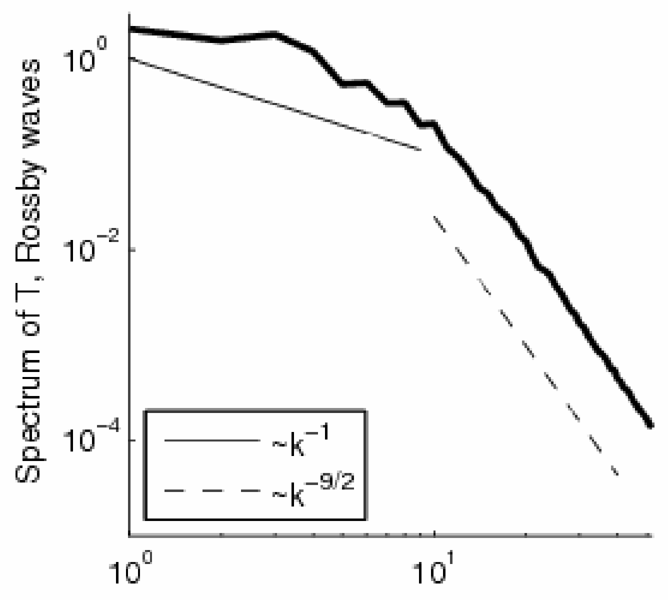
spectrum of v



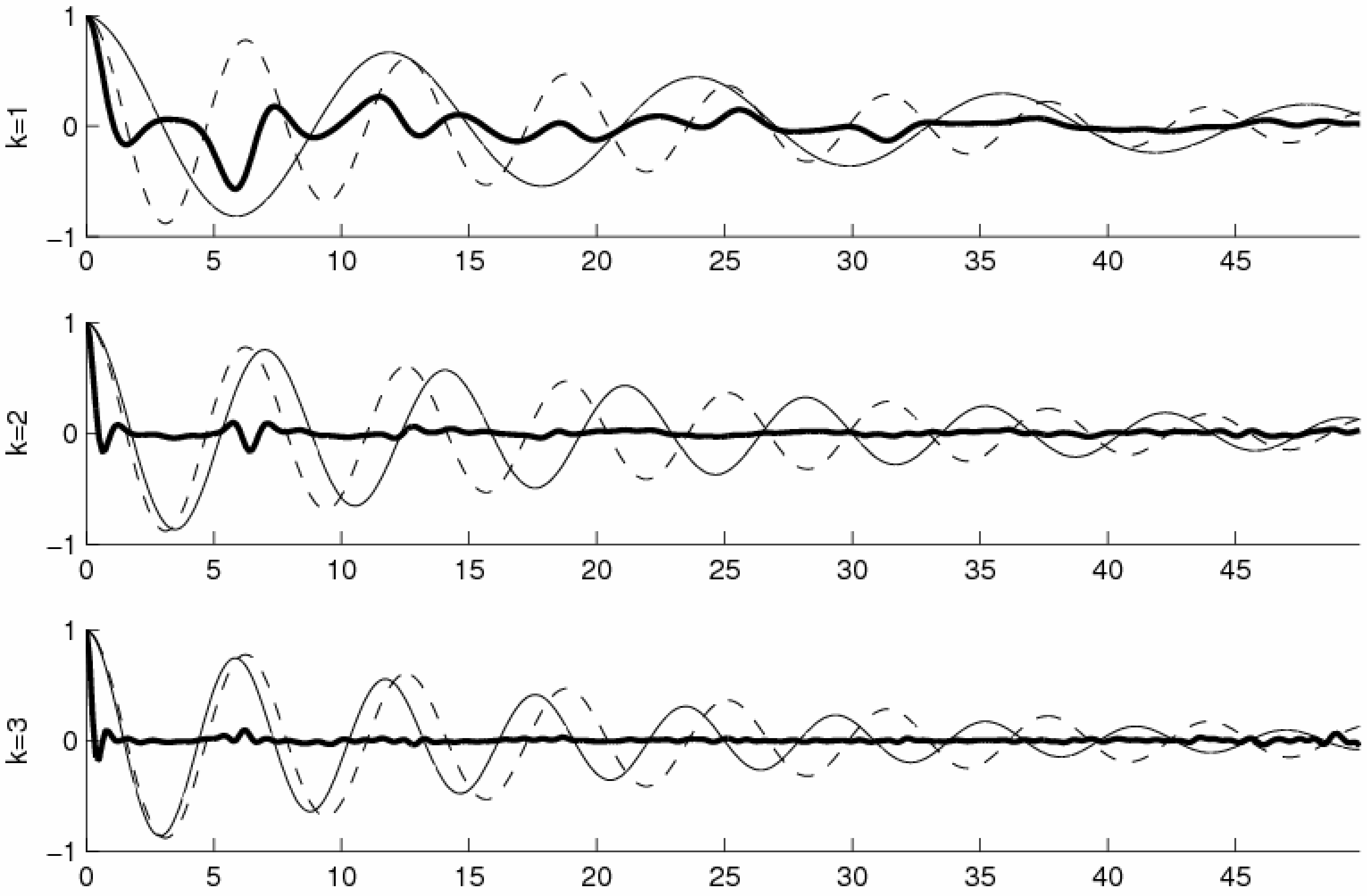
$$E_k = \begin{cases} E, & |k| \le k_0, \\ E\left(\frac{k}{k_0}\right)^{-5/3}, & |k| > k_0, \end{cases}$$

spectrum of T

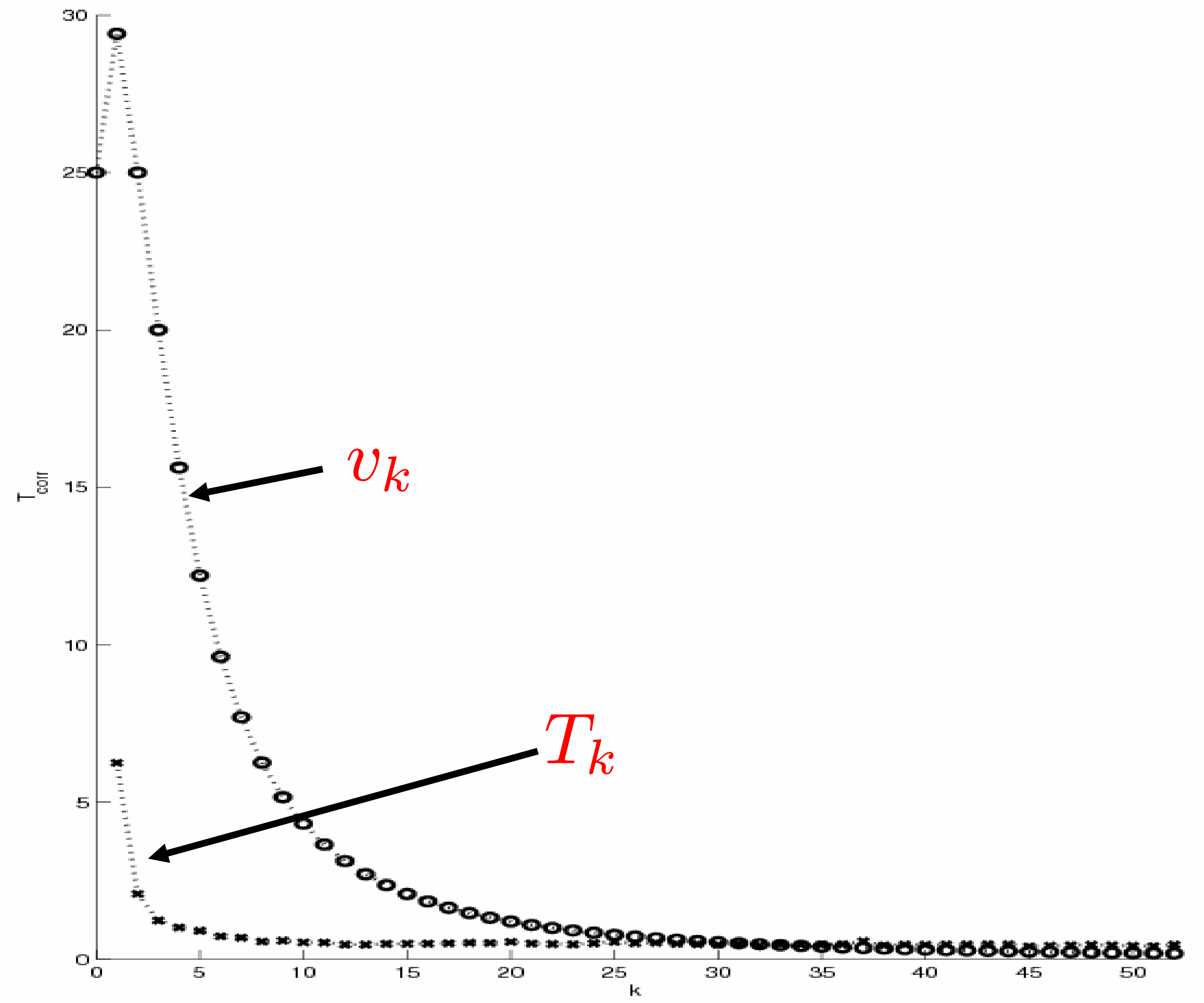




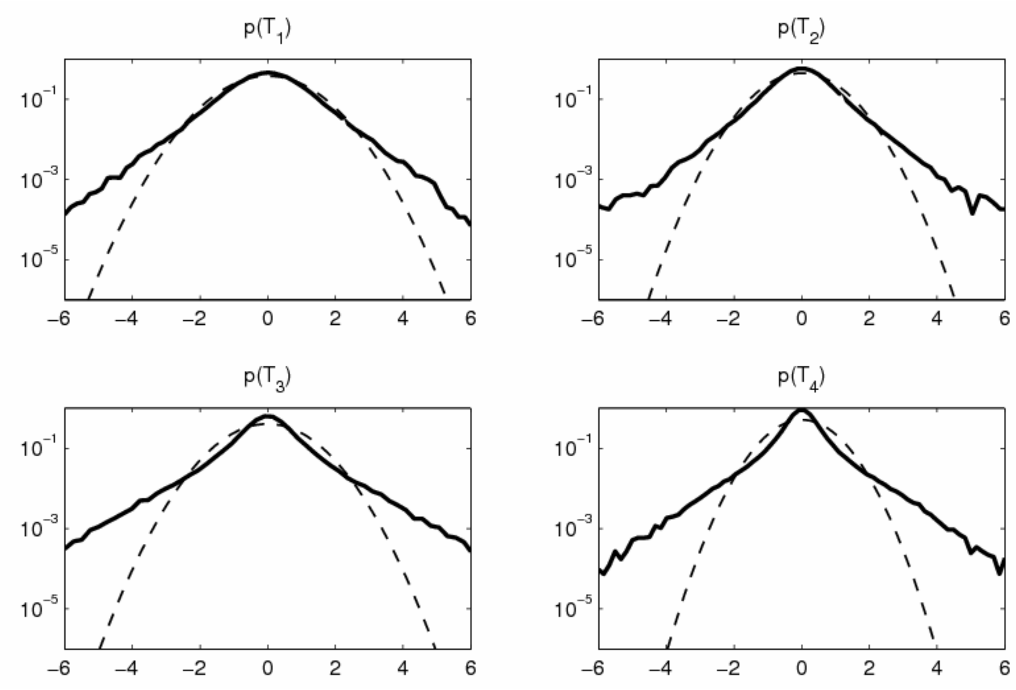
Correlation functions



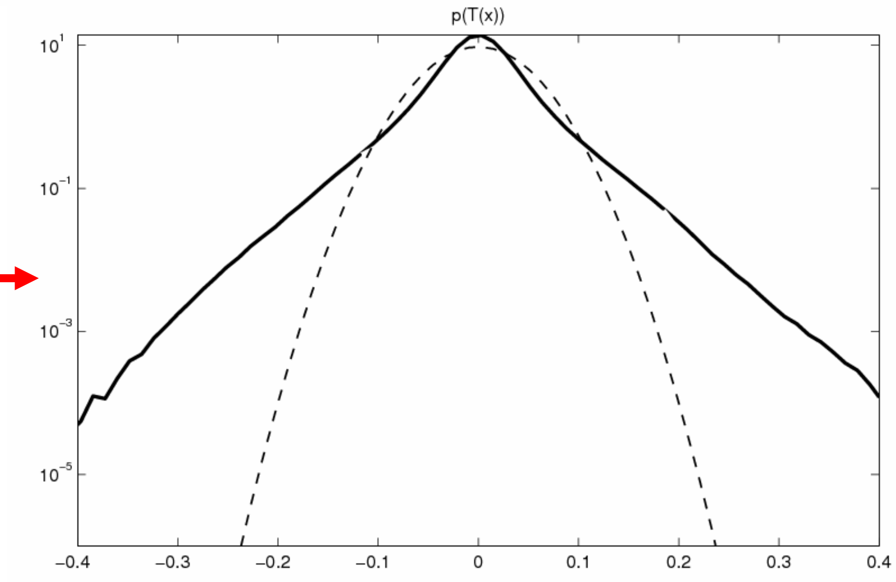
Correlation time



Strong intermittency



pdfs of Fourier modes



pdf in physical space

Neelin et al (2010) atmospheric tracers

Bourlioux, Majda (2002) different mechanism

Cross-sweep statistics

$U(t)$: zonal jet (east-west)

$$U(t) = Re[V(t)]$$

$$\frac{dV(t)}{dt} = \lambda_U V(t) + f_U(t) + \sigma_U \dot{W}_U(t)$$

$$U(t) = \bar{U}(t) + Re[V'(t)]$$
$$\bar{U}(t) = U_0 + A_U \sin(\eta t)$$

$$\frac{dV'(t)}{dt} = \lambda_U V'(t) + \sigma_U \dot{W}_U(t)$$

$$V'(t) = e^{\lambda_U (t-t_0)} V(t_0) + \sigma_U \int_{t_0}^{t} e^{\lambda_U (t-s)} dW_U(s)$$

Gaussian statistics

$$\begin{aligned}
\langle V'(t) \rangle &= e^{\lambda_U (t-t_0)} \langle V'(t_0) \rangle \\
Var(V'(t)) &= e^{-2\gamma_U (t-t_0)} Var(V'(t_0)) + \frac{\sigma_U^2}{2\gamma_U} (1 - e^{-2\gamma_U (t-t_0)}) \\
Cov(V'(t), V'(t)^*) &= e^{2\lambda_U (t-t_0)} Cov(V'(t_0), V'(t_0)^*)
\end{aligned}$$

Waves statistics

$v(x,t)$: waves $$\frac{dv_k(t)}{dt} = \lambda_k v_k(t) + f_k(t) + \sigma_k \dot{W}_k(t)$$

$$v_k(t) = e^{\lambda_k(t-t_0)} v_k(t_0) + \int_{t_0}^{t} e^{\lambda_k(t-s)} f_k(s) ds + \sigma_k \int_{t_0}^{t} e^{\lambda_k(t-s)} dW_k(s)$$

Gaussian statistics

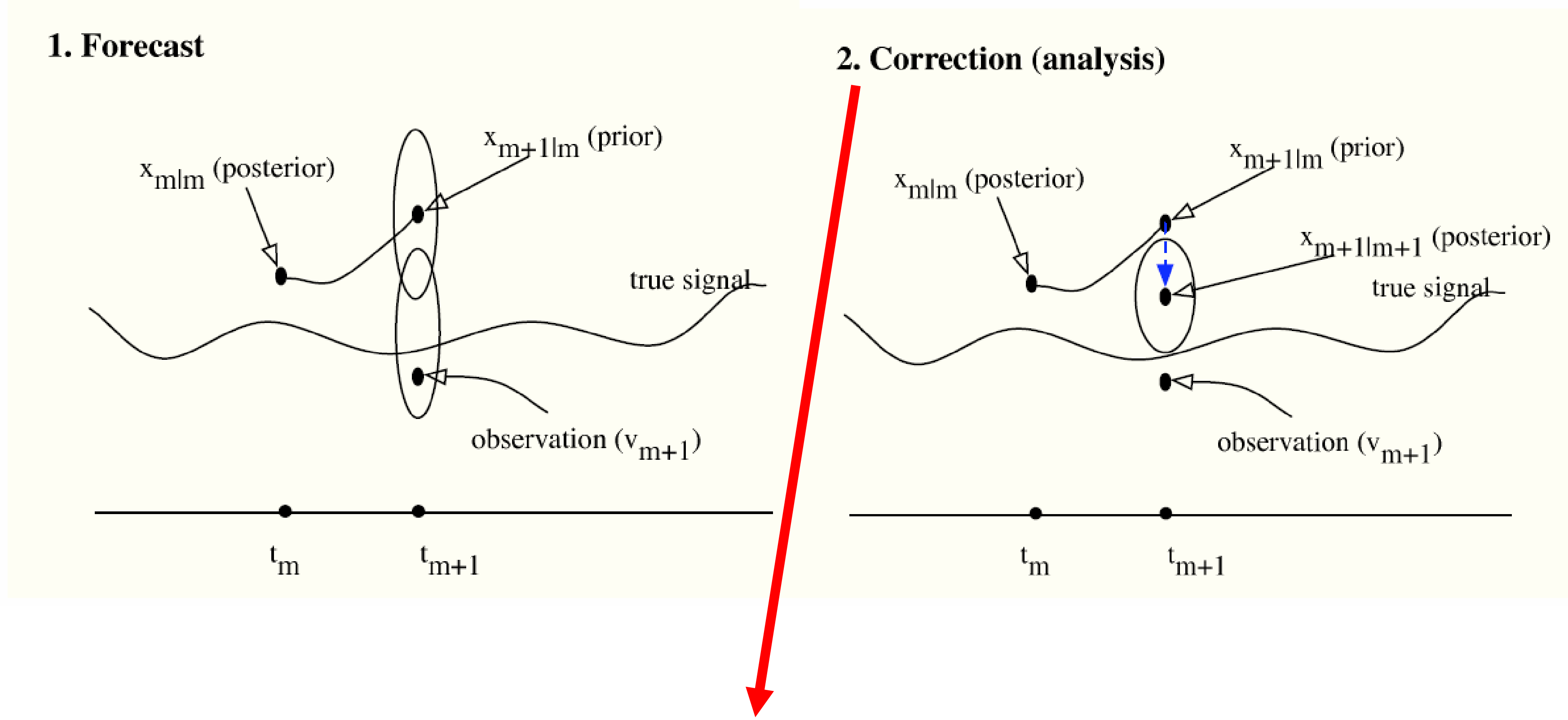
$$\langle v_k(t) \rangle = e^{\lambda_k(t-t_0)} \langle v_k(t_0) \rangle + F_k(t_0, t),$$

$$Cov(v_k(t), v_l(t)) =$$
$$e^{(\lambda_k+\lambda_l^*)(t-t_0)} \left(Cov(v_k(t_0), v_l(t_0)) + \frac{\sigma_k^2}{2\gamma_k} \delta_l^k \left(e^{2\gamma_k(t-t_0)} - 1 \right) \right).$$

$$Cov(V'(t), v_k(t)^*) = e^{(\lambda_U+\lambda_k)(t-t_0)} Cov(V'(t_0), v_k(t_0)^*)$$

First and second order statistics for the velocity field

Kalman filter



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$$r_{m+1|m+1} = (\mathcal{I} - K_{m+1}G)r_{m+1|m}$$
$$K_{m+1} = r_{m+1|m}G^*(Gr_{m+1|m}G^* + r^o)^{-1}$$

Nonlinear Extended Kalman filter

Dynamics is NONLINEAR but
the exact mean and covariance
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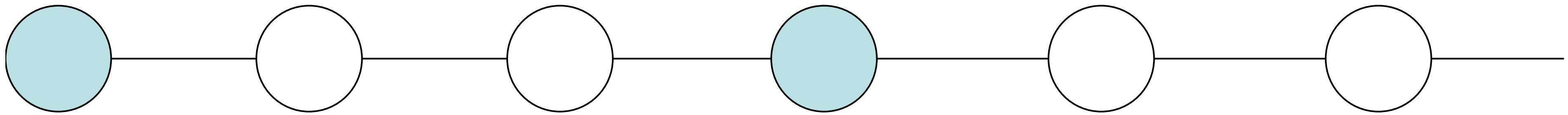
(no linearization applied)

Use the linear Kalman filter
formalism for constructing posterior

Observations

State vector

$$\left[U, v_{x_1}, v_{x_2}, \ldots, v_{x_{2K}}, v_{x_{2K+1}}, T_{x_1}, T_{x_2}, \ldots, T_{x_{2K}}, T_{x_{2K+1}}\right]^T$$



every 3d point observed:

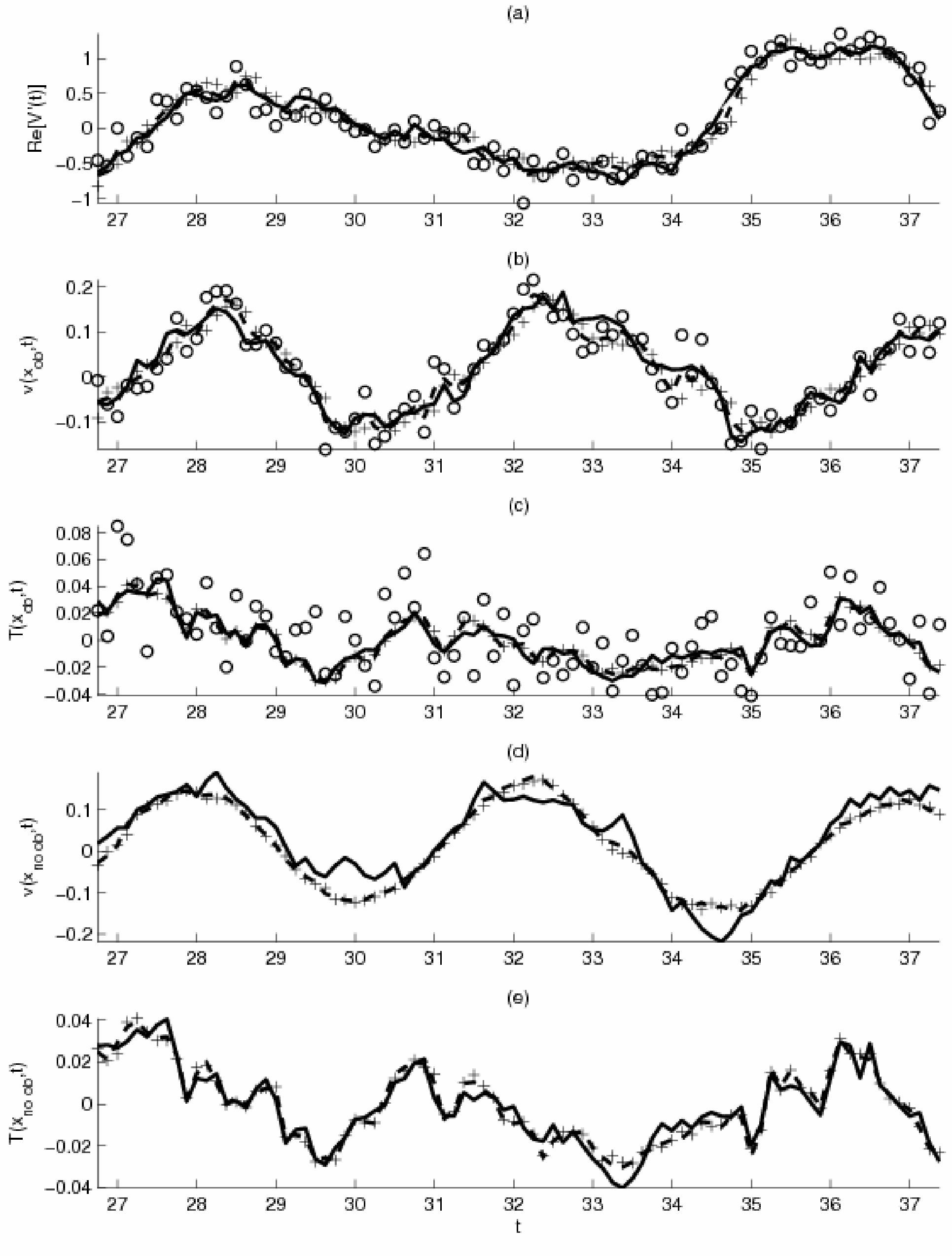
$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

\# of observations

$4K + 4$

can consider irregular grid

Path-wise filtering: time



zonal jet

waves at the observed location

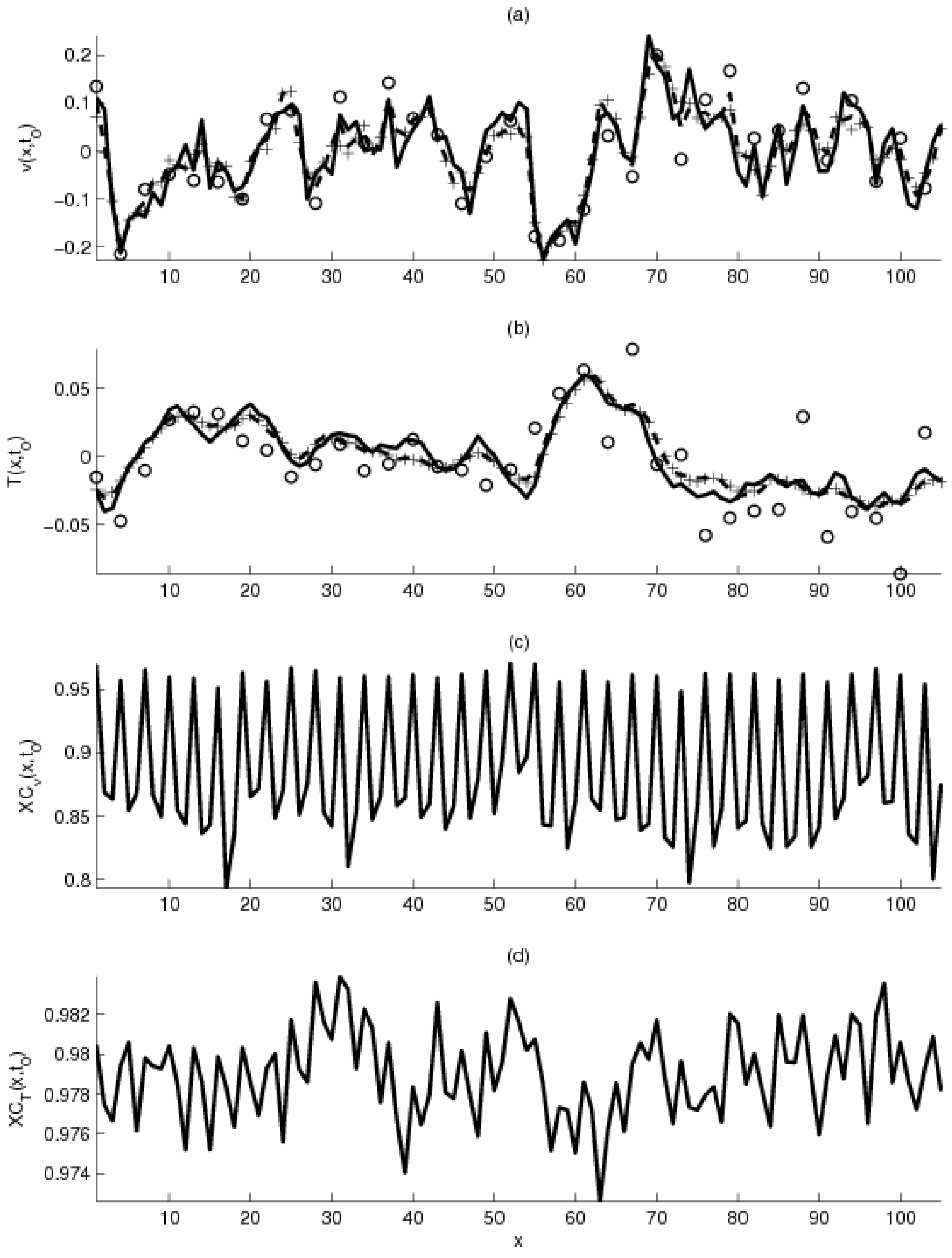
tracer at the observed location

waves at the unobserved location

tracer at the unobserved location

observe
U, v, T
every 3d point

Path-wise filtering: space



waves

observe
U, v, T
every 3d point

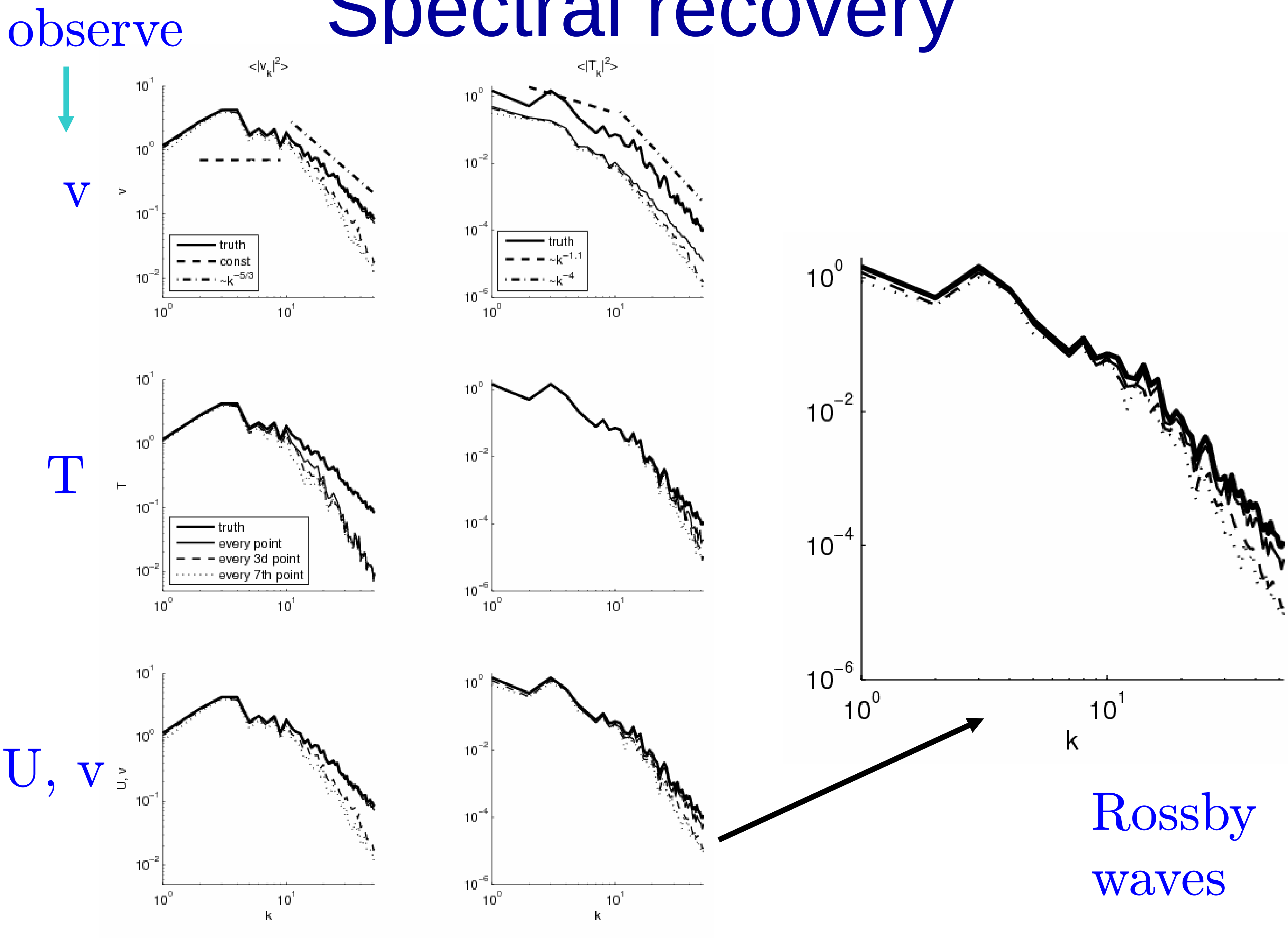
tracer

$$XC(\vec{x},\vec{y}) = \frac{\vec{x}\cdot\vec{y}}{\sqrt{|\vec{x}|^2|\vec{y}|^2}}.$$

pattern correlation
for the waves

pattern correlation
for the tracer

Spectral recovery



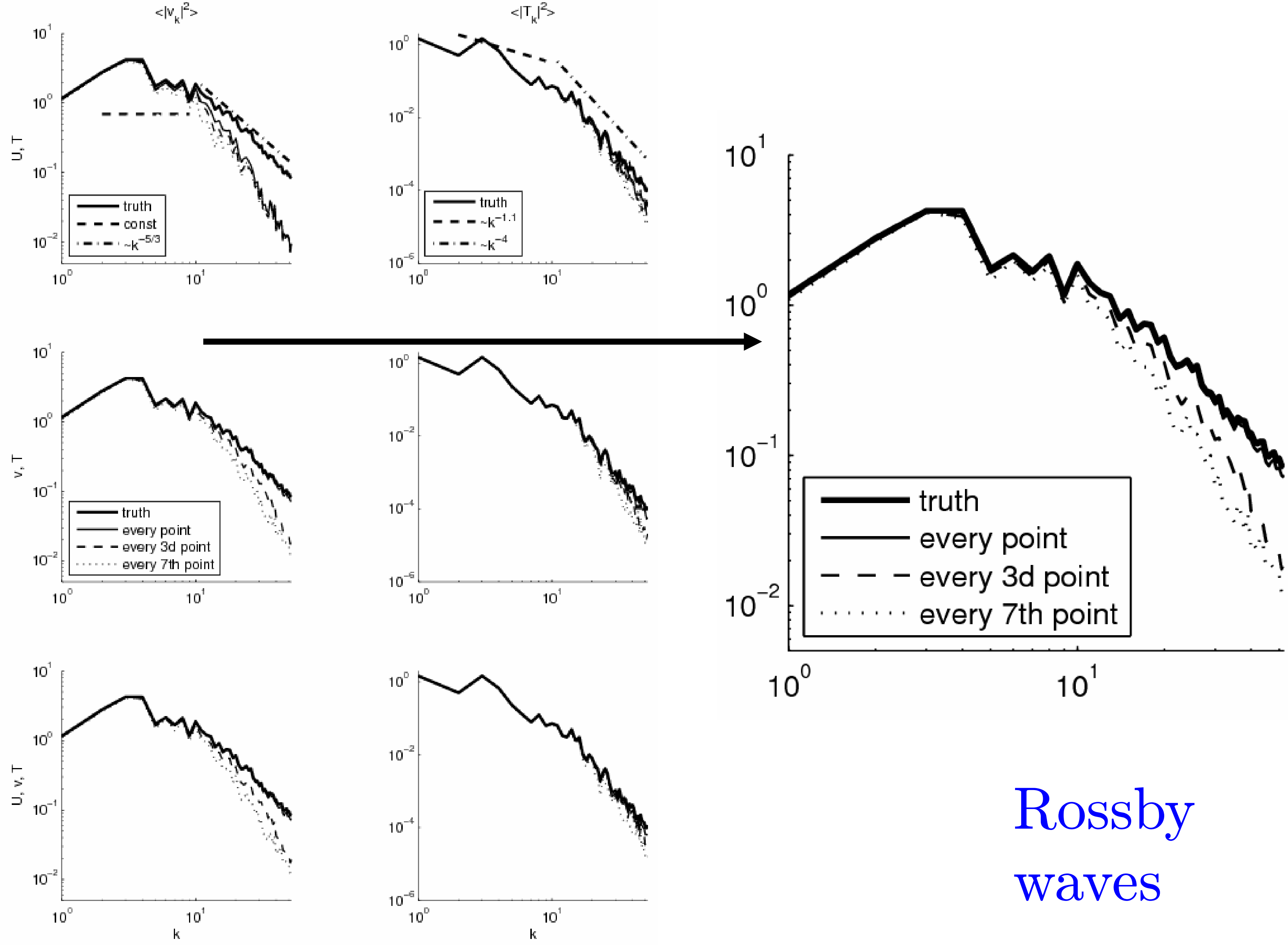
Spectral recovery

observe

U, T

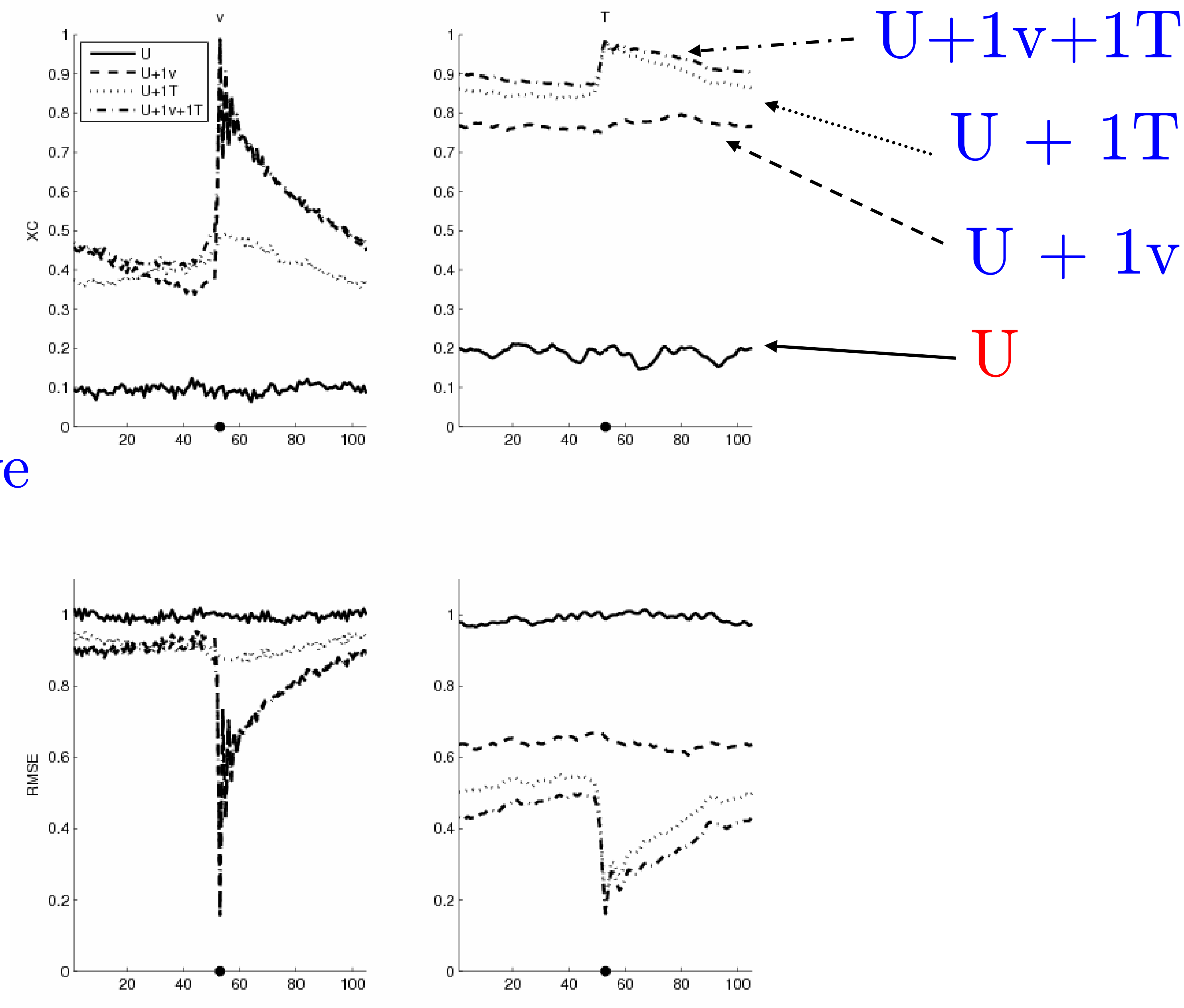
v, T

U, v, T



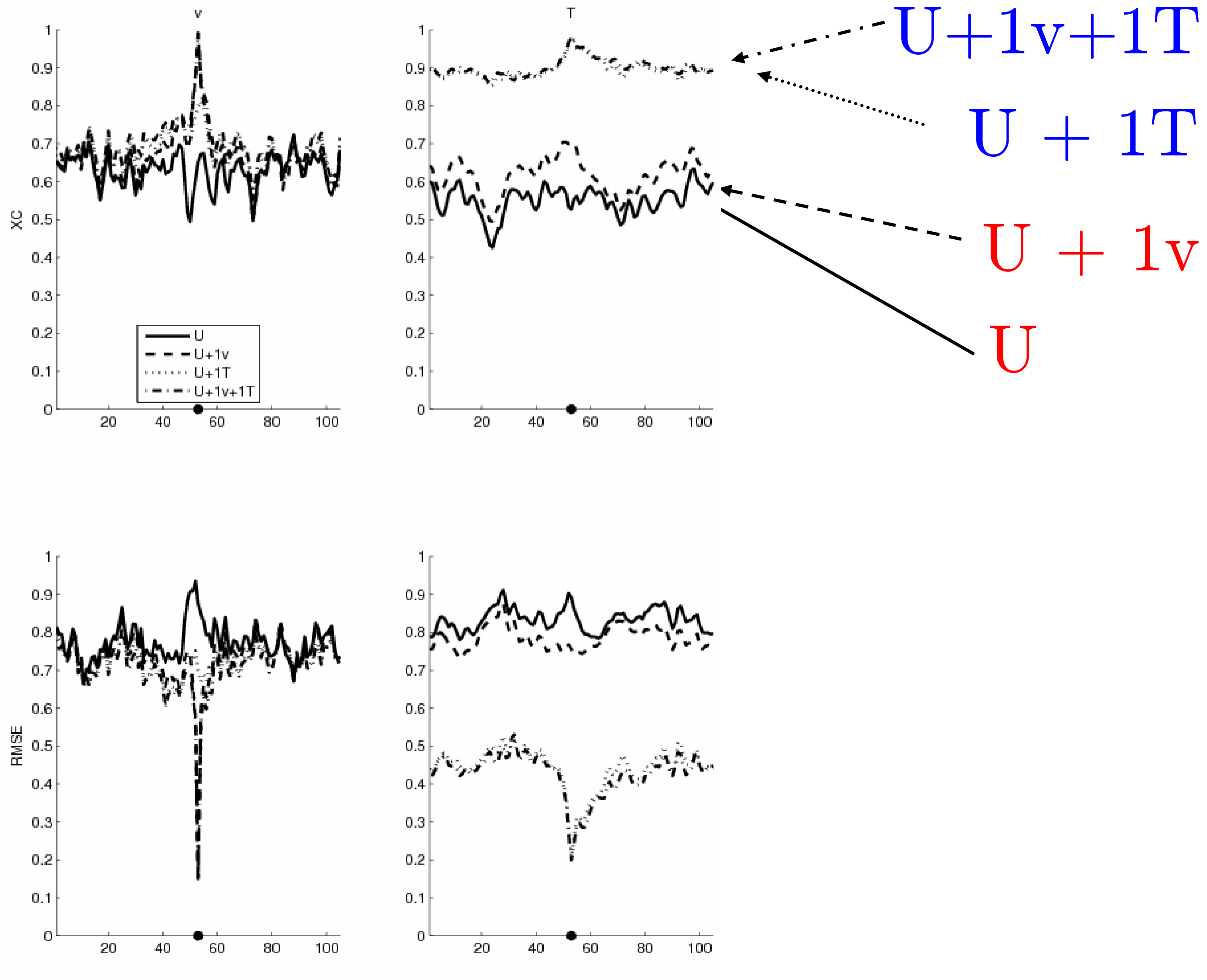
Rossby waves

Add one observations



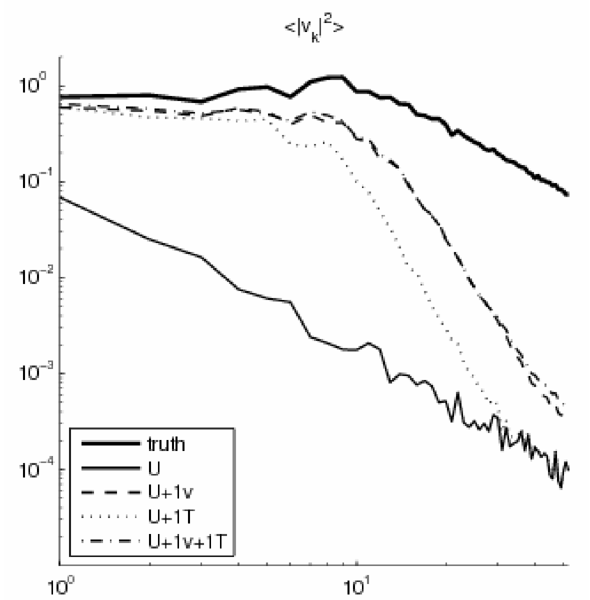
nondispersive
waves

Add one observations

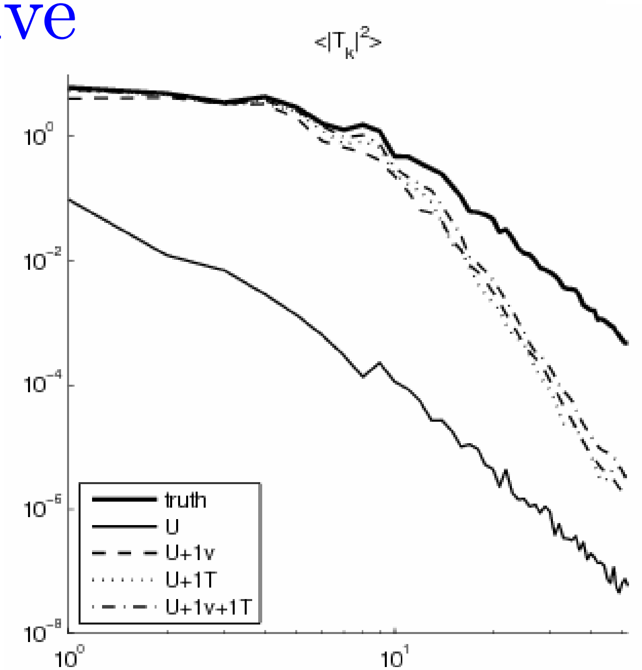


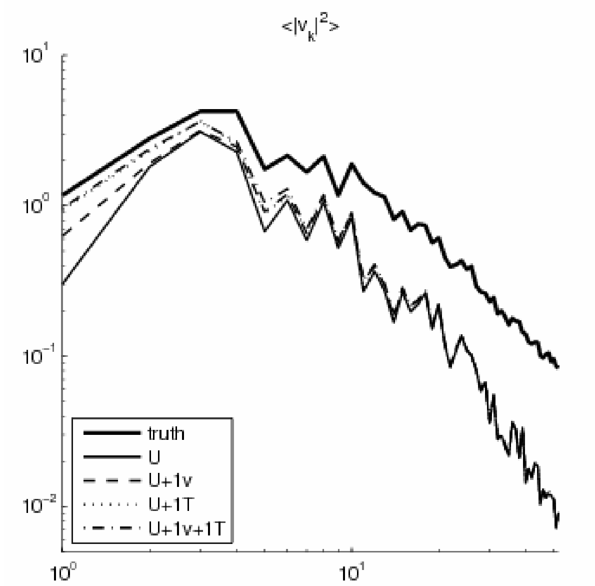
Rossby waves

Spectral recovery with one observation

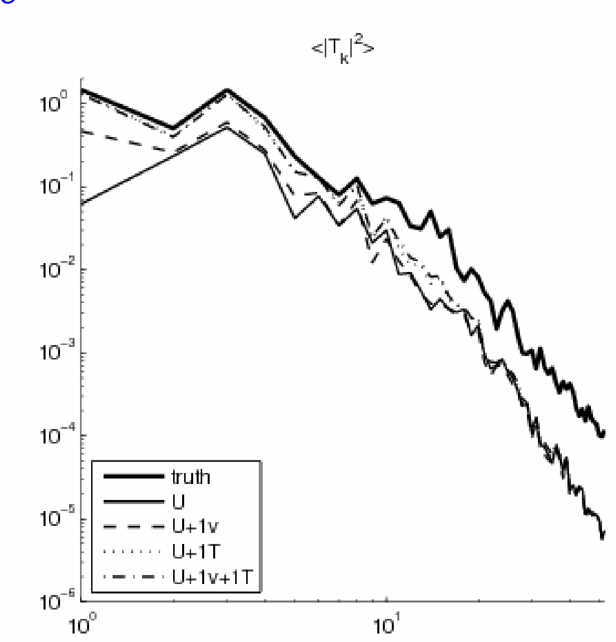


nondispersive
waves



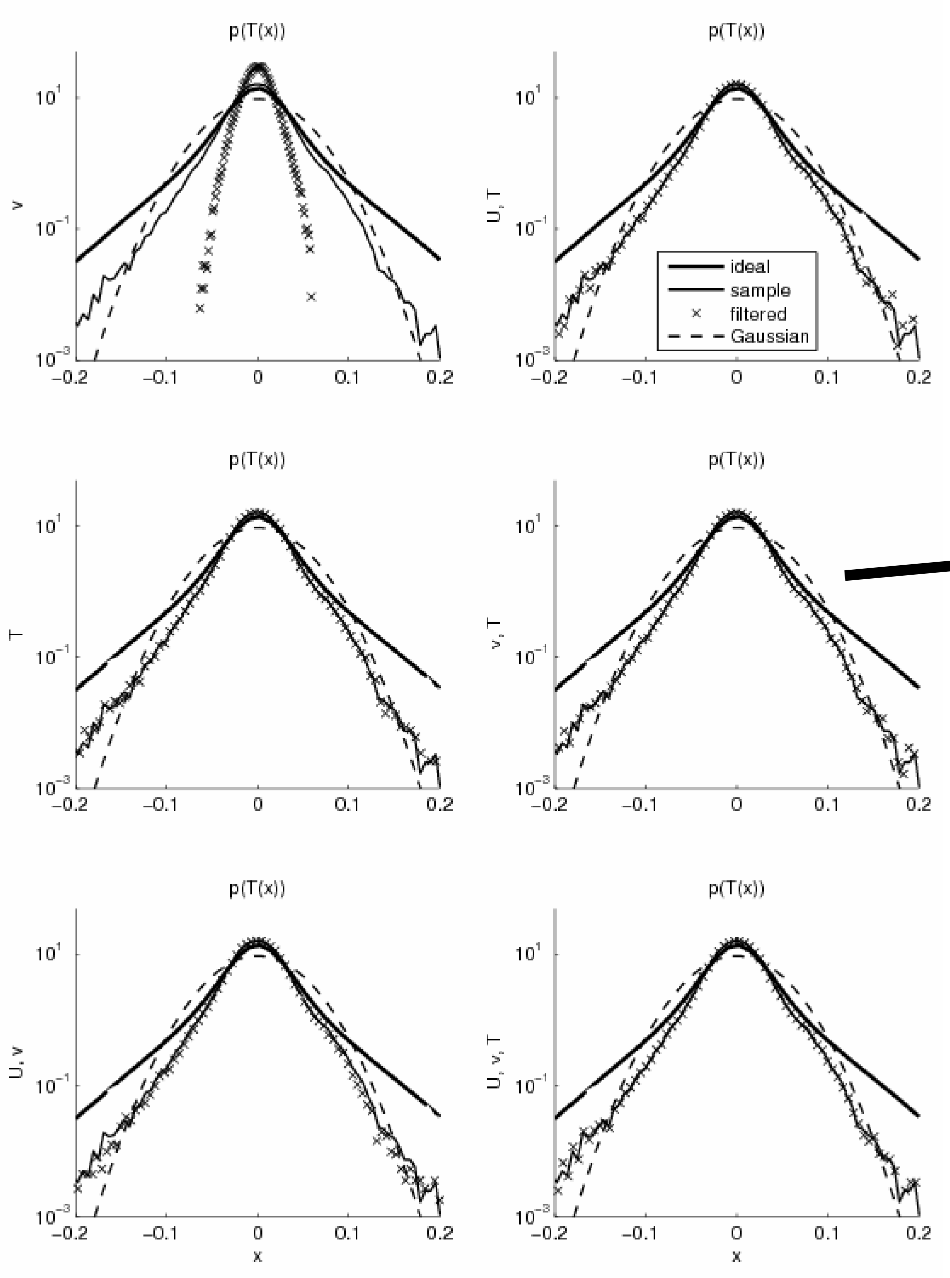


Rossby
waves

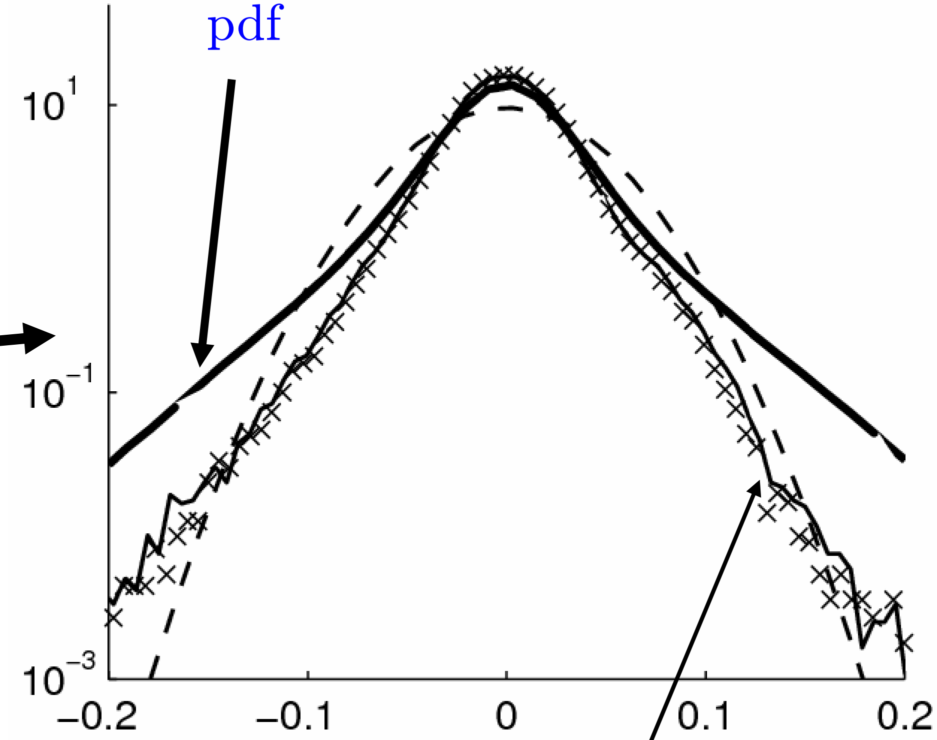


Pdf recovery

Rossby waves

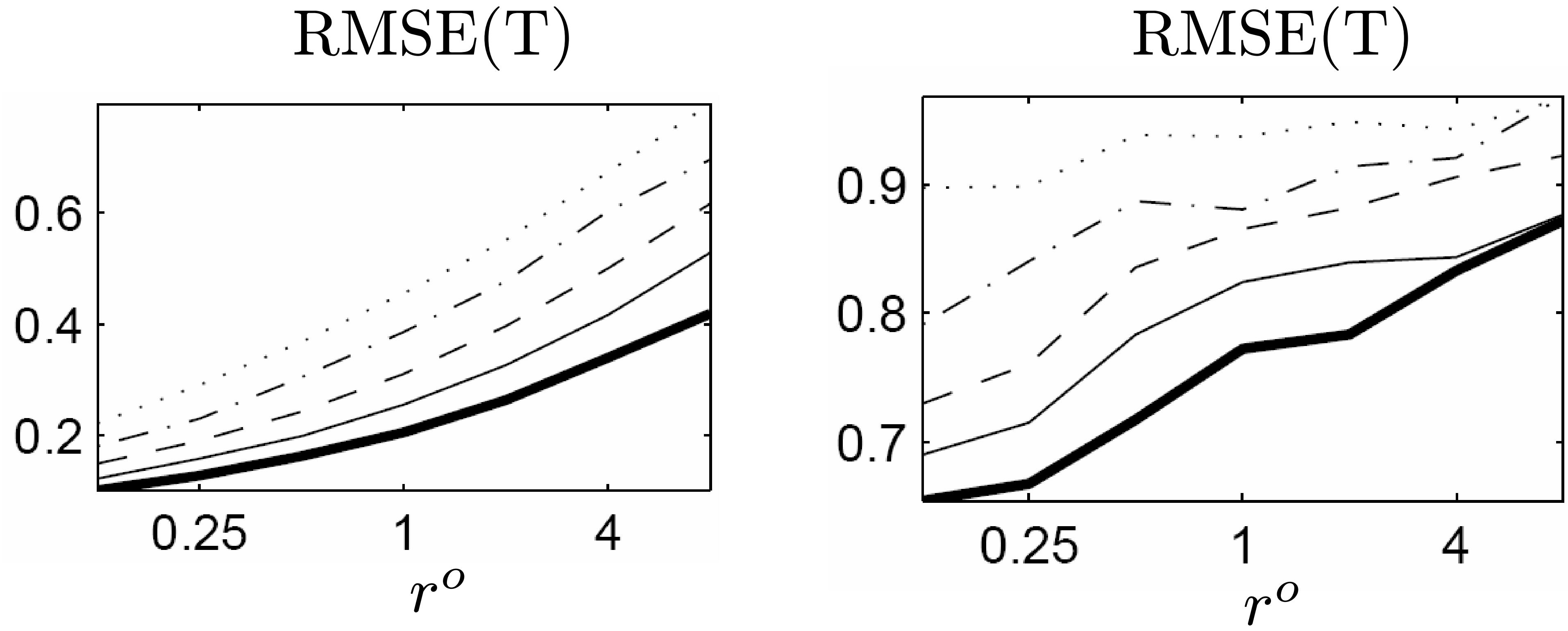


ideal pdf

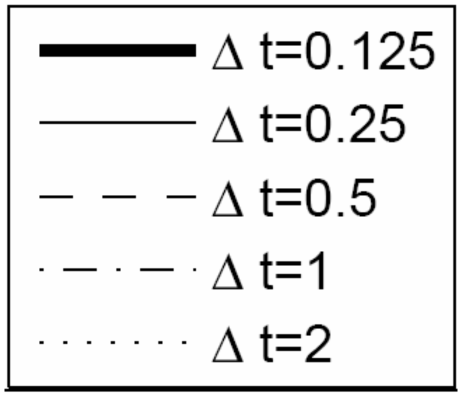


pdf of a short time sample

Skill as a function of noise



observe T
at every point



observe U and v
at every 7th point

Conclusions and future directions

- Exactly solvable model for turbulent tracers
- General interest for climate science, engineering, environmental science…
- Build NEKF using exact statistics
- Role of partial sparse observations
- Future: model error via eddy diffusivity, parameter estimation
- Future: climate response to external perturbations
- Future: empirical information theory to quantify model error, optimize observations