

# Model Hierarchies, Ergodicity and Scaling

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# Collaborators

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# Questions

There are two related questions we address:

- What is the relation between thermal critical phenomena and percolation?
- How are scaling and critical phenomena related?

We discuss these questions in simple models.

# Outline

- ★ Motivation
- ★ Percolation in Ising models
- ★ Earthquake models, equilibrium and critical points.
- ★ Damage and scaling
- ★ Summary and Conclusions

# Motivation - Critical Phenomena, Percolation and Climate

- ★ Several approaches to climate involving, critical phenomena and percolation.

O. Peters, C. Hertlein, K. Christensen, PRL **88**, 018701 (2002)

S. Lovejoy, Science **216**, 185 (1982)

A. P. Siebesma and H. J. J. Jonker, PRL **85**, 214 (2000)

O. Peters, J. D. Neelin, S. W. Nesbitt, J. Atmos. Sci. **66**, 2913 (2009)

A. G. Hunt, Vadose Zone Journal, **4** 134 (2005)

- ★ Two ideas-Scaling  $\rightarrow$  criticality  $\rightarrow$  percolation.
- ★ Ideas are subtle and require caution

## Critical Point $\rightarrow$ percolation

- ★ Critical point  $\rightarrow$  percolation - Ising model

Hamiltonian

$$H = -J \sum_{ij} s_i s_j - h \sum_i s_i \quad (1)$$

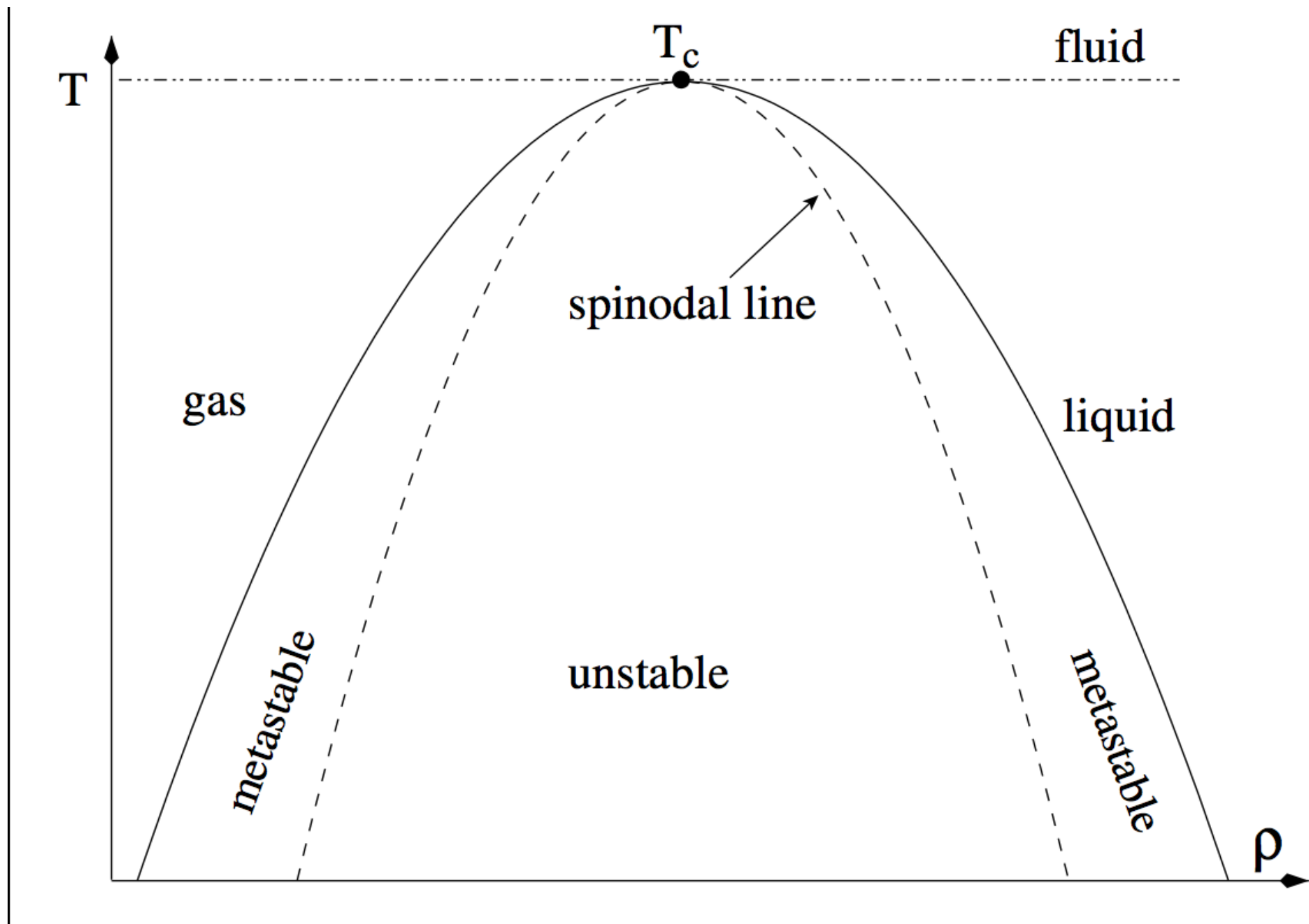
$J$  is coupling constant,  $h$  is magnetic field,  $\sum_{ij}$  is sum over pairs.

The partition function is

$$Z = \sum_{\{s\}} \exp[-\beta H] \quad (2)$$

## Meanfield

- \* For our purposes we can consider meanfield systems.
- \* Meanfield-Kac-Coupling constant( $J$ ) of the form  $\gamma^d$   
Sum over pairs within  $\gamma^{-d} = R^d$ .
- \* Kac - Calculate the free energy in the thermodynamic limit  
Take the limit  $\gamma \rightarrow 0 (R \rightarrow \infty)$ .  
Calculate equation of state etc.
- \* In the limit  $R \rightarrow \infty$  the system is meanfield.  
W. Klein, H. Gould, N. Gulbahce, J. B. Rundle and  
K. F. Tiampo, Phys. Rev. E **75**, 031114 (2007)  
Phase diagram.



\* Can we map the critical point onto a percolation problem? What does that mean?

\* Take  $h = 0$ . Start with spins randomly oriented at temperature  $T = \infty$ .

Lower temperature until percolation threshold is reached.

\* Want percolation temperature = critical temperature.

Correlation length = connectedness length etc.

**Percolation is defined by two characterizations**

\* What are the items and how are they distributed?

\* What is meant by connected?

Triangular lattice in  $d = 2$ . Items are down spins distributed according to Boltzmann factor.

The probability of a configuration of spins  $p(\{s_i\})$  is

$$p(\{s_i\}) = \frac{\exp[-\beta H(\{s_i\})]}{Z} \quad (3)$$

- \* Two sites are connected if they are within  $R$  of each other
- \* Percolation threshold for  $R \rightarrow \infty \sim 1/q$ .
- \* For  $T = \infty$  we are above the percolation threshold.
- \* Then the random, and correlated, percolation transition has nothing to do with the phase transition.
- \* To fix problem - thin out clusters.

- \* Place random bonds between pairs of down spins with probability

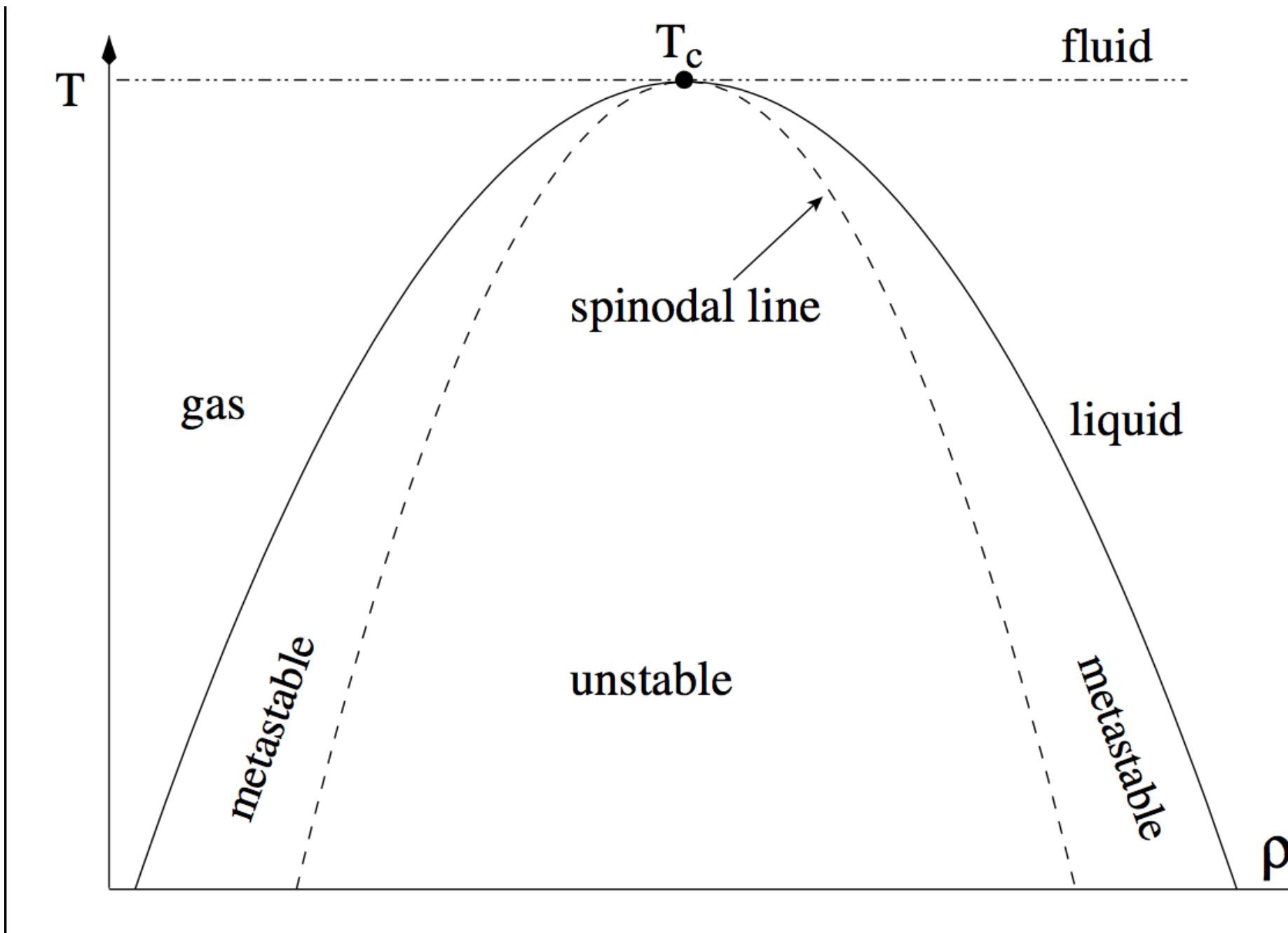
$$p_b = 1 - \exp[-2\beta J] \quad (4)$$

- \* Critical point is percolation transition.
- \* Correlation length is connectedness length.
- \* Susceptibility is mean cluster size.
- \* True for all interaction ranges, not just meanfield.

A. Coniglio and W. Klein, J. Phys. **A 13**, 2775 (1980)

W. Klein et al

- \* Spinodal line is line of critical points



- \* Spinodal maps onto percolation transition with

$$p_{bs} = 1 - \exp[-\beta 4J(1 - \rho)] \quad (5)$$

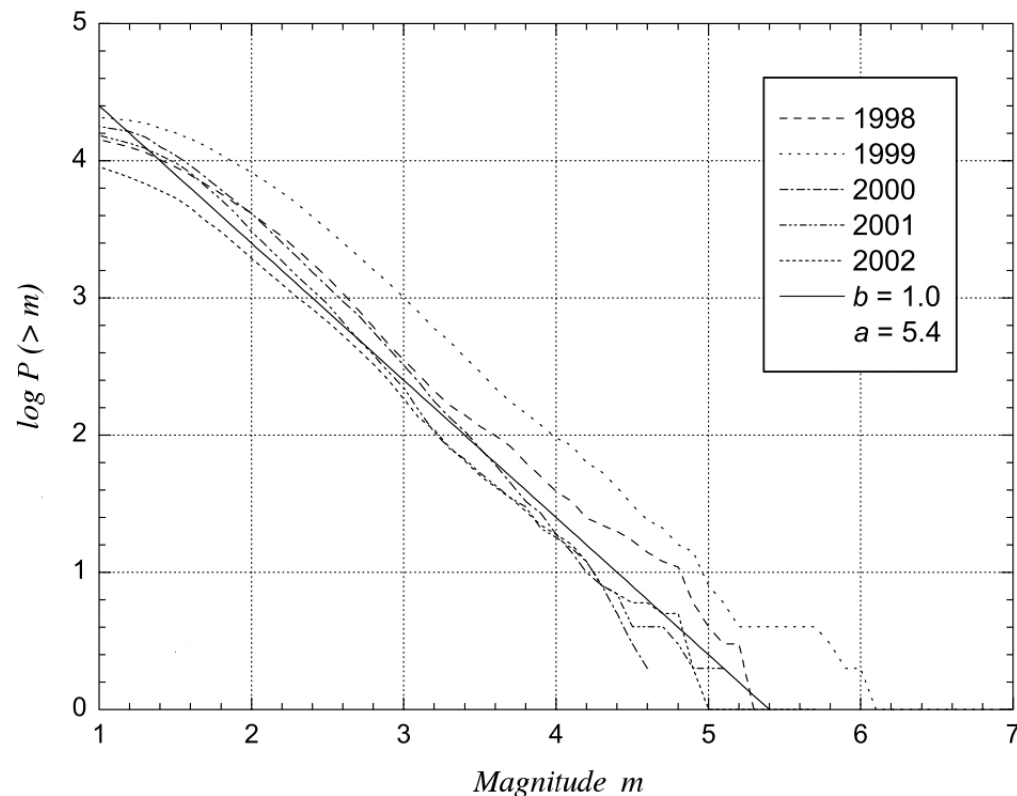
where  $\rho$  is the density of spins in the direction of the stable phase.

**Conclusion:** Ising model critical point and spinodal can be mapped onto percolation transition but mapping is not straightforward. Cluster scaling - critical point

- \* Have not been able to map systems with frustration onto percolation - Long range anti ferromagnet - fluid. spin glass etc.

The second question we want to address is the relation between scaling and critical points. We will do this with a model for earthquake faults and fault systems.

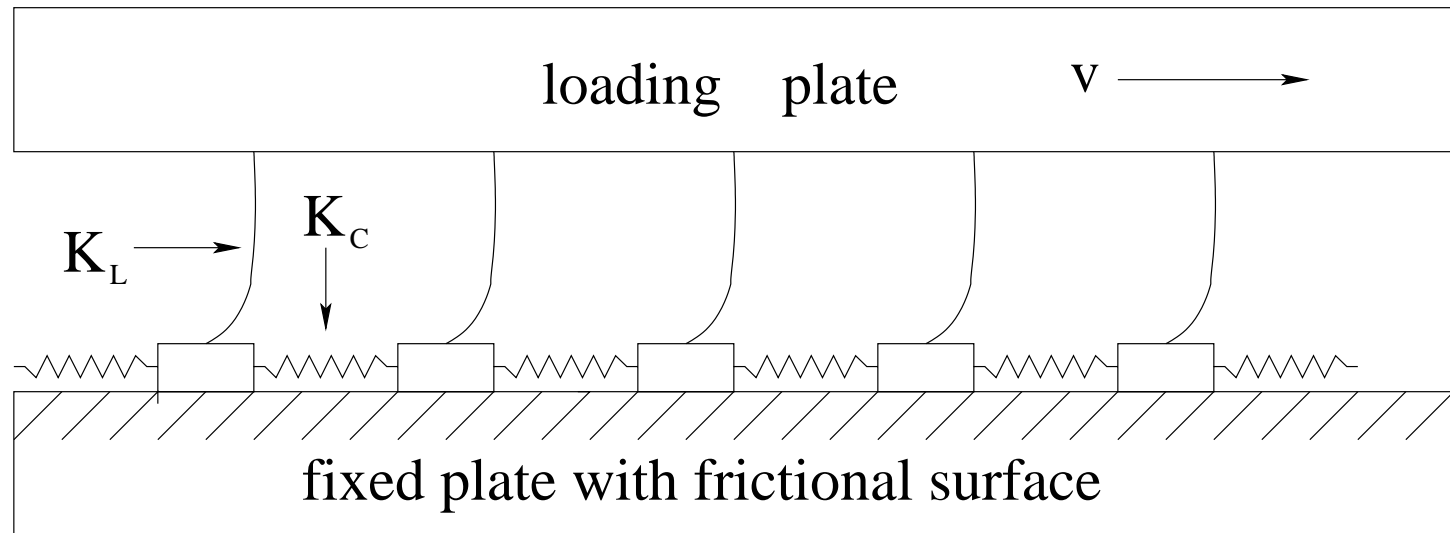
- \* We want to describe the Gutenberg-Richter scaling in real fault systems.



- \* We will make use of cluster scaling in the model of earthquake faults.
- \* We will also introduce quench randomness.

## CA Model for Earthquake Faults

Model of a fault



R. Burridge and L. Knopoff, Bull. Seismol. Soc. Amer. 57, 341 (1967)

- \* Blocks have inertia, friction force, Newton's equations  
MD

- \* CA version J. B. Rundle and D. Jackson, Bull. Seismol Soc. Amer. 67, 1363 (1977).

Same blocks and springs - failure threshold  $\sigma^F$ ,  
residual stress  $\sigma^R$ .

- \* If  $\sigma_i < \sigma^F$  move to next block.

- \* Distribute stress at random, If  $\sigma_i \geq \sigma^F$   
move block

$$\Delta x = \frac{\sigma_i - \sigma^R}{K_L + qK_c} + noise \quad (6)$$

- \* Continue until  $\sigma_i < \sigma^F \forall i$ .

- \* Move plate so that block with largest stress fails
- \* Measure avalanches.
- \* OFC model Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. A 46, 1829 (1992)  
( $\sim$  identical to the RJB model).
- \* Square lattice with stress on each site. Assign failure threshold  $\sigma^F$  and residual stress  $\sigma^R$  to each site.  
Choose dissipation coefficient  $\alpha$  and stress transfer range  $R$ . ( $\alpha = K_L/K_L + qK_c$ )
- \* Initially distribute stress at random. If  $\sigma_j < \sigma^F$  skip.
- \* If  $\sigma_j > \sigma^F$ , set stress  $\sigma_j = \sigma^R$  and

distribute  $(1 - \alpha)(\sigma_j - \sigma^R) + \eta$  to the  $(2R + 1)^d$  neighbors

- \*  $\eta$  the noise (flat random)
- \* Continue until  $\sigma_j < \sigma^F \forall j$
- \* Find site with largest stress – add stress to bring this site to failure. (Add same amount of stress to each site.)

Many variations (  $\alpha$  can vary, lower failure threshold)

Clusters or “earthquakes” are sites that fail between updates.

- \* Is the system in equilibrium?

Measure the Thirumalai-Mountain metric,

D. Thirumalai and R. Mountain,

Phys. Rev. A 42, 4574 (1990)

and Phys. Rev. E 47, 479 (1996).

\* Take  $f_j(t)$  to be a quantity of one of the  $N$  sites (stress).

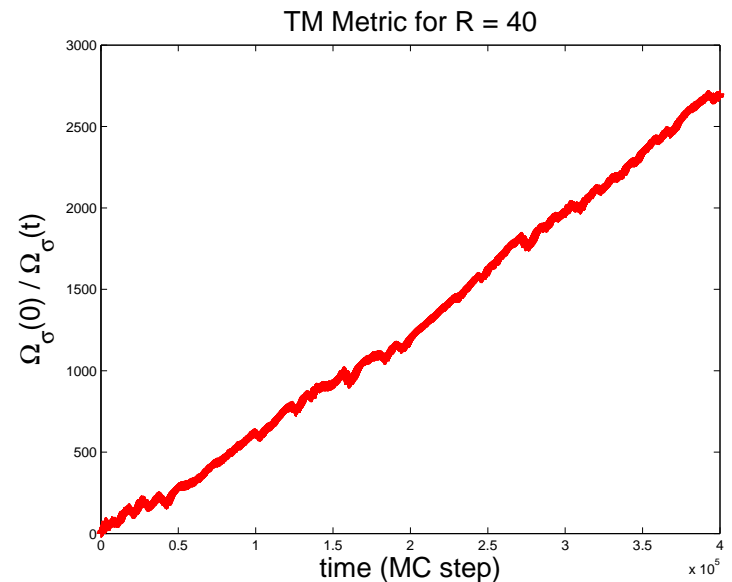
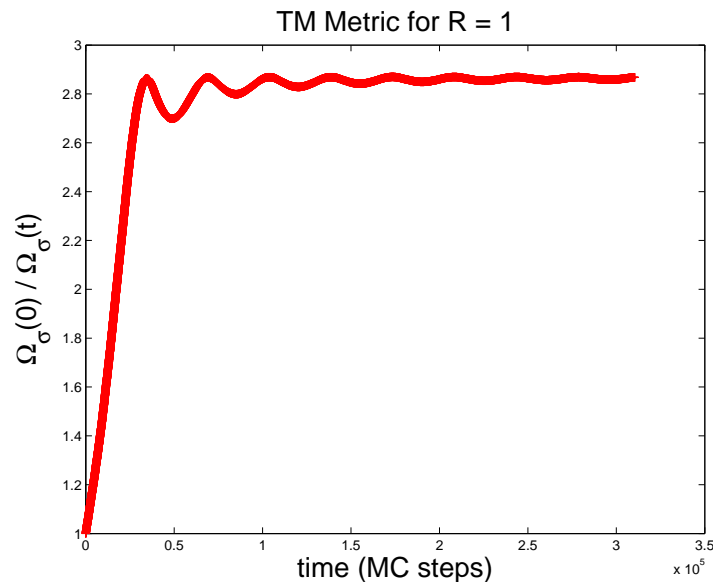
$$\bar{f}_j(t) = \frac{1}{t} \int_0^t f_j(t') dt'$$

$$\langle f(t) \rangle = \frac{1}{N} \sum_{j=1}^N \bar{f}_j(t).$$

\* The metric is defined as

$$\Omega_f(t) = \frac{1}{N} \sum_{j=1}^N [\bar{f}_j(t) - \langle f(t) \rangle]^2.$$

- \* If the system is ergodic, then  $\Omega_f(t) \propto 1/t$ .
- \* For  $R = 1$  the system is not ergodic. For  $R = \infty$  system is ergodic. If  $R \gg 1$  but finite, system is punctuated ergodic. (W. Klein et al)



\* We can also show that the RJB model has a Boltzmann description.

J. B. Rundle, W. Klein, S. Gross and D. L. Turcotte, PRL **75**, 1658 (1995)

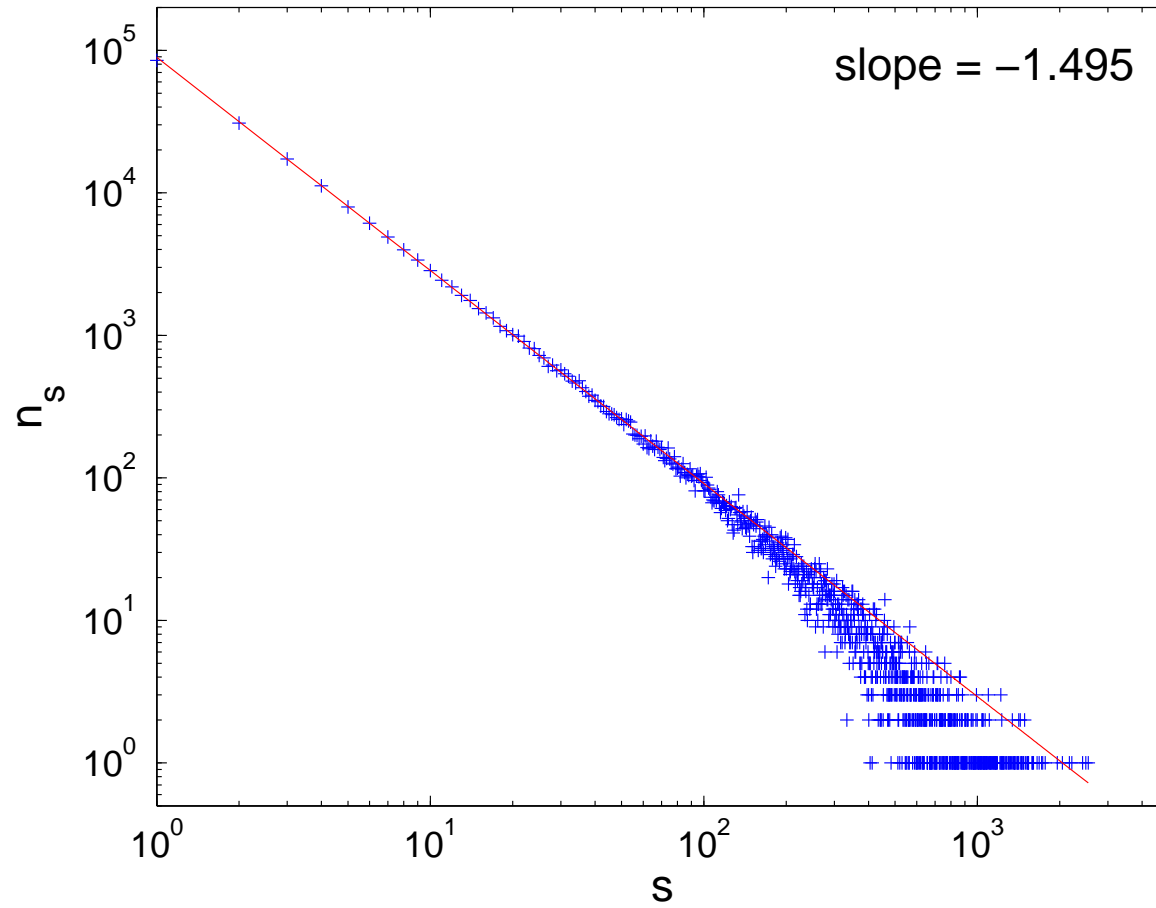
\* In addition we can derive Langevin equation with driving force the functional derivative of a free energy.

W. Klein, J. B. Rundle and C. D. Ferguson, PRL **78**, 3793 (1997)

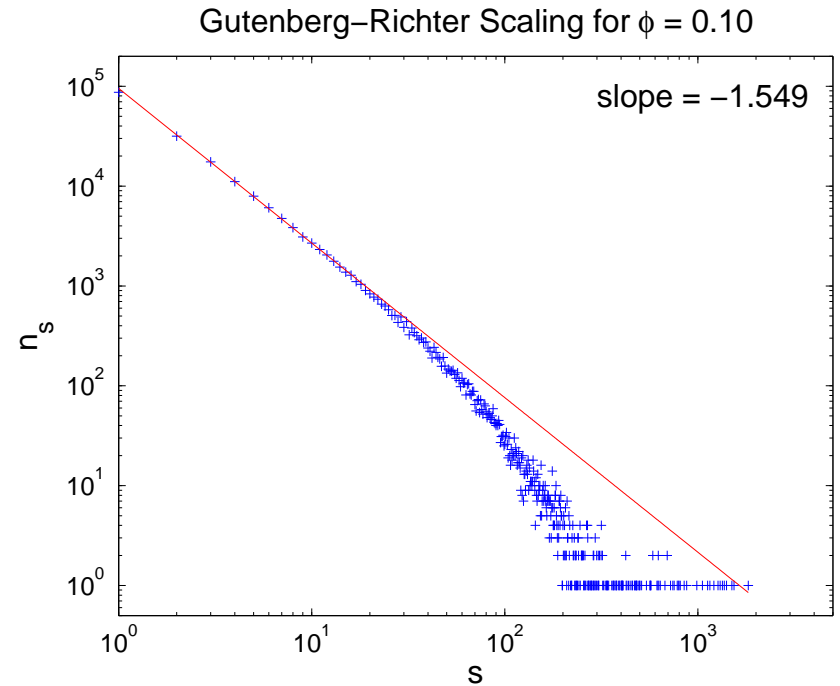
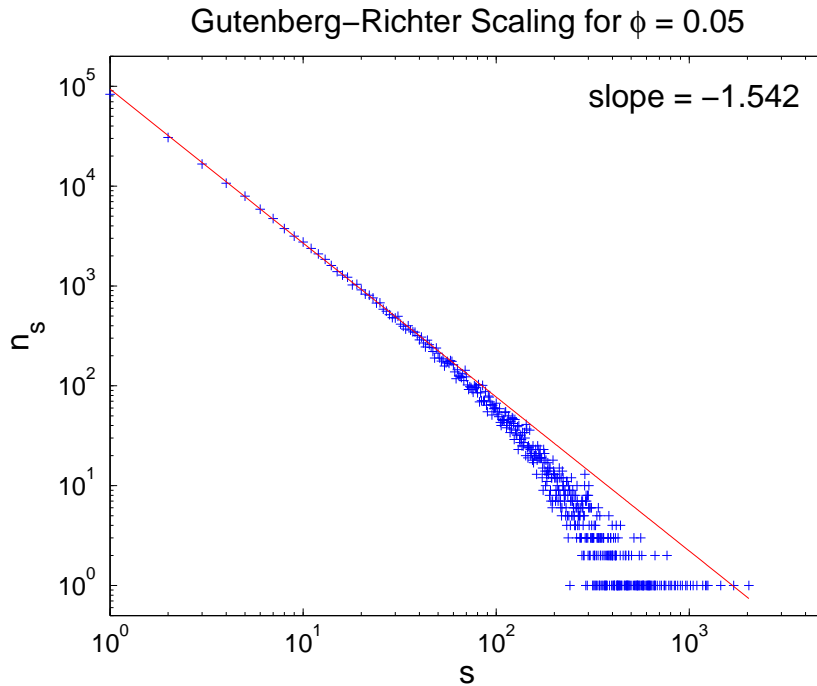
\* Fault systems have GR scaling. Model?

$$n_s \sim \frac{\exp[-\Delta h s^\sigma]}{s^\tau} \quad (7)$$

## Gutenberg–Richter Scaling

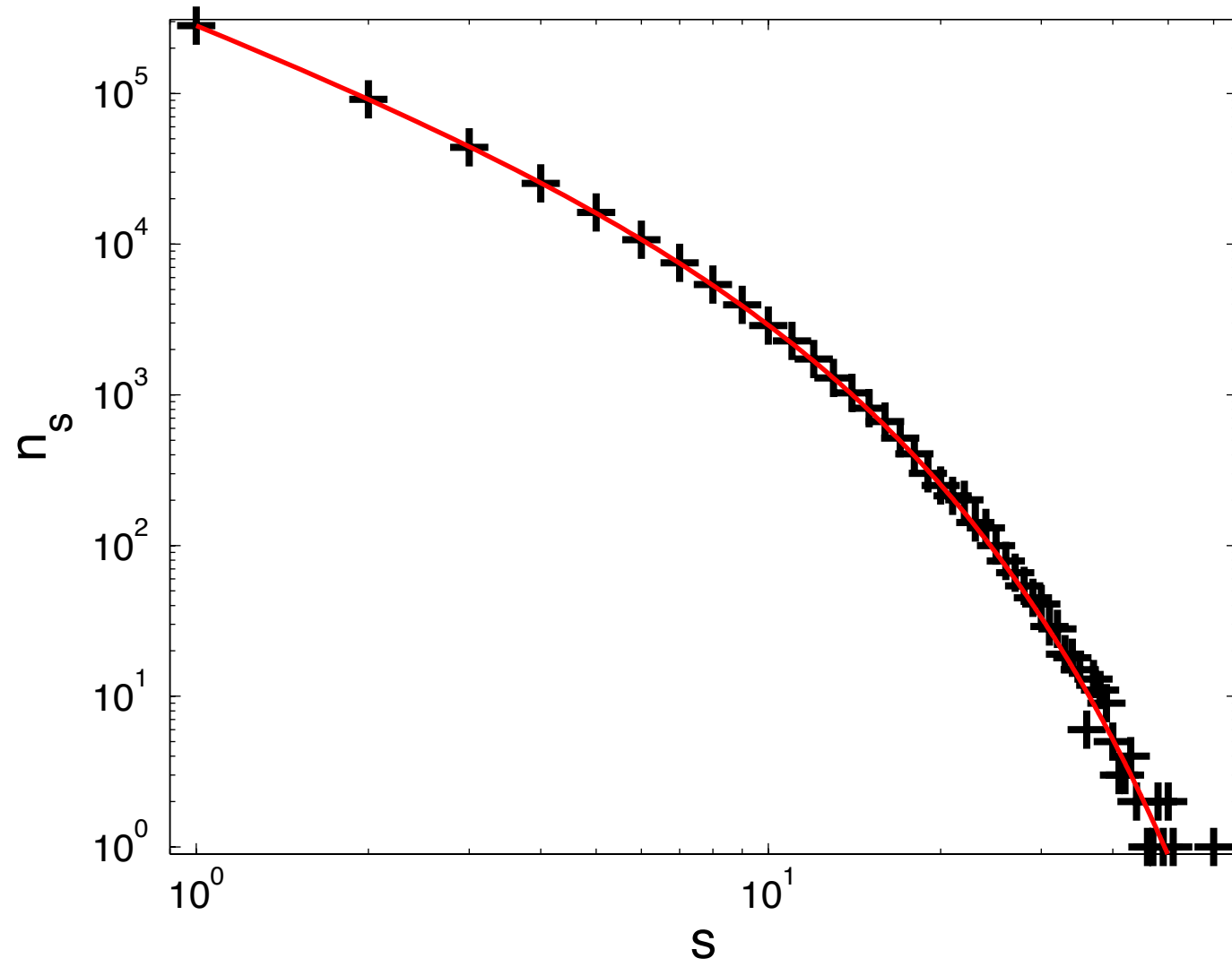


- \* Fits to the model data indicate  $\tau = 1.5(0.5)$  and  $\sigma = 1.0$
- \* Fault systems - GR scaling  $b = 1$
- \* Where does the scaling come from?
- \* In long range system - punctuated equilibrium -  
System has pseudo - spinodal  $\rightarrow$  cluster scaling.
- \* Exponents can be obtained from cluster scaling at spinodal  
W. Klein et al., Phys. Rev. E 75, 031114 (2007).
- \* Real faults - system has scaling.  
Model is **not** a fault system
- \* Damage fault by removing fraction  $\phi$  of sites.
- \* Empty sites dissipate transferred stress.(small cracks)

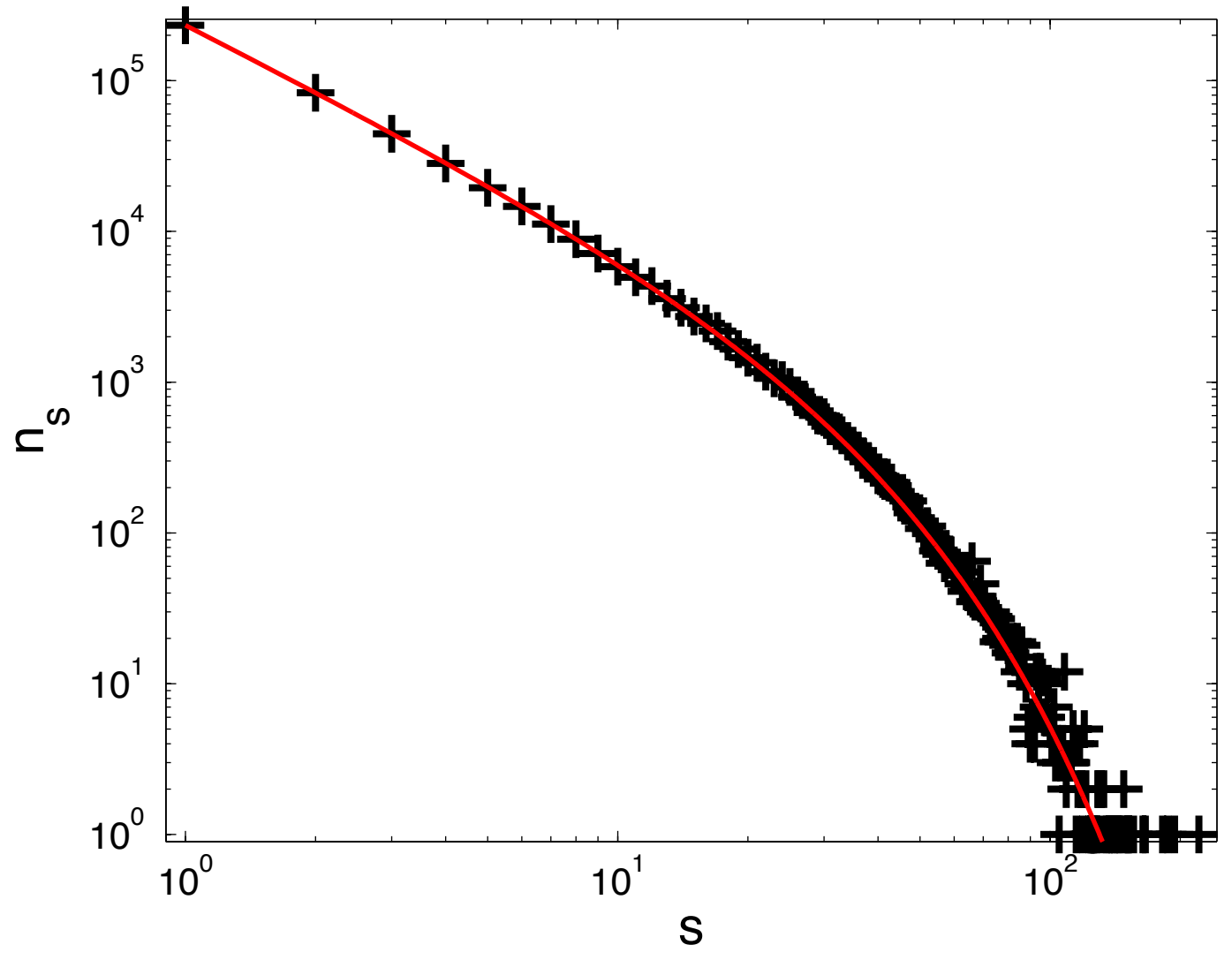


- ★ Damage drives system from spinodal - worse scaling.
- ★ Fit to form.

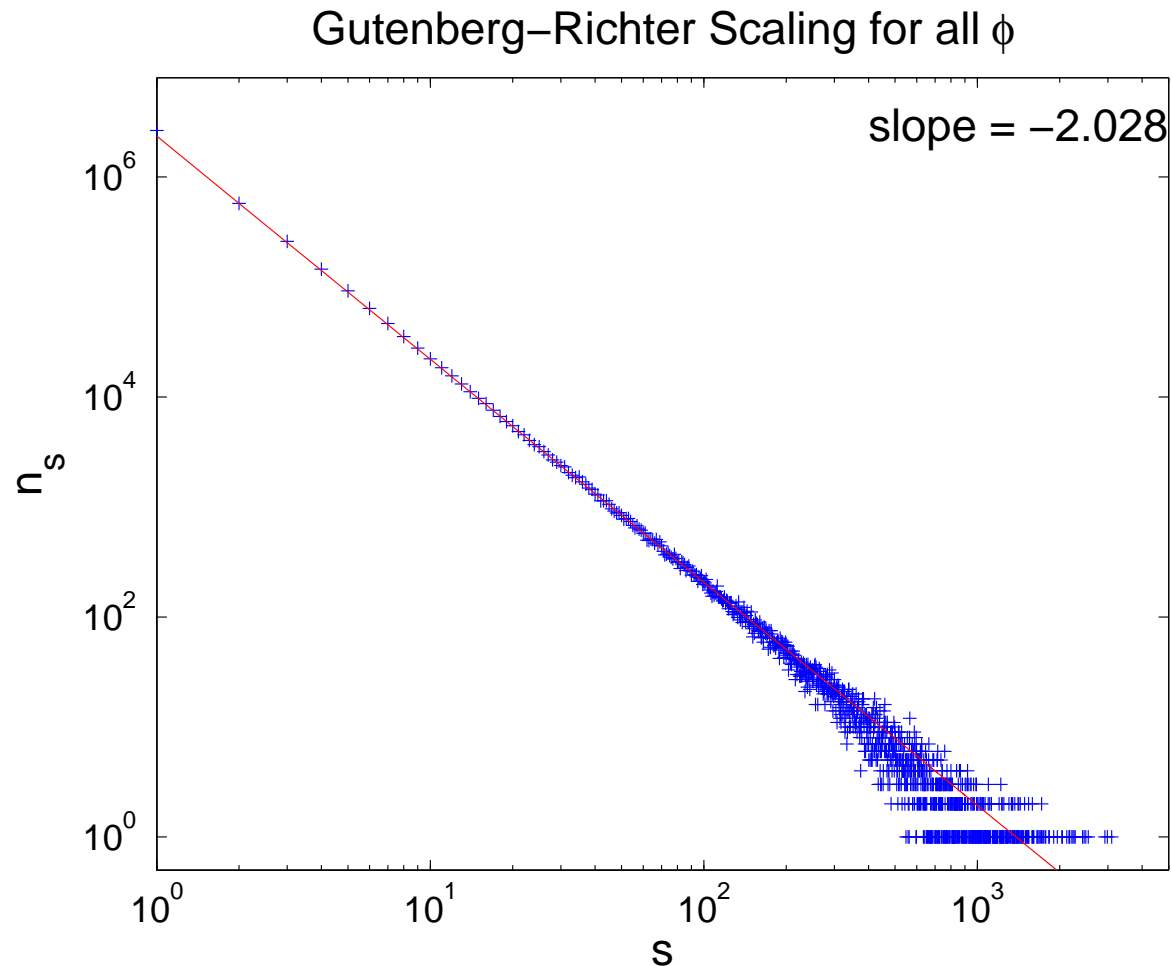
$$n_s \sim \frac{N_o \exp[-\Delta h s]}{p s^{3/2}} \quad (8)$$

Cluster Scaling for  $p = 0.4$ 

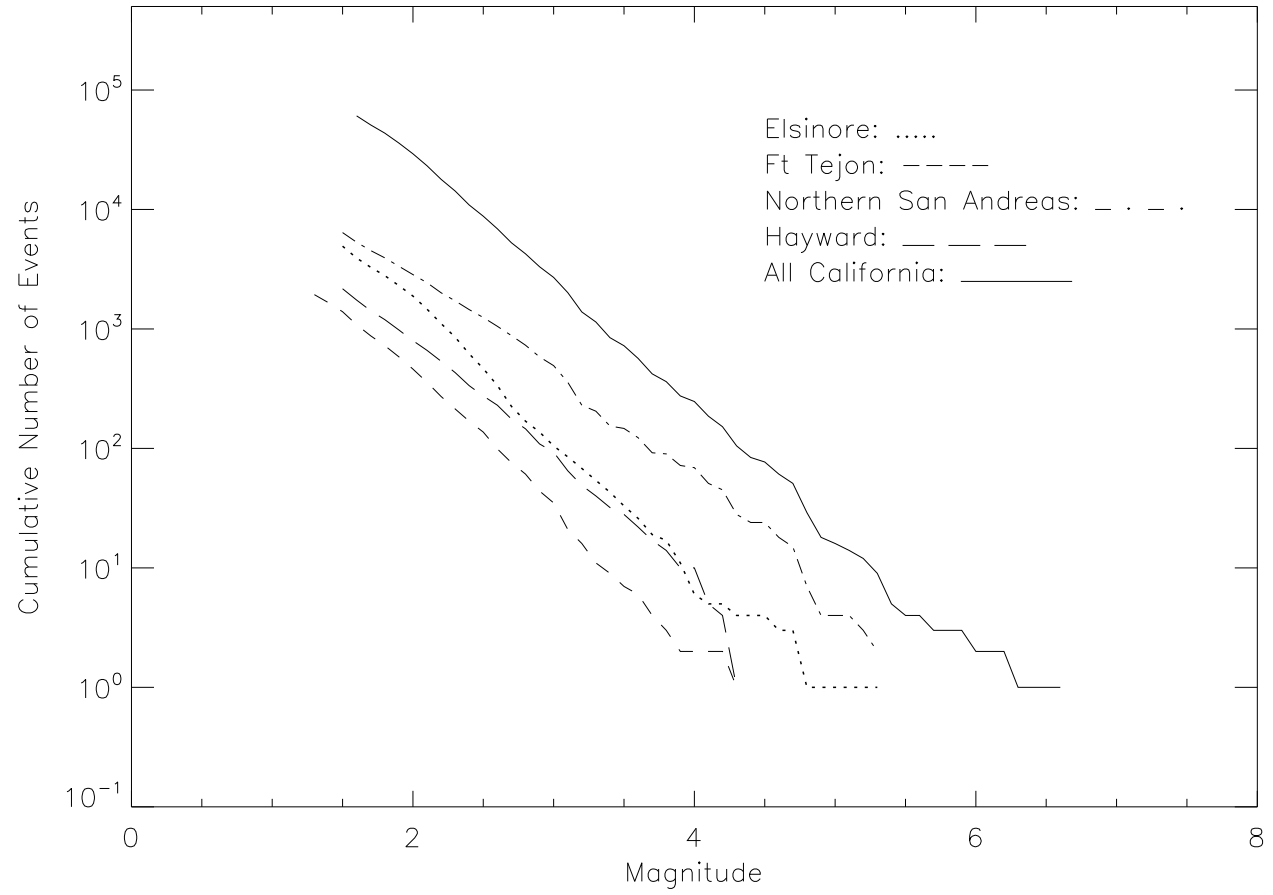
### Cluster Scaling for $p = 0.2$



- ★ Plot  $n_s$  versus  $s$  on a cumulative plot for all  $\phi$



- ★ Appears to be near a critical point. Many non-interacting systems ( $b = 1$ )



How do we understand the cumulative scaling?

- ★ The scaling for a single system is

$$n_s \sim N_o \frac{\exp[-\Delta h s]}{s^{3/2}} \quad (9)$$

- ★ This is number of clusters per site.
- ★ To compare we need to divide by the number of sites for each system.

$$n_s \sim N_o \frac{\exp[-\Delta h s]}{q s^{3/2}} \quad (10)$$

where  $q = 1 - p$ .

We want to integrate over all  $\Delta h$ .

- ★ For scaling assume  $s \gg 1$ .

$$n_{sT} \sim \int_0^{\infty} d\Delta h \frac{\exp[-\Delta h s]}{q s^{3/2}} \quad (11)$$

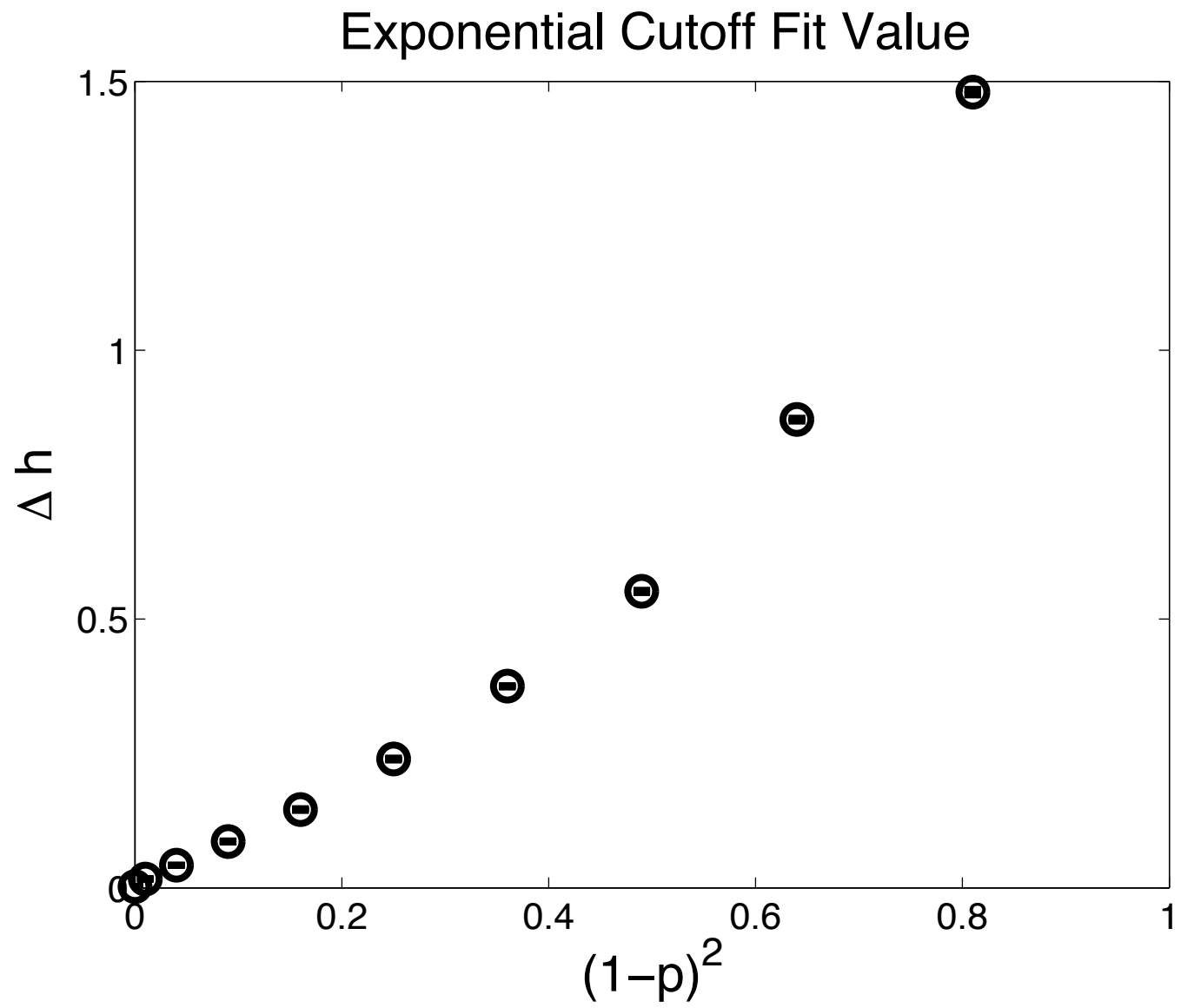
- ★ However  $q$  depends on  $\Delta h$
- ★ Integral restricted to small  $\Delta h$  for  $s \gg 1$ .
- ★ Fit shows that for small  $\Delta h$   $q \sim \Delta h^{1/2}$

Integral becomes

$$n_{sT} \sim \int_0^{\infty} d\Delta h \frac{\exp[-\Delta h s]}{\Delta h^{1/2} s^{3/2}} = \frac{s^{1/2}}{s^{3/2} s} \int_0^{\infty} dy \frac{\exp[-y]}{y^{1/2}} \quad (12)$$

where  $y = s\Delta h$ .

- ★ This gives a slope of  $-2$  or a  $b$  value of  $1$ .



# Conclusions

- ★ Long range systems in punctuated equilibrium.
- ★ Damage drives system from spinodal in long-range systems.
- ★ Scaling persists in cumulative plots.(Gutenberg-Richter b value)
- ★ Correlation length small fraction of system size. → system is not in critical state.