Freie Universität Berlin

Scale-Dependent Models for Atmospheric Flows

Regime(s) of validity of sound-proof models

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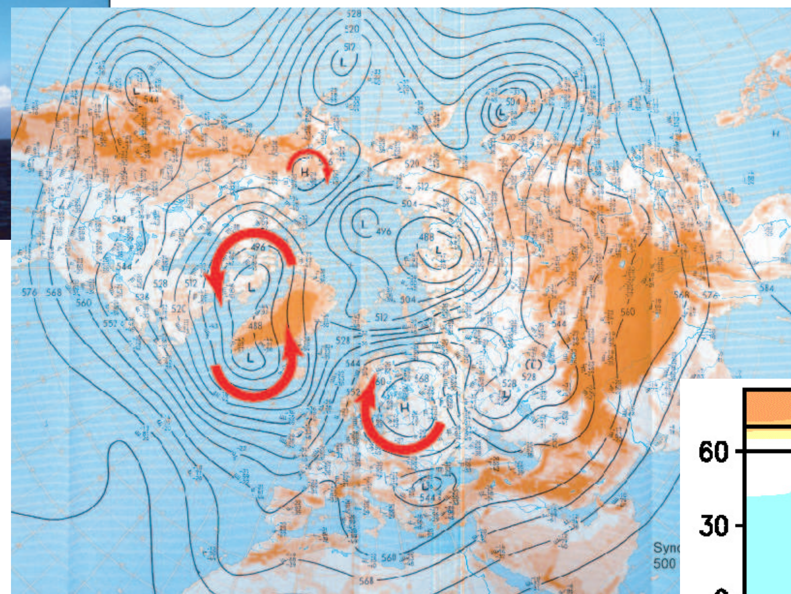
Scale-Dependent Models for Atmospheric Flows

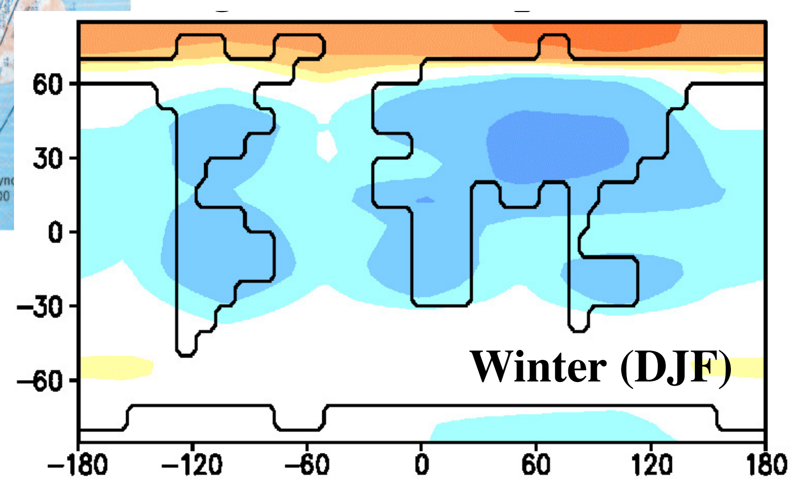
Regime(s) of validity of sound-proof models

Scale-Dependent Models

10 km / 20 min



1000 km / 2 days



10000 km / 1 season

Scale-Dependent Models

$$
\begin{aligned}
\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z + \nabla\pi &= \boldsymbol{S}_{\boldsymbol{u}}\\
w_t + \boldsymbol{u}\cdot\nabla w + w w_z + \pi_z &= -\theta' + S_w\\
\theta'_t + \boldsymbol{u}\cdot\nabla\theta' + w\theta'_z &= S'_\theta\\
\nabla\cdot(\rho_0\boldsymbol{u}) + (\rho_0 w)_z &= 0
\end{aligned}
$$

$$\theta = 1 + \varepsilon^4\theta'(\boldsymbol{x}, z, t) + o(\varepsilon^4)$$

Anelastic Boussinesq Model

10 km / 20 min

$$\left(\partial_\mathcal{T} + \boldsymbol{u}^{(0)}\cdot\nabla\right) q = 0$$

$$q = \zeta^{(0)} + \Omega_0\beta\eta + \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{d\Theta/dz}\theta^{(3)}\right)$$

$$\zeta^{(0)} = \nabla^2\pi^{(3)}, \qquad \theta^{(3)} = -\frac{\partial\pi^{(3)}}{\partial z}, \qquad \boldsymbol{u}^{(0)} = \frac{1}{\Omega_0}\boldsymbol{k}\times\nabla\pi^{(3)}$$

Quasi-geostrophic theory

1000 km / 2 days

$$
\begin{aligned}
\frac{\partial Q_T}{\partial t} + \nabla\cdot\boldsymbol{F}_T &= S_T\\
\frac{\partial Q_q}{\partial t} + \nabla\cdot\boldsymbol{F}_q &= S_q
\end{aligned}
$$

$$Q_\varphi = \int_{z_s}^{H_a}\rho\,\varphi\,dz\,, \qquad \boldsymbol{F}_\varphi = \int_{z_s}^{H_a}\rho\left(\boldsymbol{u}\,\varphi + (\widehat{\boldsymbol{u}'\varphi'}) + \boldsymbol{D}^\varphi\right)dz\,, \qquad \left(\varphi\in\{T,q\}\right)$$

$$T = T_s(t,\boldsymbol{x}) + \Gamma(t,\boldsymbol{x})\Big(\min(z, H_T) - z_s\Big)\,, \qquad q = q_s(t,\boldsymbol{x})\exp\left(-\frac{z-z_s}{H_q}\right)$$

$$\rho = \rho_*\exp\left(-\frac{z}{h_{\rm sc}}\right)\,, \qquad p = p_*\exp\left(-\frac{\gamma z}{h_{\rm sc}}\right) + p_0(t,\boldsymbol{x}) + g\rho_*\int_0^z\frac{T}{T_*}\,dz'$$

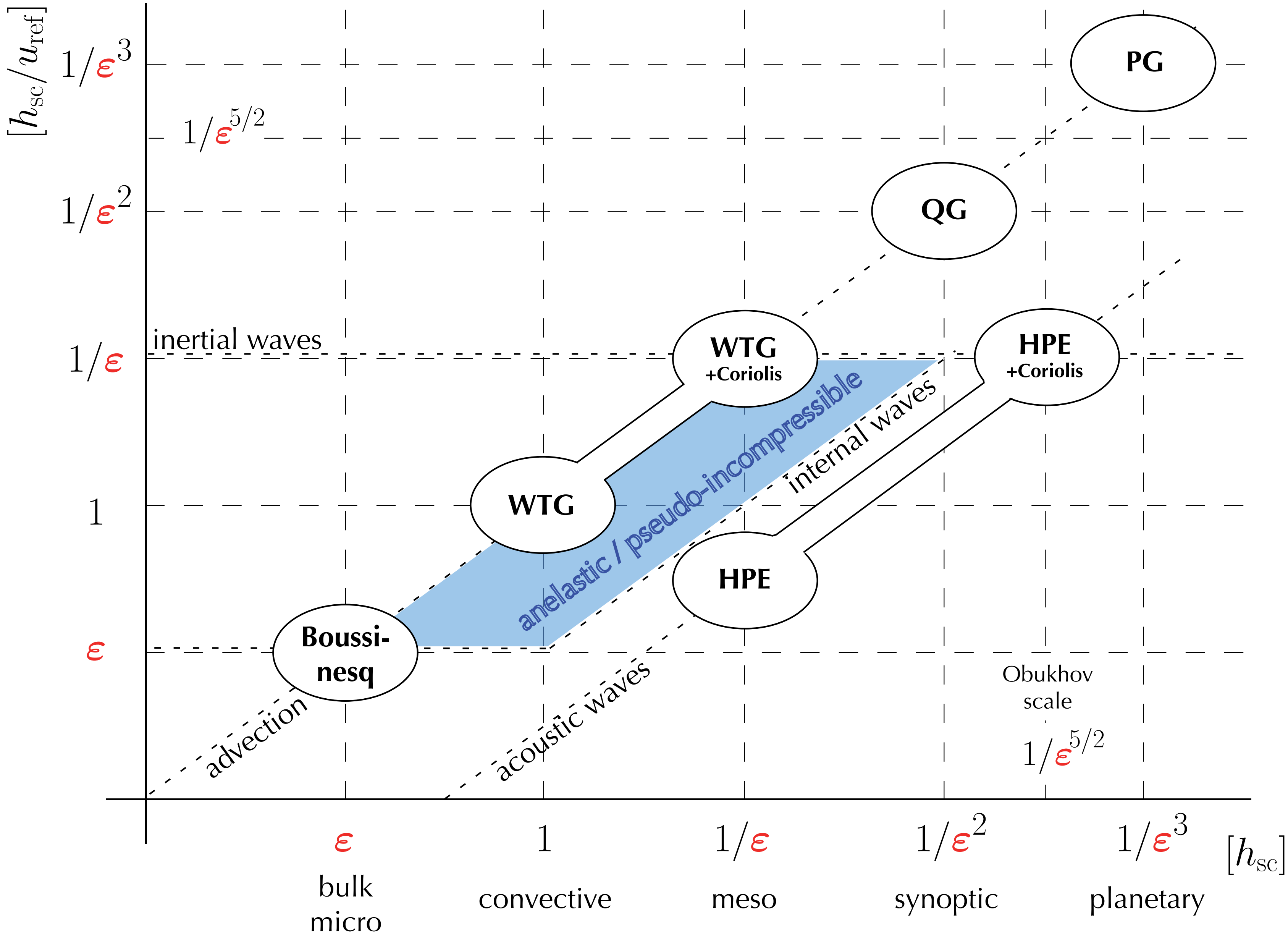
$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a\,, \qquad f\rho_*\boldsymbol{k}\times\boldsymbol{u}_g = -\nabla_{\boldsymbol{x}}p \qquad \boldsymbol{u}_\alpha = \alpha\nabla p_0$$

V. Petoukhov et al., *CLIMBER-2 ...*, Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

10000 km / 1 season

Scale-Dependent Models



R.K., Ann. Rev. Fluid Mech., **42**, 2010

Scale-Dependent Models

Earth's radius	a	$=$	$6 \cdot 10^6$	m
Earth's rotation rate	Ω	$\sim$	10^{-4}	s^{-1}
Acceleration of gravity	g	$=$	9.81	ms^{-2}
Sea level pressure	p_{ref}	$=$	10^5	$\mathrm{kgm}^{-1}\mathrm{s}^{-2}$
H_2O freezing temperature	T_{ref}	$\sim$	273	K
Troposph. potential temp. var.	$\Delta\Theta$	$\sim$	40	K
Dry gas constant	R	$=$	287	$\mathrm{m}^2\mathrm{s}^{-2}\mathrm{K}^{-1}$
Dry isentropic exponent	γ	$=$	1.4	

$$\Pi_1 = \frac{h_{\mathrm{sc}}}{a} \sim 1.6 \cdot 10^{-3} \sim \varepsilon^3$$

$$\Pi_2 = \frac{\Delta\Theta}{T_{\mathrm{ref}}} \sim 1.5 \cdot 10^{-1} \sim \varepsilon$$

$$\Pi_3 = \frac{c_{\mathrm{ref}}}{\Omega a} \sim 4.7 \cdot 10^{-1} \sim \sqrt{\varepsilon}$$

where

$$h_{\mathrm{sc}} = \frac{RT_{\mathrm{ref}}}{g}$$

$$c_{\mathrm{ref}} = \sqrt{RT_{\mathrm{ref}}}$$

R.K., Ann. Rev. Fluid Mech., **42**, 2010

Scale-Dependent Models

Nondimensionalization

$$(\boldsymbol{x}, z) = \frac{1}{h_{\rm sc}} (\boldsymbol{x}', z')\,, \qquad t = \frac{u_{\rm ref}}{h_{\rm sc}} t'$$

$$(\boldsymbol{u}, w) = \frac{1}{u_{\rm ref}} (\boldsymbol{u}', w')\,, \qquad (p, T, \rho) = \left(\frac{p'}{p_{\rm ref}}, \frac{T'}{T_{\rm ref}}, \frac{\rho' R T_{\rm ref}}{p_{\rm ref}} \right)$$

where

$$u_{\rm ref} = \frac{2}{\pi} \frac{g h_{\rm sc}}{\Omega a} \frac{\Delta\Theta}{T_{\rm ref}} \qquad \text{(thermal wind scaling)}$$

Scale-Dependent Models

Classical length scales and dimensionless numbers

$$
\begin{aligned}
L_{\text{meso}} &= \varepsilon^{-1} \; h_{\text{sc}} \\
L_{\text{Ro}} &= \varepsilon^{-2} \; h_{\text{sc}} \\
L_{\text{Ob}} &= \varepsilon^{-5/2} h_{\text{sc}} \\
L_{\text{p}} &= \varepsilon^{-3} \; h_{\text{sc}}
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{Fr}_{\text{int}} &\sim \varepsilon \\
\text{Ro}_{h_{\text{sc}}} &\sim \varepsilon^{-1} \\
\text{Ro}_{L_{\text{Ro}}} &\sim \varepsilon \\
\text{Ma} &\sim \varepsilon^{3/2}
\end{aligned}
$$

Compressible flow equations with general source terms

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w\frac{\partial}{\partial z}\right) \boldsymbol{v}_{\shortparallel} &+ \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\shortparallel} + \frac{1}{\varepsilon^3 \rho} \nabla_{\|} p &= \boldsymbol{S}_{\boldsymbol{v}_{\shortparallel}}\,, \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w\frac{\partial}{\partial z}\right) w &+ \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\perp} + \frac{1}{\varepsilon^3 \rho} \frac{\partial p}{\partial z} &= S_w - \frac{1}{\varepsilon^3}\,, \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w\frac{\partial}{\partial z}\right) \rho &+ \rho \nabla \cdot \boldsymbol{v} &= 0\,, \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w\frac{\partial}{\partial z}\right) \Theta & &= S_{\Theta}\,.
\end{aligned}
$$

Scale-Dependent Models

Recovered classical single-scale models:

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$	Linear small scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)$	Anelastic & pseudo-incompressible models
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon t, \varepsilon^2 \boldsymbol{x}, z)$	Linear large scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Mid-latitude Quasi-Geostrophic Flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Equatorial Weak Temperature Gradients
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1} \xi(\varepsilon^2 \boldsymbol{x}), z)$	Semi-geostrophic flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\underline{\varepsilon^{3/2} t}, \underline{\varepsilon^{5/2} x}, \underline{\varepsilon^{5/2} y}, z)$	Kelvin, Yanai, Rossby, and gravity Waves

... and many more

R.K., Ann. Rev. Fluid Mech., **42**, 2010

Scale-Dependent Models for Atmospheric Flows

Regime(s) of validity of sound-proof models

Motivation

Stratification limit in the design-regime

Wave-breaking regime with strong stratification

Thanks to ...

Ulrich Achatz	(Goethe-Universität, Frankfurt)
Didier Bresch	(Université de Savoie, Chambéry)
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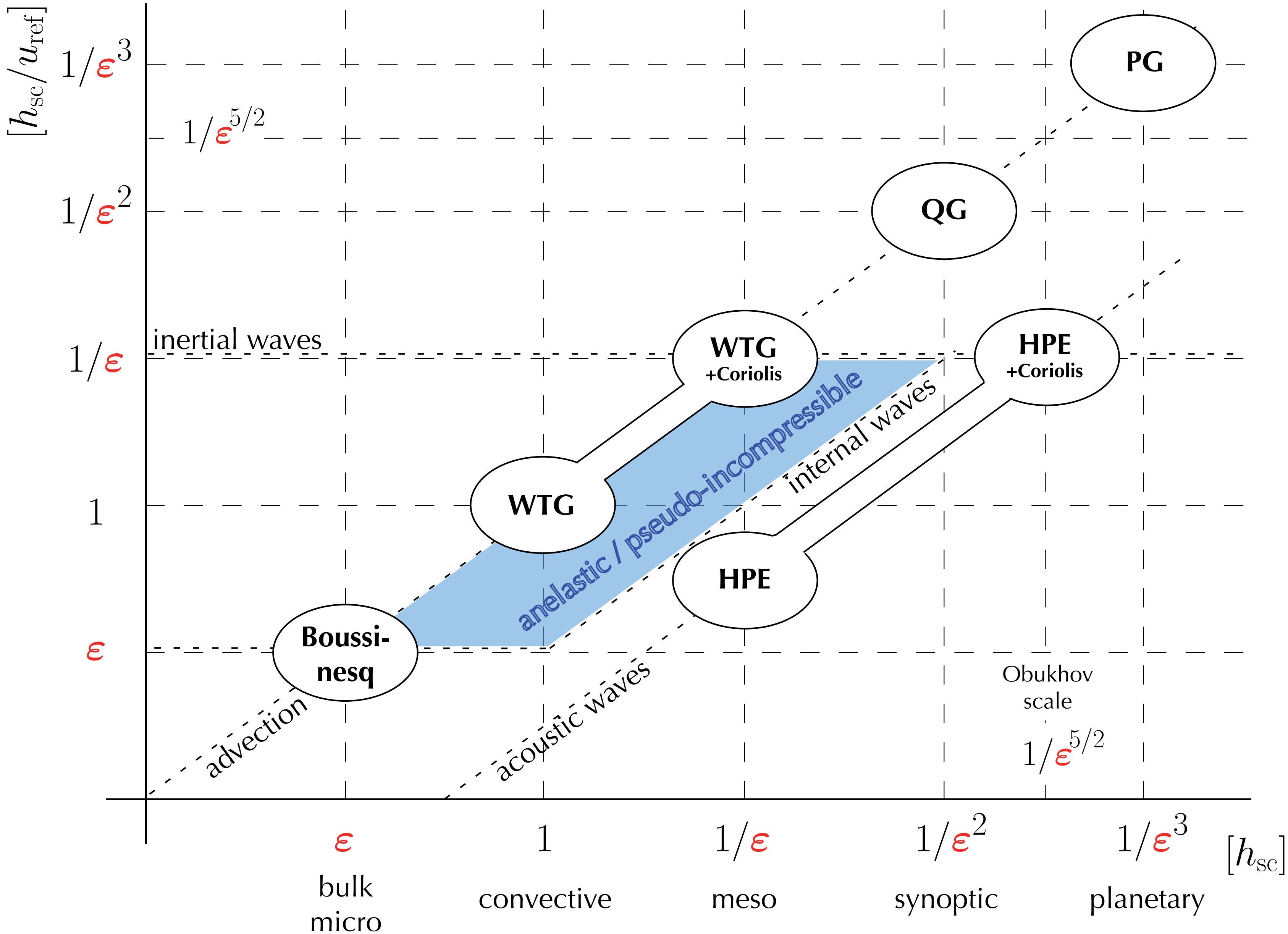
Deutsche
Forschungsgemeinschaft

MetStröm DFG

Regimes of Validity ... Motivation



Regimes of Validity ... Motivation

Compressible flow equations

$$
\begin{aligned}
\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi &= 0 \\
(\rho w)_t \; + \nabla \cdot (\rho \boldsymbol{v} w) + P \pi_z &= -\rho g \\
\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

drop term for:

anelastic (approx.)

pseudo-incompressible

$$
P = p^{\frac{1}{\gamma}} = \rho\theta \,, \qquad \pi = p / \Gamma P \,, \qquad \Gamma = c_p / R \,, \qquad \boldsymbol{v} = \boldsymbol{u} + w \boldsymbol{k} \,, \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)
$$

Regimes of Validity ... Motivation

Anelastic

$$
\begin{aligned}
\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) &= 0 \\
(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi &= \frac{\theta'}{\overline{\theta}} \overline{\rho} g \boldsymbol{k} \\
P_t + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

$$
\overline{\rho}(z)\theta = P \,, \qquad \theta = \overline{\theta}(z) + \theta'
$$

$$
\begin{aligned}
\nabla \cdot (\overline{\rho} \boldsymbol{v}) &= 0 \\
\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \pi &= \frac{\theta'}{\overline{\theta}} g \boldsymbol{k} \\
\theta_t + \boldsymbol{v} \cdot \nabla \theta &= 0
\end{aligned}
$$

$$
\theta' = \theta(z) - \overline{\theta}(z)
$$

Pseudo-incompressible

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \pi &= \frac{\theta'}{\overline{\theta}} \rho g \boldsymbol{k} \\
\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) &= 0
\end{aligned}
$$

$$
\rho(z)\theta = \overline{P} \,, \qquad \theta = \overline{\theta}(z) + \theta'
$$

baroclinic torque / modified divergence

$$
\begin{aligned}
(1/\theta)_t + \boldsymbol{v} \cdot \nabla (1/\theta) &= 0 \\
\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \underline{\theta \nabla \pi} &= \frac{\theta'}{\overline{\theta}} g \boldsymbol{k} \\
\underline{\nabla \cdot (\overline{P} \boldsymbol{v})} &= 0
\end{aligned}
$$

relevant for deep atmospheres / large scales*

*see, e.g., Smolarkiewicz & Dörnbrack (2007)

Regimes of Validity ... Motivation

Anelastic

$$\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$(\overline{\rho}\boldsymbol{v})_t + \nabla \cdot (\overline{\rho}\boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho}\nabla\pi = \frac{\theta'}{\overline{\theta}}\,\overline{\rho} g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P\boldsymbol{v}) = 0$$

$$\overline{\rho}(z)\theta = P\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla\pi = \frac{\theta'}{\overline{\theta}}\, g \boldsymbol{k}$$

$$\theta_t + \boldsymbol{v} \cdot \nabla\theta = 0$$

$$\theta' = \theta(z) - \overline{\theta}(z)$$

Pseudo-incompressible

zero-Mach, variable density flow

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho\boldsymbol{v})_t + \nabla \cdot (\rho\boldsymbol{v} \circ \boldsymbol{v}) + \overline{P}\nabla\pi = \frac{\theta'}{\overline{\theta}}\,\rho g \boldsymbol{k}$$

$$\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) = 0$$

$$\rho(z)\theta = \overline{P}\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\rho_t + \boldsymbol{v} \cdot \nabla\rho = 0$$

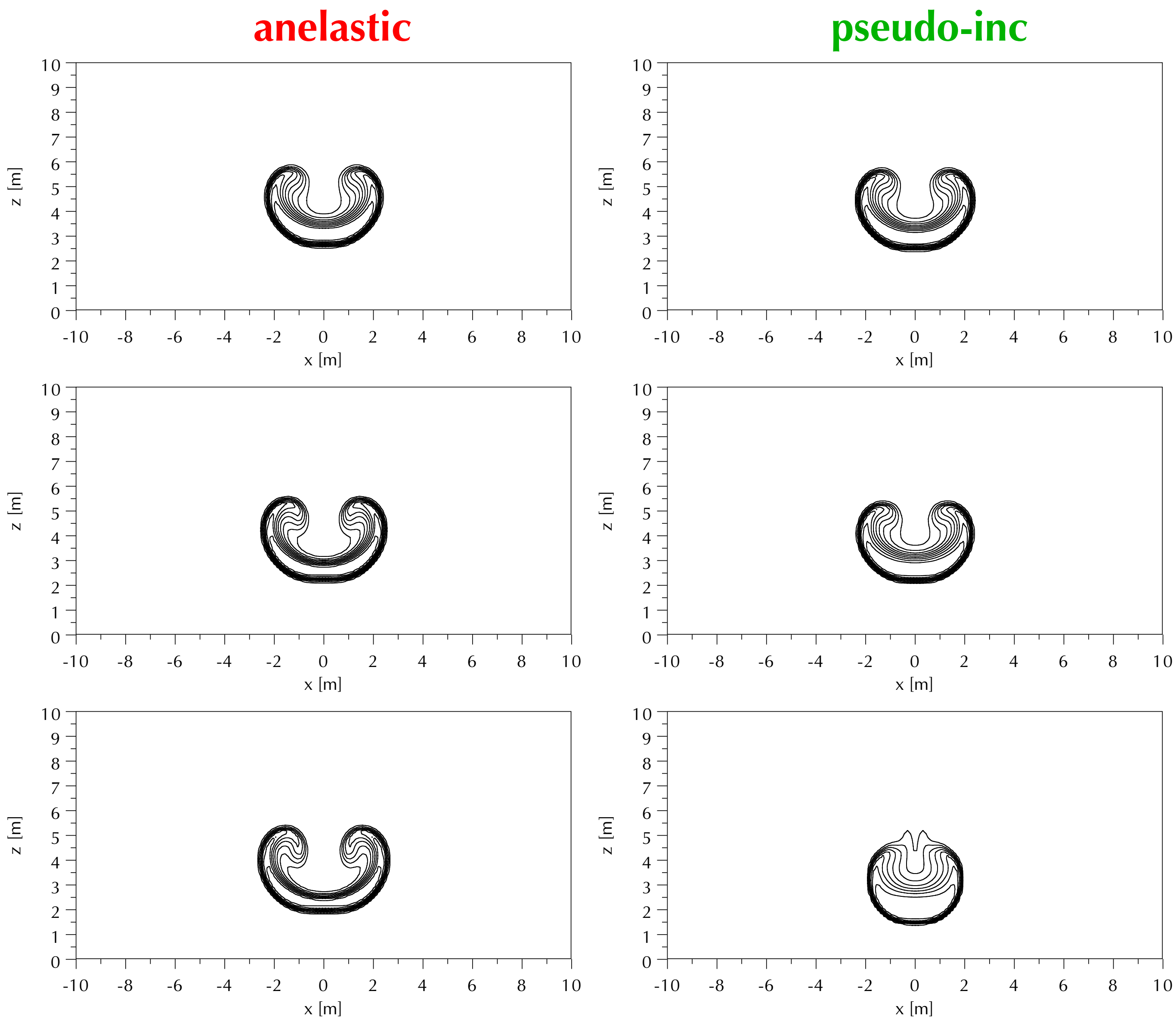
$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho}\nabla\pi = (\rho - \overline{\rho})\, g \boldsymbol{k}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

Small scale limits

Regimes of Validity ... Motivation

Cold air blobs at small scales



$$\theta_1/\theta_2 = 0.9$$

$$\theta_1/\theta_2 = 0.5$$

$$\theta_1/\theta_2 = 0.1$$

Scale-Dependent Models for Atmospheric Flows

Regime(s) of validity of sound-proof models

Motivation

Stratification limit in the design-regime

Wave-breaking regime with strong stratification

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{gh_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2)$$

* Ogura & Phillips (1962)

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu}\widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Regimes of Validity ... Design Regime

Desirable:

1. **Sound-proof model** which
2. accurately represents the **(fast) internal waves**, and
3. remains accurate over **advective time scales**.

Regimes of Validity ... Design Regime

$$
\begin{aligned}
\tilde{\theta}_\tau \; + \; \frac{1}{\varepsilon^{\nu}}\,\tilde{w}\,\frac{d\widehat{\theta}}{dz} \qquad\qquad\qquad\qquad\qquad & = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau \; + \; \frac{1}{\varepsilon^{\nu}}\,\frac{\tilde{\theta}}{\overline{\overline{\theta}}}\,\boldsymbol{k} \quad + \; \frac{1}{\varepsilon}\,\overline{\theta}\nabla\tilde{\pi} \qquad\qquad & = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi} \quad \cdot \\
\tilde{\pi}_\tau \qquad\qquad\qquad + \; \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) & = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

For the linear variable coefficient system:

- ✓ Conservation of weighted quadratic energy
- ✓ Control of time derivatives by initial data $(\tau = O(1))$

... consider internal wave scalings for $\tau = O(\varepsilon^{\nu})$:

$$
\vartheta = \frac{\tau}{\varepsilon^{\nu}}\,, \qquad \pi^{*} = \varepsilon^{\nu-1}\tilde{\pi}\,,
$$

Klainerman, Majda (1982) ... Dutrifoy, Majda, Schochet (2006,2008)

Regimes of Validity ... Design Regime

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_\vartheta \; + \; \tilde{w}\,\frac{d\overline{\theta}}{dz} \qquad\qquad = \; 0$$

$$\tilde{\boldsymbol{v}}_\vartheta \; + \; \frac{\tilde{\theta}}{\overline{\theta}^{\varepsilon}}\,\boldsymbol{k} + \overline{\theta}^{\varepsilon}\nabla\pi^* \qquad = \; 0$$

$$\varepsilon^{\mu}\,\pi^*_\vartheta + \left(\gamma\Gamma\overline{\pi}^{\varepsilon}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}^{\varepsilon}}{dz}\right) \; = \; 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^* \end{pmatrix}(\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^* \\ \boldsymbol{U}^* \\ W^* \\ \Pi^* \end{pmatrix}(z)\;\exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Regimes of Validity ... Design Regime

Relation between compressible and pseudo-incompressible vertical modes

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}}}\frac{1}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^*$$

$\varepsilon^{\mu} = 0$: pseudo-incompressible case

regular Sturm-Liouville problem for internal wave modes

$\varepsilon^{\mu} > 0$: compressible case

nonlinear Sturm-Liouville problem ...

$\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}} = O(1)$: perturbations of pseudo-incompressible modes & EVals

Regimes of Validity ... Design Regime

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}}}\frac{1}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^*$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}} = O(1)\right)$

- pseudo-incompressible modes/EVals $=$ compressible modes/EVals + $O(\varepsilon^{\mu})$ †
- phase errors remain small **over advection time scales** for $\mu > \frac{2}{3}$

The anelastic and pseudo-incompressible models remain relevant for stratifications

$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{sc}} \lesssim 40\ \text{K}$$

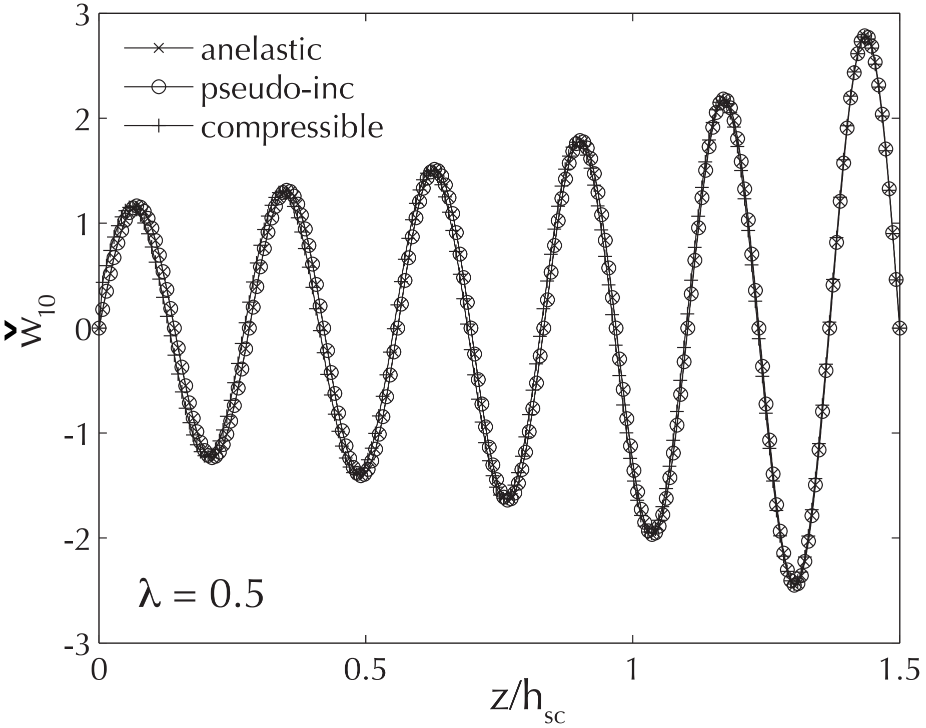
not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

R.K., U. Achatz et al., JAS, submitted

† rigorous proof with D. Bresch

Regimes of Validity ... Design Regime

A typical vertical structure function $(L \sim \pi h_{\rm sc} \sim 30 \text{ km})$



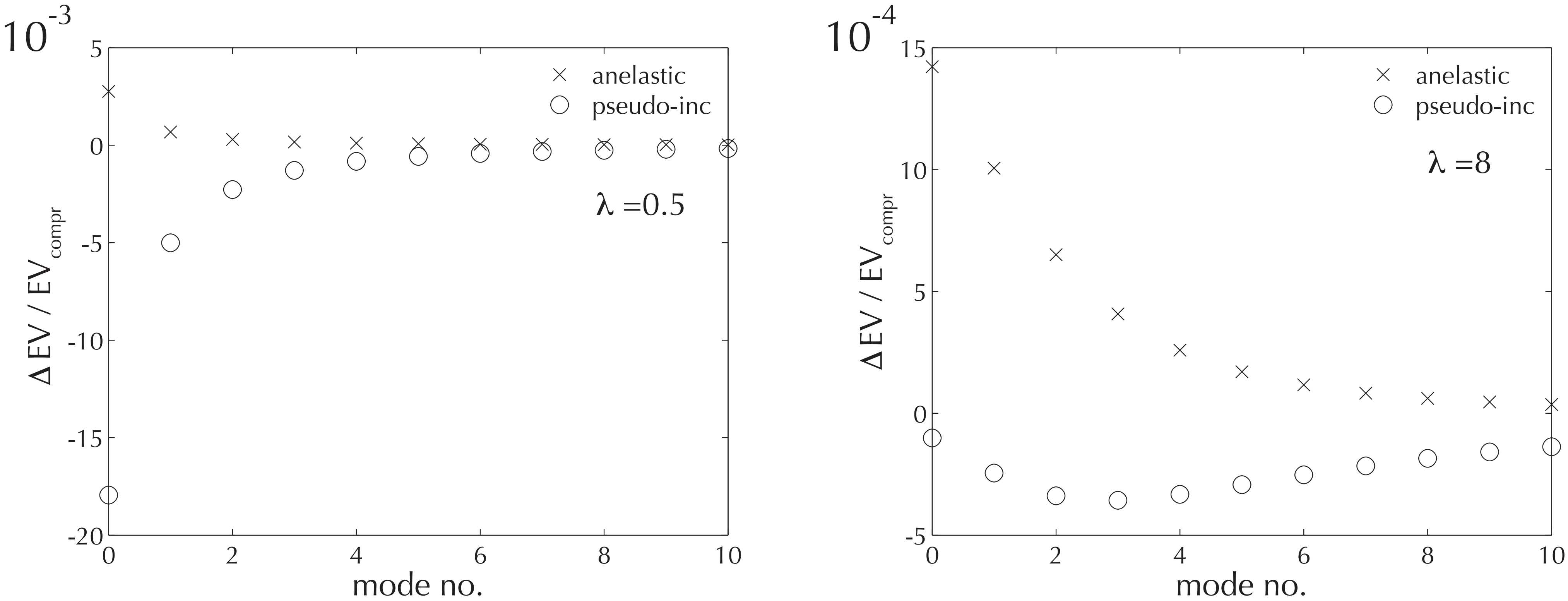
R.K., U. Achatz et al., JAS, submitted

thanks to Dr. Veerle LeDoux, Ghent, for the SL-solver MATSLISE!

Regimes of Validity ... Design Regime

Eigenvalue errors

$$\frac{EV_{sproof} - EV_{compr}}{EV_{compr}}$$



R.K., U. Achatz et al., JAS, submitted

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Regime(s) of validity of sound-proof models

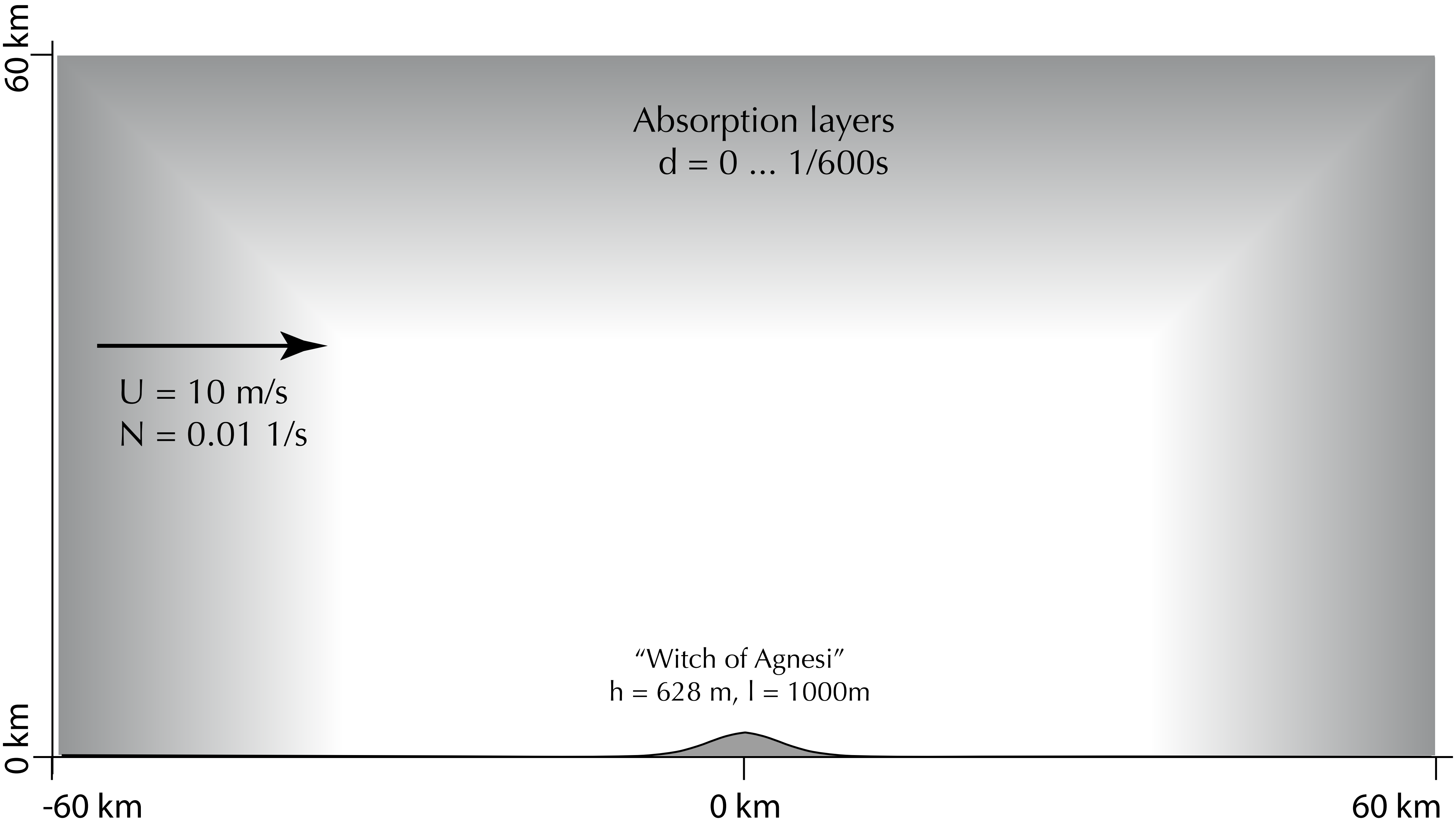
Motivation

Stratification limit in the design-regime

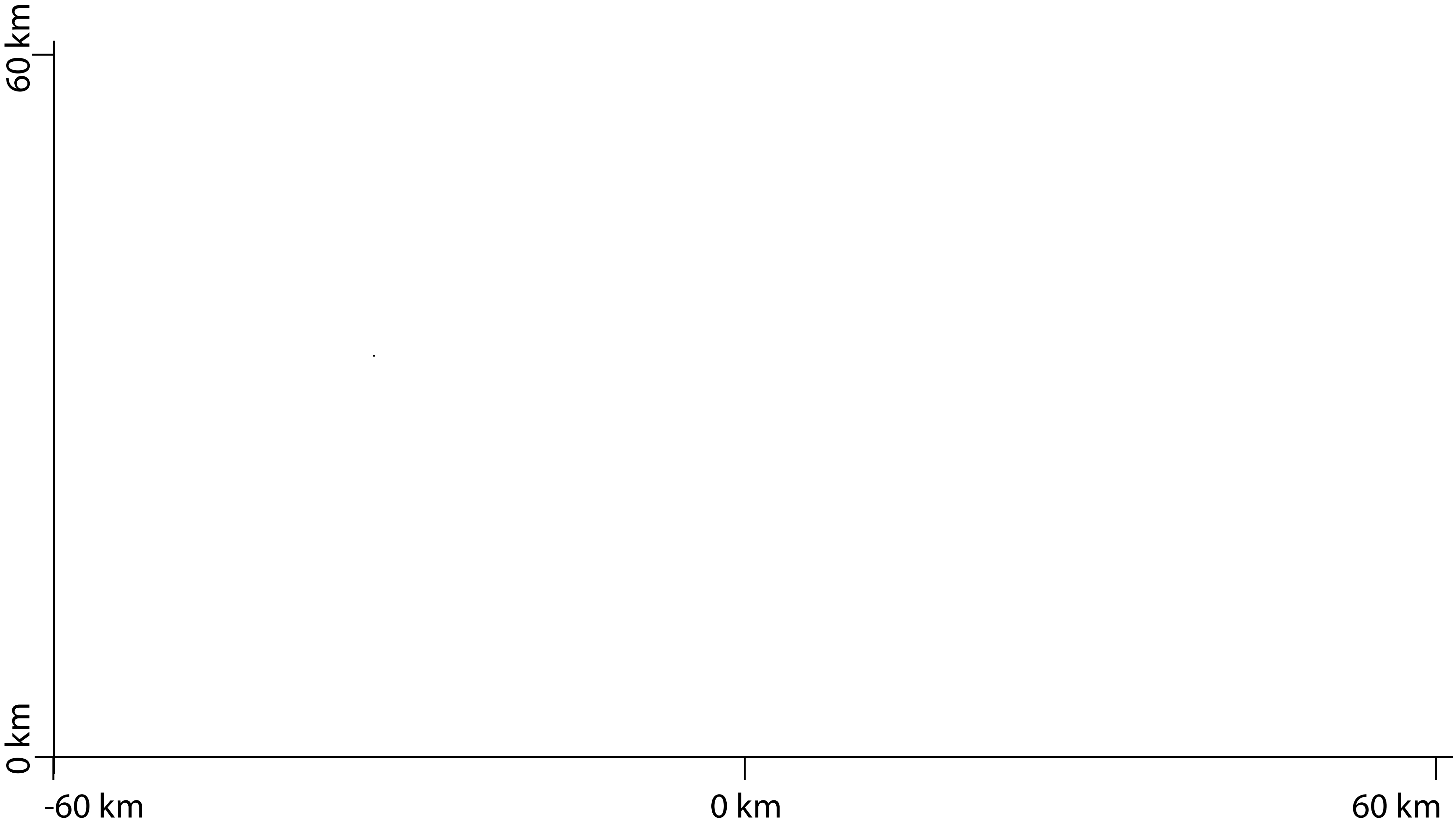
Wave-breaking regime with strong stratification

Achatz, K., Senf, JFM, to be submitted

Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Remarks

- Test case involves isothermal stratification
  at $\gamma = 1.067$: isothermal $\approx$ isentropic

- In the real atmosphere, internal waves tend to break in the stratosphere or higher up
  yet at $\gamma \sim 1.4$: isothermal $\not\approx$ isentropic

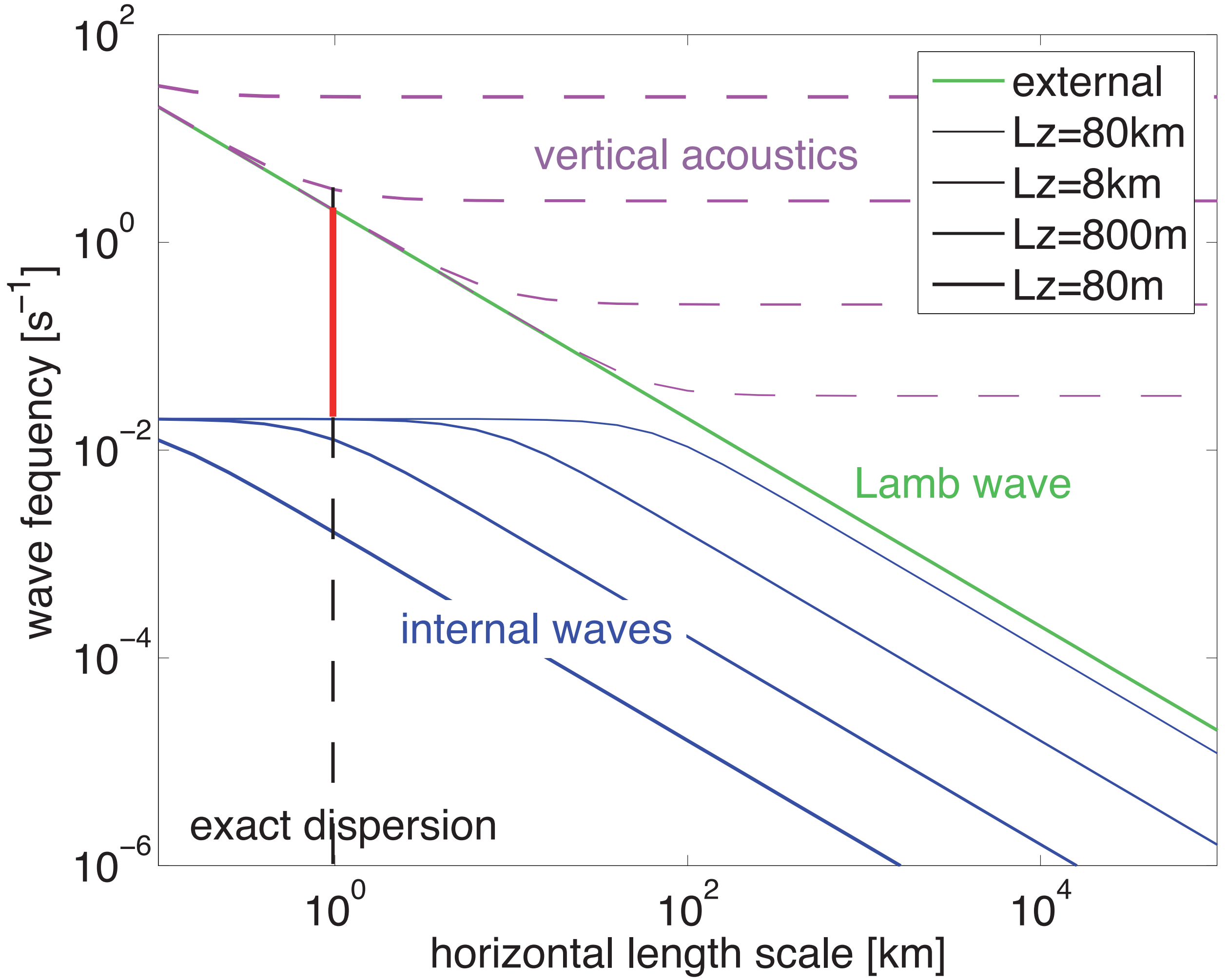
- Modifying Durran (1988)
  The pseudo-incompressible approximation should be valid whenever the time scale, T, of the relevant flow phenomena* satisfies $T \gg T_{\text{sound}}$.

$$\Downarrow$$

Consider short-wavelength internal modes in a strongly stratified atmosphere

* Durran (1998): "lagrangian time scale"

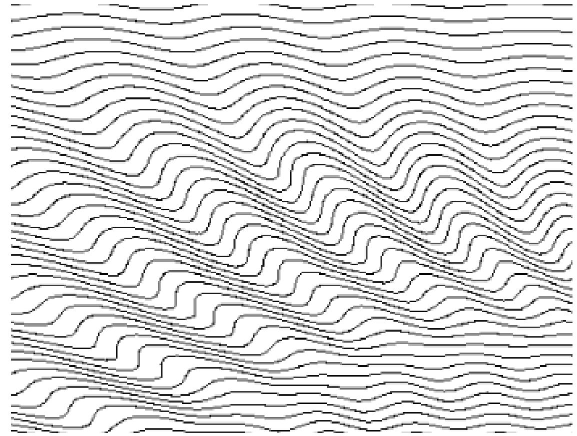
In fact there is a time scale gap for short waves $\sim 1\ \mathrm{km}$



Wave breaking regime, strong stratification

WKB theory:

- ~ 1 km wave packets
- modulated over ~ 10 km distances
- stratification of order $O(1)$
- scalings allow overturning of θ-contours



Expansion scheme:

$$U(t, \boldsymbol{x}, z; \boldsymbol{\varepsilon}) = \overline{U}(z) + U_1^{(0)} \exp\left(i\frac{\varphi^{\varepsilon}}{\boldsymbol{\varepsilon}}\right) + \boldsymbol{\varepsilon} \sum_{n=0}^{\mathbf{2}} U_n^{(1)} \exp\left(in\frac{\varphi^{\varepsilon}}{\boldsymbol{\varepsilon}}\right)$$

$$\varphi^{\varepsilon} = \varphi^{(0)} + \boldsymbol{\varepsilon}\varphi^{(1)} + o(\boldsymbol{\varepsilon})$$

$$\left(U_n^{(i)}, \varphi^{(i)}\right) \equiv \left(U_n^{(i)}, \varphi^{(i)}\right)(t, \boldsymbol{x}, z)$$

Wave breaking regime, strong stratification

Leading order: — classical Boussinesq / ray tracing theory

$$\underbrace{\begin{pmatrix} -i\hat{\omega} & 0 & 0 & ik \\ 0 & -i\hat{\omega} & -N & im \\ 0 & N & -i\hat{\omega} & 0 \\ ik & im & 0 & 0 \end{pmatrix}}_{M(\hat{\omega},k,m)} \begin{pmatrix} \hat{U}^{(0)} \\ \hat{W}^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}^{(2)} \end{pmatrix} = 0 \qquad \text{where} \qquad \begin{cases} \hat{\omega} = -\dfrac{\partial \varphi^{(0)}}{\partial t} - k u_0^{(0)} \\ k = \dfrac{\partial \varphi^{(0)}}{\partial \boldsymbol{x}} \\ m = \dfrac{\partial \varphi^{(0)}}{\partial z} \end{cases}$$

Wave breaking regime, strong stratification

First order: — **pseudo-incompressible** corrections

$$
M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \dfrac{1}{N}\dfrac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \dfrac{1}{N}\dfrac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\dfrac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\dfrac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\dfrac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\dfrac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\dfrac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\dfrac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\dfrac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \dfrac{1}{N}\left[-\dfrac{\partial}{\partial \tau}\left(\dfrac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\dfrac{\partial}{\partial \chi}\left(\dfrac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\dfrac{\partial U_1^{(0)}}{\partial \chi} - \dfrac{\partial W_1^{(0)}}{\partial \zeta} - \dfrac{1-\kappa}{\kappa}\dfrac{W_1^{(0)}}{\pi^{(0)}}\dfrac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}
$$

Wave breaking regime, strong stratification

First order: — **pseudo-incompressible** corrections

$$M^{(0)}\begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)}\begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

- First-order Hamilton-Jacobi-eqn. for $\varphi^{(1)}$

Wave breaking regime, strong stratification

First order: — **pseudo-incompressible** corrections

$$
M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \dfrac{1}{N}\dfrac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \dfrac{1}{N}\dfrac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\dfrac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\dfrac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\dfrac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\dfrac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\dfrac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\dfrac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\dfrac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \dfrac{1}{N}\left[-\dfrac{\partial}{\partial \tau}\left(\dfrac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\dfrac{\partial}{\partial \chi}\left(\dfrac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\dfrac{\partial U_1^{(0)}}{\partial \chi} - \dfrac{\partial W_1^{(0)}}{\partial \zeta} - \dfrac{1-\kappa}{\kappa}\dfrac{W_1^{(0)}}{\pi^{(0)}}\dfrac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}
$$

- Wave action conservation law

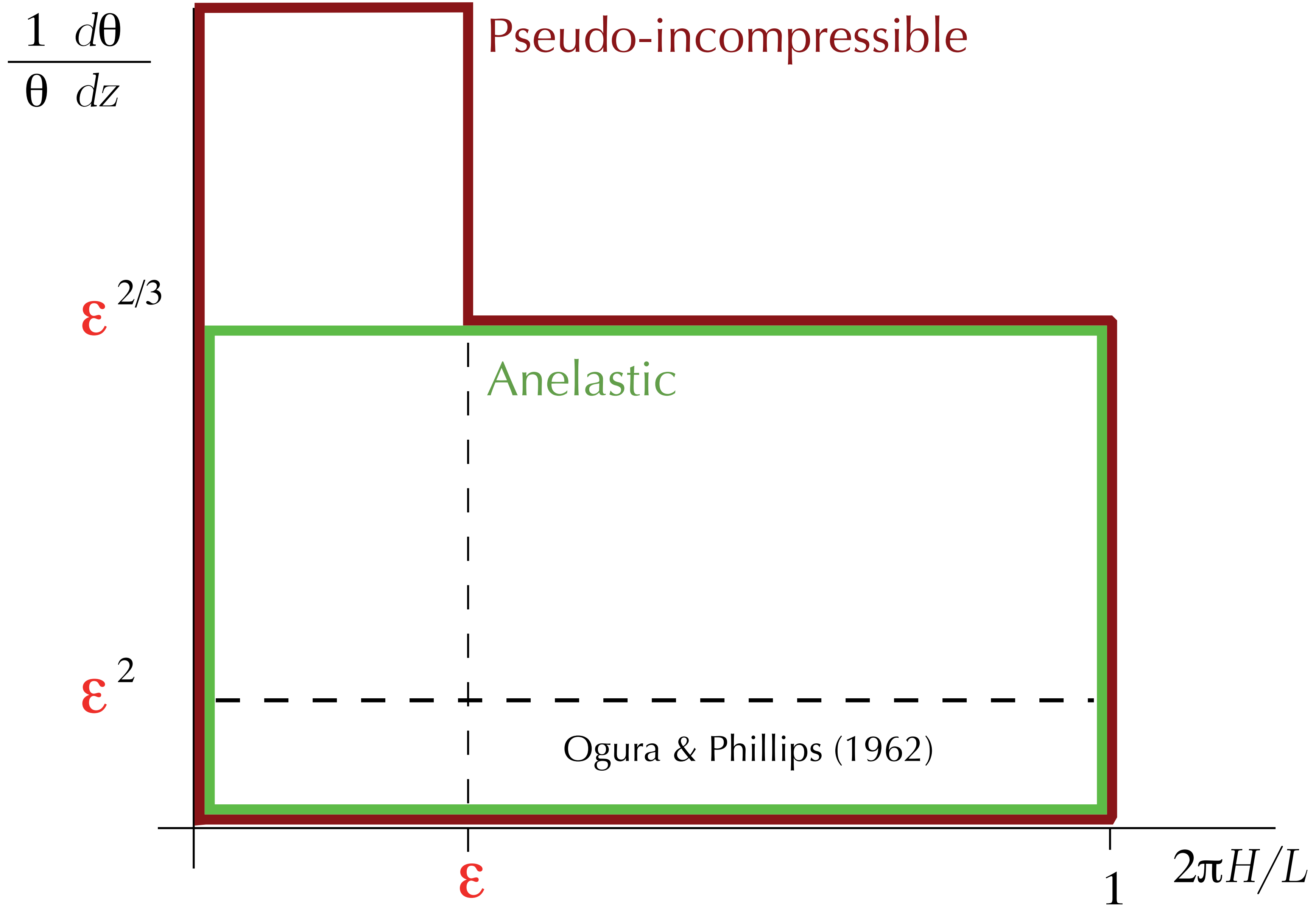
Wave breaking regime, strong stratification

First order: — **pseudo-incompressible** corrections

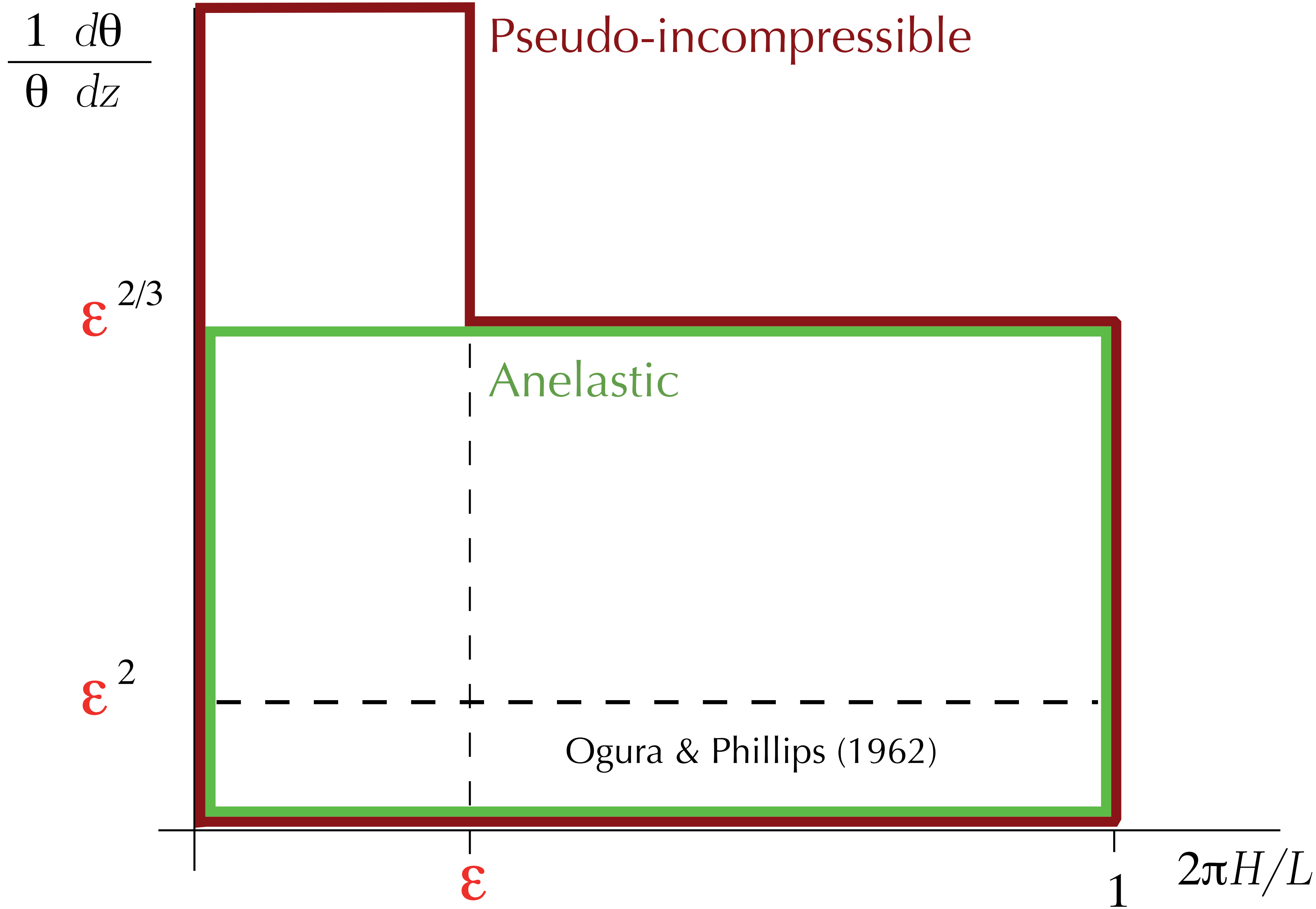
$$M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

+ Explicit solutions for all higher-oder modes $\sim \exp\left(i\,n\,\varphi^{(1)}/\varepsilon\right)$, $\qquad (n = 1, 2, \ldots)$

Summary


$\frac{1}{\theta}\frac{d\theta}{dz}$
Pseudo-incompressible
$\varepsilon^{2/3}$
Anelastic
ε^{2}
Ogura & Phillips (1962)
ε
1
$2\pi H/L$

Summary



$\frac{1}{\theta}\frac{d\theta}{dz}$
Pseudo-incompressible
$\varepsilon^{2/3}$
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1
$2\pi H/L$