Quantifier elimination, decision algorithm, effectivenesss vs non-effectiveness in semialgebraic geometry

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Outline

- Presentation of quantifier elimination and decision problem
- Quantifier elimination algorithm for real closed fields
- How to obtain effective uniform bounds for free: example of Hauptvermutung
- Semialgebraic geometry is not always effective

ref.: Acquistapace-Benedetti-Broglia 1990, C.

Two problems

1. Does there exist four disjoint unit spheres in $\mathbb{R}^3$ which have 12 real common tangents ?
2. Describe the set of $(A, B, C, D) \in (\mathbb{R}^3)^4$ which are the centers of distinct unit spheres in $\mathbb{R}^3$ having 12 real common tangents.

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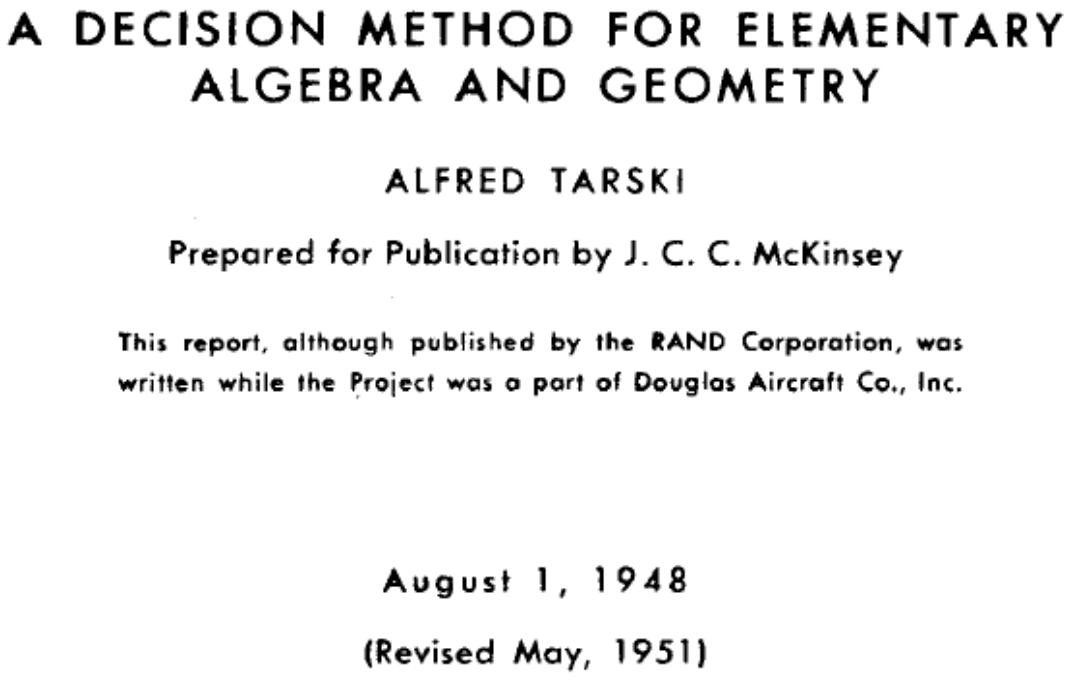
Tarski has an algorithm to solve both problems (in principle), result obtained in the 30's:



A DECISION METHOD FOR ELEMENTARY ALGEBRA AND GEOMETRY

ALFRED TARSKI

Prepared for Publication by J. C. C. McKinsey

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Elementary algebra and geometry

All that can be expressed by a formula (of the first order language of ordered fields), obtained as follows :

1. If $P \in \mathbb{Z}[X_1, \ldots, X_n]$, then $P = 0$ et $P > 0$ are formulas.
2. If Φ and Ψ are formulas, then "Φ and Ψ", "Φ or Ψ", "not Φ" are formulas.
3. If Φ is a formula and X a variable (intended to range over $\mathbb{R}$), then $\exists X\ \Phi$ and $\forall X\ \Phi$ are formulas.

Quantifier-free formula: obtained only with 1 and 2.
Closed formula: without free variable.

- Exercise: express by a formula the fact that the line through $P = (p_1, p_2, p_3)$ directed by the unit vector $U = (u_1, u_2, u_3)$ is tangent to the unit sphere with center $A = (a, b, c)$; express that the line through P directed by U is different from the line through P' directed by U'; express that the two unit spheres with center A and A' are disjoint.
- The existence of four unit spheres as in problem 1 can be expressed by a closed formula. Decision problem: decide whether it is true or not.
- The set of (A, B, C, D) as in problem 2 can be described by a formula with 12 free variables. Quantifier elimination: produce a description by a quantifier-free formula (boolean combination of polynomial equations and inequalities)

Real-closed fields

Axioms of real closed fields:

- ordered field
- intermediate value theorem for polynomials

Note: "For every polynomial ..." is not a formula. "For every polynomial of degree $\leq d$ ..." is a formula (quantification on $d+1$ coefficients). Intermediate value theorem for polynomials is expressed by infinitely many axioms.

Examples of real closed fields: $\mathbb{R}$, $\mathbb{R}_{\mathrm{alg}}$, Puiseux series $\sum_{i \geq i_0} a_i \, \epsilon^{i/q}$ with coefficients in a real-closed field...

The only archimedean real-closed fields are subfields of $\mathbb{R}$.
Archimedeanity cannot be expressed by a formula:
$\forall x \, \exists n \in \mathbb{N} \; x \leq n$ contains a quantification on a natural number, and no formula describes the set of natural numbers.

Quantifier elimination (QE)

Theorem

There is an algorithm which takes as input any formula Φ and returns a quantifier-free formula which is provably equivalent to Φ in the theory of real-closed fields.

Well known example: $\exists X\ AX^2 + BX + C = 0$ is equivalent to

$$(A \neq 0 \quad \text{and} \quad B^2 - 4AC \geq 0)$$
$$\text{or} \quad (A = 0 \quad \text{and} \quad B \neq 0) \quad \text{or} \quad (A = B = C = 0)\,.$$

Decision method

As a consequence (quantifier free closed formulas are boolean combinations of $n > 0$ or $n = 0$ where n is integer):

Theorem

There is an algorithm which takes as input any closed formula Φ (without free variable) and decides whether Φ or its negation is a theorem of the theory of real-closed fields.

In contrast, there cannot be a decision method for elementary arithmetic, i.e. the arithmetic of natural numbers with addition and multiplication (Gödel-Church-Rosser).

QE via counting roots (Tarski)

- Algorithmic reduction to an equivalent prenex form

$$Q_1 X_1 \; Q_2 X_2 \ldots Q_\ell X_\ell \; \Psi(Y_1, \ldots, Y_k, X_1, \ldots, X_\ell) \; ,$$

  where Q_i are quantifiers and Ψ is quantifier-free.
- It suffices to eliminate one quantifier at a time, and to deal with an existential quantifier.
- $\exists X \, \mathcal{S}(Y_1, \ldots, Y_k, X)$ where $\mathcal{S}$ is a system of polynomial equations and inequalities.
- Case of $P(\underline{Y}, X) = 0, Q_1(\underline{Y}, X) > 0, \ldots, Q_r(\underline{Y}, X) > 0$: variants of Sturm to produce Boolean combinations $\mathcal{T}_c$ of sign conditions on the coefficients (depending on $\underline{Y}$) satisfied iff the number of solutions in X is c.
- Primitive recursive complexity.

QE via CAD (Cylindrical Algebraic Decomposition - Collins)

- Starting with $Q_1 X_1\ Q_2 X_2 \ldots Q_\ell X_\ell\ \Psi(Y_1, \ldots, Y_k, X_1, \ldots, X_\ell)$, where Ψ is a Boolean combination of sign conditions on polynomials in a finite family $\mathcal{P}$.
- Construct a CAD of $R^{k+\ell}$ adapted to $\mathcal{P}$. The formula describes a union of cells in R^k. Already OK for decision algorithm.
- For quantifier elimination, a quantifier-free description of cells is needed. This is provided by Thom's lemma: if $\mathcal{Q} \subset R[X]$ is a finite family of polynomials closed under derivation, $\bigcap_{Q \in \mathcal{Q}} \{x \in R \mid Q(x)\ ?_Q\ 0\}$ (where $?_Q$ is either $>$, $=$, $<$) is empty, or a point or an open interval.
- Better complexity: doubly exponential in the number of variables (free and bound).

QE via critical points

- Instead of eliminating one quantifier after the other, eliminate one block of existential quantifiers at a time using a parametric version of critical point method.
- Complexity: doubly exponential in the number of alternations of quantifiers (Grigoriev-Vorobjov, Renegar, Basu-Pollack-Roy).

Semialgebraic sets

By definition, a *semialgebraic subset* in R^n (R a real closed field) is described by a quantifier-free formula with parameters in R (some free variables replaced with elements of R). Semialgebraic mapping = mapping with semialgebraic graph.

Quantifier elimination: any subset described by a formula with parameters is semialgebraic.

Coarse complexity for semialgebraic sets

A semialgebraic subset $S \subset R^n$ has *complexity* $\leq p$ if it can be described as

$$S = \bigcup_{i=1}^{q} \bigcap_{j=1}^{r_i} \{x \in R^n \mid f_{i,j}(x) \ ?_{i,j} 0\} \ ,$$

where $q \leq p$, $r_i \leq p$, $f_{i,j} \in R[X_1, \ldots, X_n]$ have degrees $\leq p$ and $?_{i,j}$ is $=$ or > 0

Quantification over semialgebraic sets or mapping is not possible within elementary algebra, but quantification on semialgebraic sets or mappings of complexity $\leq p$ is possible.

Semialgebraic triangulation

Theorem

Let S be a closed and bounded semialgebraic subset of R^n. Then there exists a finite simplicial complex K in R^n and a semialgebraic homeomorphism $\phi : |K| \to S$

Elementary proof (in the theory of real closed fields) using CAD + control on adjacency of cells.

Effective uniform bound for free

- One can algorithmically produce from (n, p, q) a formula $\Theta(n, p, q)$ which says "For every closed and bounded semialgebraic subset of R^n of complexity $\leq p$, there exists a simplical complex K in R^n with $\sharp K \leq q$ (number of simplices) and a semialgebraic homeomorphism $\phi : |K| \to S$ of complexity $\leq q$."
- Elementary proof + compactness of first-order logic $\Rightarrow$ uniform bound: for every (n, p) there is a q such that $\Theta(n, p, q)$ is a theorem.
- Decision method applied to $\Theta(n, p, 1), \Theta(n, p, 2), \Theta(n, p, 3), \ldots \Rightarrow$ this bound is effective, i.e. a recursive function of (n, p).
- Very weak result: with a little more work, doubly exponential bound .

Semialgebraic Hauptvermutung

Semialgebraic triangulations are essentially unique:

Theorem (Shiota-Yokoi 1984)

Let K, L be simplicial complexes and let $\phi : |K| \to |L|$ be a semialgebraic homeomorphism. Then there are subdivisions K' of K and L' of L such that K' is simplicially isomorphic to L'. This amounts to say that $|K|$ and $|L|$ are PL (piecewise linear)-homeomorphic.

The proof is not elementary: uses integration of vector fields, working over $\mathbb{R}$, not in the theory of real closed fields.

Note: there are homeomorphic polyhedra $|K|$ and $|L|$ which are homeomorphic but not PL-homeomorphic.

Expressing Hauptvermutung by a formula

$\Phi(n, d, p, s)$: "For every couple (K, L) of simplicial complexes in affine n-space with $\sharp K, \sharp L \leq d$ and for every semialgebraic homeomorphism $f : |K| \to |L|$ of complexity $\leq p$, there exist simplicially isomorphic subdivisions K' of K and L' of L with $\sharp K', \sharp L' \leq s$."

There is an algorithm with input (n, d, p, s) and output $\Phi(n, d, p, s)$ written as a formula (of the first order language of ordered fields).

Effective Hauptvermutung

Lemma

There exists $s(n,d)$ such that for every PL-homeomorphic polyhedra $|K|$ et $|L|$ with $\sharp K, \sharp L \leq d$, there exist simplicially isomorphic subdivisions K' of K and L' of L whith $\sharp K', \sharp L' \leq s(n,d)$.

Proof: There are finitely many such couples (K, L) up to simplicial isomorphisms.

Shiota-Yokoi + Lemma : for every n, d, p, there is s such that $\mathbb{R}$ satisfies $\Phi(n, d, p, s)$.

Effective Hauptvermutung

For such s, $\Phi(n, d, p, s)$ is a theorem of the theory of real closed fields. Applying the decision algorithm to $\Phi(n, d, p, 1), \Phi(n, d, p, 2), \Phi(n, d, p, 3), \ldots$:

Theorem

There is a recursive function $s(n, d, p)$ such that $\Phi(n, d, p, s(n, d, p))$ holds for any real closed field.

The number of vertices of the common subdivision is bounded by a recursive function of n, d, p.

- It is easy to find example of finiteness results without uniform bounds (typically, for results using archimedeanity). For instance, the fact that for every open covering $\mathcal{U}$ of $|K|$ there is a finite iterate of barycentric subdivision of K which is finer than $\mathcal{U}$. Such result does not hold over an arbitrary real closed field.
- More surprisingly, there are uniform bounds which are not effective, i.e. cannot be bounded by any recursive functions.

Semialgebraic families

- A semialgebraic family of subsets of R^n parametrized by a semialgebraic set M is a semialgebraic subset $S \subset R^n \times M$. The fiber of the family at $t \in M$ is $S_t = \{x \in R^n \mid (x, t) \in S\}$.
- If $S \subset R^n \times M$ is a semialgebraic family, there is p such that every fiber has complexity $\leq p$. The semialgebraic subsets of complexity $\leq p$ of R^n can be organized in a semialgebraic family $S(n, p) \subset R^n \times M(n, p)$.

Theorem (Hardt)

If $S \subset R^n \times M$ is a semialgebraic family, there is a finite semialgebraic partition $M = \bigcup_{i \in I} M_i$ such that there is a semialgebraic trivialization $S|_{M_i} \simeq F_i \times M_i$ over each M_i.

A non-effective uniform bound

Corollary

Given n, p, there is q such that for every couple T, U of semialgebraic subsets of R^n of complexity $\leq p$, if T and U are semialgebraically homeomorphic, then there is a semialgebraic homeomorphism $T \to U$ of complexity $\leq q$.

Proof: apply Hardt's theorem to $S(n, p) \subset R^n \times M(n, p)$.
So q is a uniform bound, but it is not effective:

Theorem

The $q(n, p)$ above cannot be bounded by any recursive function of n, p.

Proof of non-effectiveness

If $q(n,p)$ were effective, then one would have effective $s(n,d)$ such that for every simplicial complex K in $\mathbb{R}^n$ with $\sharp K \leq d$ such that $|K|$ is a PL n-ball (PL-homeomorphic to the standard simplex Δ_n), there is a semialgebraic homeomorphism $|K| \to \Delta_n$ of complexity $s(n,d)$. By the effective Hauptvermutung, there would be an effective $D(n,d)$ such that for every K as before, there would be a subdivision K' of K with $\sharp K' \leq D(n,d)$, simplicially isomorphic to a subdivision of Δ_m. This would provide an algorithm for deciding whether a polyhedron $|K|$ is a PL n-ball. By a result of Novikov, no such algorithm can exist for $n \geq 6$.