Tutorial on the Calculus of Variations: Part II, Hamiltonians

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CAARMS ACKNOWLEDGEMENTS

Thanks to ***Prof. Rudy L. Horne*** for pointing out to me the connections between optimization and Hamiltonian systems.

These notes also benefited from many discussions with ***Prof. Earl Barnes***.

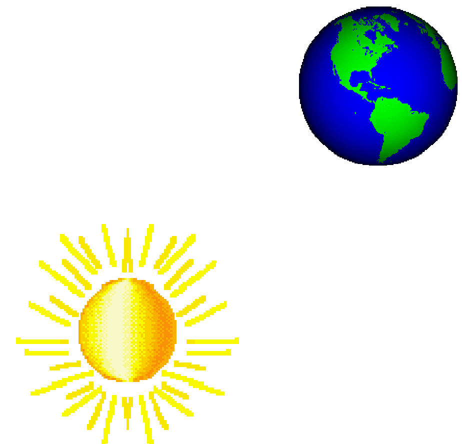
Application Interpretation for Hamiltonians: Classical Mechanics

The curve $x \equiv \left\{ x(t) \middle| 0 \le t \le T \right\}$ is the evolution of the *position of a system* and the Hamiltonian $\mathcal{H}$ is the *energy of the system*.

A *least action principle* for a Hamiltonian with no explicit time dependence yields a *conservation of energy principle*.

Similary, a Hamiltonian with no explicit position dependence yields a *conservation of momentum principle*.

Classical Mechanics Example: The Two Body Problem

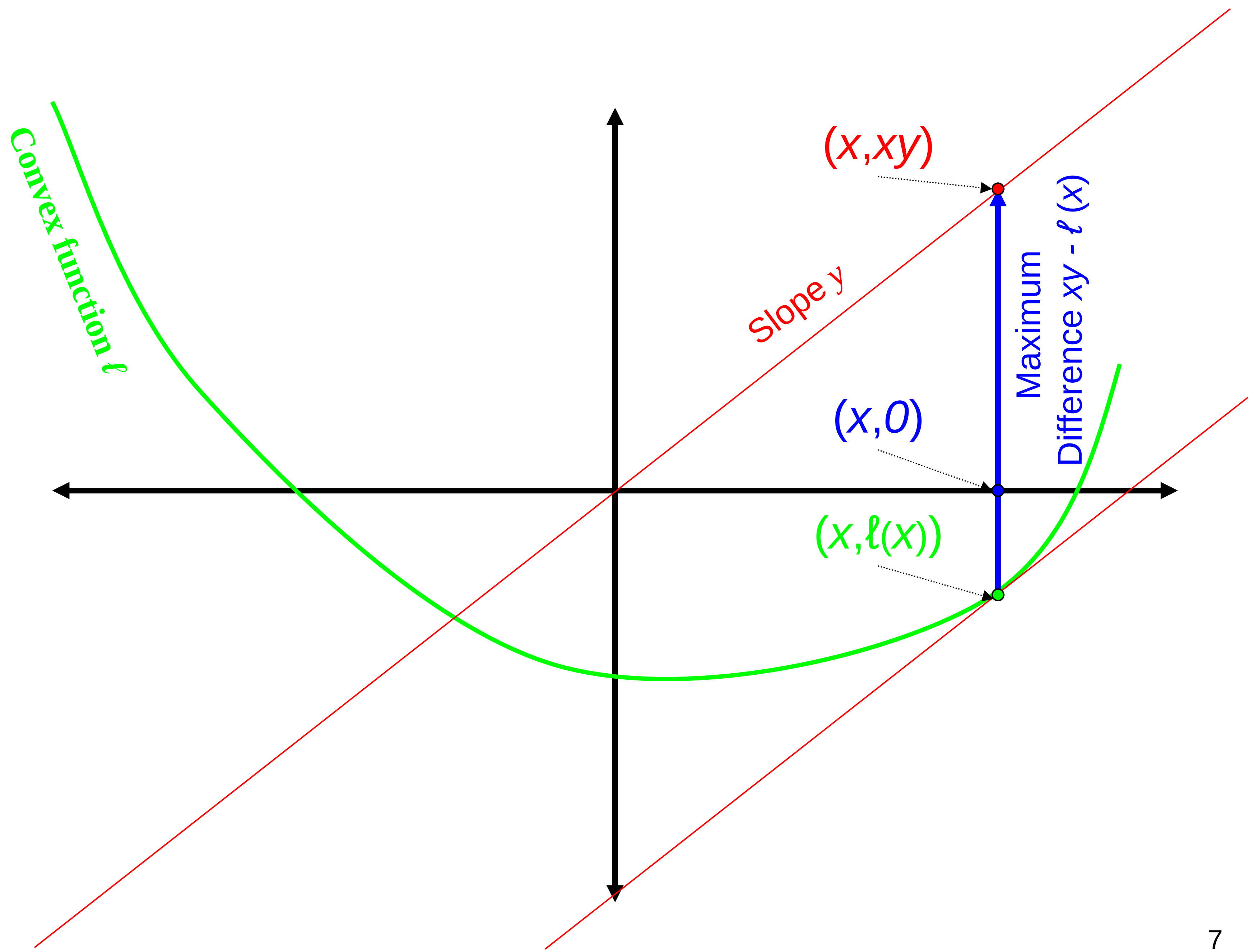


Extremal Notation

$$\text{ext} \equiv \min \text{ or } \max$$

Legendre Transforms

$$h(y) \equiv \operatorname*{ext}_{x}\Big(x \cdot y - \ell\,(x)\Big)$$



Convex function ℓ
(x,xy)
Slope y
Maximum
Difference xy - ℓ (x)
(x,0)
(x,ℓ(x))

Legendre Transform Example

$$\ell(x) = \frac{1}{2} m \cdot x^2 \quad \Leftrightarrow \quad h(y) = \frac{y^2}{2m}$$

$$h(y) = \max_x \left(x \cdot y - \frac{1}{2} m \cdot x^2 \right) = \frac{y}{m} \cdot y - \frac{1}{2} m \cdot \frac{y^2}{m^2} = \frac{y^2}{2m}$$

Fundamental Result for Legendre Transforms

If $\ell : \mathbb{R} \to \mathbb{R}$ is a *nice* function, then so is h and

$$h(y) = \operatorname*{ext}_{x} \Big(x \cdot y - \ell(x) \Big)$$

$$\Updownarrow$$

$$\ell(x) = \operatorname*{ext}_{y} \Big(x \cdot y - h(y) \Big)$$

When we have min or max, then strict concavity or respectively strict convexity is nice.

$$h(y) \equiv \operatorname*{ext}_{x}\Big(x \cdot y - \ell\,(x)\Big)$$

$$y = \ell'\big(x(y)\big)$$

$$h(y) = x(y) \cdot y - \ell\,\big(x(y)\big)$$

$$h'(y) = x(y) + x'(y) \cdot y - \ell'\big(x(y)\big) \cdot x'(y)$$

$$y = \ell' \circ h'(y)$$

$$x = h' \circ \ell'(x)$$

$$\ell(x) = \operatorname*{ext}_{y}\Big(x \cdot y - h\,(y)\Big)$$

The Hamiltonian as the Legendre Transform of the Lagrangian

$$\mathcal{H}(p,q) \equiv \operatorname*{ext}_{x}\Big(p \cdot x - \mathcal{L}(q,x)\Big)$$

$$\Updownarrow$$

$$\mathcal{L}(q,\dot{q}) = \operatorname*{ext}_{y}\Big(y \cdot \dot{q} - \mathcal{H}(y,q)\Big)$$

where q is the *position variable* and p is the *generalized momentum* variable.

Hamilton's Equations

The extremal behavior for the action integral of $p \cdot \dot{q} - \mathcal{H}(p,q)$ satisfies the dynamical system

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}(p,q) \quad \text{and} \quad \dot{q} = \frac{\partial \mathcal{H}}{\partial p}(p,q).$$

$$\operatorname*{ext}_{p,q} \int_0^T \left[p \cdot \dot{q} - \mathcal{H}(p,q) \right] dt$$

$$\Downarrow$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{p}} \left[p \cdot \dot{q} - \mathcal{H}(p,q) \right] = \frac{\partial}{\partial p} \left[p \cdot \dot{q} - \mathcal{H}(p,q) \right]$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}} \left[p \cdot \dot{q} - \mathcal{H}(p,q) \right] = \frac{\partial}{\partial q} \left[p \cdot \dot{q} - \mathcal{H}(p,q) \right]$$

$$\Downarrow$$

$$0 = \dot{q} - \frac{\partial \mathcal{H}}{\partial p}(p,q)$$

$$\frac{d}{dt} p = -\frac{\partial \mathcal{H}}{\partial q}(p,q)$$

Equivalence of the Lagrangian and Hamiltonian Formulations

The extremal behavior for q and $\dot{q}$ as governed by the Lagrangian $\mathcal{L}$ gives us

$$\mathcal{H}(p, q) = p \cdot \dot{q} - \mathcal{L}\left(q, \dot{q}\right),$$

where p is given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}\left(q, \dot{q}\right).$$

Endpoint Conditions

Moreover, if $q(T)$ has no fixed value, then

$$p(T) = 0.$$

Finally, if $q(0)$ has no fixed value, then

$$p(0) = 0.$$

$$
\begin{aligned}
&\operatorname*{ext}_{p,q}\int_0^T\left[p\cdot\dot q-\mathcal{H}(p,q)\right]dt\\
&=\operatorname*{ext}_{q}\left(\operatorname*{ext}_{p}\int_0^T\left[p\cdot\dot q-\mathcal{H}(p,q)\right]dt\right)\\
&=\operatorname*{ext}_{q}\left(\int_0^T\operatorname*{ext}_{p}\left[p\cdot\dot q-\mathcal{H}(p,q)\right]dt\right)\\
&=\operatorname*{ext}_{q}\int_0^T\mathcal{L}(q,\dot q)\,dt.
\end{aligned}
$$

$$\mathcal{H}(p, q, \dot{q}) \equiv p \cdot \dot{q} - \mathcal{L}(q, \dot{q})$$

$$\frac{\partial \mathcal{H}}{\partial \dot{q}}(p, q, \dot{q}) = p - \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) = 0.$$

$$\frac{\partial \mathcal{H}}{\partial p}(p, q) = \dot{q}.$$

$$\frac{\partial \mathcal{H}}{\partial q}(p, q) = -\frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) = -\frac{d}{dt}p = -\dot{p}.$$

Conservation of Momentum?

The extremal behavior for p and q as governed by the Hamiltonian $\mathcal{H}$ gives us (by Hamilton's equations)

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}(p, q)$$

So if the Hamiltonian $\mathcal{H}$ has *no* explicit dependence on position q, then the generalized momentum p is a *conserved* quantity (constant over time).

Arc Length Example

$$\mathcal{L}\left(q(t), \dot{q}(t), t\right) = \ell\left(\dot{q}(t)\right) \equiv \sqrt{1 + \dot{q}(t)^2}$$

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \Rightarrow \dot{q}(t) = v \ \text{(constant)}$$

$$\mathcal{H}\left(p(t), q(t), t\right) = p(t) \cdot \frac{\partial}{\partial \dot{q}} \sqrt{1 + \dot{q}(t)^2} - \sqrt{1 + \dot{q}(t)^2}$$

$$= p(t) \cdot \ell'(v) - \ell(v)$$

$$\Rightarrow \dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial q}(p, q) = 0.$$

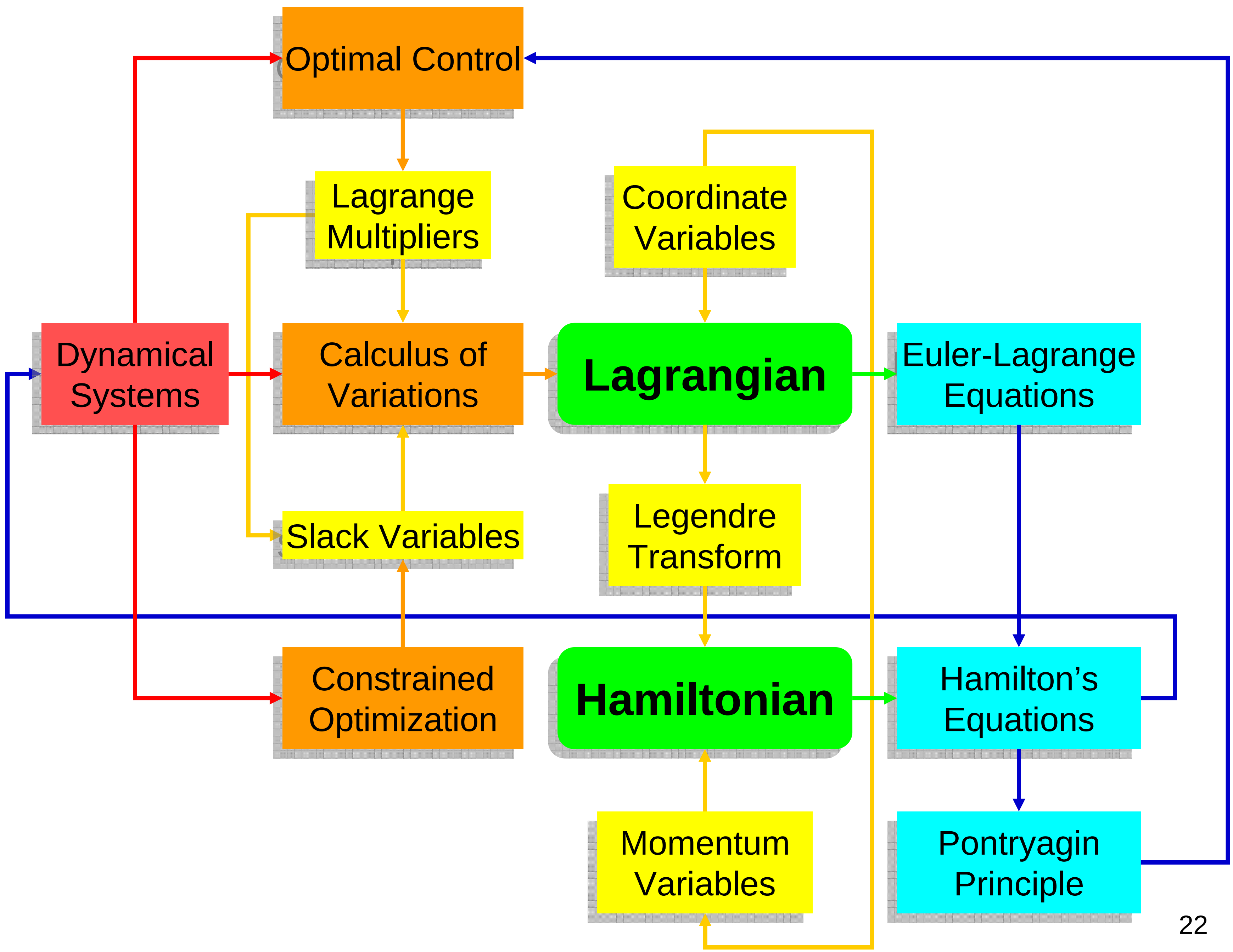
Conservation of Energy?

The extremal behavior for p and q as governed by the Hamiltonian $\mathcal{H}$ gives us

$$\frac{d}{dt}\mathcal{H}(p,q) = \frac{\partial \mathcal{H}}{\partial t}(p,q).$$

Moreover, if the Hamiltonian $\mathcal{H}$ has *no* explicit time dependence, then it is a *conserved* quantity (constant over time).

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H}(p,q) &= \dot{p}\,\frac{\partial\mathcal{H}}{\partial p}(p,q) + \dot{q}\,\frac{\partial\mathcal{H}}{\partial q}(p,q) + \frac{\partial\mathcal{H}}{\partial t}(p,q) \\
&= \dot{p}\cdot\dot{q} + \dot{q}\cdot\left(-\dot{p}\right) + \frac{\partial\mathcal{H}}{\partial t}(p,q) \\
&= \frac{\partial\mathcal{H}}{\partial t}(p,q).
\end{aligned}
$$


Optimal Control
Lagrange Multipliers
Coordinate Variables
Dynamical Systems
Calculus of Variations
Lagrangian
Euler-Lagrange Equations
Slack Variables
Legendre Transform
Constrained Optimization
Hamiltonian
Hamilton's Equations
Momentum Variables
Pontryagin Principle