Volume-Preserving
vs
Symplectic Dynamics

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Hamiltonian Flow is Symplectic

- Hamiltonian Dynamics:

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$\frac{dq}{dt} = \frac{\partial}{\partial p} H$$

$$\frac{dp}{dt} = -\frac{\partial}{\partial q} H$$

$$\dot{z} = J \nabla H$$

- Flow φ on $z = (q,p)$

$$z(t) = \varphi_t(z(0)) \Rightarrow D\varphi_t^T J D\varphi_t = J$$

- A map is symplectic if:

$$Df^T J Df = J \qquad Df = \frac{\partial(x', y')}{\partial(x, y)}$$

- ⇒ preservation of the Poincaré loop action $\mathcal{A} = \oint_{\mathcal{L}} p \cdot dq$

Symplectic Maps

- A map $f(x,y) \rightarrow (x',y')$ is symplectic if

coordinates
momenta

$$Df^T J Df = J \qquad J = \begin{pmatrix} 0 & I \end{pmatrix}$$

insert “new” *y* here!

- Standard (Chirikov-Froeschlé) form on $\mathbb{T}^n \times \mathbb{R}^n$

$$\begin{aligned} x' &= x + \Omega(y') \mod 1 \\ y' &= y + F(x) \end{aligned}$$

Angle-Action Form

- Symplectic only if frequency & force are gradients

$$\begin{aligned} \Omega(y) &= \nabla S(y) \\ F(x) &= -\nabla V(x) \end{aligned}$$

Frequency Map

Force

Symplectic Shears

$$Df^T J Df = J$$

$$x' = x + \Omega(y') \mod 1$$
$$y' = y + F(x)$$

- The Chirikov-Froeshlé form is a composition of “shears”

$$f(x,y) = s_x \circ s_y(x,y)$$
$$s_x(x,y) = (x + \Omega(y), y)$$
$$s_y(x,y) = (x, y + F(x))$$

- each is symplectic:

$$Ds_x^T J Ds_x = \begin{pmatrix} I & 0 \\ D\Omega^T & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & D\Omega \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} 0 & I \\ -I & D\Omega^T - D\Omega \end{pmatrix} = J$$

- whenever $D\Omega^T = D\Omega$, $\Rightarrow$ $\Omega = \nabla S$

Note: it is possible to generalize by changing the symplectic form

Volume-Preserving Maps

- A map $f(x,y) \to (x',y')$ is volume preserving if

$$\det(Df) = 1$$

m actions

- Standard (angle-action) form on $\mathbb{T}^n \times \mathbb{R}^m$

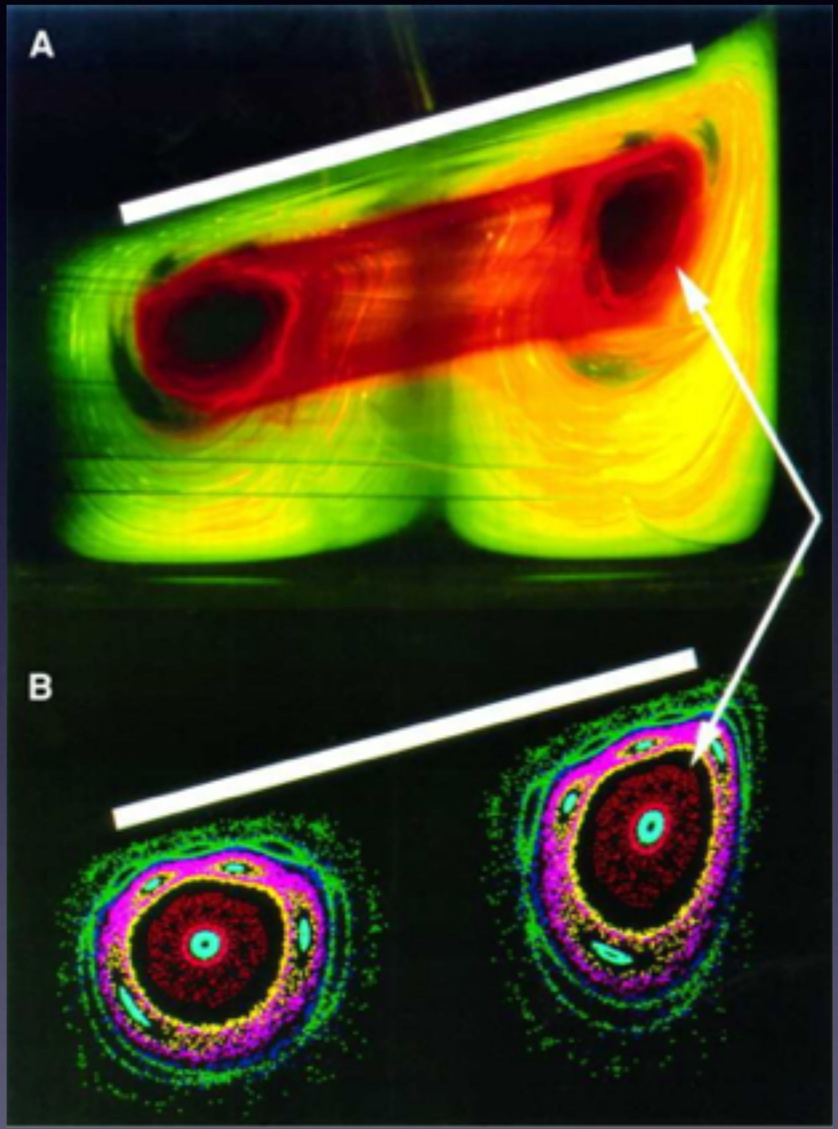
$$\begin{aligned} x' &= x + \Omega(y') \mod 1 \\ y' &= y + F(x) \end{aligned}$$

Angle-Action Form

- Note: number of angles and actions can be different, $n \neq m$
- Volume preserving for *any* frequency map $\Omega : \mathbb{R}^m \to \mathbb{T}^n$ and force $F : \mathbb{T}^n \to \mathbb{R}^m$

Why Volume-Preserving?

- Mixing (stirring) in incompressible fluids $\nabla \cdot v = 0$
    - Chaotic advection of dye $\dot{x} = v(x,t)$
- Magnetic fields $\nabla \cdot B = 0$
    - stellerators, solar flares, Earth's magnetotail
- 3D is simpler than 4D symplectic case!
- What is the impact of lack of symplecticity on dynamics?



Fountain, G. O., F. V. Khakhar and J. M. Ottino (1998). "Visualization of Three-Dimensional Chaos." Science 281: 683.

Curl Forces

- Berry & Shukla note that odes on $\mathbb{R}^2 \times \mathbb{R}^2$ of the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= F(x) \end{aligned} \qquad \nabla \times F \neq 0$$

arise in dynamics of polarizable particles in a polarized electric field $E(x, y, z) = e^{ikz}\psi(x, y)\hat{e}_p$

$$F(x) = -\nabla|\psi|^2 + a\,\mathrm{Im}[\psi^* \nabla \psi]$$

- Such systems are incompressible, but non-Hamiltonian.
- Poincaré Maps will be Volume-preserving!

Berry, M. V. and P. Shukla (2015). "Hamiltonian Curl Forces." Proc Roy Soc Lond A 471: 20150002.

Volume-Preserving Maps

- Examples:
  - Chirikov's Standard map (1,1) $(x', y') = (x + y', y - \varepsilon \sin(2\pi x))$
  - Froeschlé map (2,2) $(x', y') = (x + y', y - \varepsilon \nabla V(x))$
- Two Angle–One Action Normal form (Rank-One Resonance)

$\Omega = (\Omega_1(y), \Omega_2(y))$

$$
\begin{aligned}
x_1' &= x_1 + y' + \gamma \\
x_2' &= x_2 + \beta y'^2 - \delta \\
y' &= y - \varepsilon[\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))]
\end{aligned}
$$

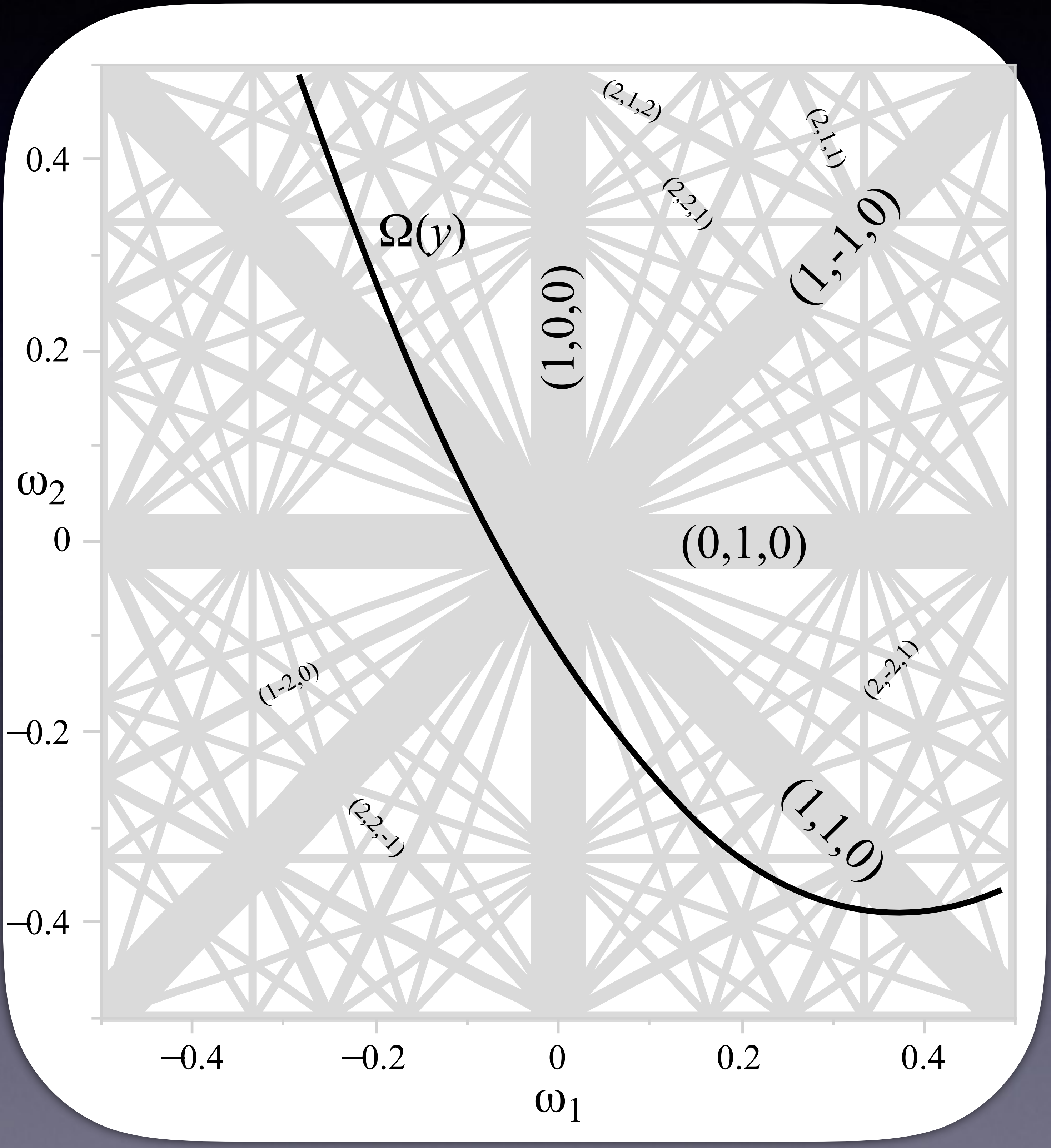
Dullin, H. R. and J. D. Meiss (2012). "Resonances and Twist in Volume-Preserving Mappings." SIAM J. Appl. Dyn. Sys. 11: 319-349.

Invariant Tori

Frequency Map

$(n,m) = (2,1)$

- Tori exist only where "true" frequency map crosses incommensurate frequency vectors
- To fix frequency: add a parameter:
  $\Omega(y,\delta) \to (\omega_1, \omega_2)$



$$\Omega(y,\delta) = (y + \gamma, \beta y^2 - \delta)$$

Volume-Preserving KAM

- Xia & Cheng/Sun: For one-action case: A Cantor set of codimension-one, Diophantine ($c > 0$, $s \geq 2$)

$$|p \cdot \omega - q| > \frac{c}{|p|^s} \quad p \in \mathbb{Z}^n \backslash 0,\ q \in \mathbb{Z}$$

tori persist in smooth, volume-preserving families, if Ω satisfies a non-degeneracy (twist) condition

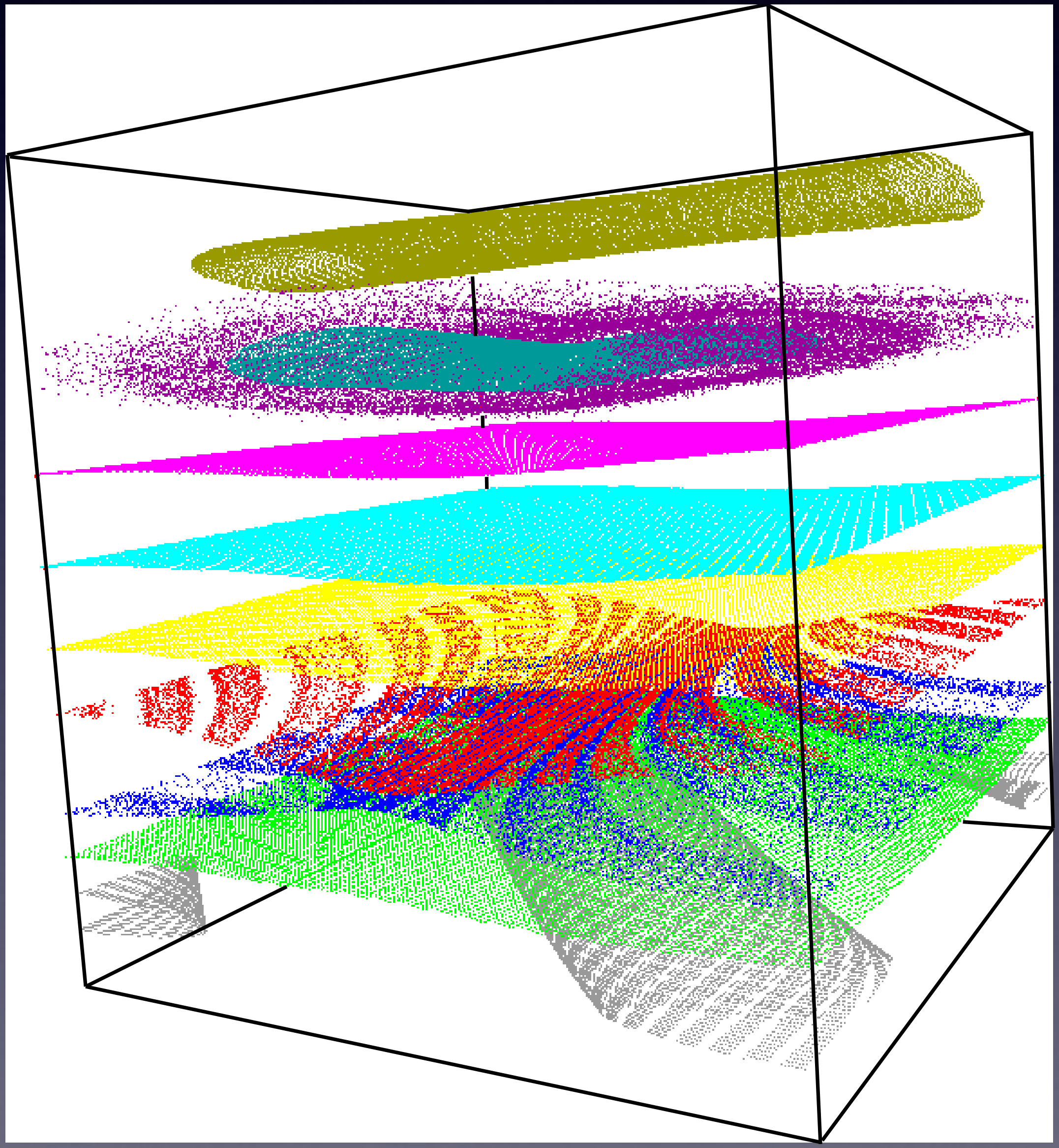
$$\mathrm{rank}(D\Omega, D^2\Omega, \ldots, D^n\Omega) = n$$

i.e., *Kolmogorov nondegeneracy*.

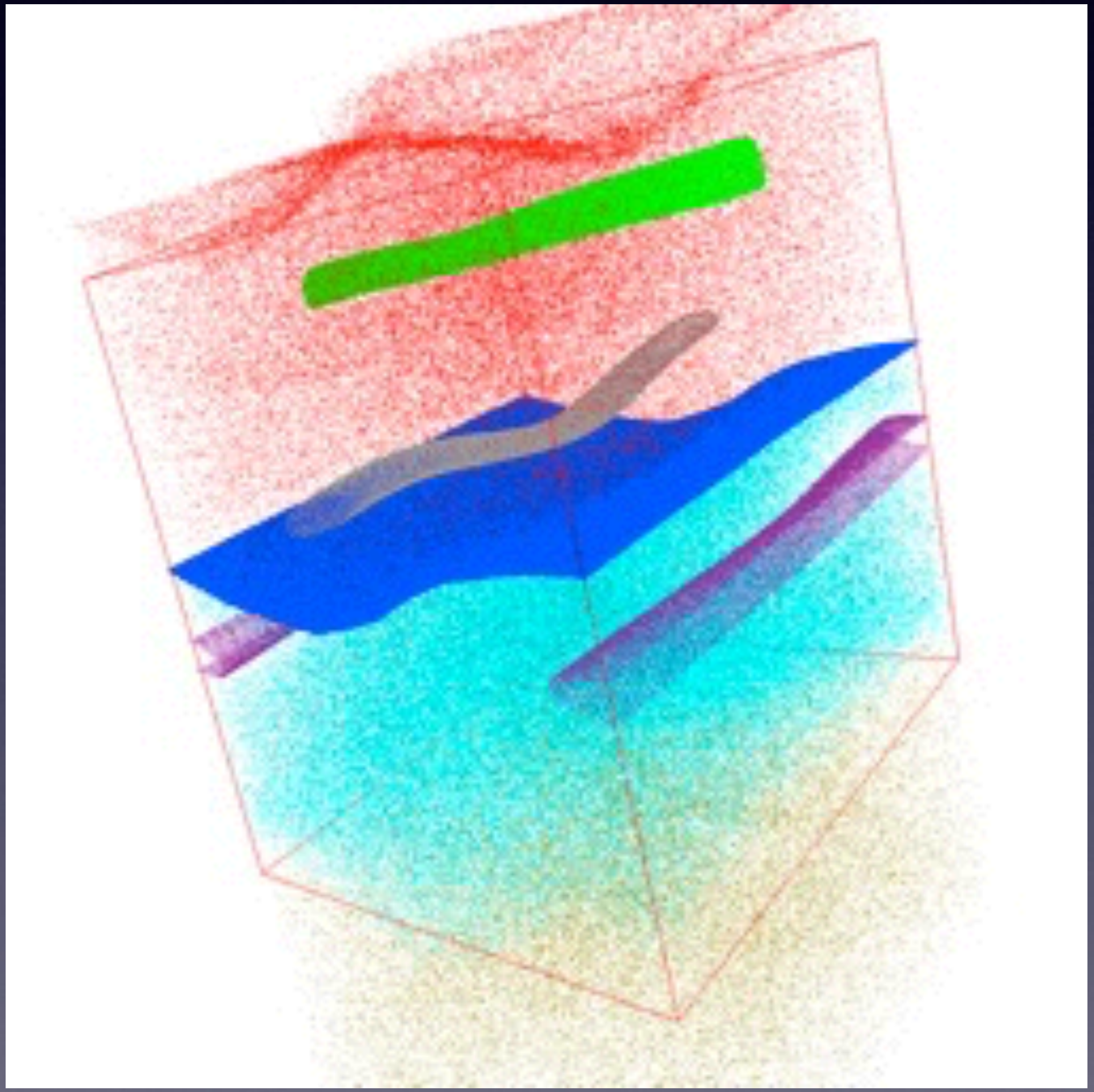
Cheng, C.-Q. and Y.-S. Sun (1989). "Invariant tori in 3D measure-preserving mappings."
Celest. Mech. 47(3): 275-292.
Xia, Z. (1992). "Existence of invariant tori in volume-preserving diffeomorphisms."
Erg. Th. Dyn. Sys. 12(3): 621-631.
Blass, T. and de la Llave, R. (2016?) KAM theory for volume-preserving maps

Tori act as Barriers

Many rotational tori



Only one rotational torus



$\varepsilon = 0.005$

$\varepsilon = 0.02725$

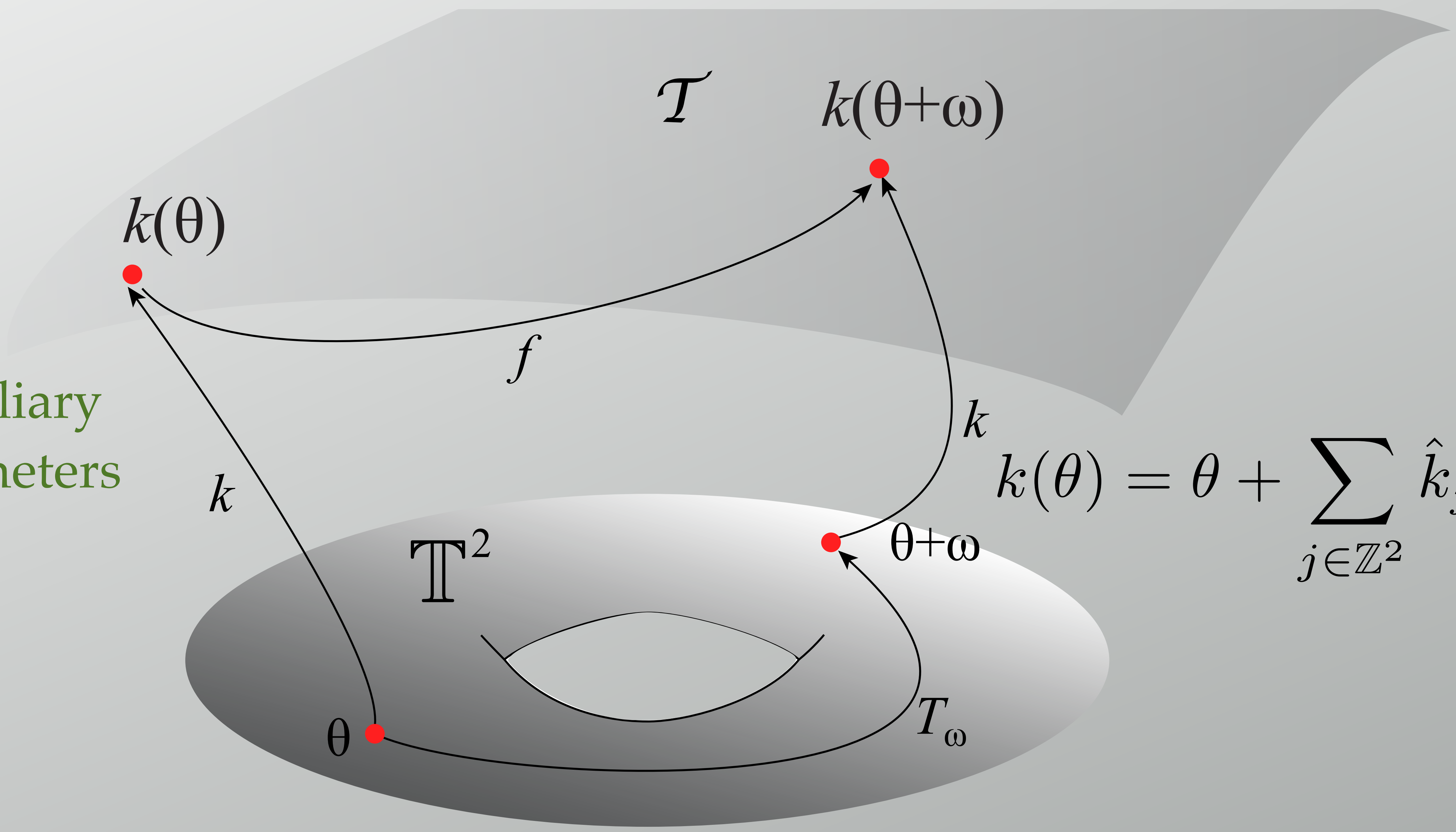
$\beta = 2, \delta = 0.1, \gamma = \frac{1}{2}(1+\sqrt{5})$

Computing Tori: Parameterization

$$f_\lambda(k(\theta)) = k(\theta + \omega)$$

Fixed Rotation Vector

Auxiliary Parameters



$$k(\theta) = \theta + \sum_{j \in \mathbb{Z}^2} \hat{k}_j e^{2\pi i j \cdot \theta}$$

Newton Iteration

- Given an approximate solution, (*k*,λ):

$$f_\lambda(k(\theta)) - k(\theta+\omega) = e(\theta)$$

error e

- Compute correction (Δ, ζ):

$$k(\theta) \to k(\theta) + \Delta(\theta)$$
$$\lambda \to \lambda + \zeta$$

- Iterative equation

$$\Delta(\theta+\omega) - Df(k(\theta))\Delta(\theta) = e(\theta) + D_\lambda f(k(\theta))\zeta,$$

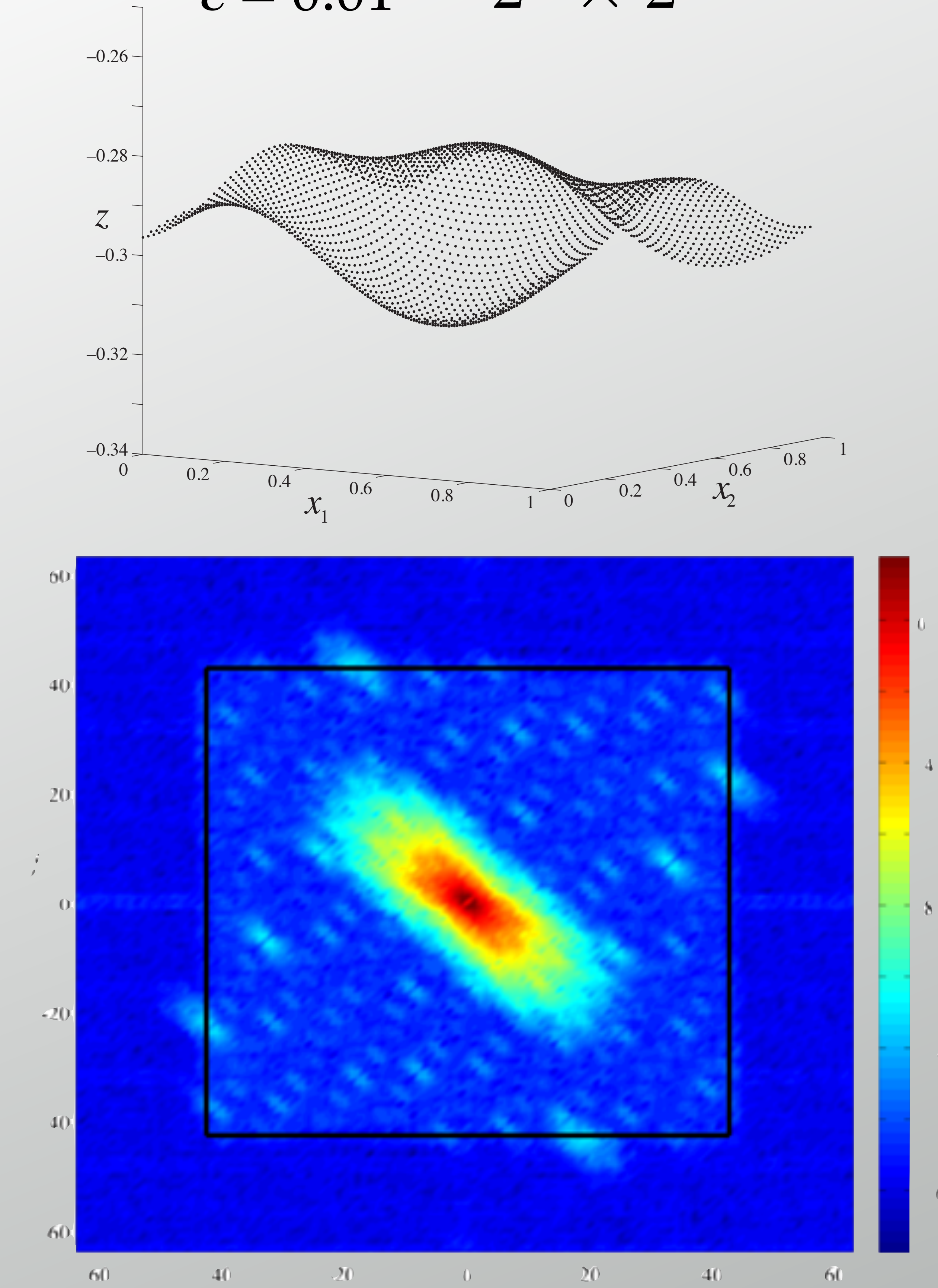
- Solved using FFT and "*automatic reducibility*"

Fox, A. M. and J. D. Meiss (2016). "Computation of Invariant Tori in Volume-Preserving Maps." SIAM Dyn.Sys. 15: 557-579.

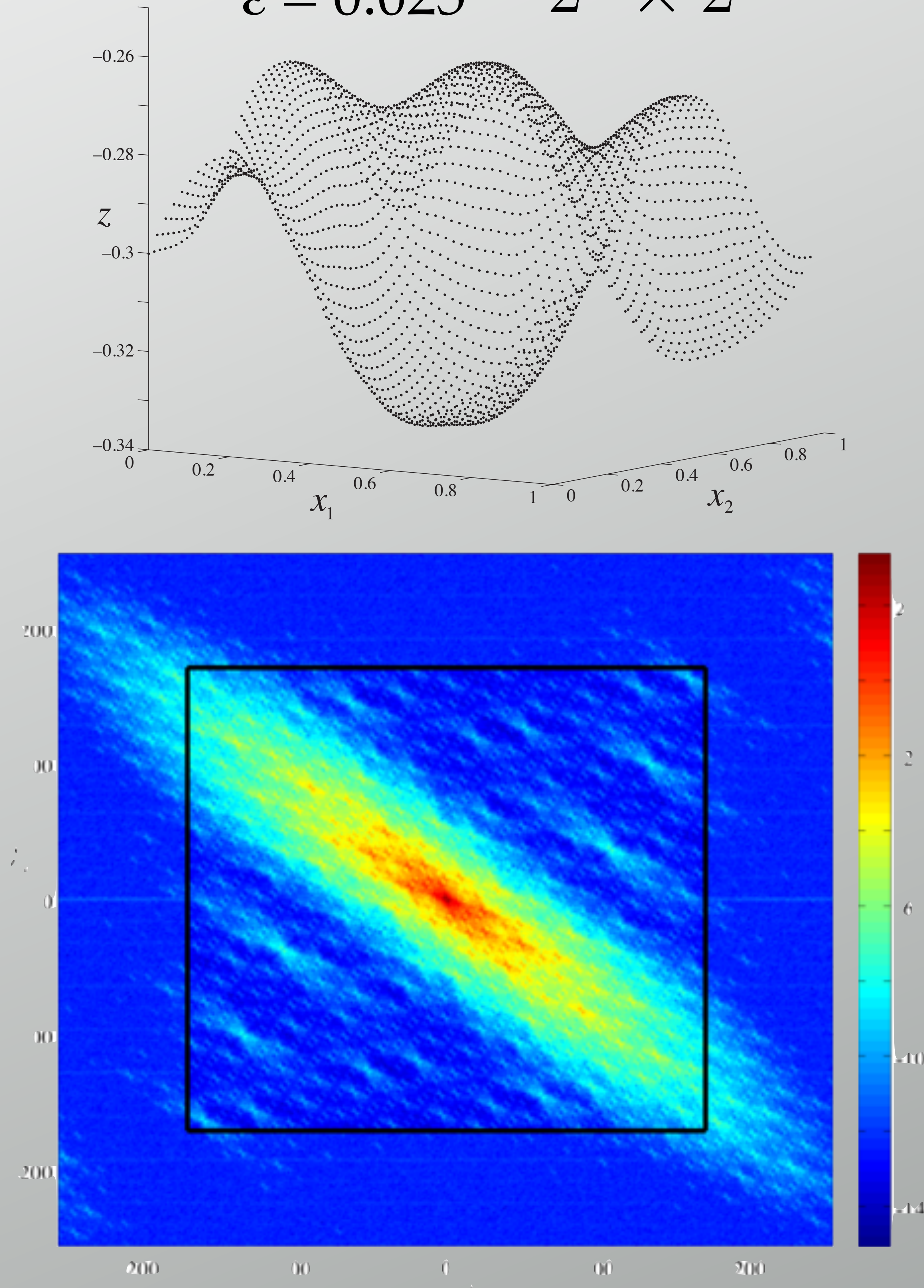
Spectral Method

$$\omega = (\sigma - 1, \sigma^2 - 1)$$

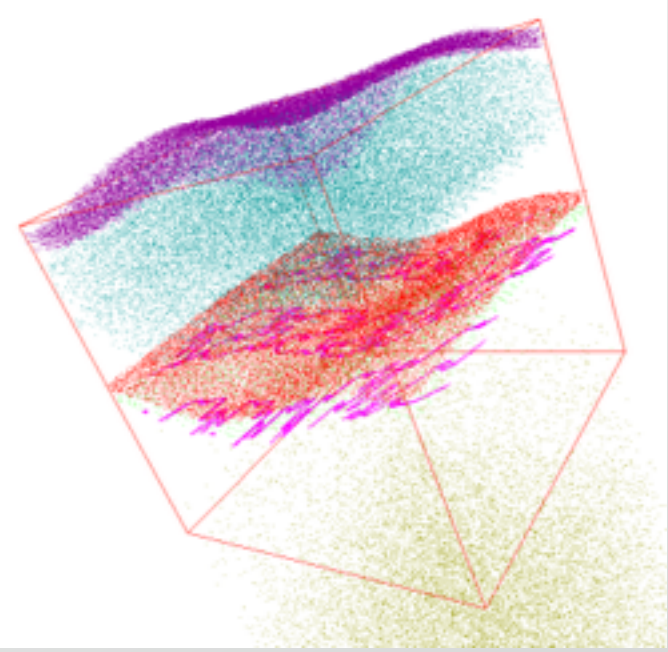
$\varepsilon = 0.01 \quad 2^7 \times 2^7$



$\varepsilon = 0.025 \quad 2^9 \times 2^9$



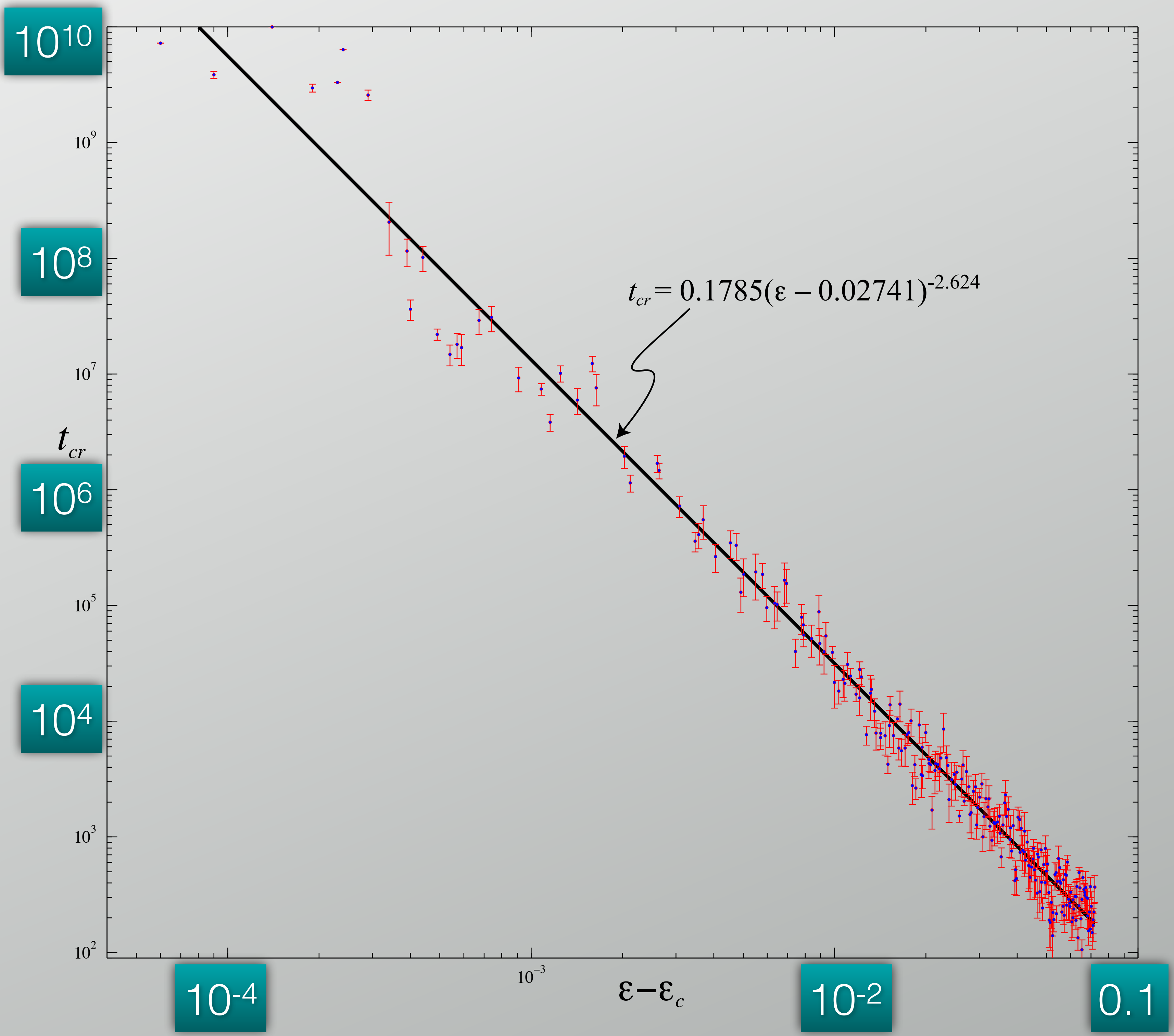
Destruction ⇒ Transport



Crossing Time vs. $\varepsilon - \varepsilon_{cr}$

$\delta = 0.1$

$\varepsilon_{cr} \approx 0.02741$



Meiss, J. D. (2012). “The Destruction of Tori in Volume-Preserving Maps.” Comm. Nonl. Sci and Num. Sim. 17: 2108-2121.

Transport

Nekhoroshev Theory

- Near integrable Symplectic Map (ε << 1)

$$
\begin{aligned}
x' &= x + \nabla S(y') \quad \mod 1 \\
y' &= y - \epsilon \nabla V(x)
\end{aligned}
$$

  - *S, V* analytic, *S* convex (though *steep* is sufficient)

- the actions do not drift far in exponentially long times:

$$\|y_t - y_0\| \leq c\varepsilon^{\alpha} \qquad t \leq T \exp\left(c/\varepsilon\right)^{\beta}$$

Guzzo, M. (2004). “Nekhoroshev Theorem for Symplectic Maps.”
Ann. Poincaré 5(6): 1013-1039.
Lochak, P. (1992). “Canonical Perturbation Theory via Simultaneous Approximation.”
Rus. Math. Sur. 47(6): 59-140.

Is there Nekhoroshev for Volume-Preserving Maps?

Near-Integrable 4D Map

$$x' = x + y' \mod 1$$
$$y' = y + F(x)$$

two angles

two actions

m = n = 2

- Froeschlé-like forces

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

  - φ = 0: Symplectic since $F = -\nabla V$
  - φ = ½: “maximally non-symplectic” coupling

- Full Spectrum Force:

$$F_{fs} = -\frac{1}{2\pi}\frac{d}{(2.1+\cos(2\pi x_1)+\cos(2\pi x_2))^2}\begin{pmatrix} \sin(2\pi x_1) \\ \sin(2\pi(x_2+\varphi)) \end{pmatrix}$$

Resonance Web

- Resonances

$$\mathcal{R} = \left\{\omega \in \mathbb{R}^n : m \cdot \omega = n, (m,n) \in \mathbb{Z}^{n+1} \setminus \{0\}\right\}$$

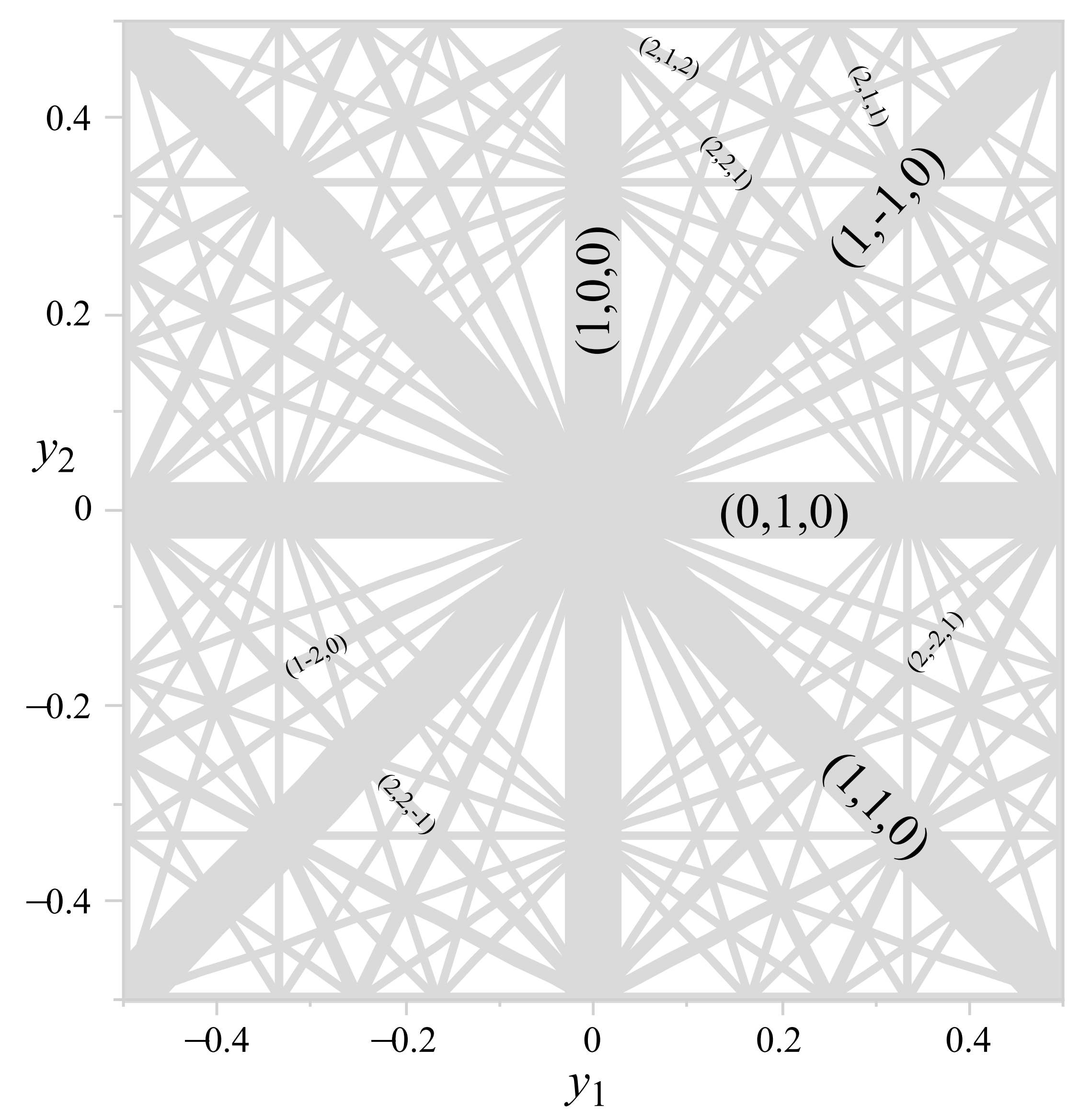
- For a resonance, the *rank* is the dimension of the module

$$\mathcal{L}_\omega = \{m \in \mathbb{Z}^n : m \cdot \omega \in \mathbb{Z}\}$$

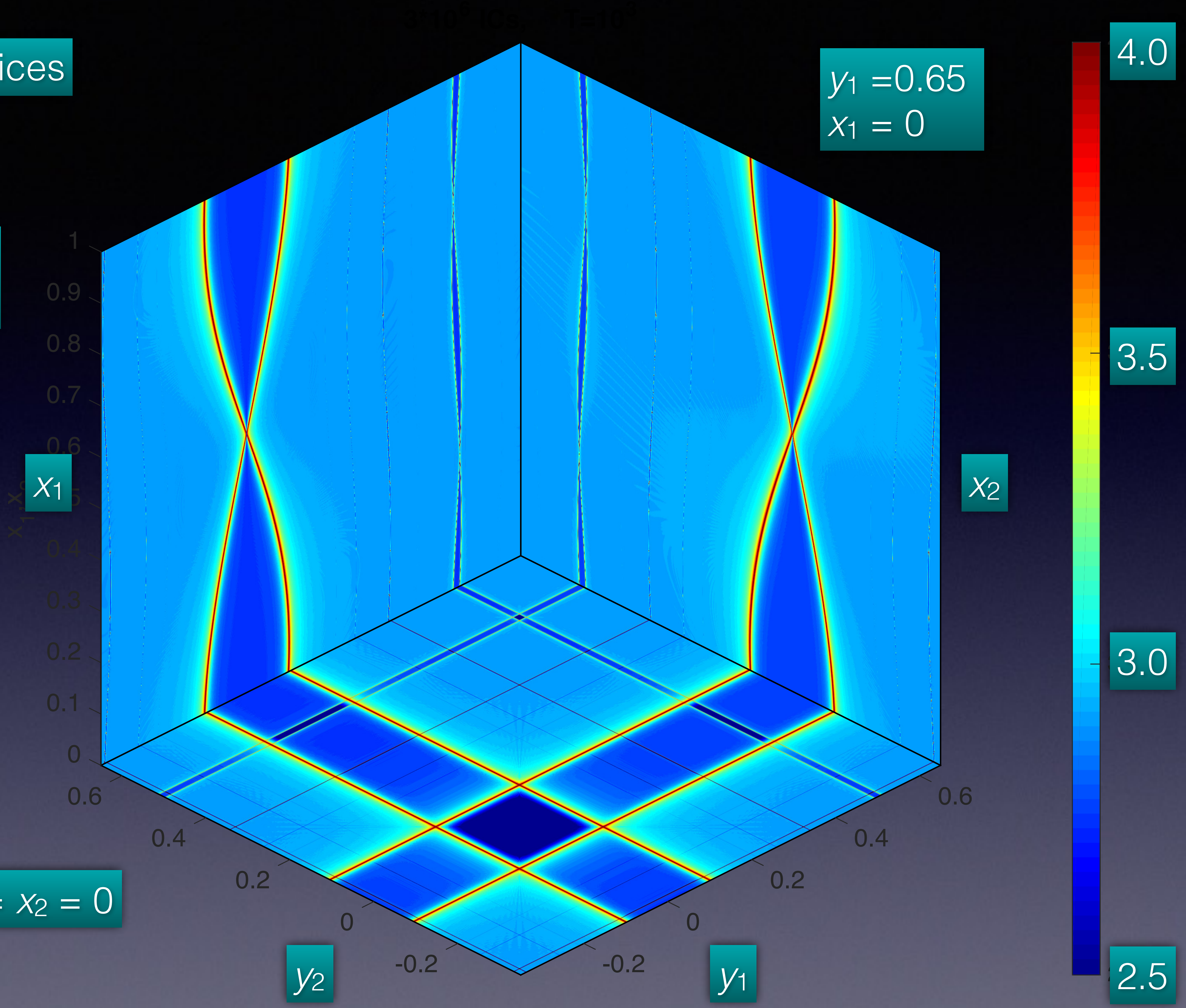
- In Action space, resonances occur at

$$m \cdot \Omega(y) = n$$

- Nonlinearity “fattens resonances



Uncoupled Slices



$y_2 = 0.65$
$x_2 = 0$

$y_1 = 0.65$
$x_1 = 0$

$x_1 = x_2 = 0$

$a = 0.1$
$b = 0.1$
$c = 0$
$d = 0$
$\varphi = 0$

FLI, $T = 1000$

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) \\ b\sin(2\pi x_2) \end{pmatrix}$$

Fast Lyapunov Indicator

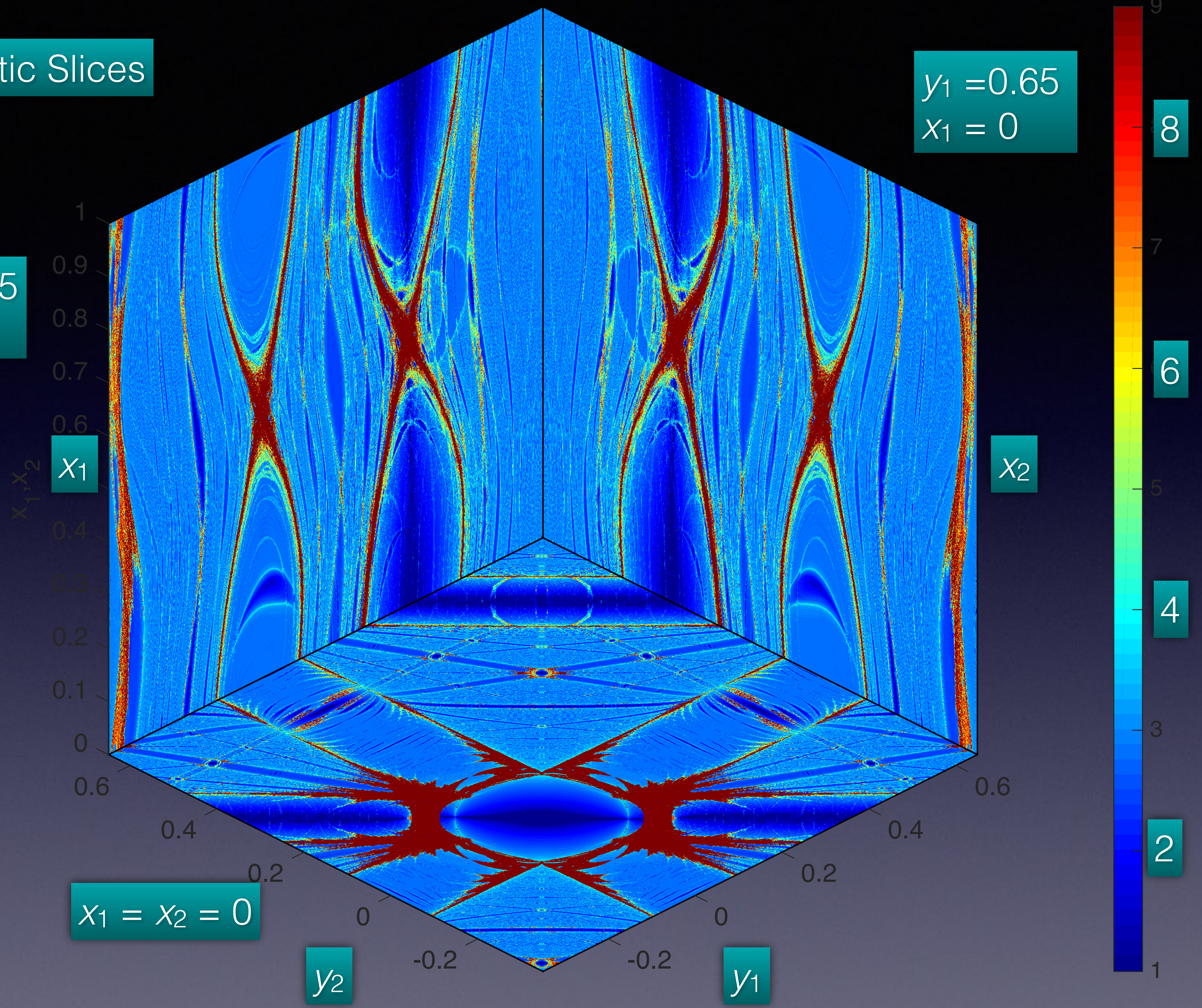
- Iterate arbitrarily chosen initial deviation v_0
- Compute the supremum up to time T

$$FLI = \sup_{t<T} \left(\log \| Df^t(x_0) v_0 \| \right)$$

  - similar to FTLE, but supremum reduces oscillations

Froeschle, C., R. Gonczi and E. Lega (1997). "The fast Lyapunov indicator: a simple tool to detect weak chaos. Planetary and Space Science 45(7): 881-886.

Symplectic Slices



$y_2 = 0.65$
$x_2 = 0$

$y_1 = 0.65$
$x_1 = 0$

$x_1 = x_2 = 0$

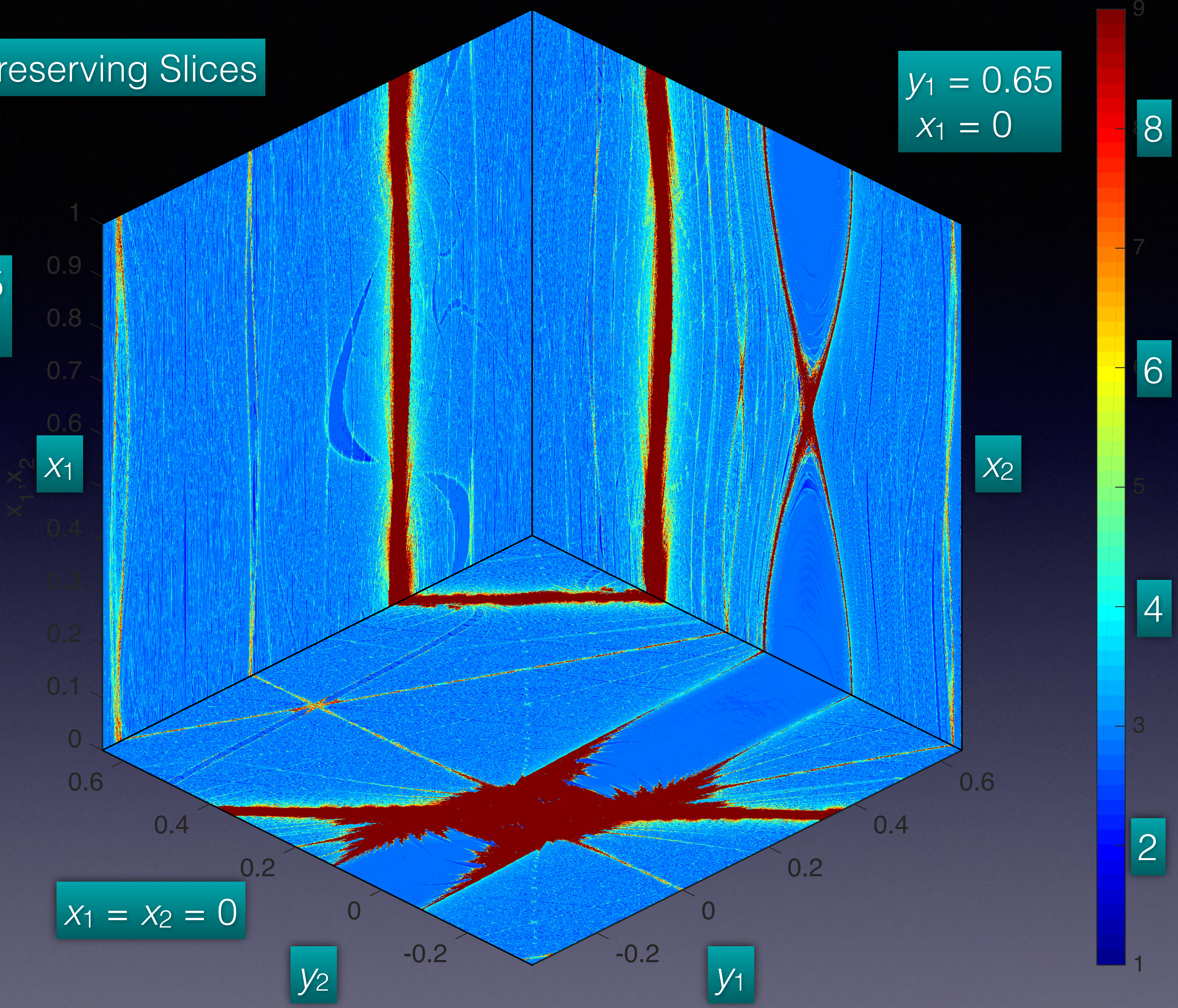
$a = 0.1$
$b = 0.1$
$c = 0.07$
$d = 0.0001$
$\varphi = 0$

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

$$F_{fs} = -\frac{1}{2\pi}\frac{d}{(2.1+\cos(2\pi x_1)+\cos(2\pi x_2))^2}\begin{pmatrix} \sin(2\pi x_1) \\ \sin(2\pi x_2) \end{pmatrix}$$

FLI, $T = 1000$

Volume-Preserving Slices



$a = 0$
$b = 0.01$
$c = 0.07$
$d = 0.0001$
$\varphi = ½$

FLI, $T = 1000$

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

$$F_{fs} = -\frac{1}{2\pi}\frac{d}{(2.1+\cos(2\pi x_1)+\cos(2\pi x_2))^2}\begin{pmatrix} \sin(2\pi x_1) \\ \sin(2\pi x_2) \end{pmatrix}$$

Resonance Web

- Near resonance

$$y = y^* + \delta y \qquad m \cdot y^* = n$$

$$F(x) = F_R(m \cdot x) + F_{NR}(x)$$

$$F(x) = \sum_{m \in \mathbb{Z}^n} \hat{F}_m e^{2\pi i m \cdot x}$$

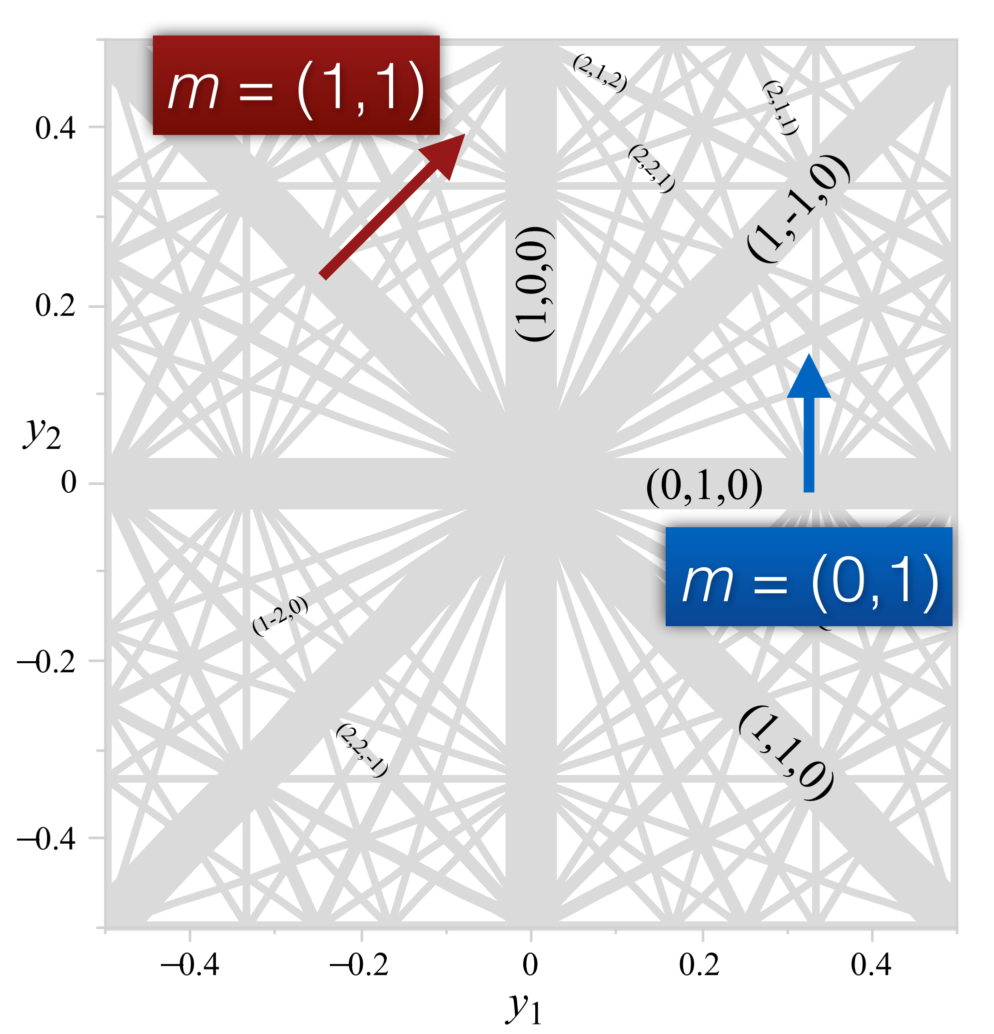
  - Note: m orthogonal to resonance channel

- Resonant Phase and action

$$\psi = m \cdot x$$

$$J_R = m \cdot \delta y$$

$$J_{||} = m_\perp \cdot \delta y$$



Resonance Web

$$\psi = m \cdot x$$

$$J_R = m \cdot \delta y$$

$$J_\parallel = m_\perp \cdot \delta y$$

- Transform away nonresonant forces F_{NR}

$$\left.\begin{aligned}\psi' &= \psi + n + J'_R \\ J'_R &= J_R + m \cdot F_R(\psi)\end{aligned}\right\} \text{2D Area-preserving map}$$

$$J'_\parallel = J_\parallel + m_\perp \cdot F_R(\psi)$$

- For Symplectic case

$$F_R = -\nabla V(m \cdot x) = -mV'(\psi)$$

- Action along channel is approximate invariant!

$$J'_\parallel = J_\parallel$$

Resonance Web

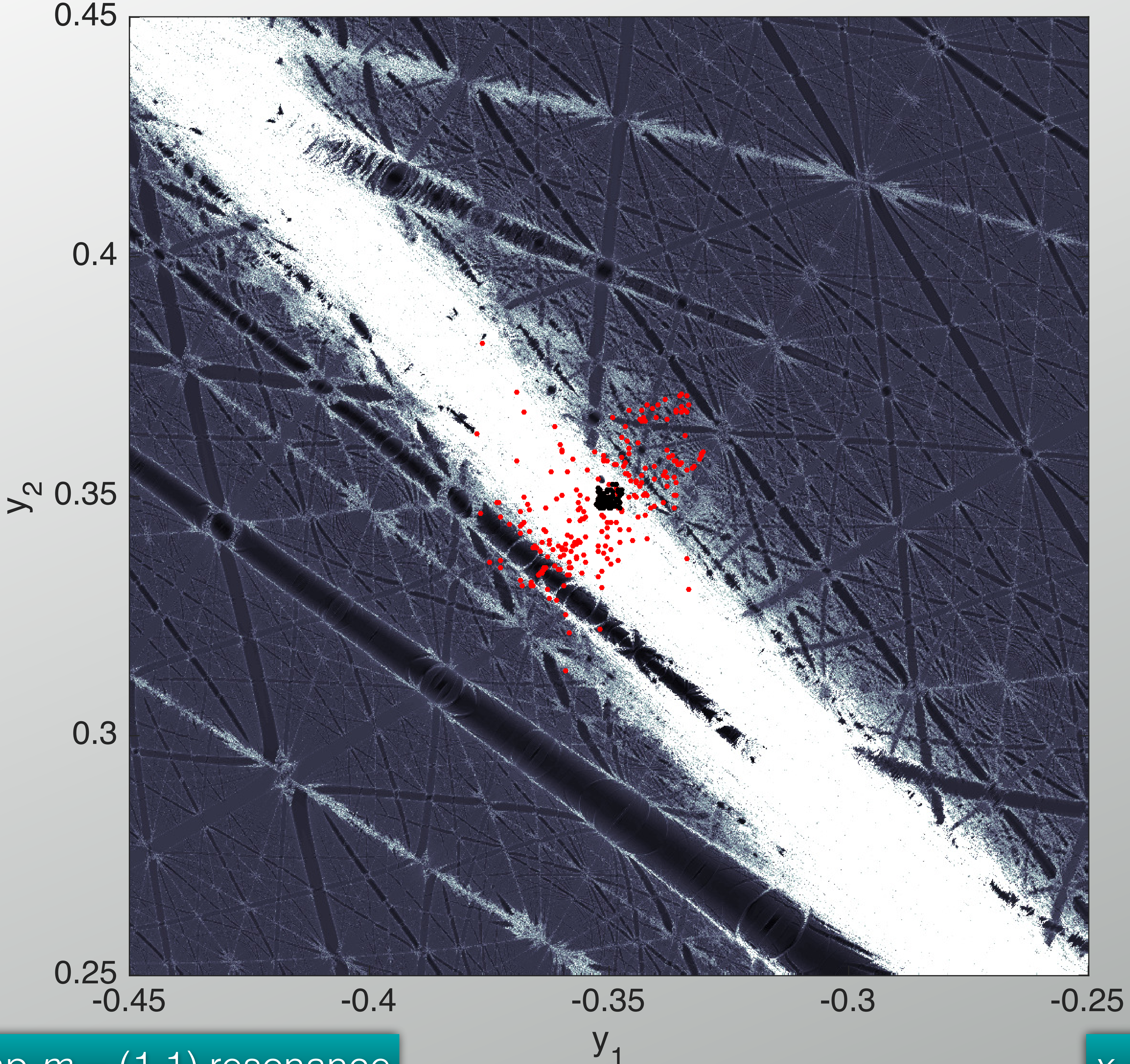
- But for a Volume-Preserving map, $J_{\|}$ can be driven!

$$\psi' = \psi + n + J_R'$$
$$J_R' = J_R + m \cdot F_R(\psi)$$
$$J_{\|}' = J_{\|} + m_{\perp} \cdot F_R(\psi)$$

Drifting Orbits

Symplectic: $(a,b,c,d) = (0,0.1,0.07,0.0001)$

$T = 10^6$

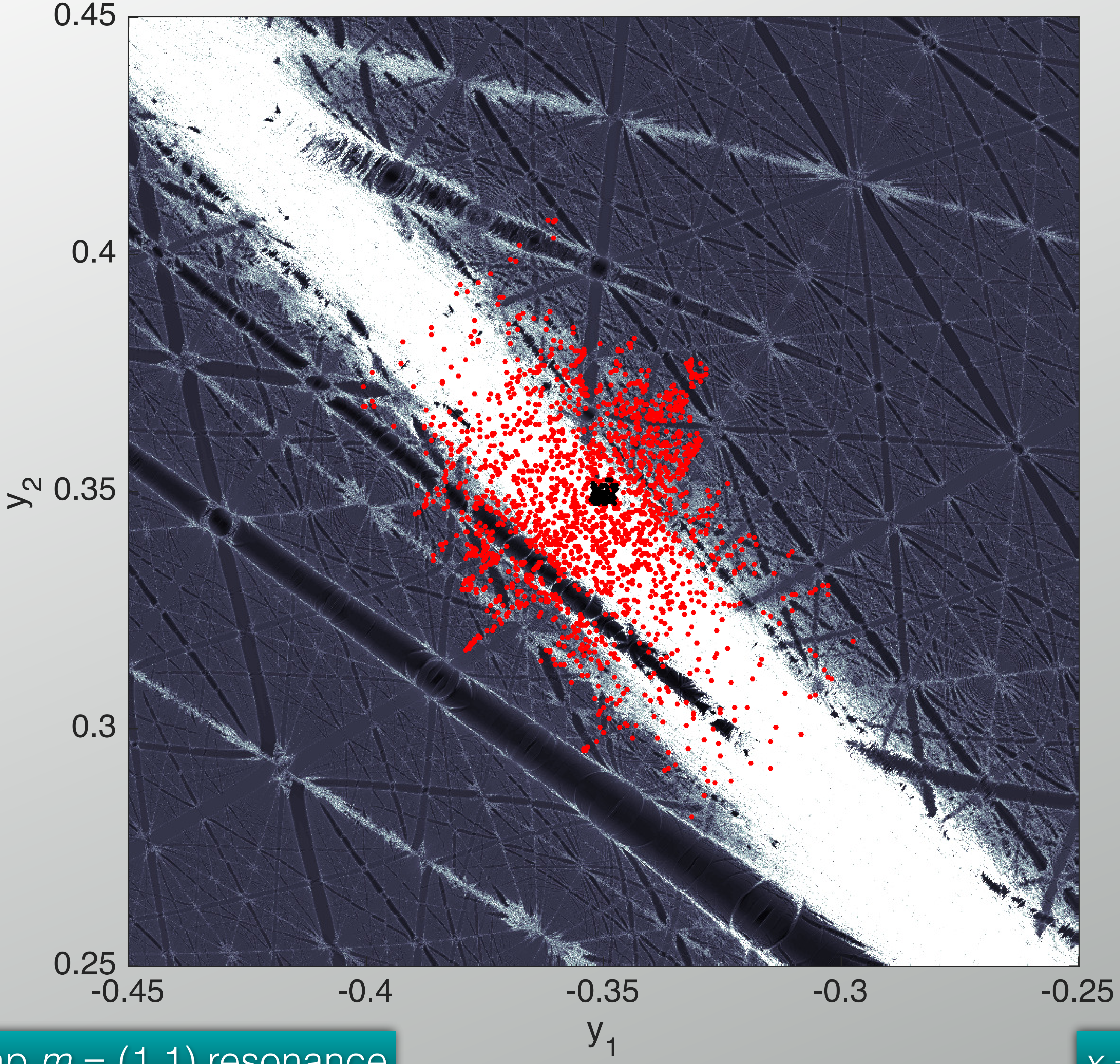


Symplectic Map $m = (1,1)$ resonance

$x = (0,0.5)$ slice

Symplectic: $(a,b,c,d) = (0,0.1,0.07,0.0001)$

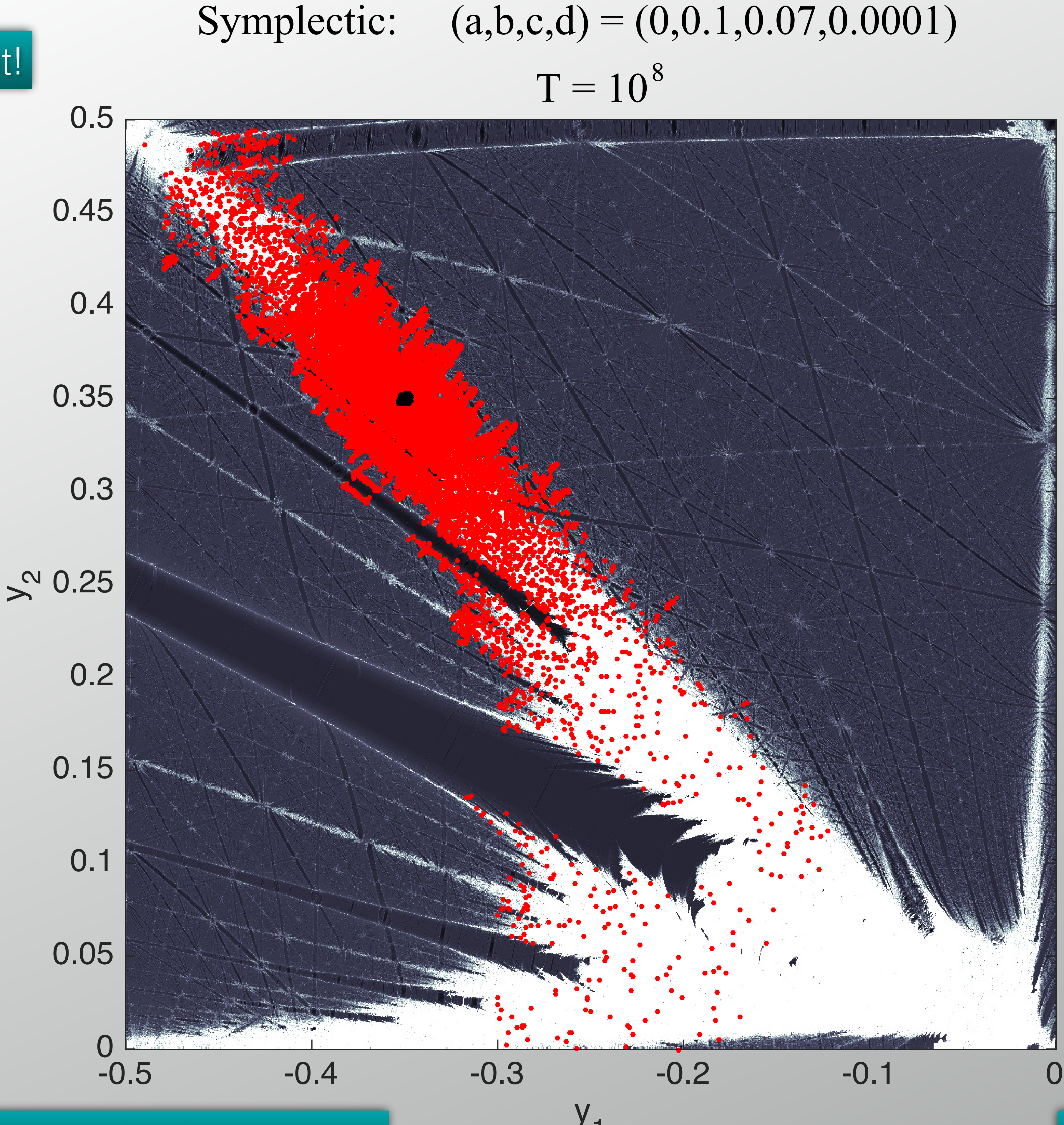
$T = 10^7$



Symplectic Map $m = (1,1)$ resonance

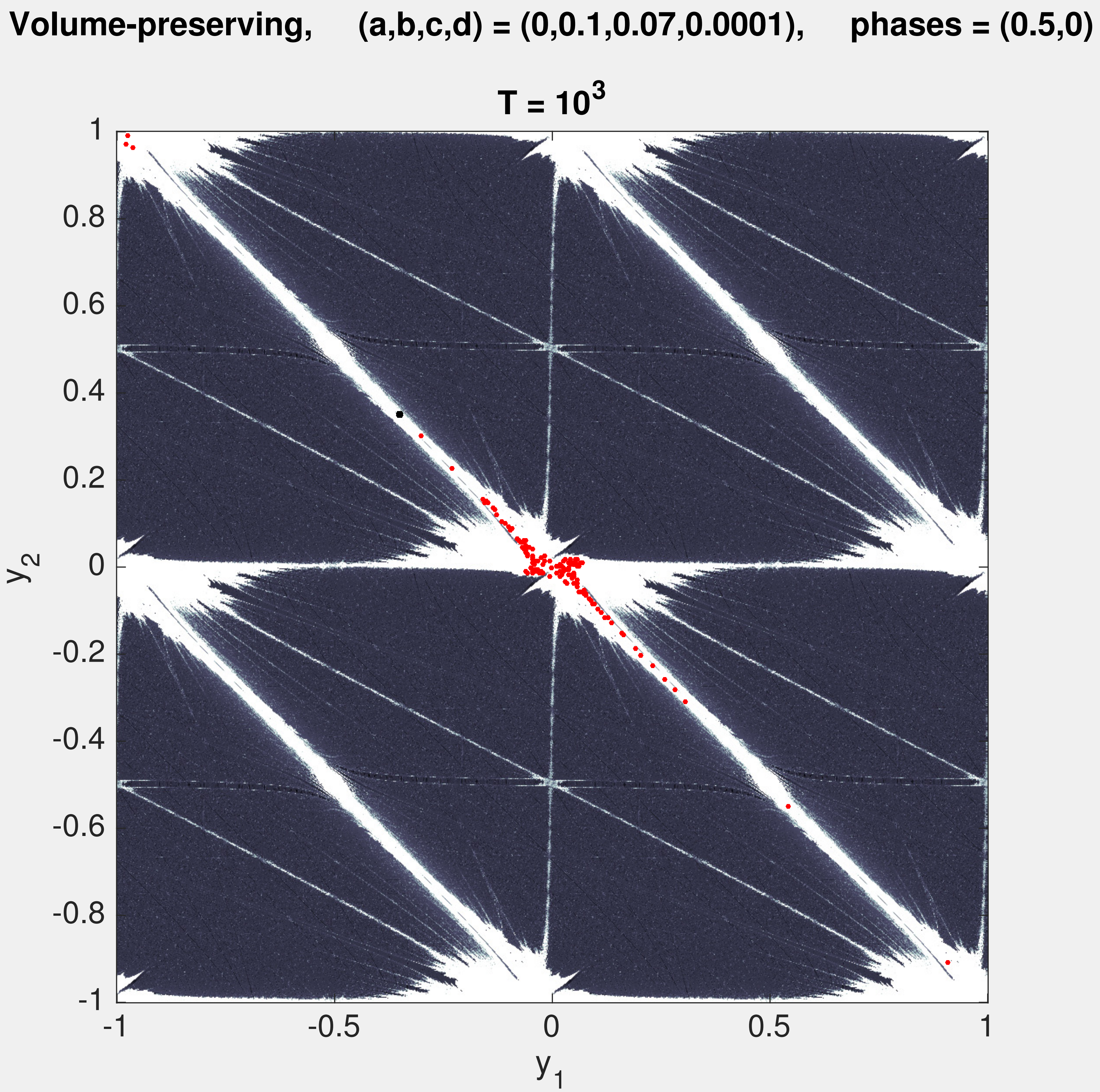
$x = (0,0.5)$ slice

Zooming Out!



Symplectic Map *m* = (1,1) resonance

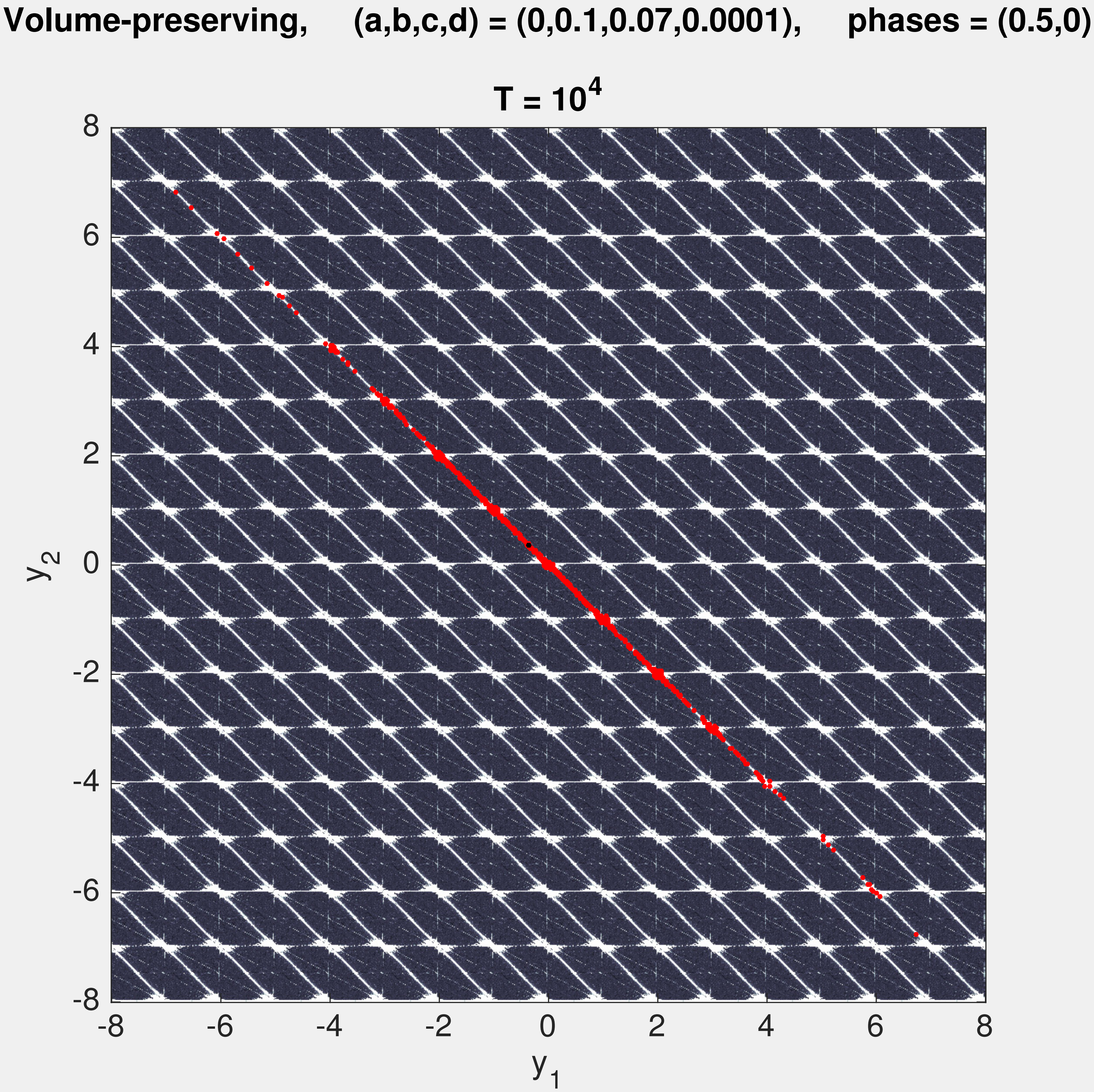
x = (0,0.5) slice



Volume-Preserving Map $m = (1,1)$ resonance

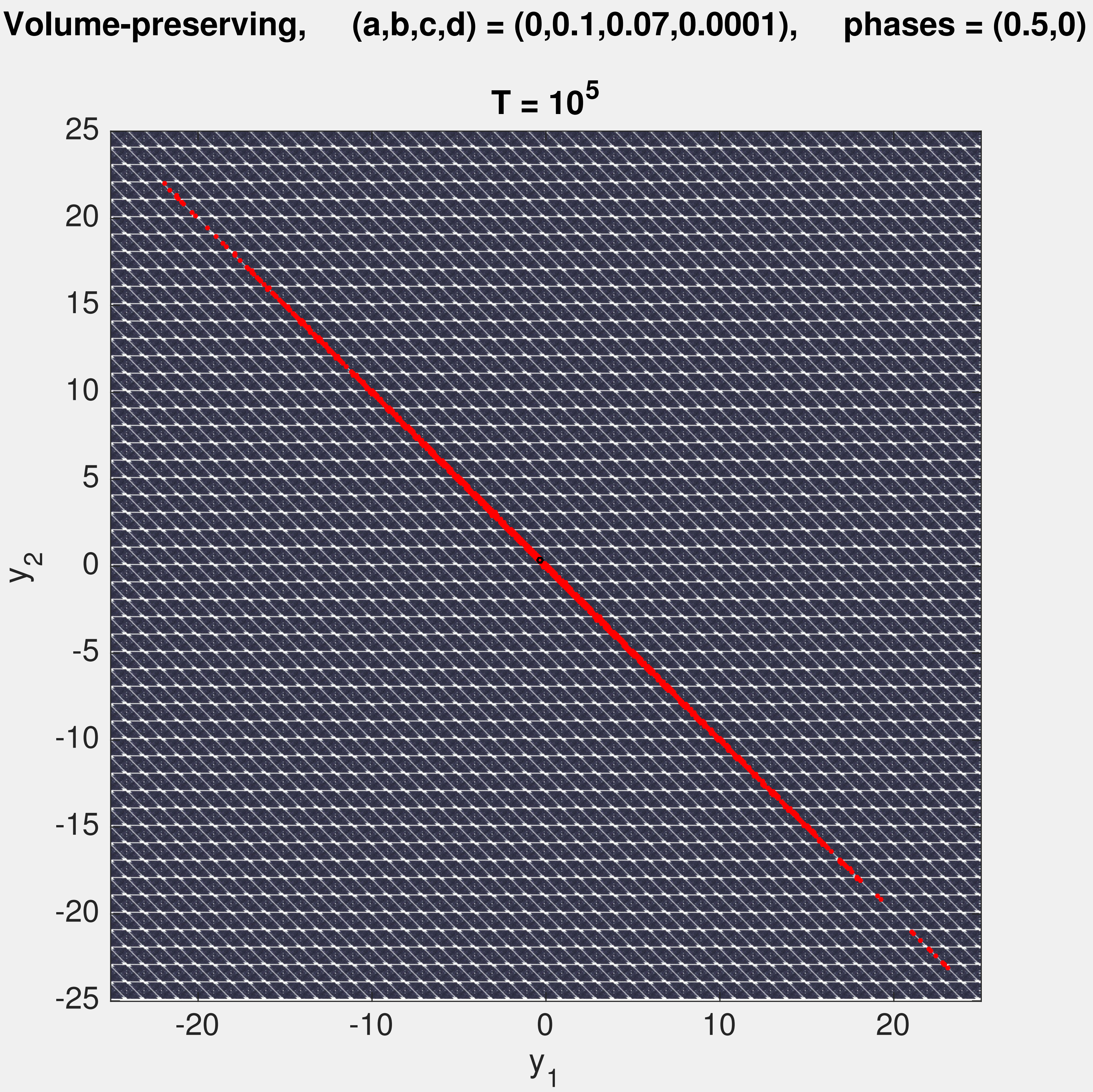
$x = (0,0.5)$ slice

Zooming Out!



Volume-Preserving Map $m = (1,1)$ resonance

Zooming Way Out!



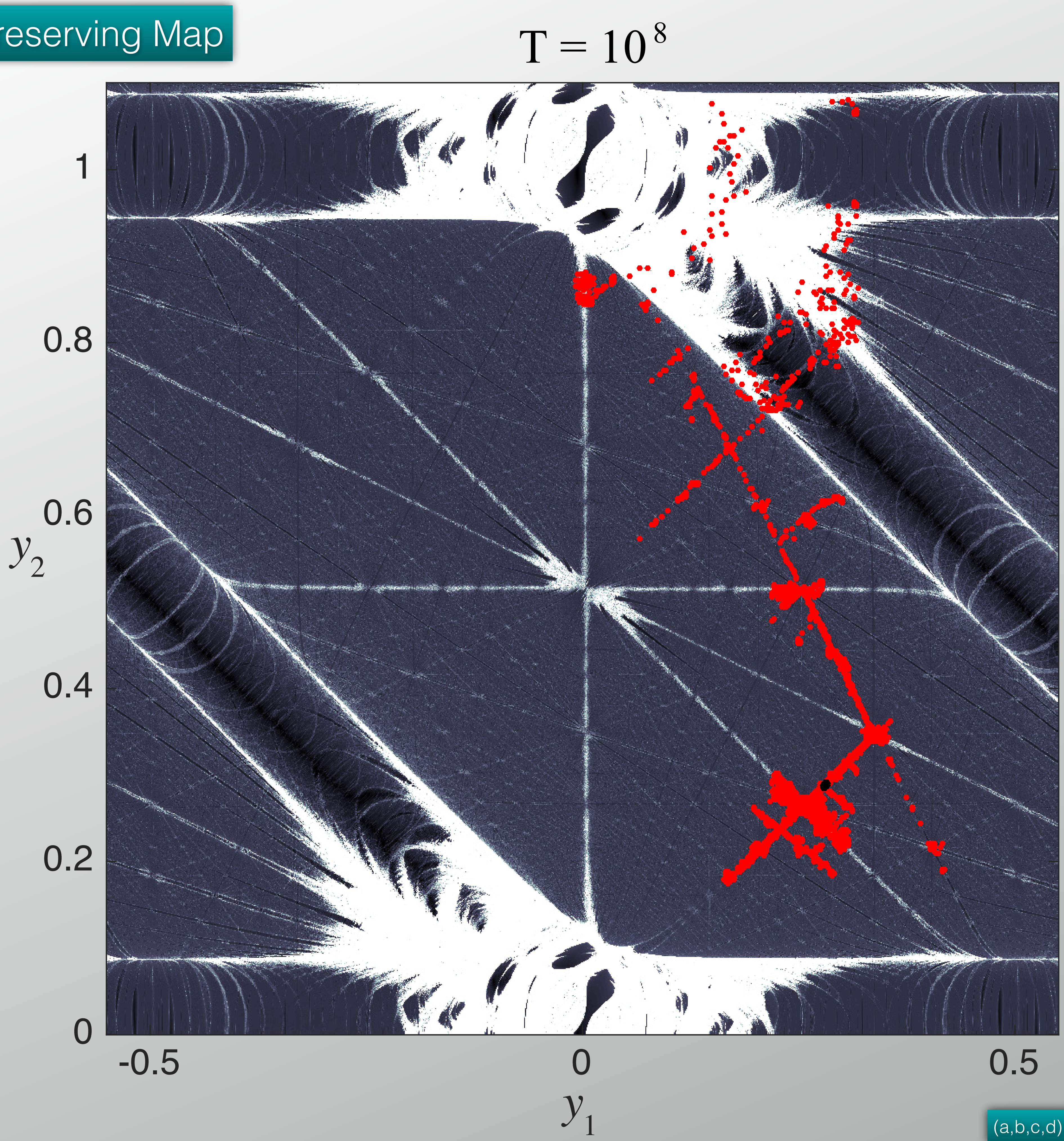
Volume-Preserving Map *m* = (1,1) resonance

x = (0,0.5) slice

Crossing Resonances

Volume-Preserving Map

$T = 10^8$



x=(0,0.25) slice

(a,b,c,d) = 0.0,0.1,0.07, 0.0001, ϕ = (0, 0..5)

More about VP Maps

- Dullin, H. R. and J. D. Meiss (2009). “Quadratic Volume-Preserving Maps: Invariant Circles and Bifurcations.” SIAM J. Appl. Dyn. Sys. 8(1): 76-128.
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