

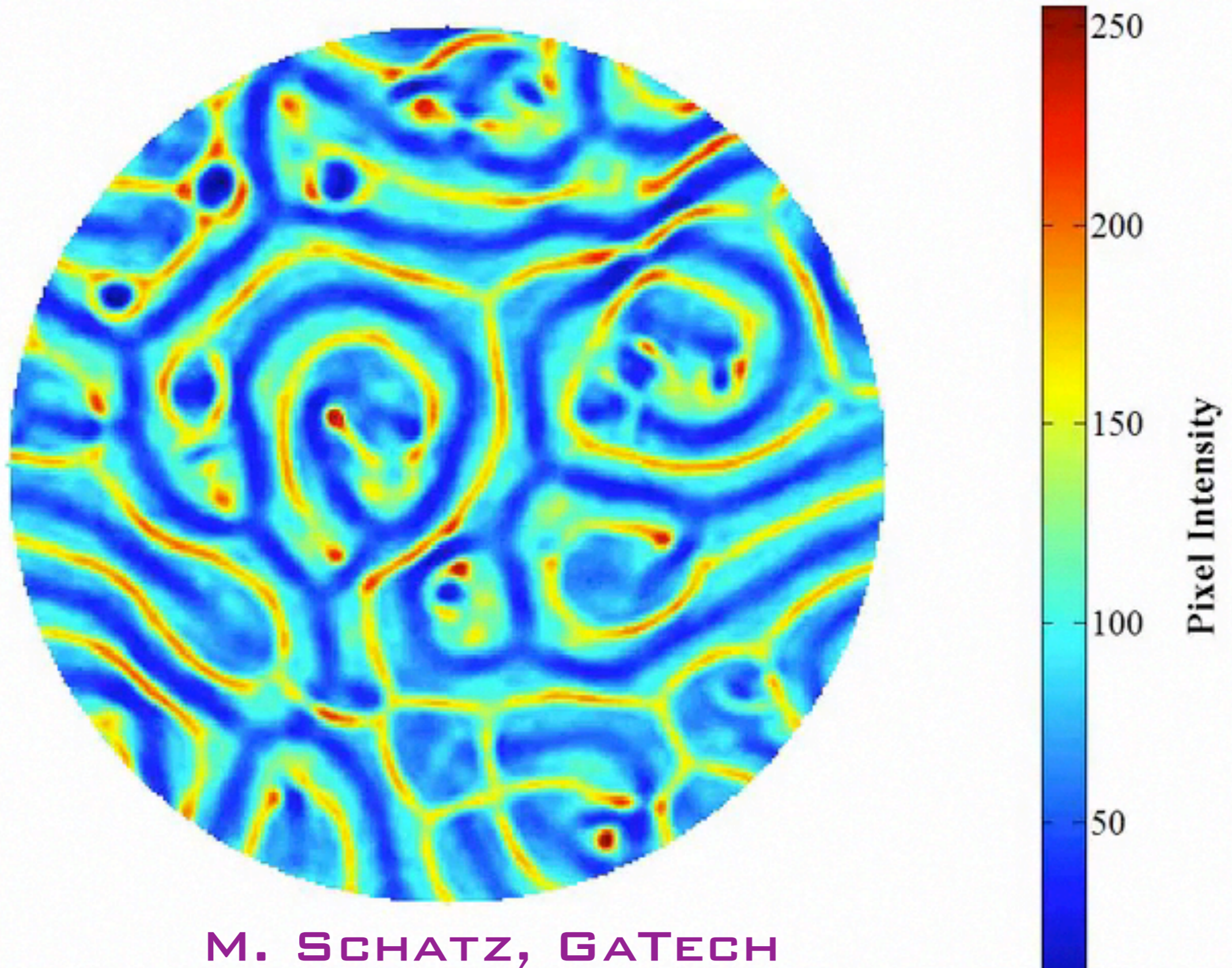
Reduction
and
Reconstruction
of
Complex SpatioTemporal Data

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RAYLEIGH-BENARD CONVECTION - EXPERIMENTS

Ra \approx 6000, Frame 00001



M. SCHATZ, GATECH

How can we characterize the dynamics in a comprehensible manner?

RAYLEIGH-BENARD CONVECTION - NUMERICS

$$Pr^{-1} \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla^2 \mathbf{u} + RaT\hat{\mathbf{z}}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T$$

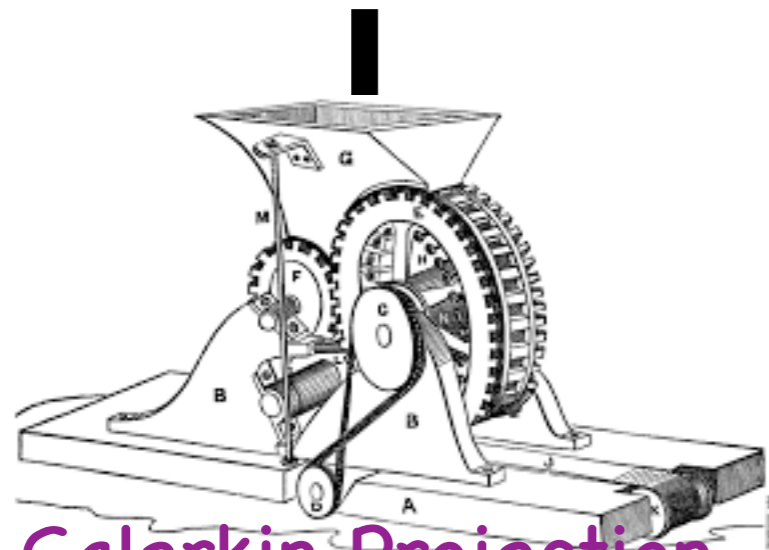
$$\nabla \cdot \mathbf{u} = 0$$

$\mathbf{u}(x, y, z, t)$ is velocity field

$p(x, y, z, t)$ is pressure field

$T(x, y, z, t)$ is temperature field

Boussinesq Equations



Galerkin Projection

$$\dot{x} = F(x), \quad x \in \mathbb{R}^d$$

$d \approx 5,000,000$



M. PAUL, VATECH

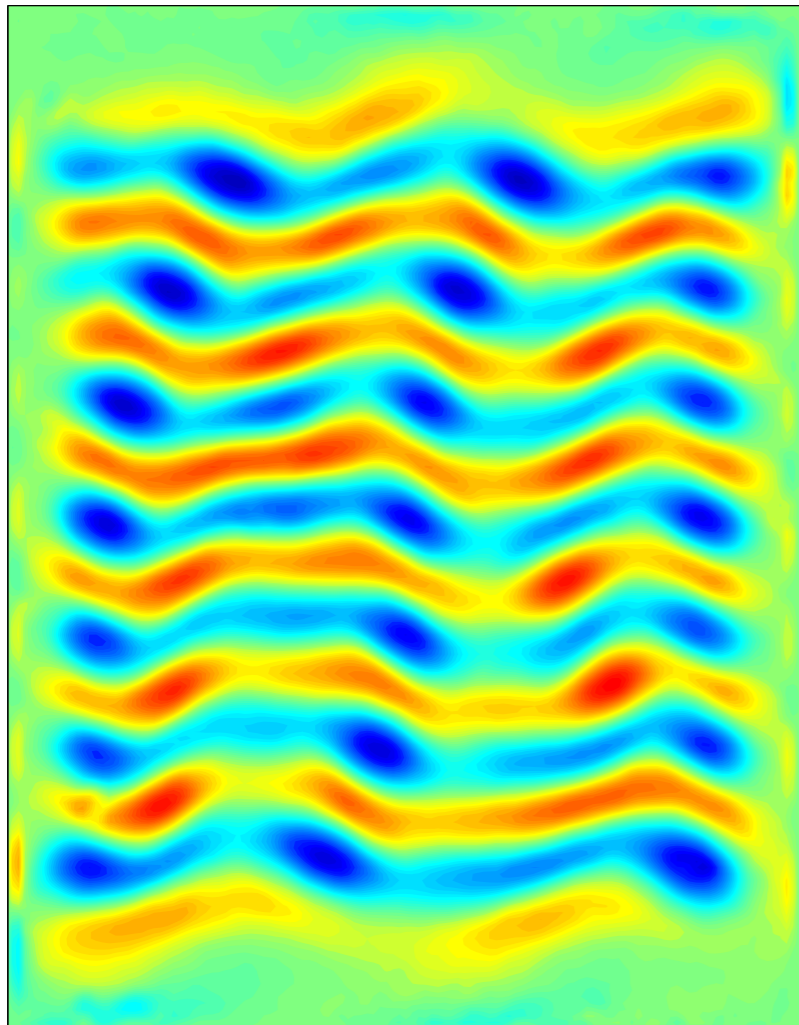
Do numerics and experiment agree?

Simple characterization????

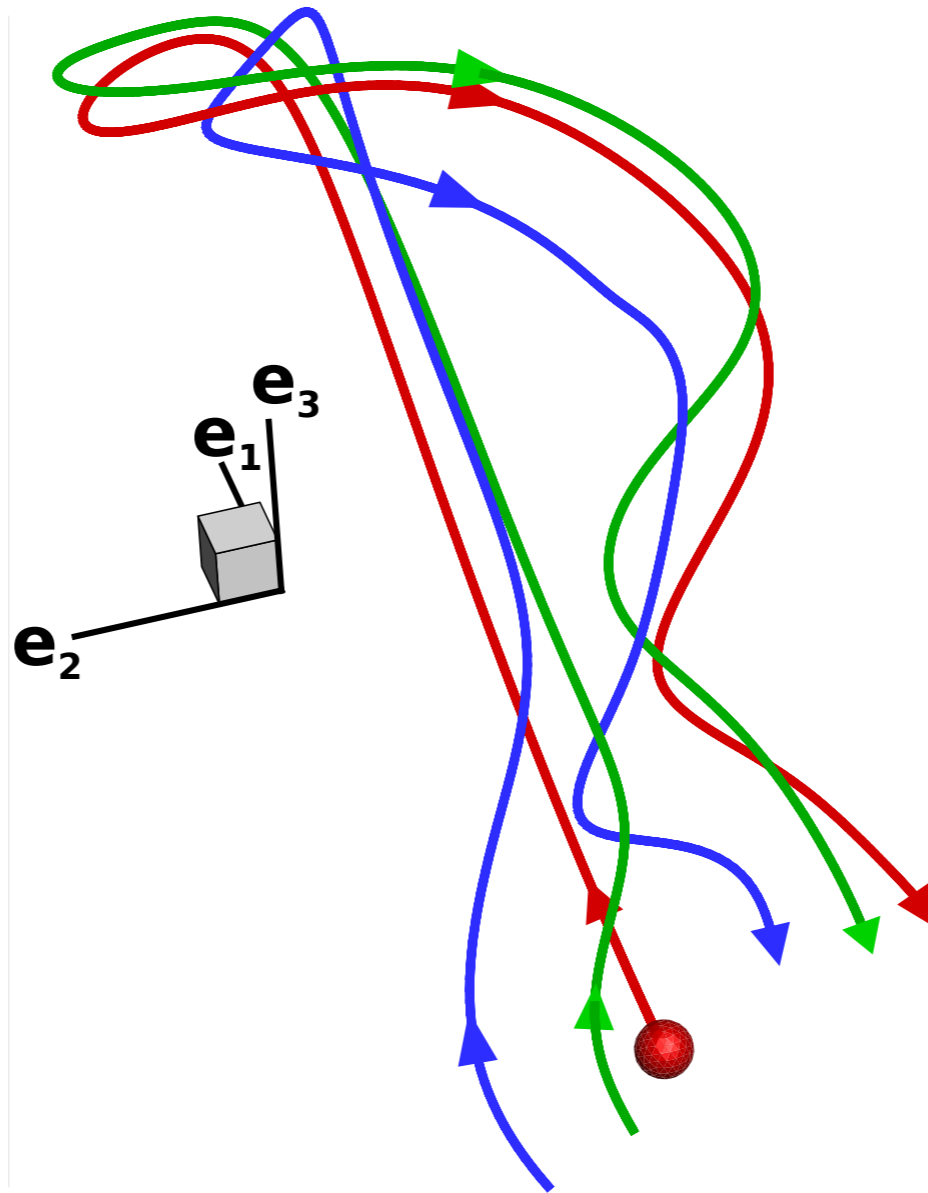
EXACT COHERENT STRUCTURES

Kolmogorov Flow

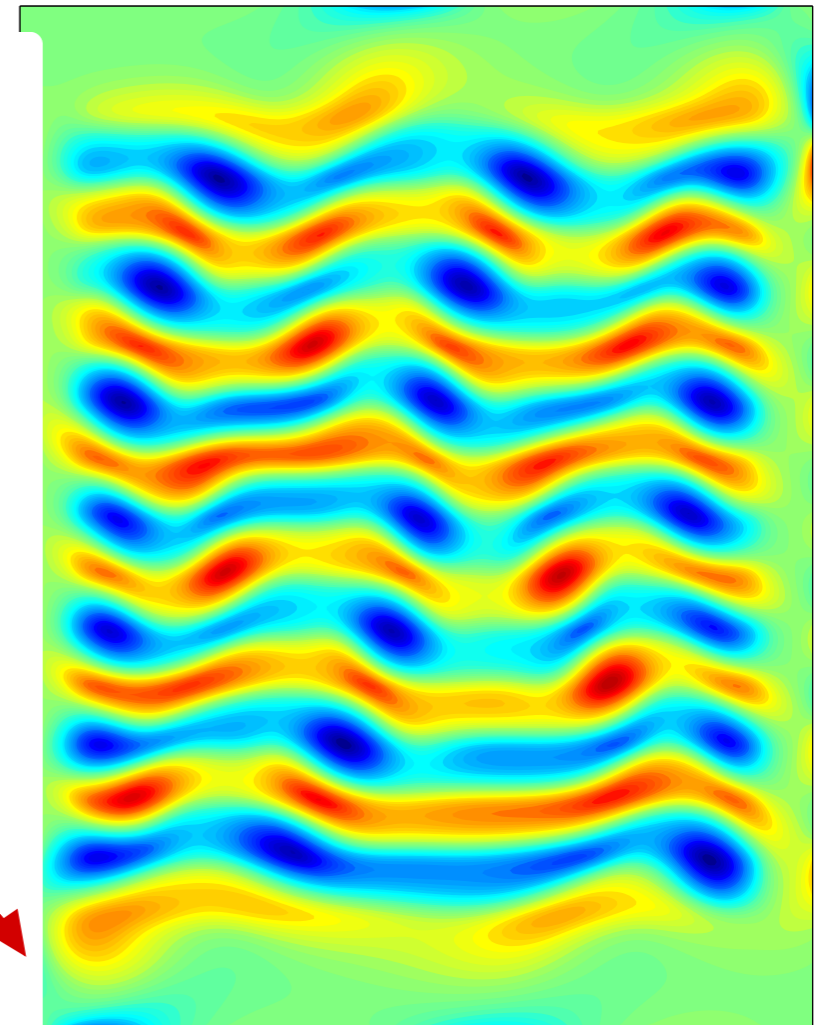
$$\frac{\partial \omega}{\partial t} + \beta \mathbf{u} \cdot \nabla \omega = \bar{\nu} \nabla^2 \omega - \alpha \omega + A \cos(2\pi y / \lambda)$$



Experimental Image of vorticity field for which small velocity is small.



Unstable manifold
Numerical trajectory
Experimental trajectory



Numerical fixed point derived from experimental image.

Summary: (1) The natural phase space for many physical systems is infinite dimensional.

(2) Approximations (both numerical and experimental) are high dimensional.

Goals: (1) Describe this dynamics in terms of low dimensional invariant structures (fixed points, periodic orbits, invariant tori, heteroclinic orbits, symbolic dynamics).

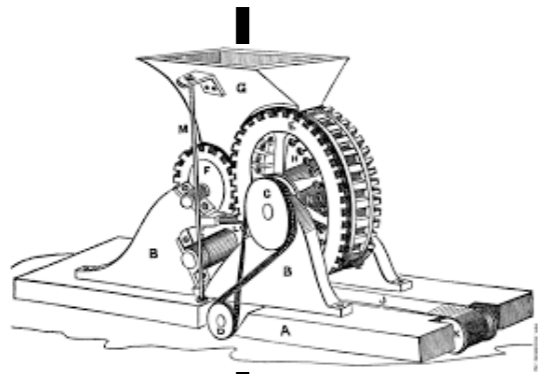
(2) Describe dynamics in terms of identifiable landmarks and probabilities of transitions between these.

(3) I would like to know that, at least theoretically, the description is mathematically rigorous.

Are these realistic goals?

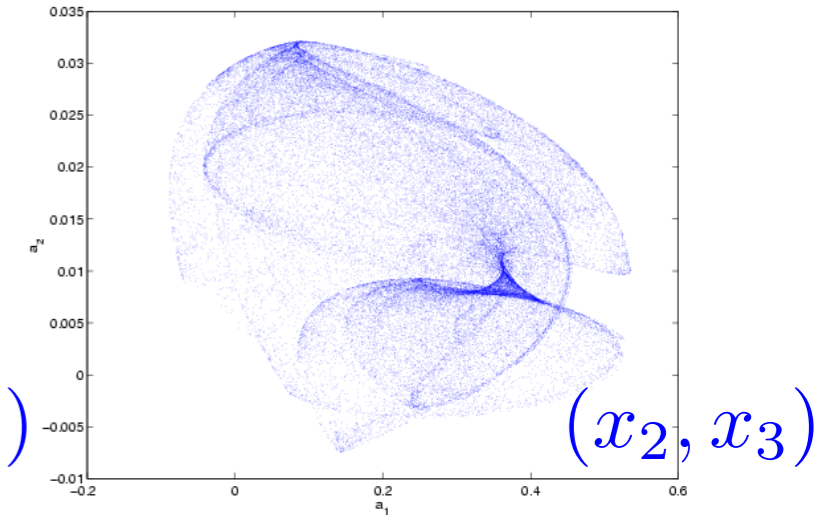
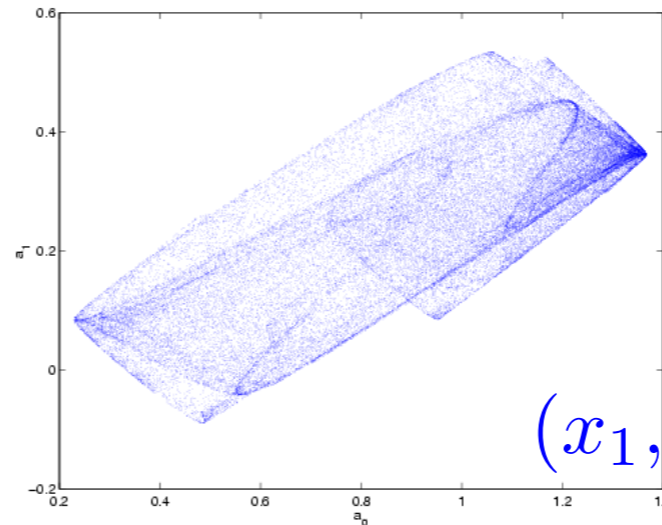
INFINITE DIMENSIONAL MAP

$$u \mapsto F[u](\bullet) = \int_{-\pi}^{\pi} D(|\bullet - y|) \mu u(y) \left[1 - \frac{u(y)}{c(y)} \right] dy \quad u: [-\pi, \pi] \rightarrow \mathbb{R}$$

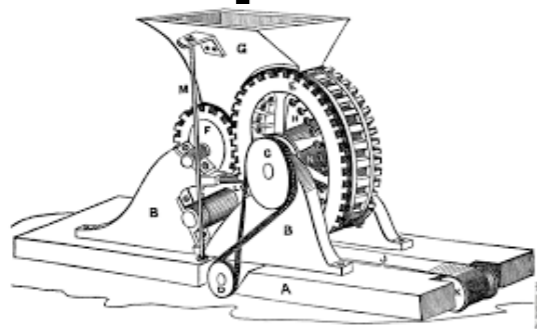


Galerkin Projection

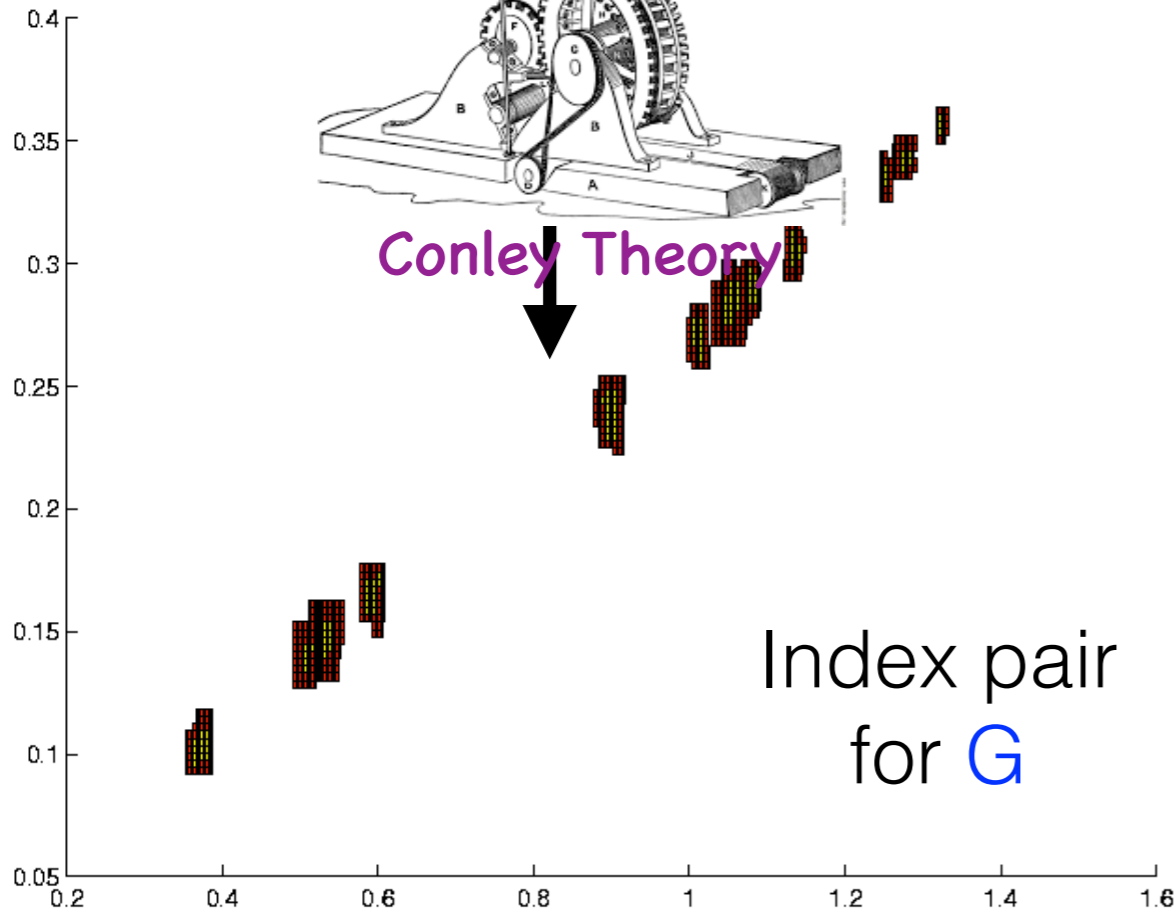
$$G: \mathbb{R}^6 \rightarrow \mathbb{R}^6$$



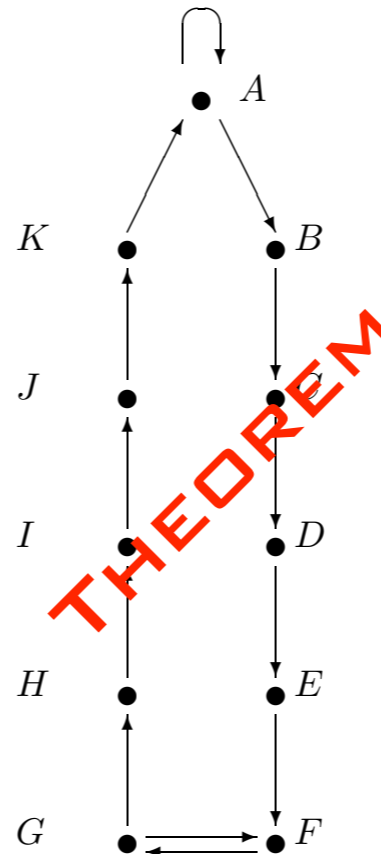
“time series data for a infinite dimensional map”



Conley Theory



Index pair
for G



THEOREM

S. Day, O. Junge, K.M.
SIADS (2004)

Goals: (1) Describe this dynamics in terms of low dimensional invariant structures (fixed points, periodic orbits, invariant tori, heteroclinic orbits, symbolic dynamics).

(2) Describe dynamics in terms of identifiable landmarks and probabilities of transitions between these.

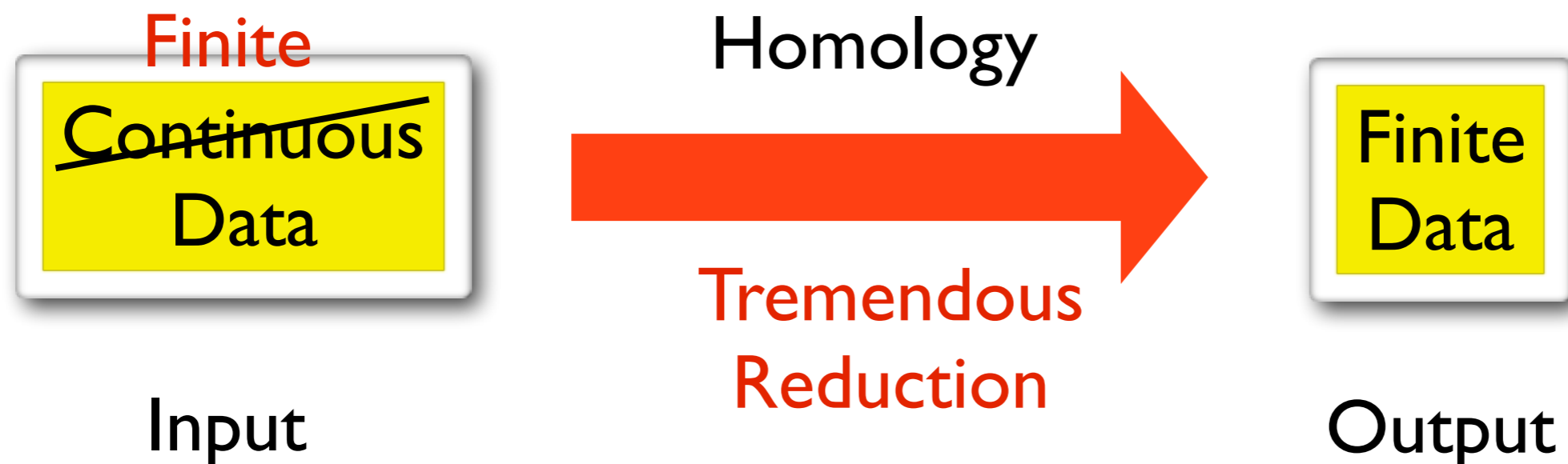
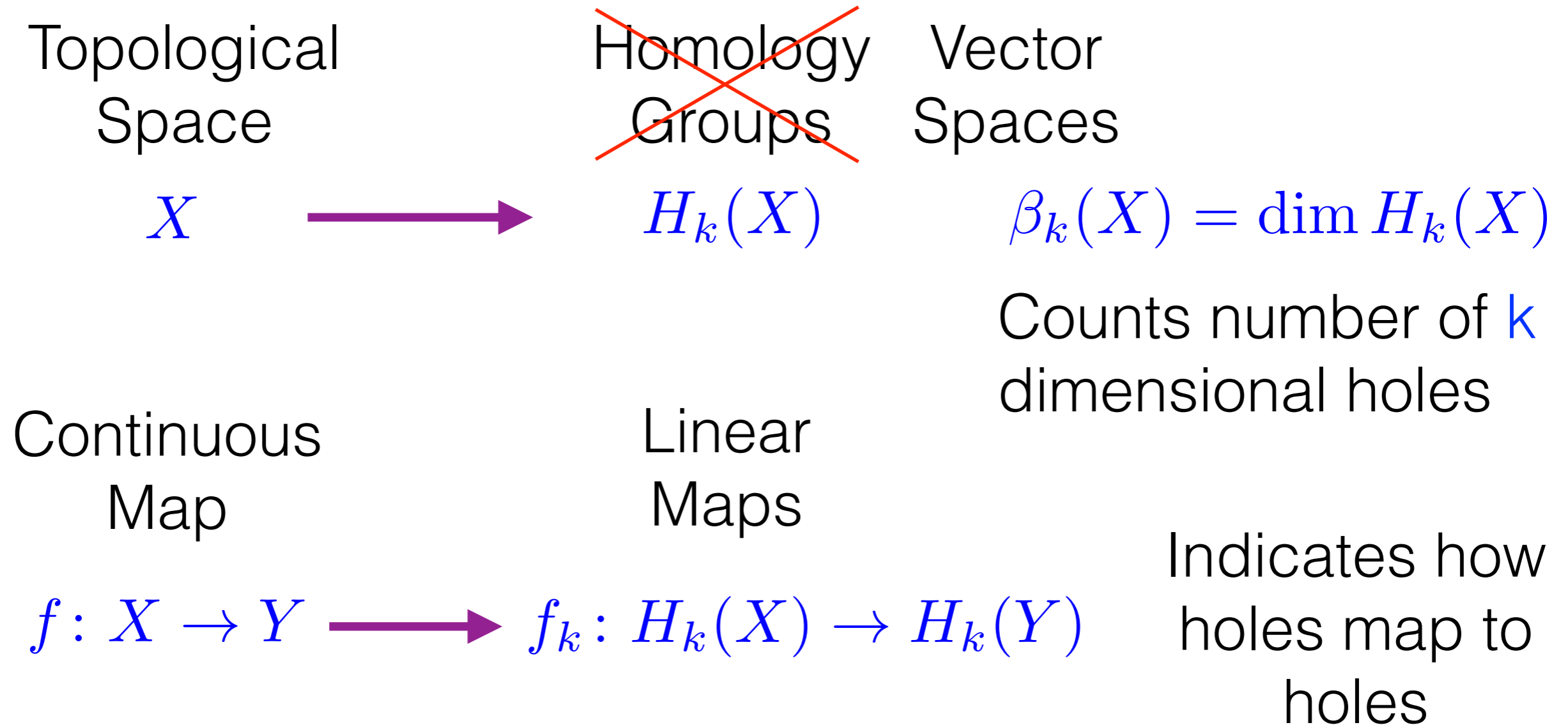
Observation: Evolution of the patterns is of central interest.

Immediate Goals: (1) Perform dimension reductions that preserve geometric information.
(2) Reconstruct dynamics from reduced system.



Persistent Homology

WHAT DO WE NEED TO KNOW ABOUT HOMOLOGY?



SHAPE OF A DENSITY FUNCTION VIA HOMOLOGY



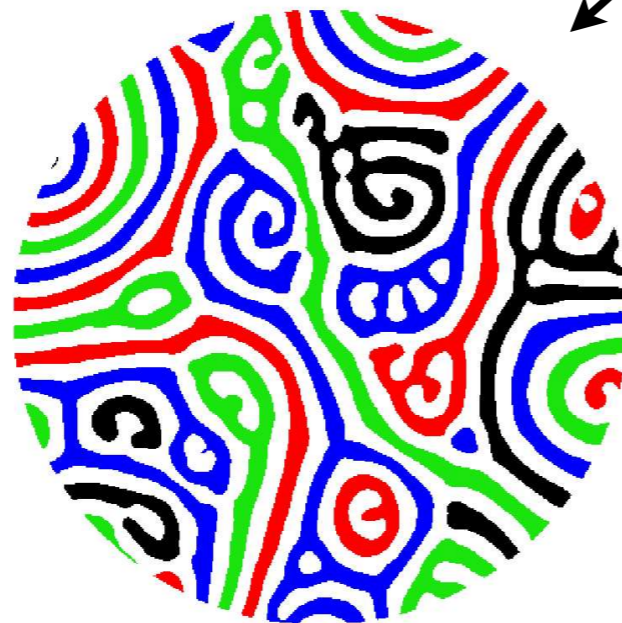
Experimental
Data

Threshold at median
grayscale
 μ



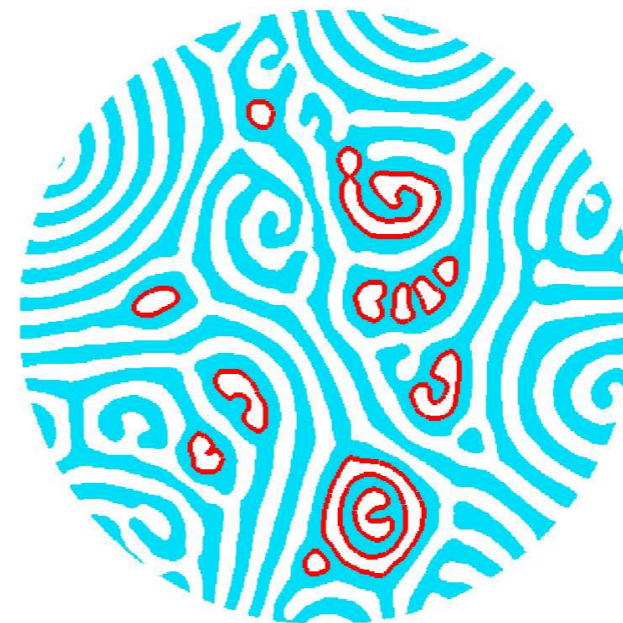
X is set
of pixels
with grayscale
 $\leq \mu$

Components



$$\beta_0(X) = 34$$

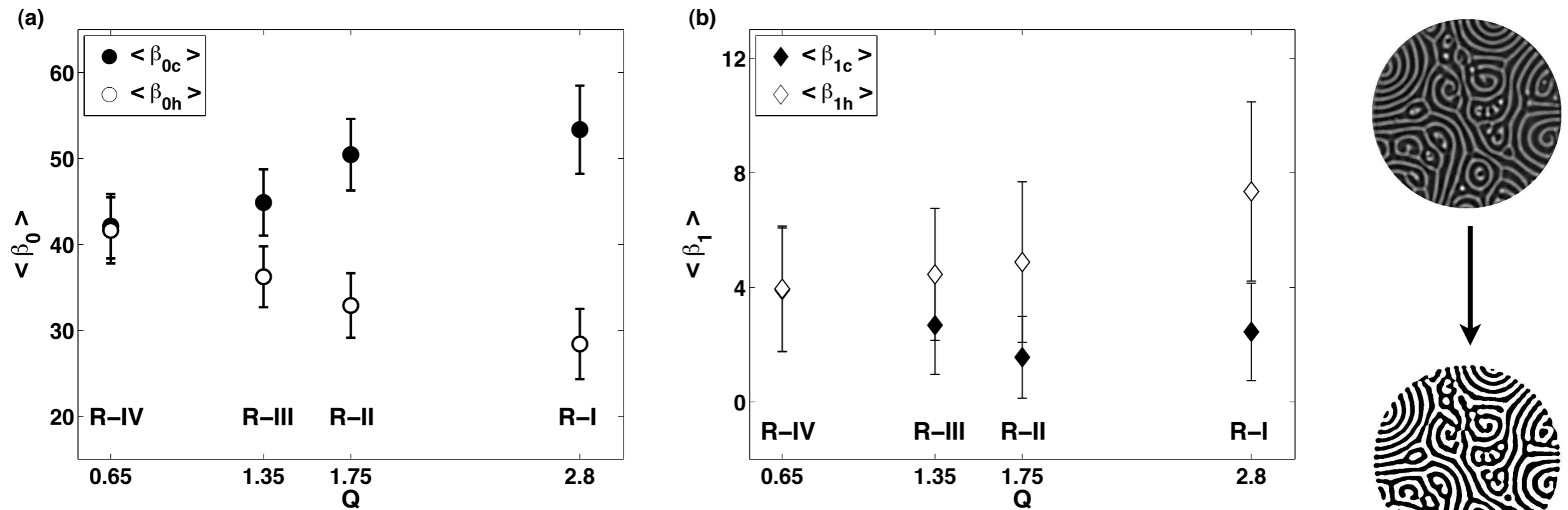
Loops



$$\beta_1(X) = 14$$

SHAPE OF A DENSITY FUNCTION VIA HOMOLOGY

Fact: Boussinesq equations exhibit an up down symmetry



Q = Busse number

Conclusion: We can use homology to quantify by how much the numerical model fails to match the physical system.

H. Kurtuldu, K. M., and M. Schatz,
Measuring the departures
from the Boussinesq
approximation in
Rayleigh-Benard
convection experiments
JFM (2011)

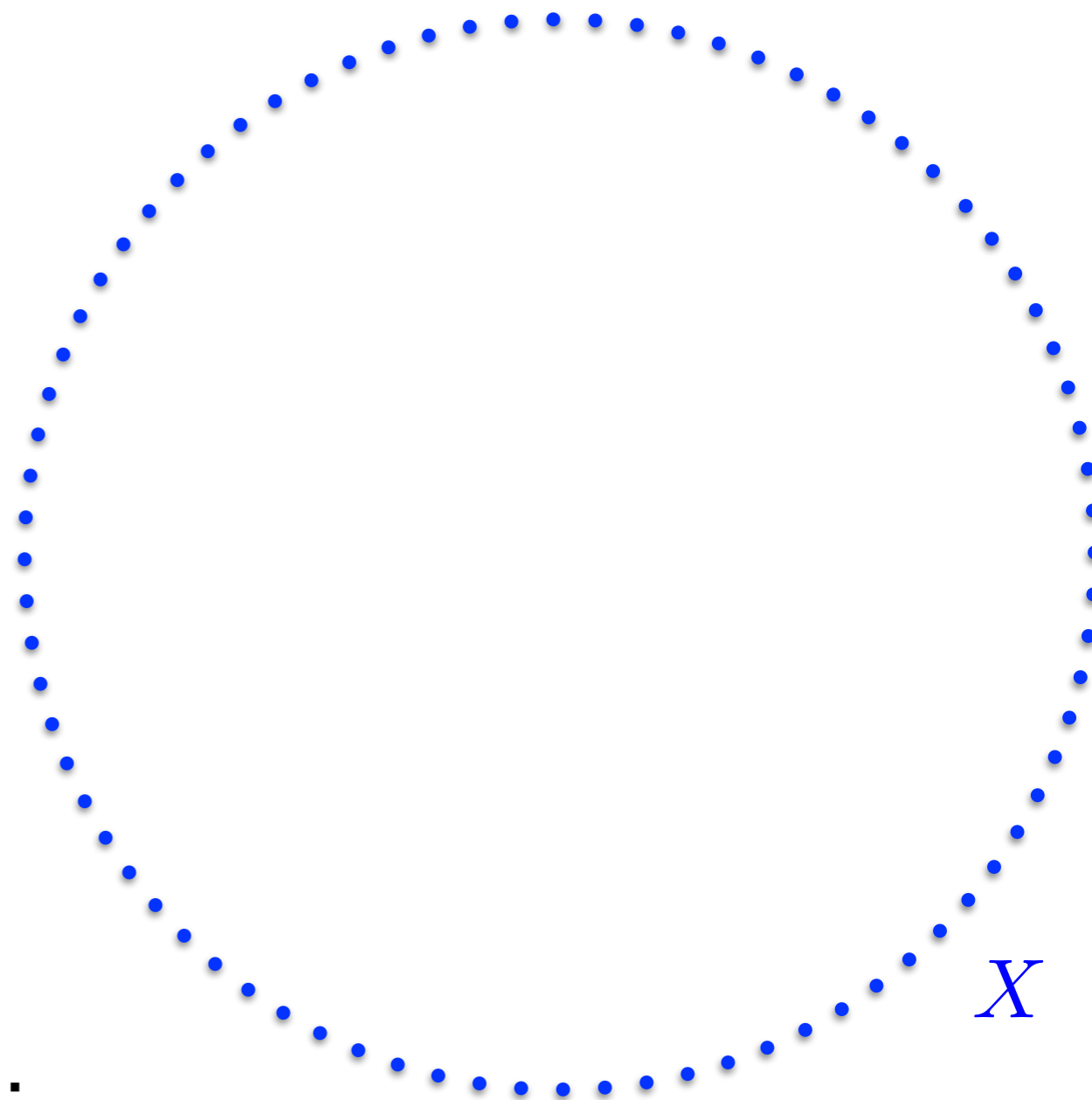
No reason a single level should characterize shape of function!

TOPOLOGY OF A TIME SERIES

Time series from

$$\dot{x} = y$$

$$\dot{y} = -x$$



Homology of time series data:

$$\beta_0(X) \approx 100$$

$$\beta_1(X) = 0$$

PERSISTENT HOMOLOGY (I)

$\text{Fn}(f, 0)$



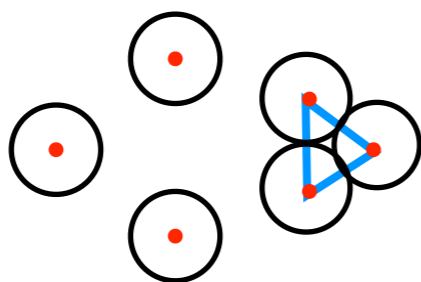
X

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(y) := \text{dist}(y, X)$$

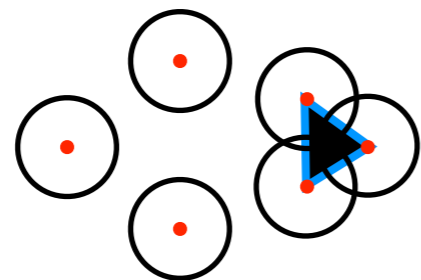
$$\text{Fn}(f, \theta) := \{y \mid f(y) \leq \theta\}$$

$\text{Fn}(f, 60)$

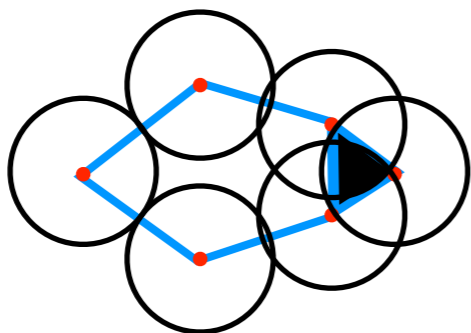


Cech
Complex

$\text{Fn}(f, 70)$

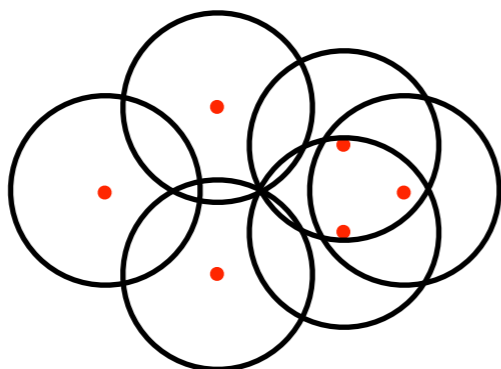


$\text{Fn}(f, 100)$

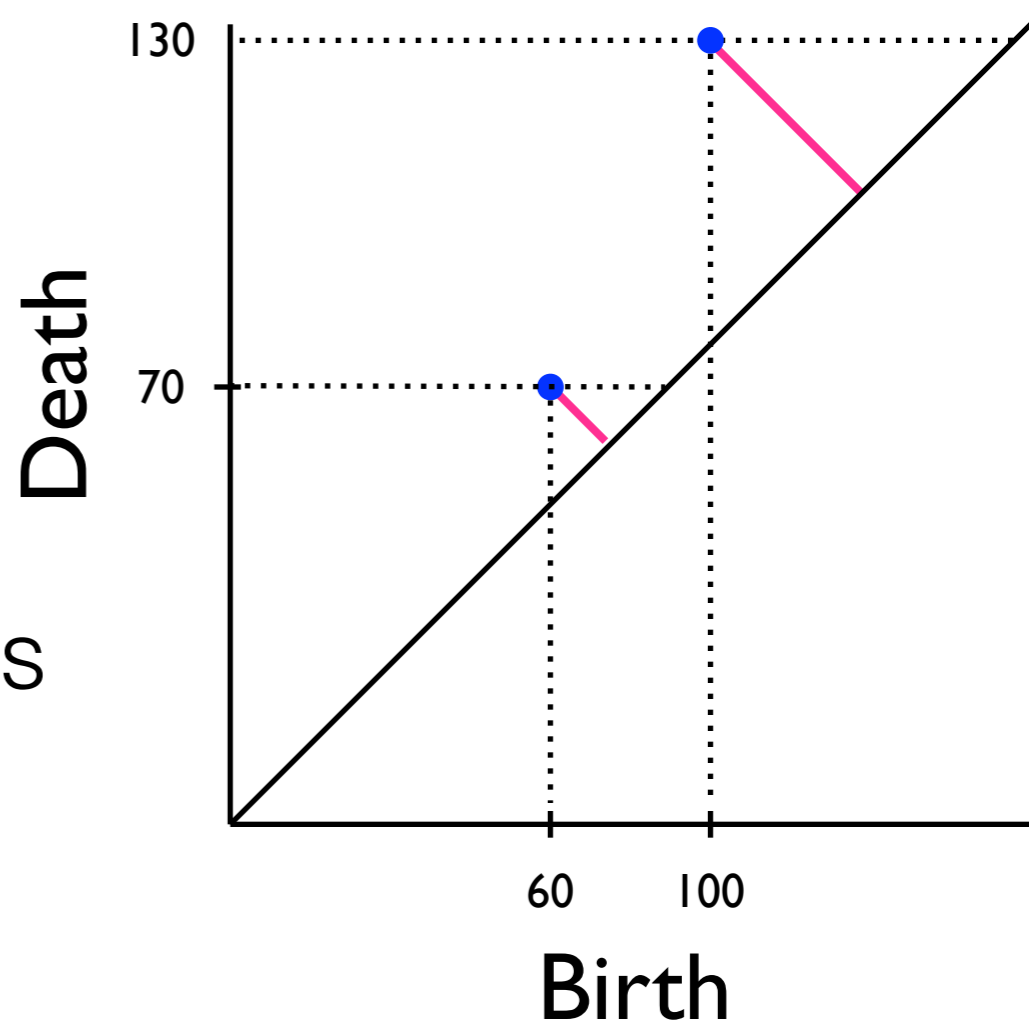


Vietoris-Rips
Complex

$\text{Fn}(f, 130)$



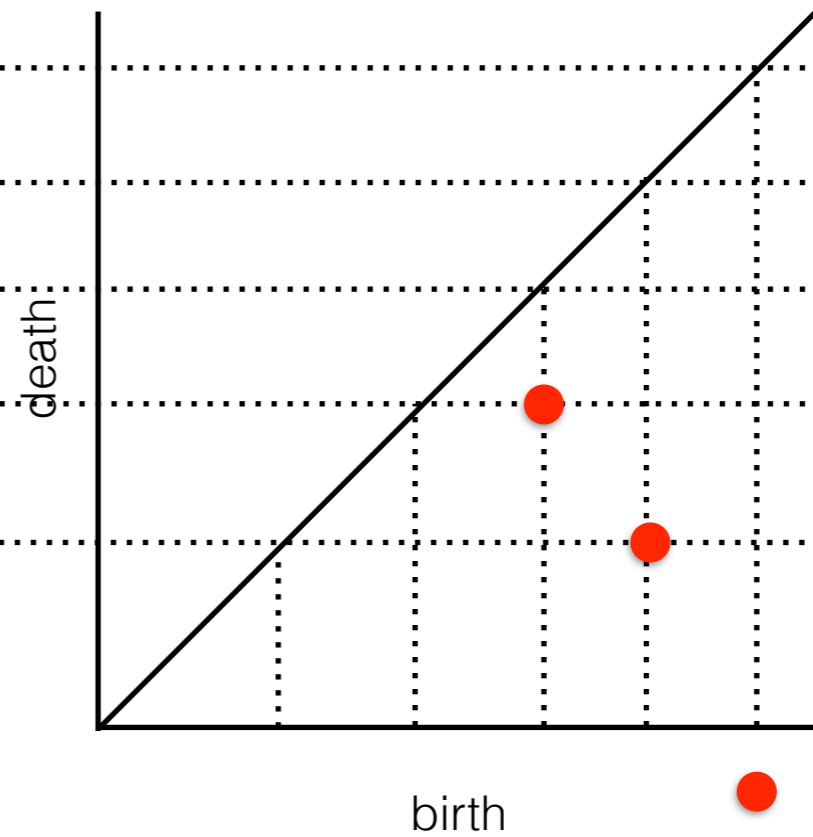
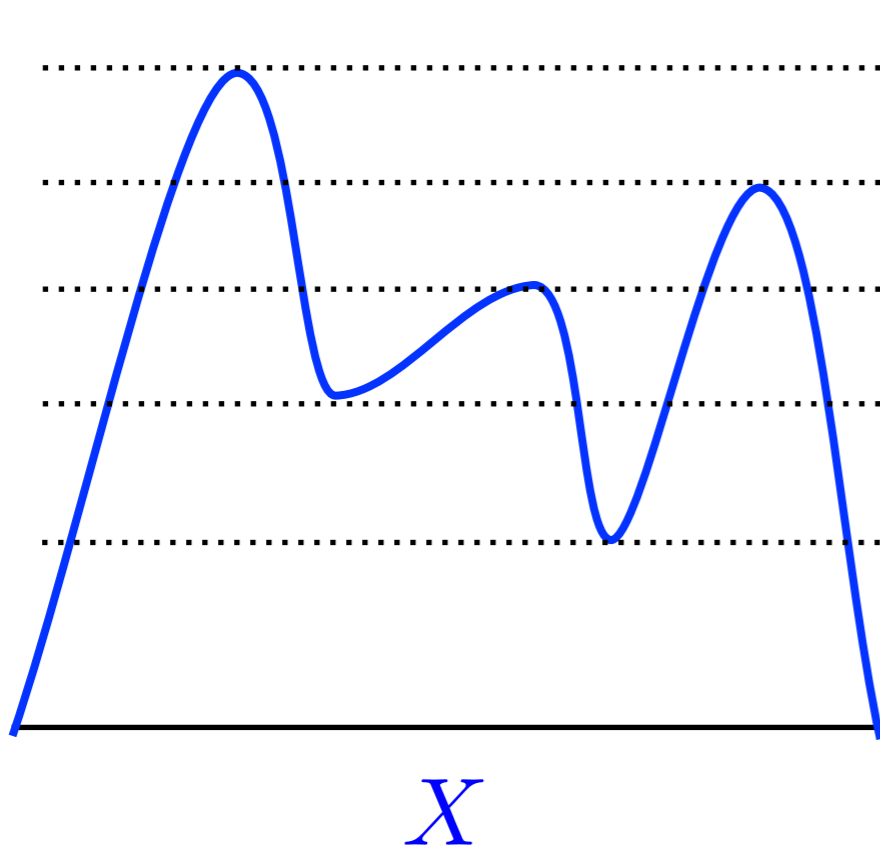
Significance of a
feature



β_1 Persistence Diagram

PERSISTENT HOMOLOGY (II)

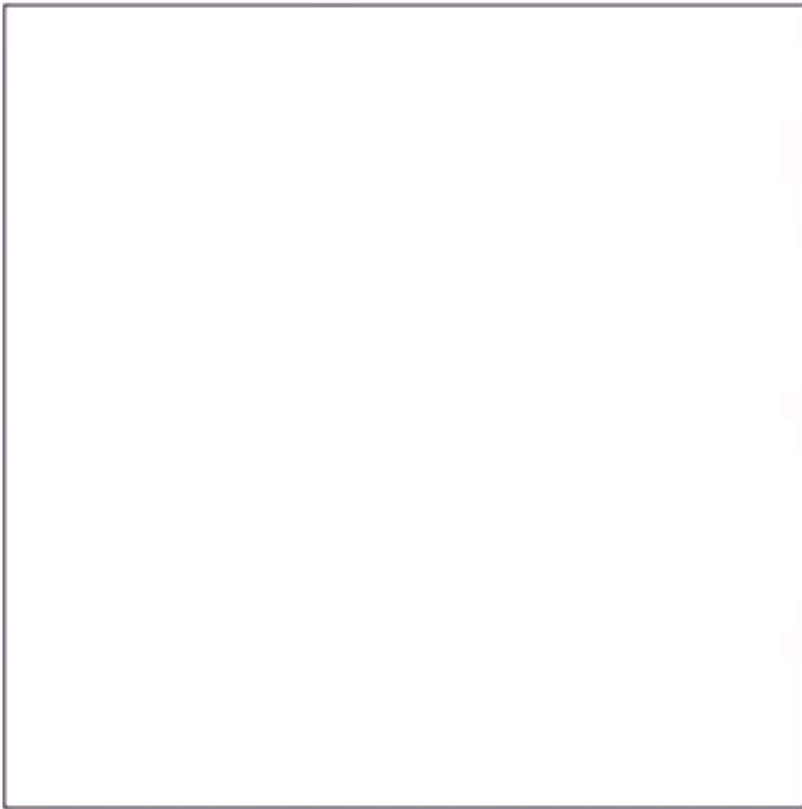
$$f: X \rightarrow \mathbb{R}$$



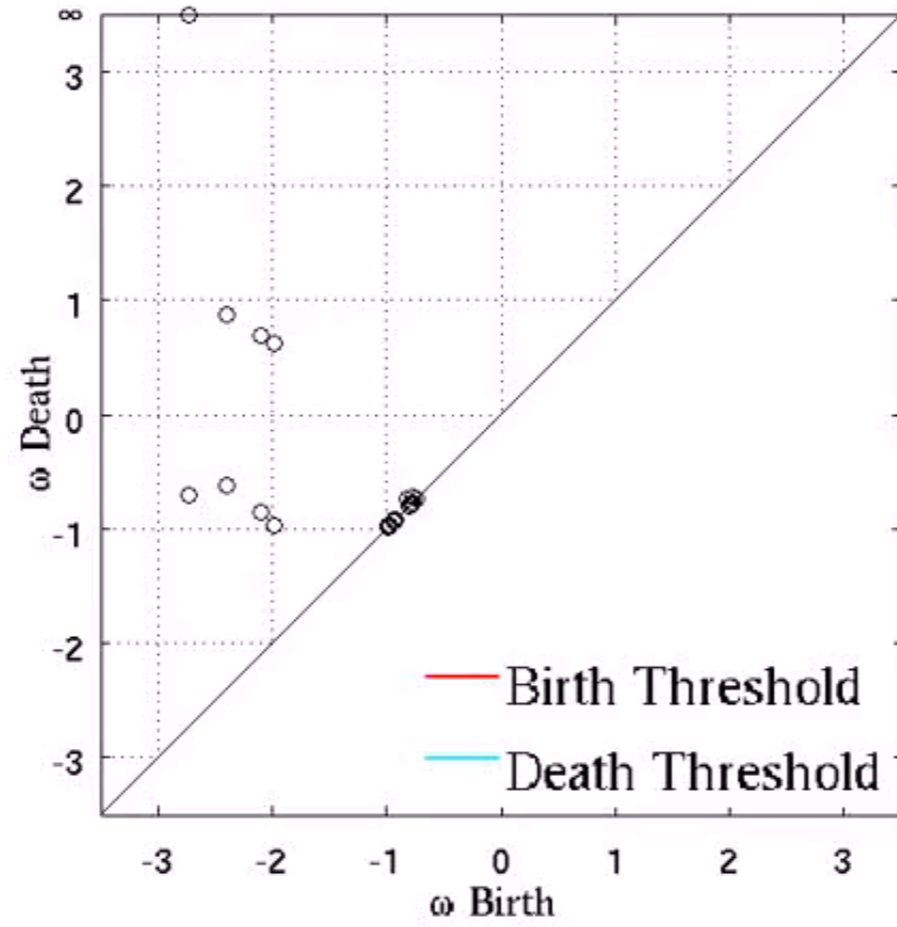
β_0 persistence diagram

$$Fn(f, \theta) := \{x \mid f(x) \geq \theta\}$$

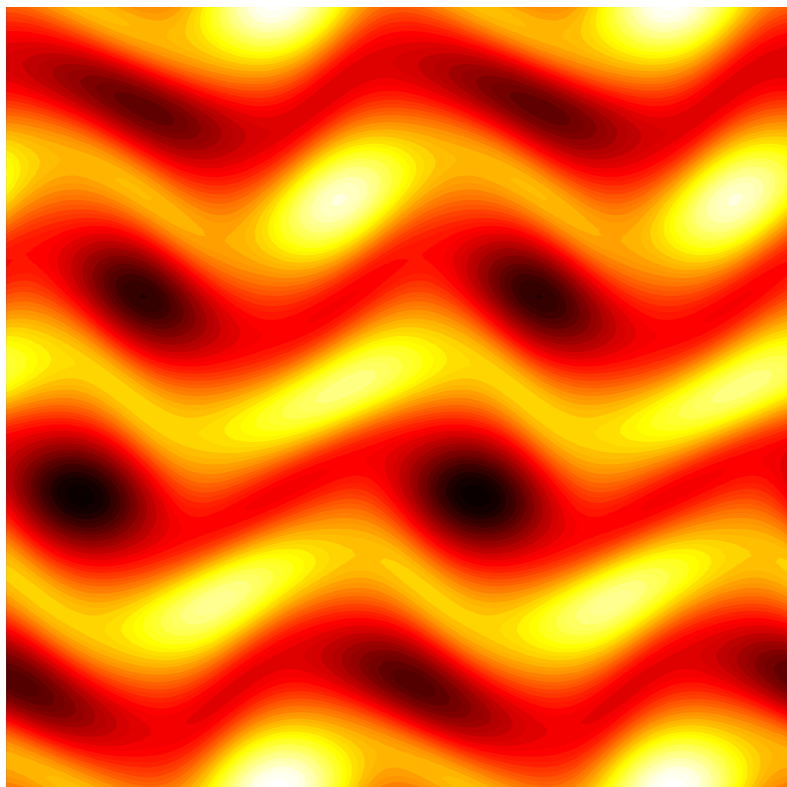
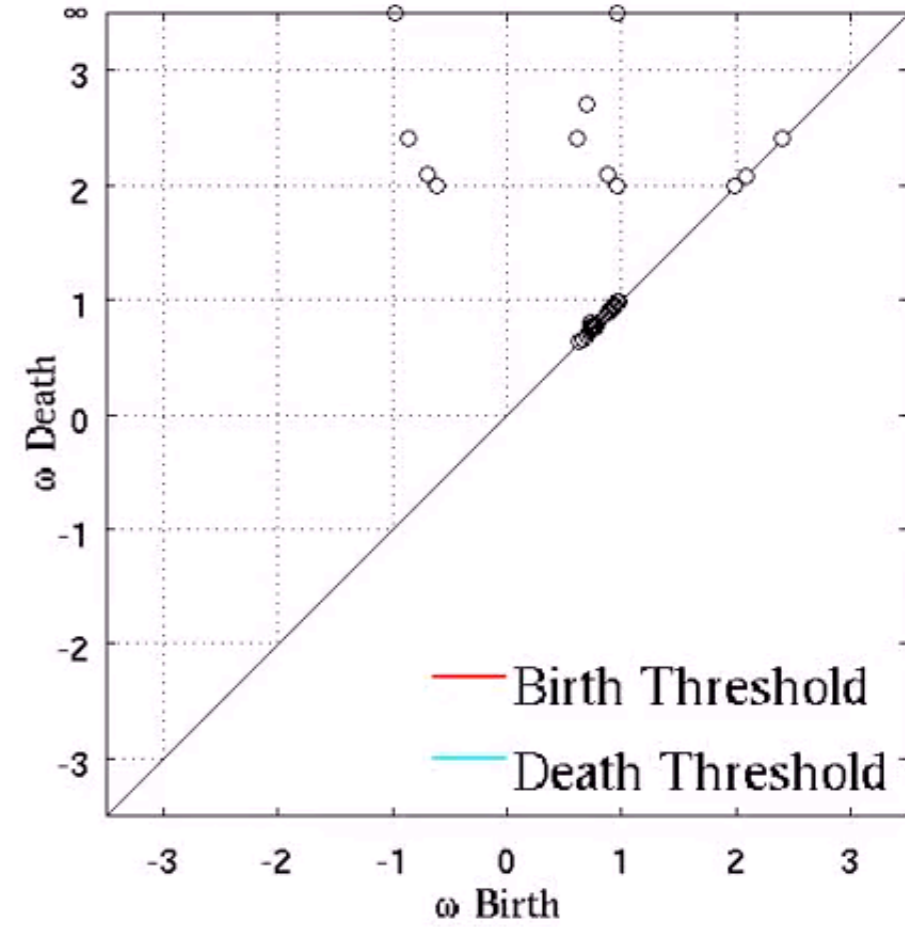
$\omega \leq -3.50$



β_0



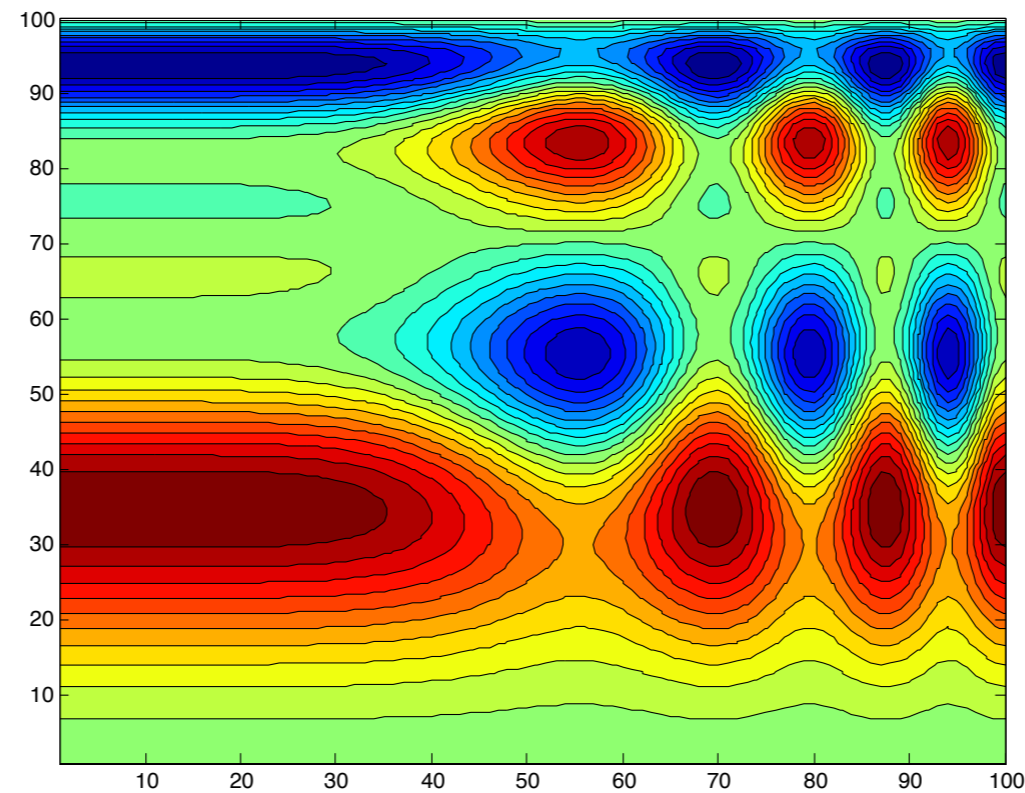
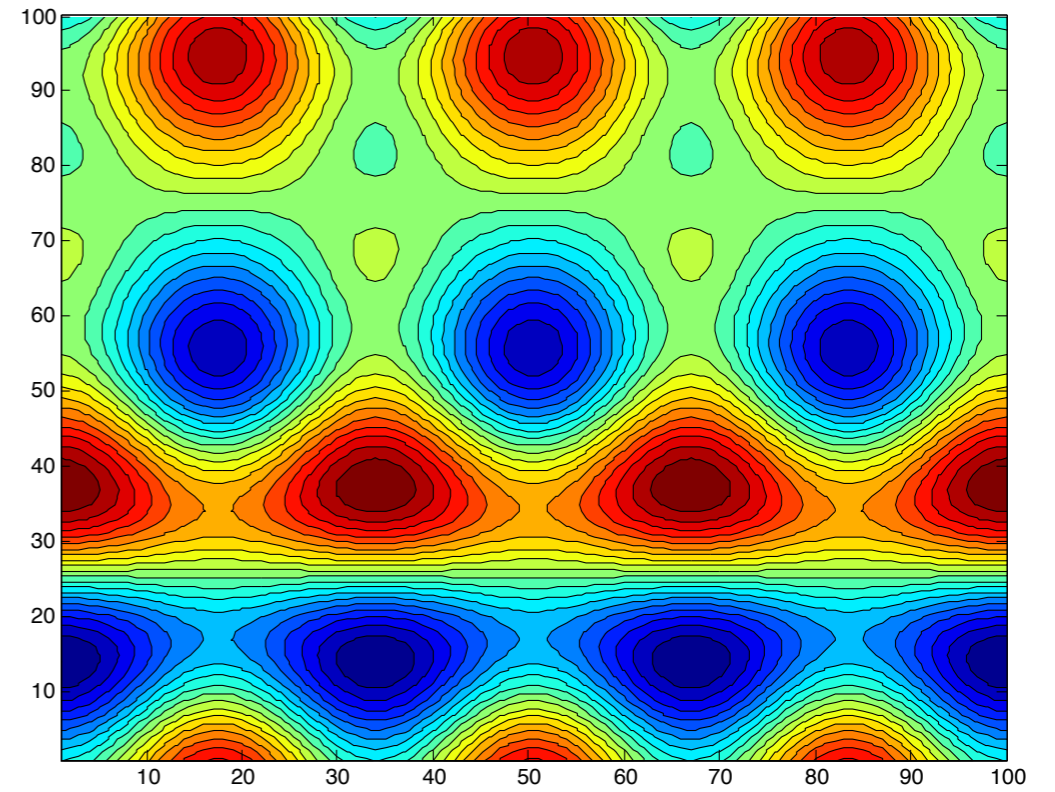
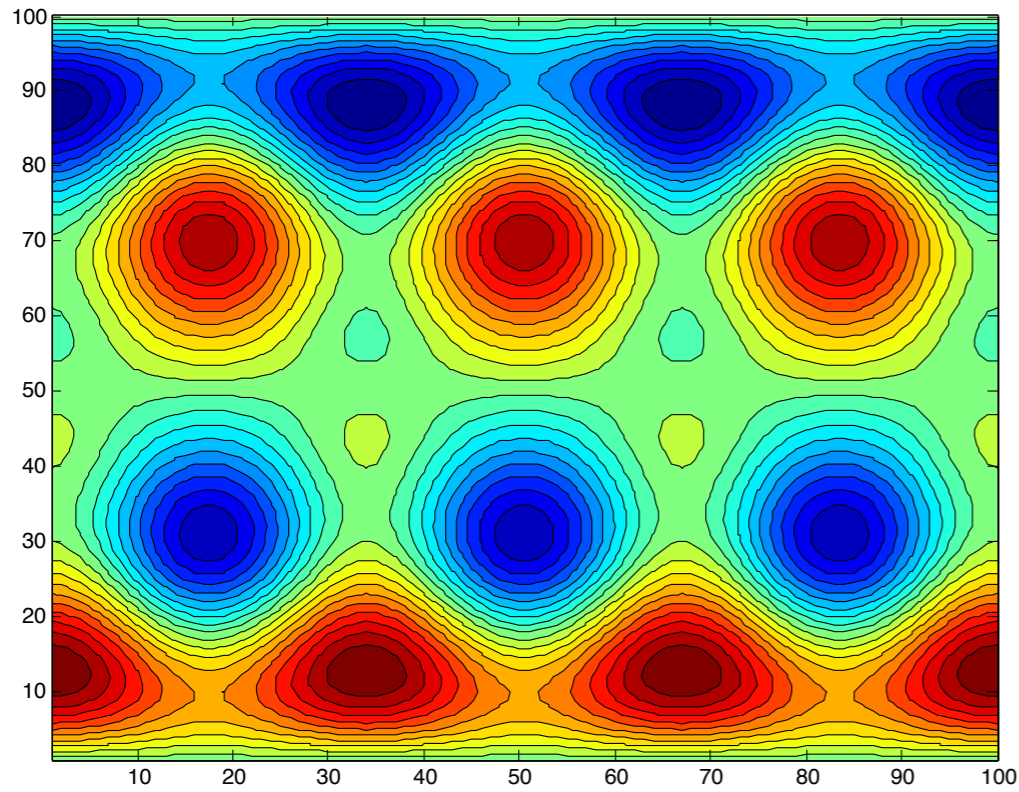
β_1



Kolmogorov Flow

$$\frac{\partial \omega}{\partial t} + \beta \mathbf{u} \cdot \nabla \omega = \bar{\nu} \nabla^2 \omega - \alpha \omega + A \cos(2\pi y / \lambda)$$

Persistent homology is invariant under homeomorphisms of the domain

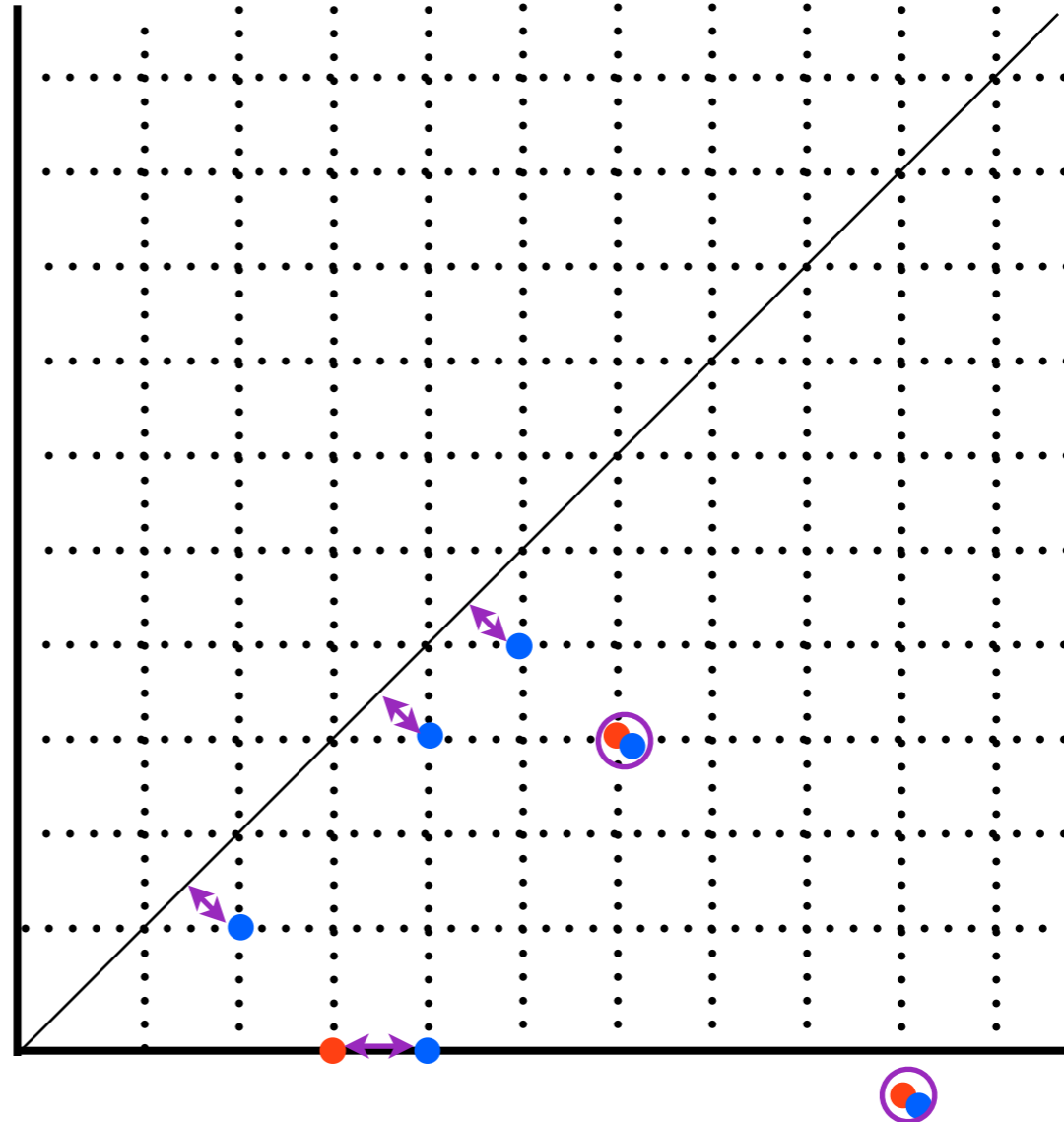
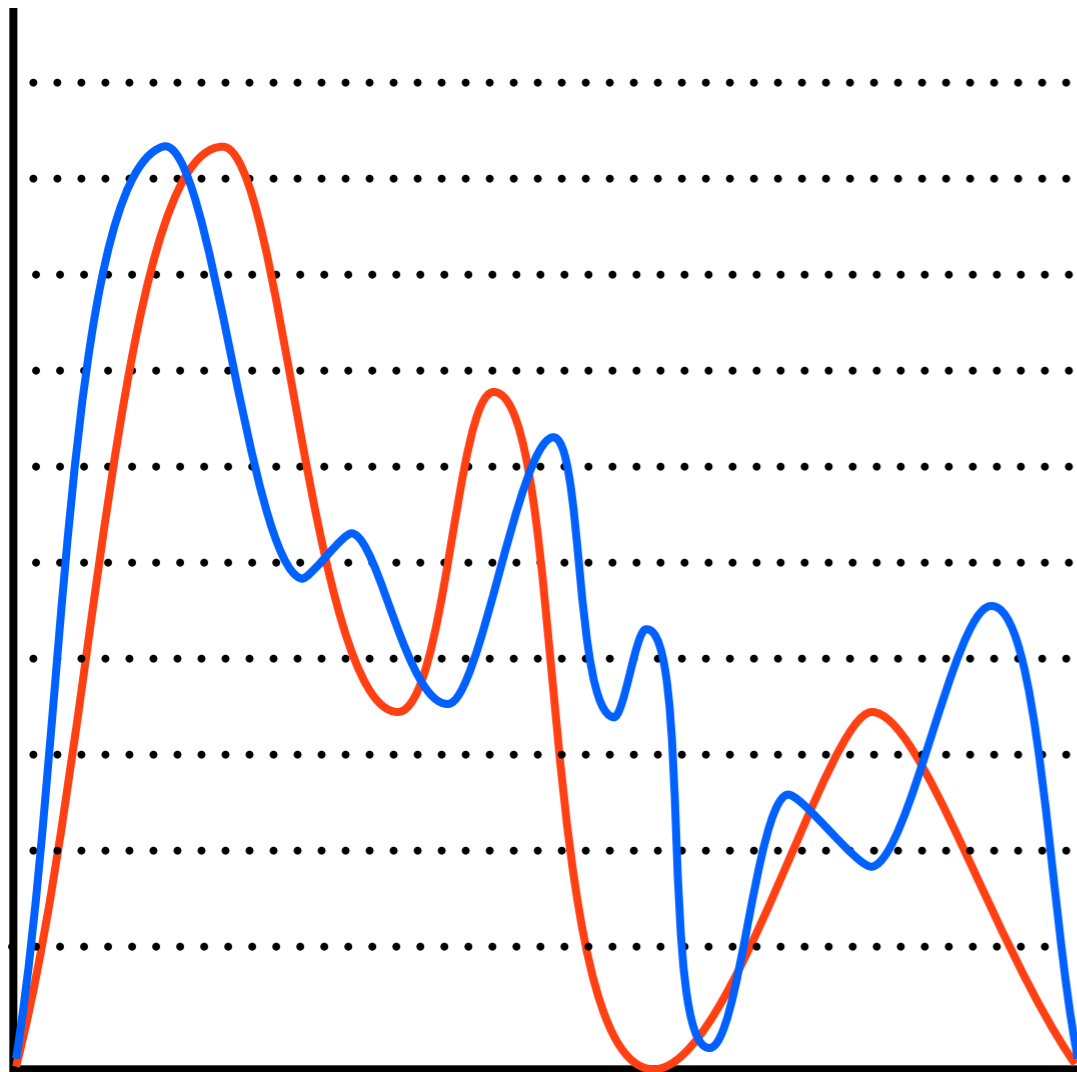


DEFINITION OF PERSISTENCE DIAGRAM

Given a scalar field $f: X \rightarrow \mathbb{R}$ and a filtration

$\Theta = \{\theta_j \mid \theta_0 < \theta_1 < \dots < \theta_N\}$ the associated **persistence diagram** $PD_i(f, \theta_1)$ is the multi set consisting of

- one point for each i -th persistence pair (θ_j, θ_k) ,
- infinitely many copies of points on the diagonal (θ_j, θ_j)



METRICS FOR PERSISTENCE DIAGRAM

Given two persistence diagrams PD_k and PD'_k let

$$W_p(PD_k, PD'_k) = \left[\inf_{\gamma} \sum_{z \in PD_i} \|z - \gamma(z)\|_{\infty}^p \right]^{\frac{1}{p}} \quad \text{Wasserstein distance}$$

where γ ranges over all bijections from PD_k to PD'_k .

Let per denote the space of all persistence diagrams.

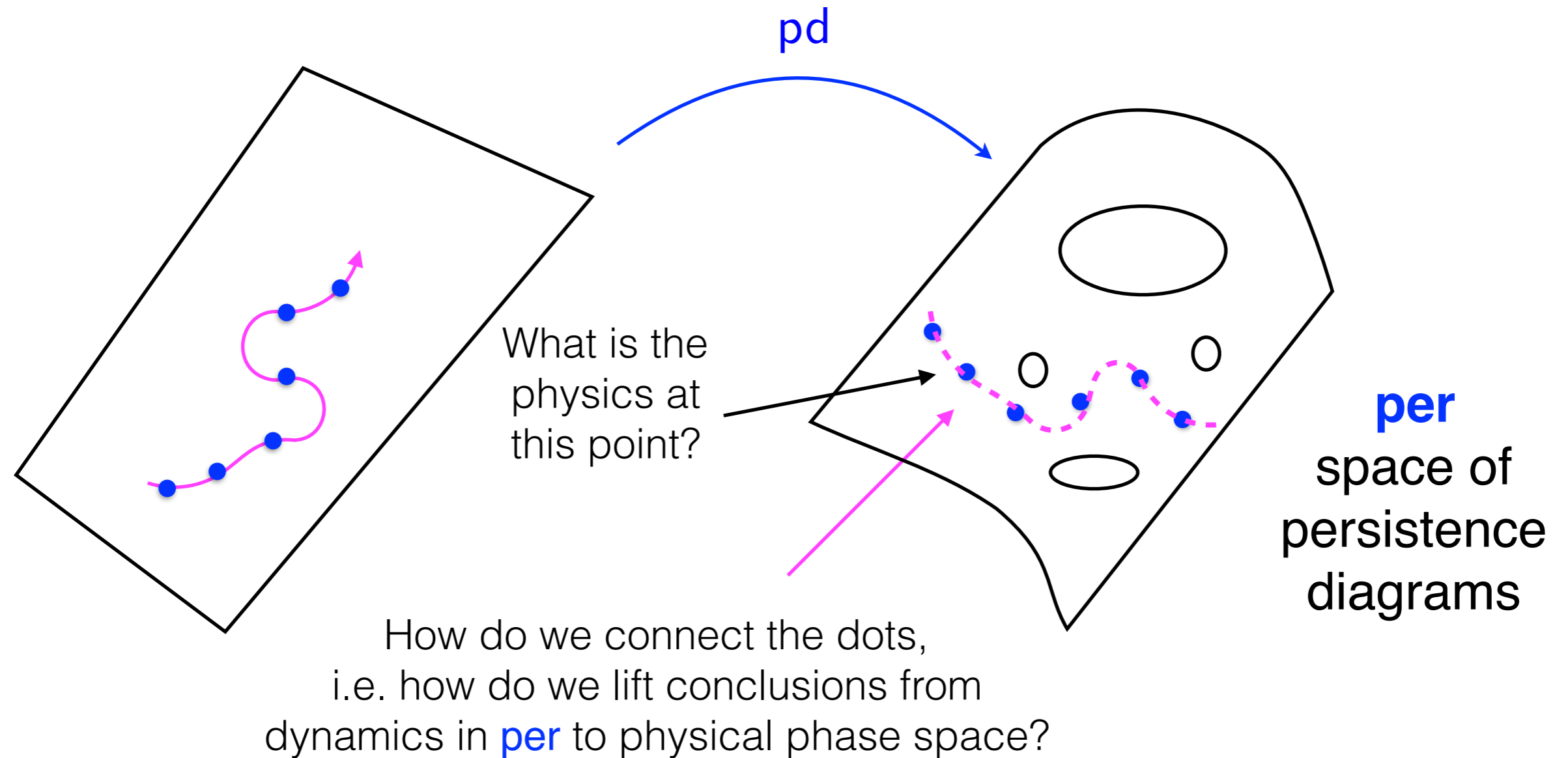
We have defined a function

$$\begin{aligned} \text{pd}: L^{\infty}(X, \mathbb{R}) &\rightarrow \text{per} \\ f &\mapsto \text{pd}(f) \quad \text{Persistence diagram of } f. \end{aligned}$$

Theorem (Cohen-Steiner, Edelsbrunner, Harer) pd is a Lipschitz continuous function

Recovering Dynamics

Strategy: Study dynamics in space of persistence diagrams

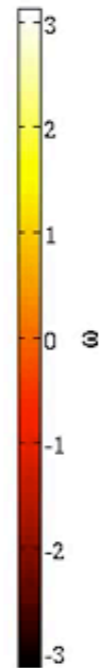
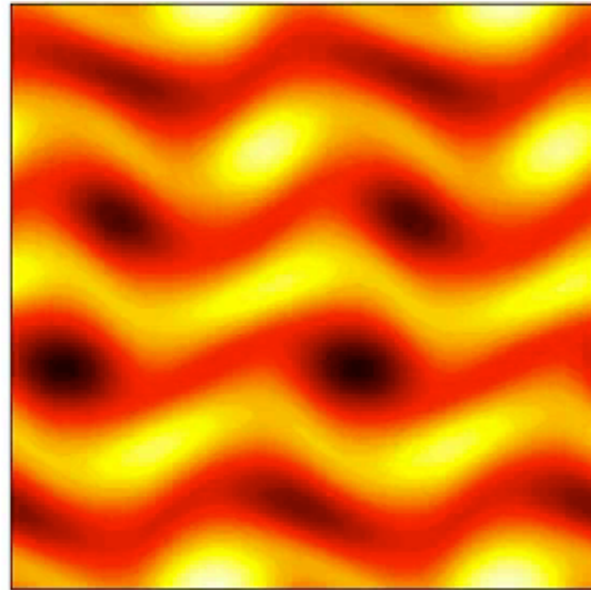


Trajectory in space of temperature fields, normal forces, etc. (high dimensional)

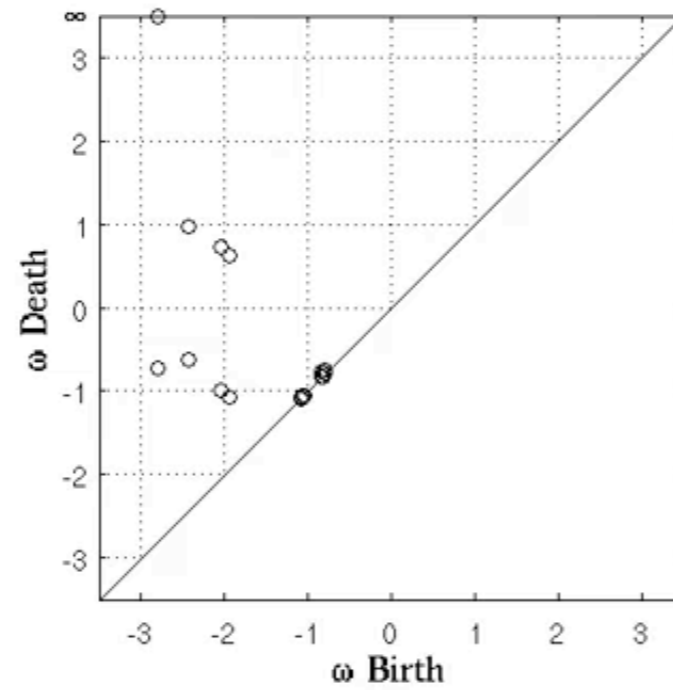
Want to do dynamics in this low dimensional space.

QUANTIFYING PATTERN DYNAMICS

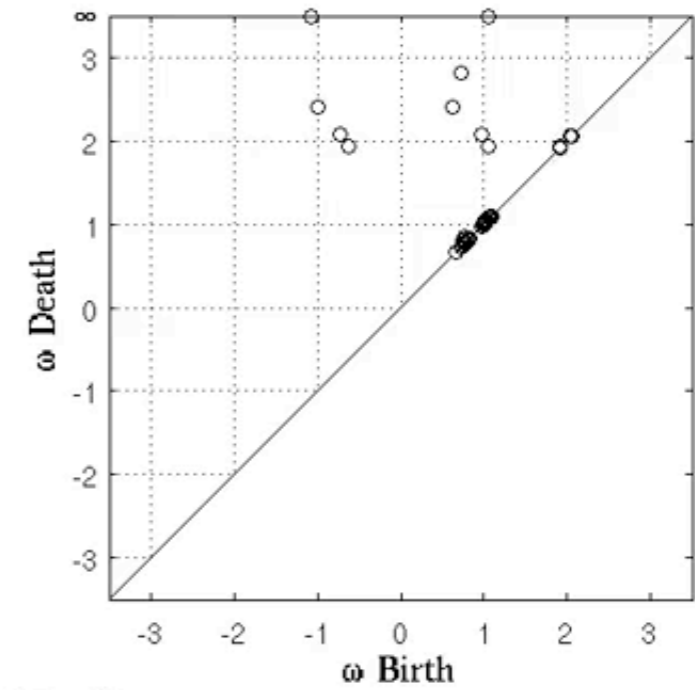
Kolmogorov Flow: $t = 1$



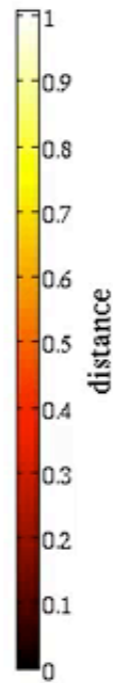
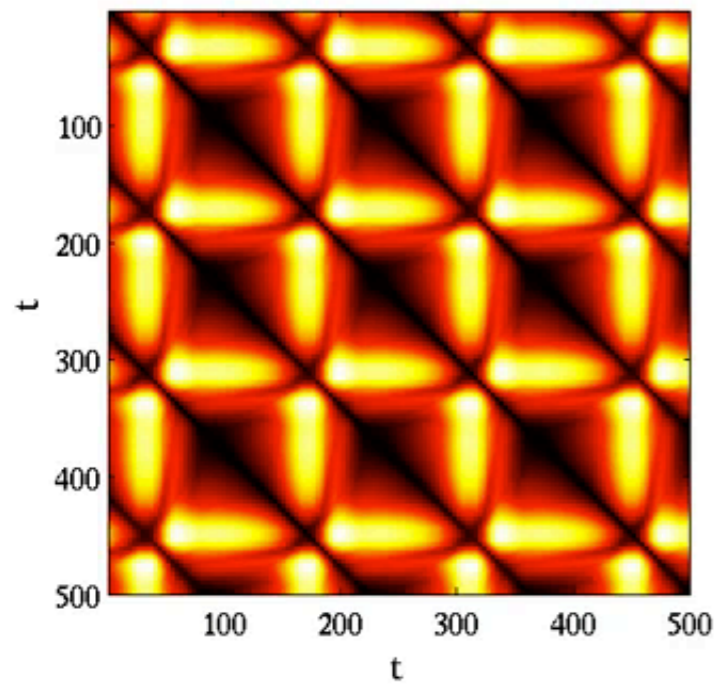
β_0 Persistence Diagram



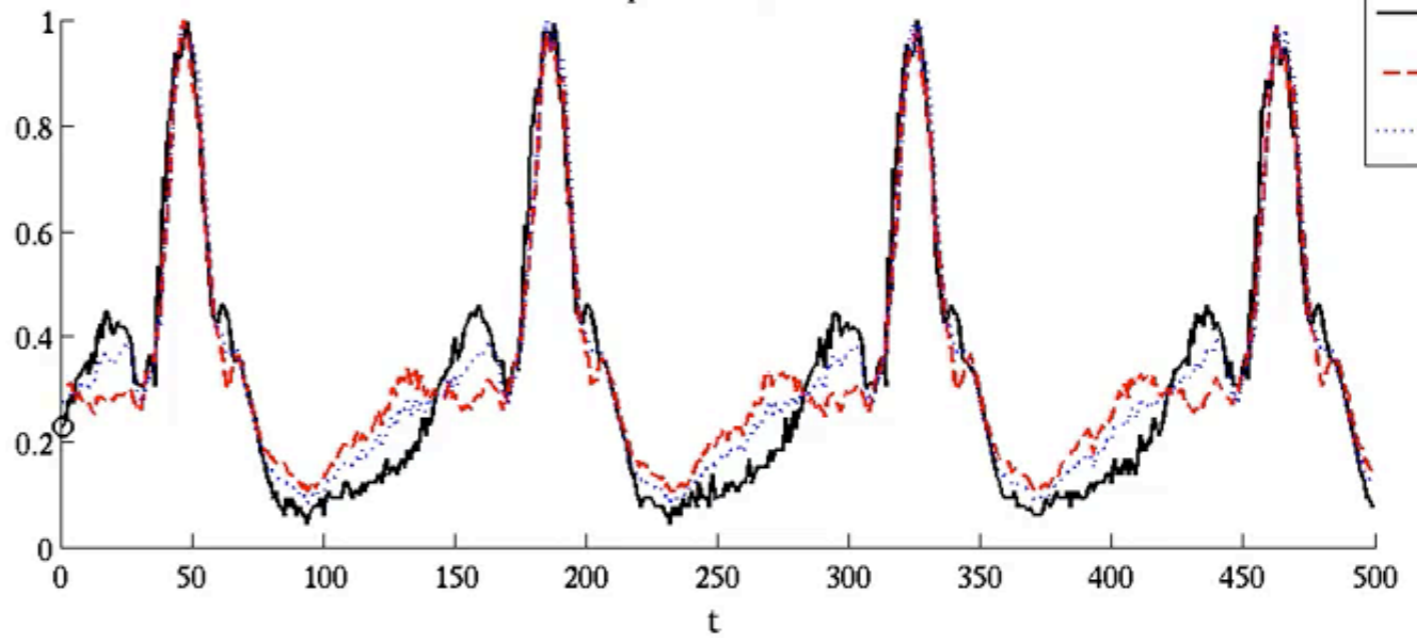
β_1 Persistence Diagram



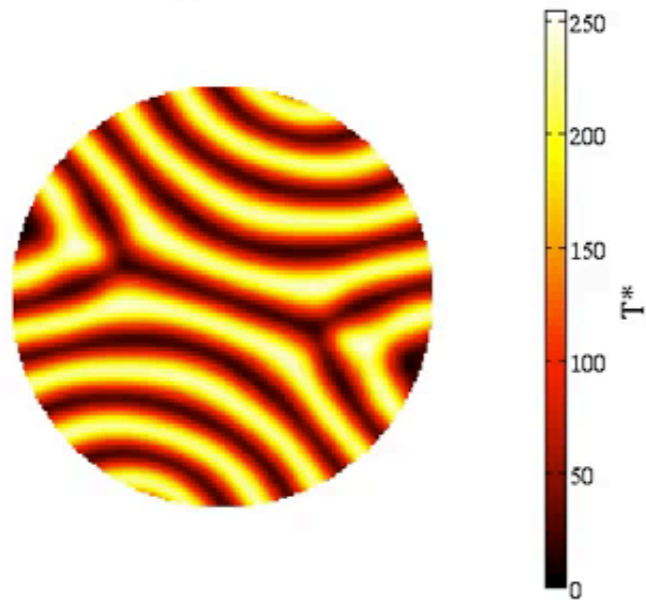
Bottleneck Distance Matrix



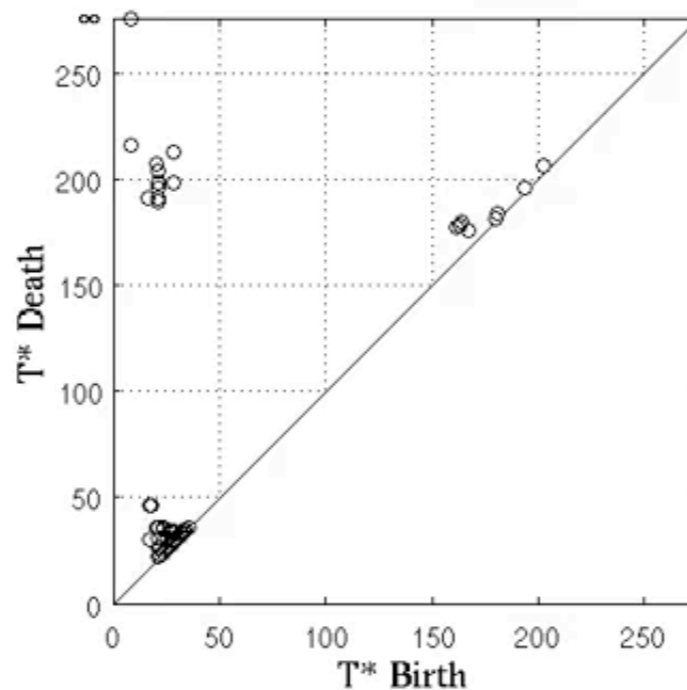
Speed Profile



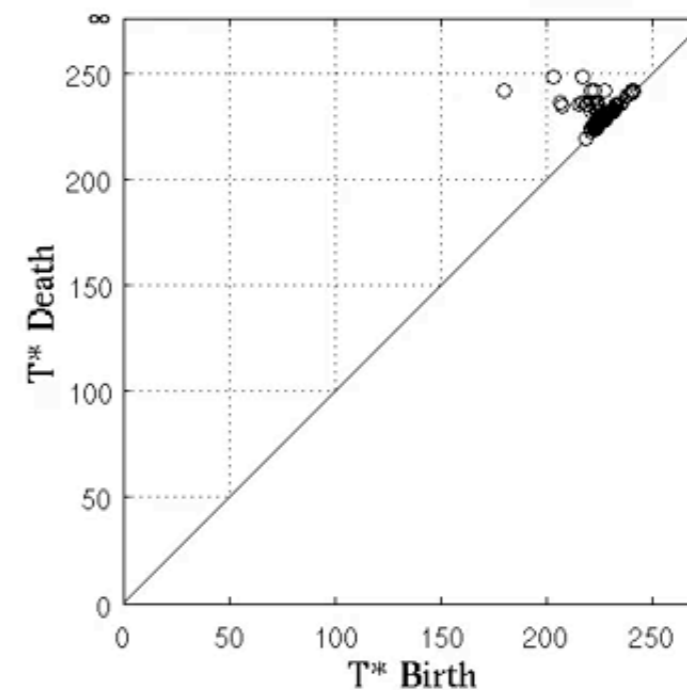
Rayleigh-Bénard Convection:
 $t = 1$



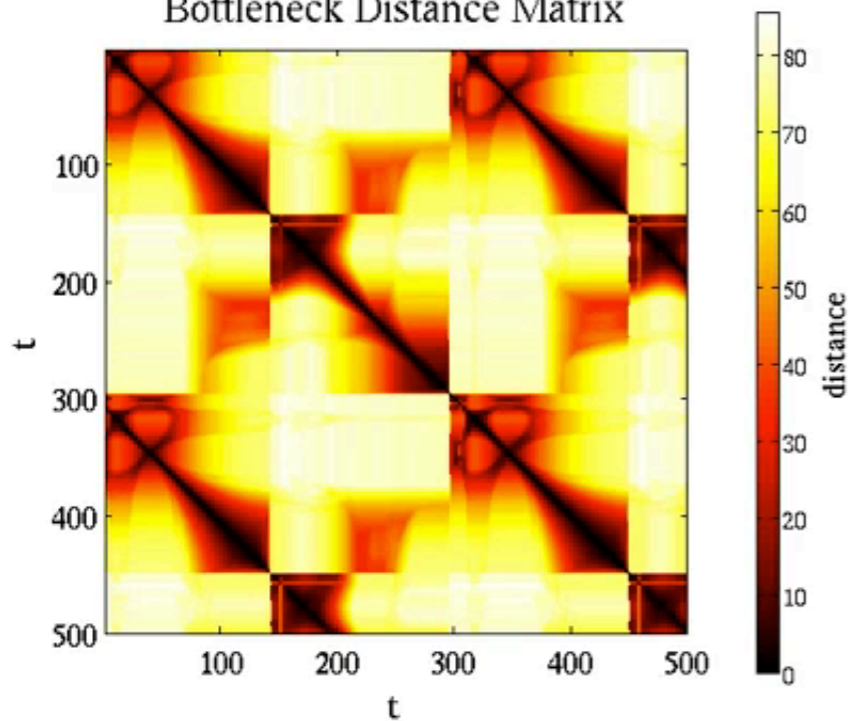
β_0 Persistence Diagram



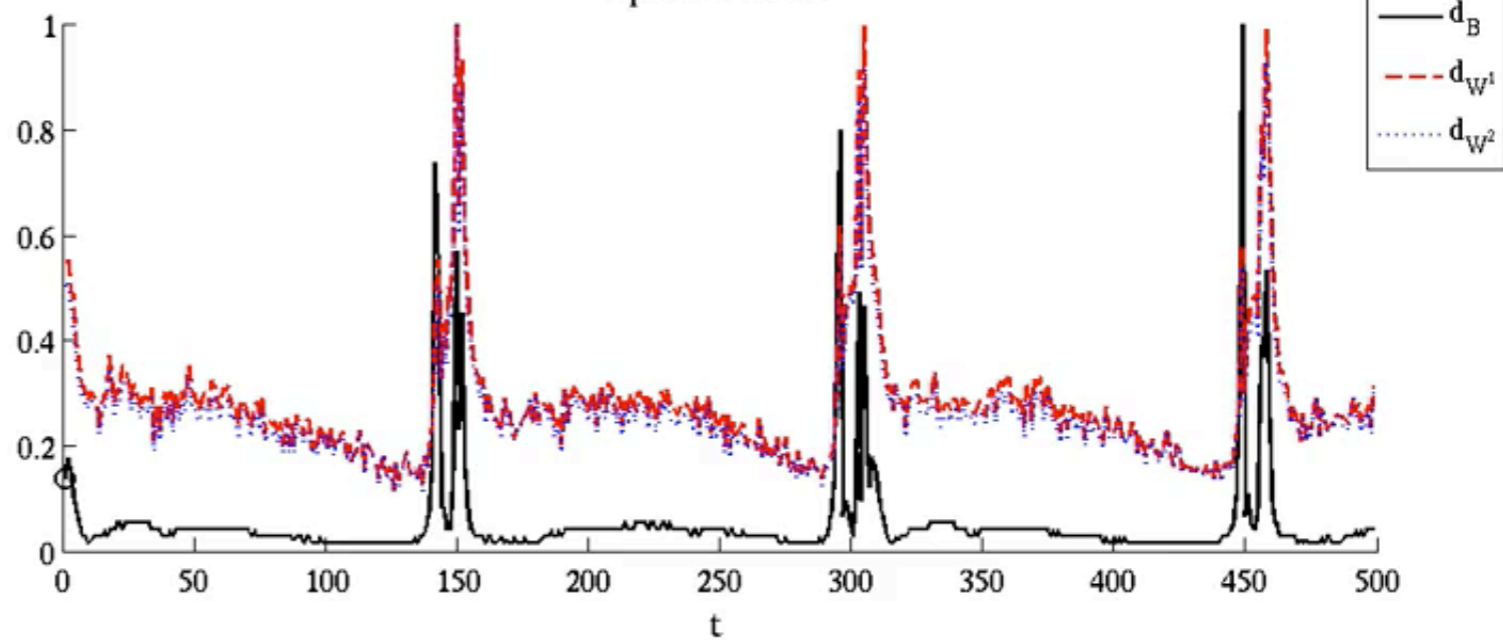
β_1 Persistence Diagram



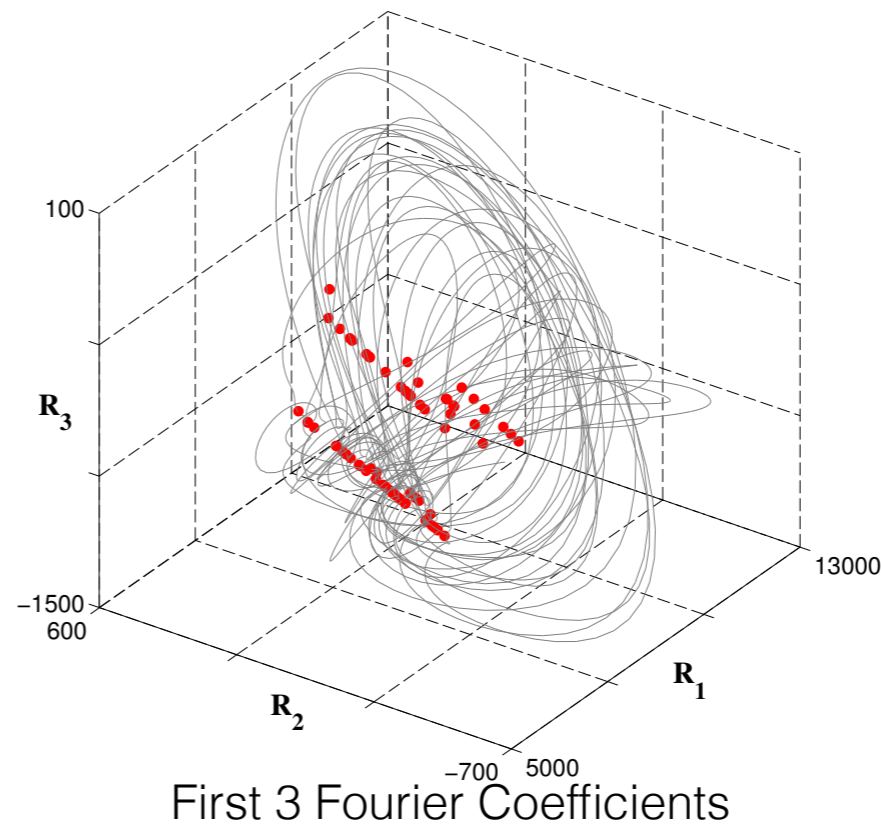
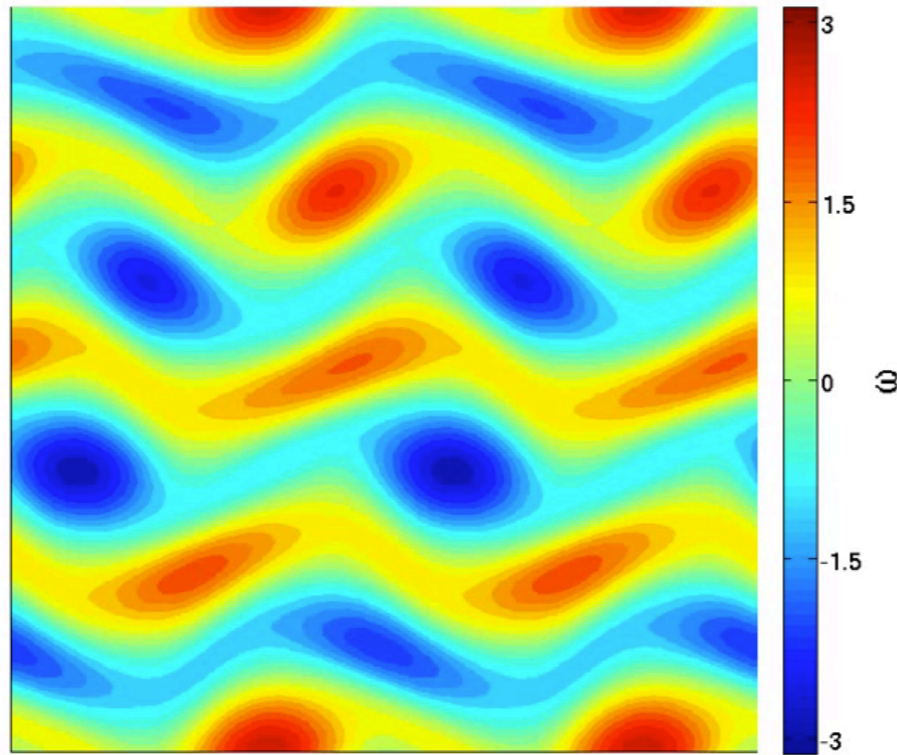
Bottleneck Distance Matrix



Speed Profile



A PROBLEM ARISING FROM SYMMETRY



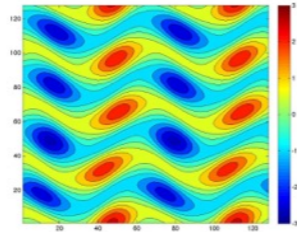
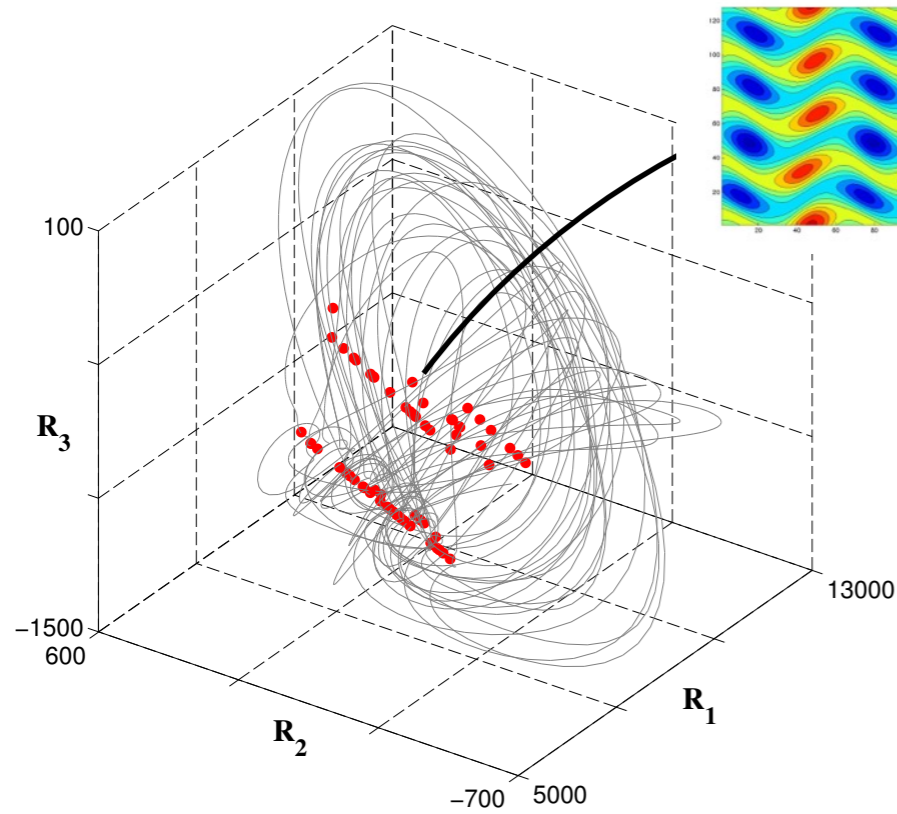
Equilibria
identified
using
Newton's
method.

Hypothesis: Chaotic Dynamics is generated by heteroclinic connections between unstable invariant sets, e.g. equilibria, periodic orbits, invariant tori.

Problem: Periodic boundary conditions lead to continuous families of symmetric fixed points.

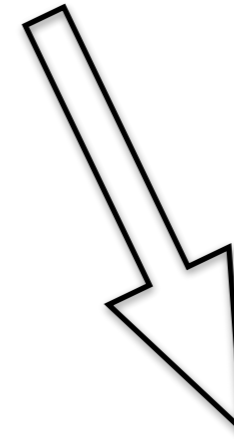
Solution: PD is invariant under domain symmetries, so cluster equilibria using persistent homology in Per.

CLUSTERING FIXED POINTS

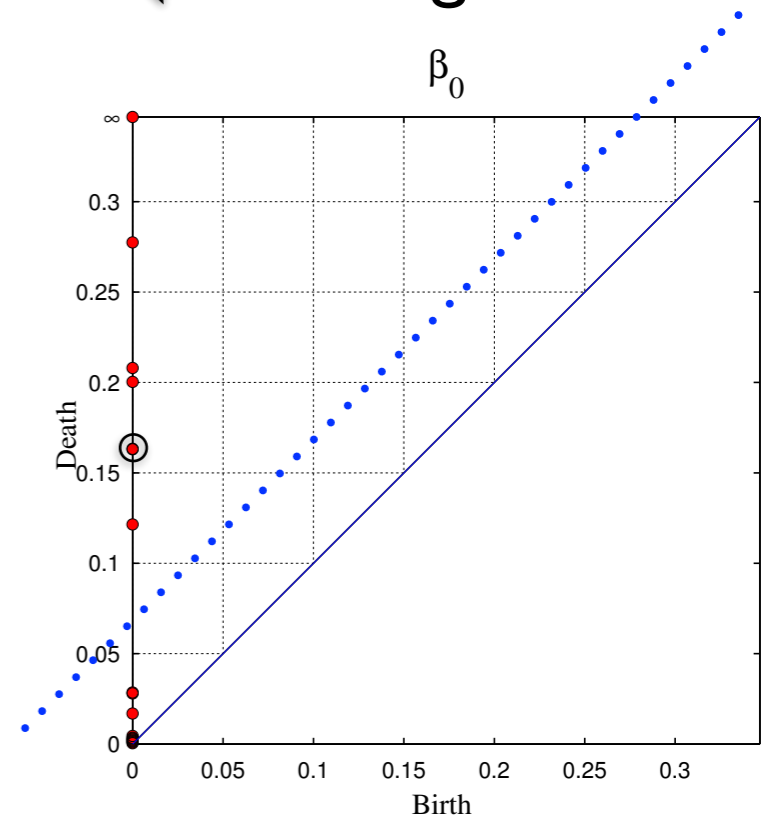


Points in
space of
Persistence
Diagrams

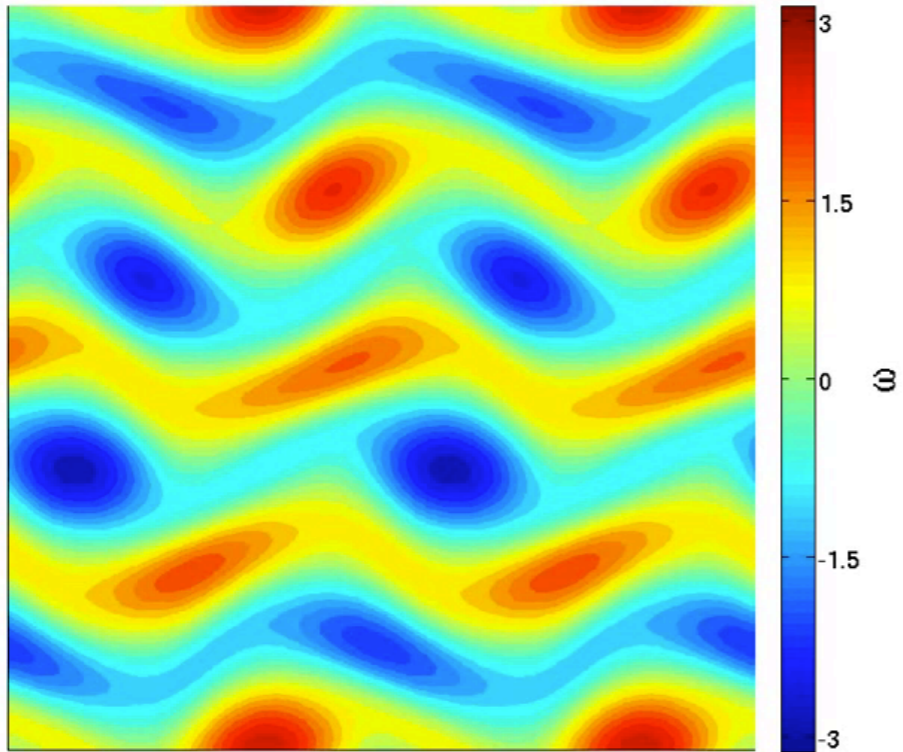
Compute
 β_0
Persistence
Diagram
of
Persistence
Diagrams



Seven Distinct Equilibria



IDENTIFYING COHERENT STRUCTURES

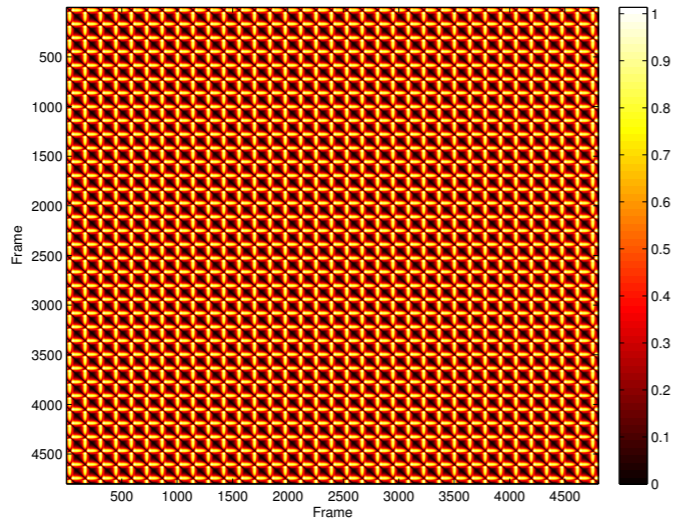


Question: Can we identify oscillations in the data?

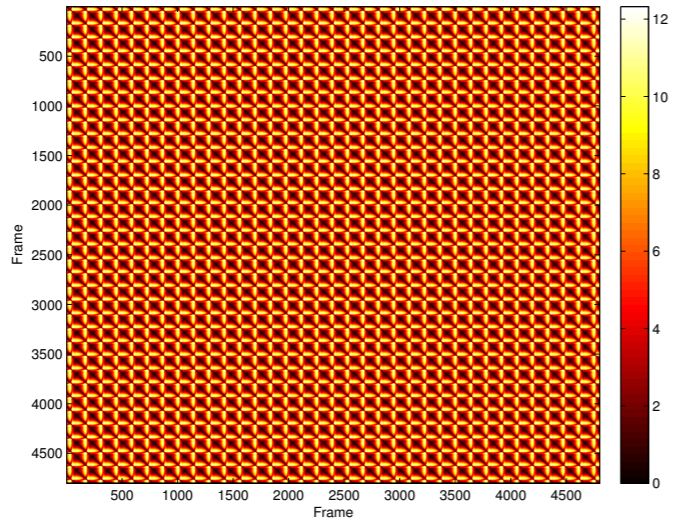
Solution: A periodic orbit is a loop, use **PD** to identify nontrivial first homology in **Per**.

Numerical Simulations of Kolmogorov flow

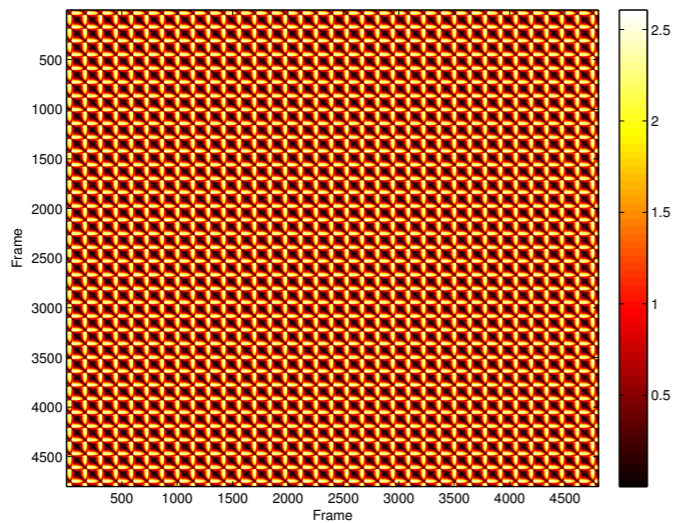
d_B



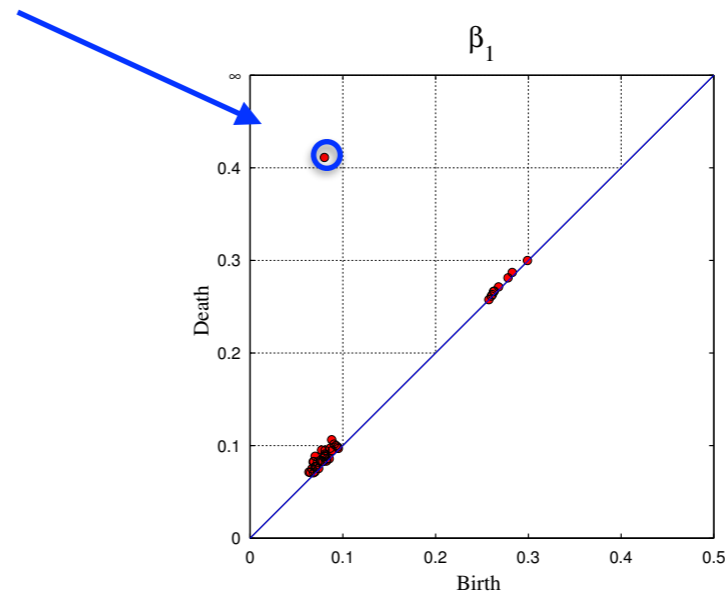
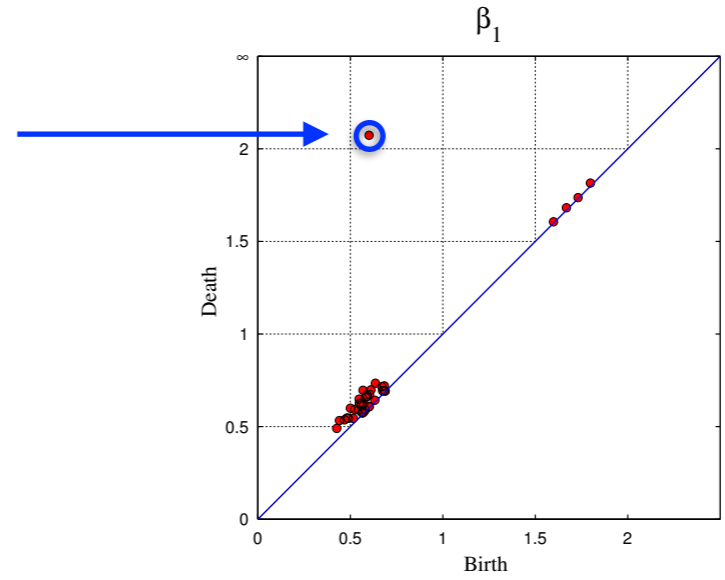
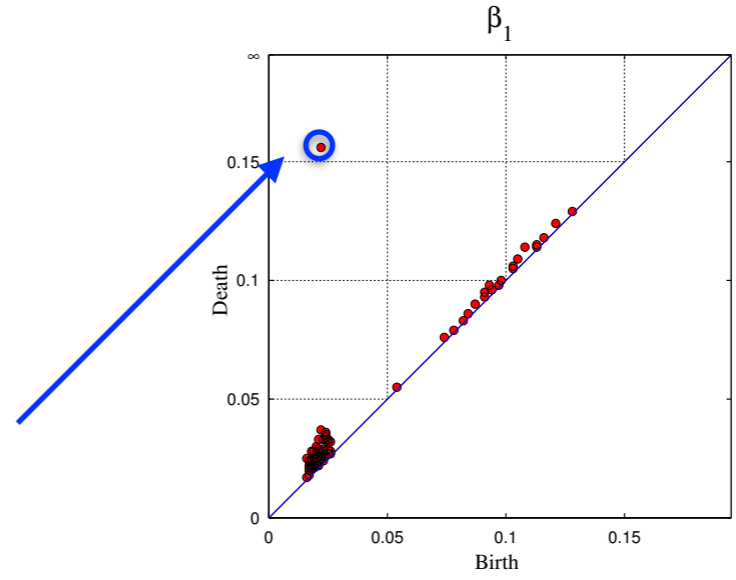
d_{W1}



d_{W2}

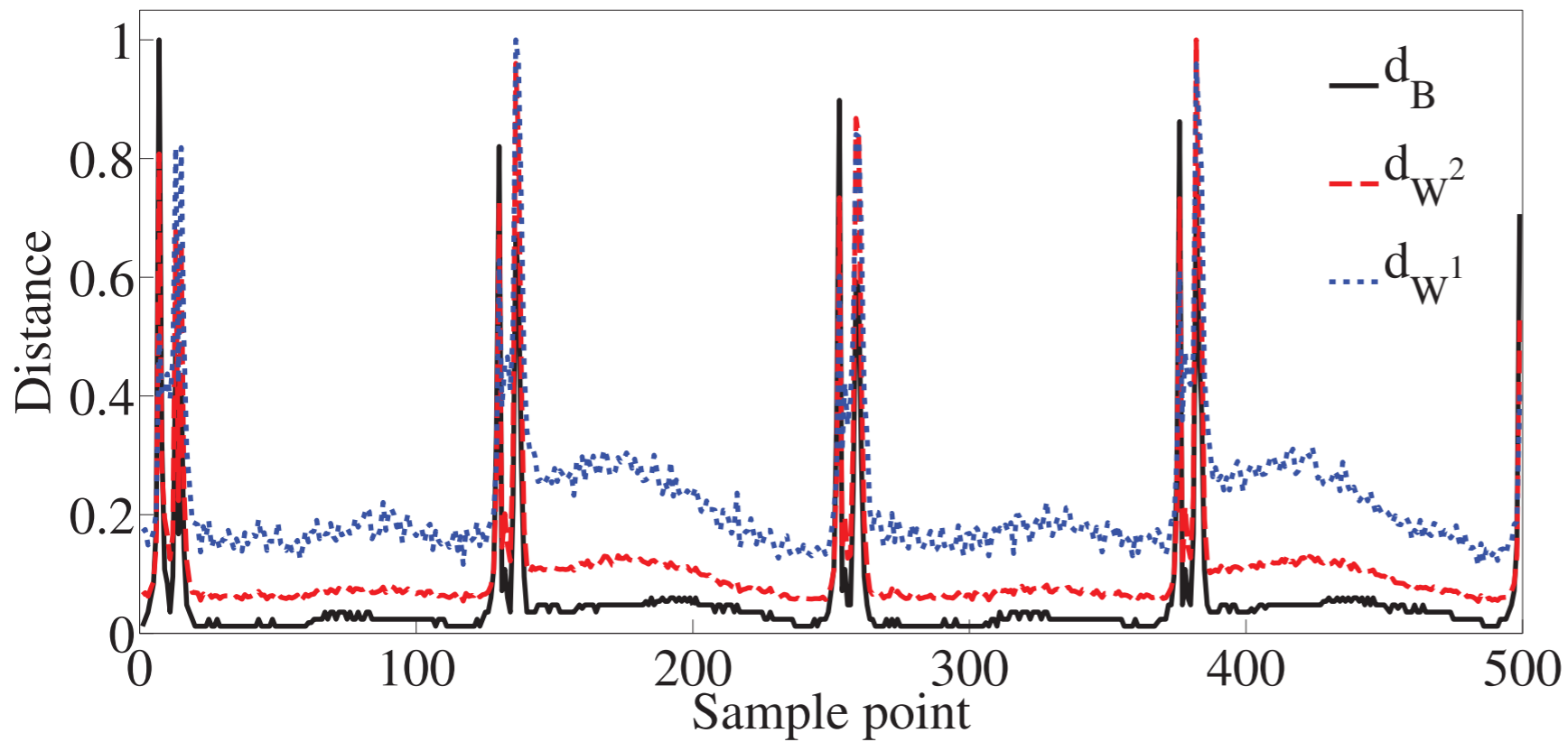
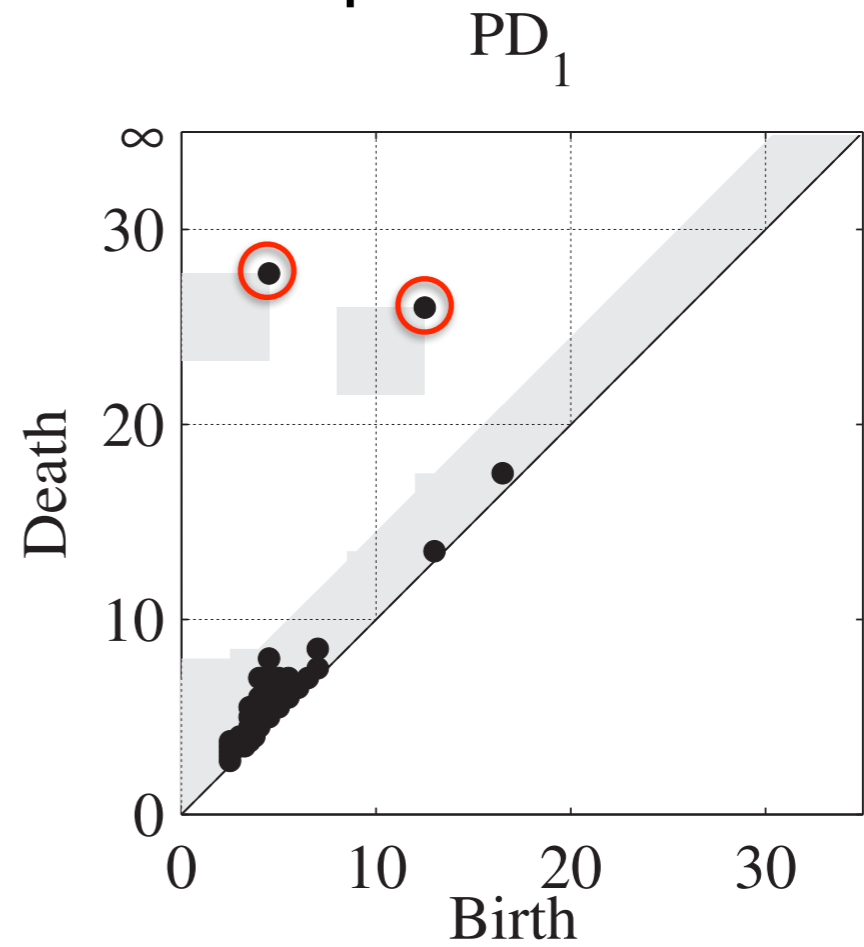
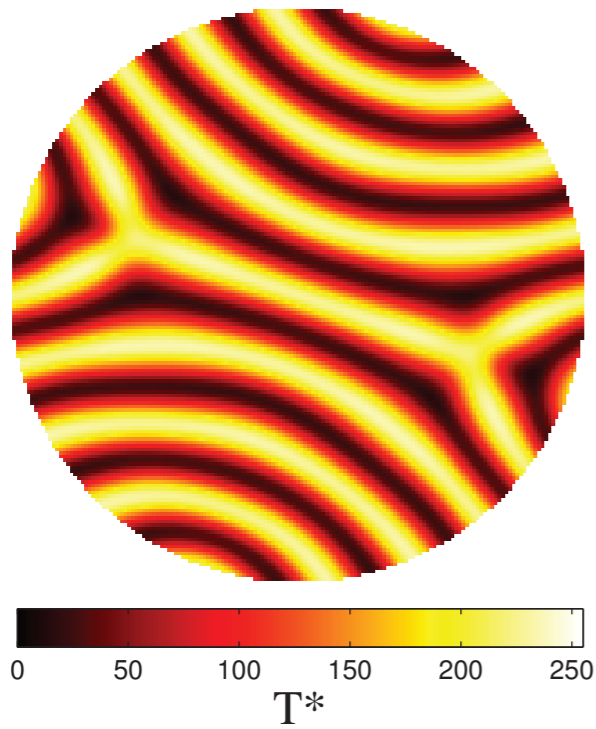


Suggests a periodic orbit



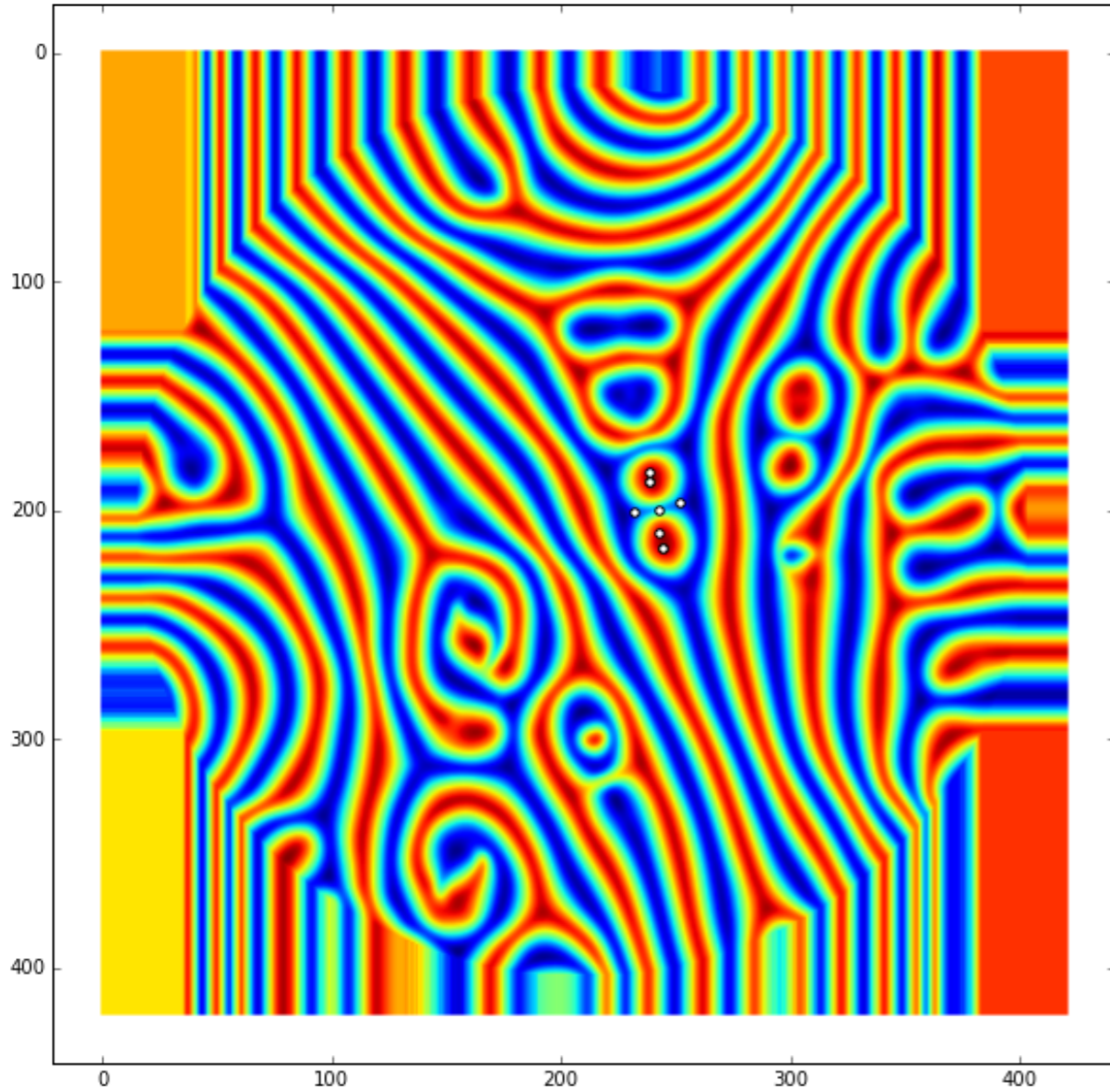
β_1 persistence diagrams of point cloud made
up of persistence diagrams of vorticity field.

Numerical Simulations of Boussinesq



β_1 persistence diagram of point cloud made up of persistence diagrams for temperature field.

IDENTIFYING LANDMARKS



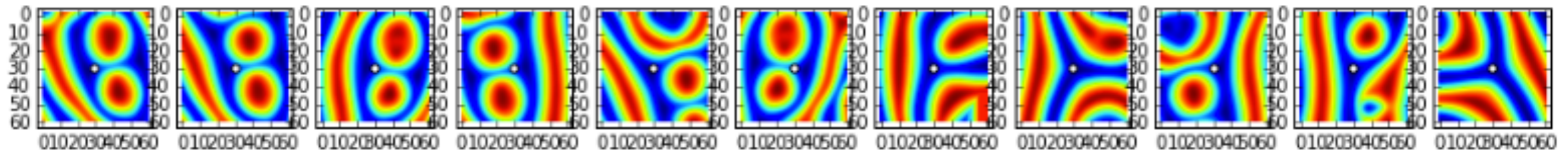
Persistent Homology

Image Analysis

Machine Learning

Feature #3

```
[[ 17.91647287  20.37154879  32.84813541  34.55430509  34.89985673
   36.11094017  36.4280112   37.68288736  37.8153408   41.70131892]
 [ 14.         8.         12.         11.         9.         25.
   27.         10.        13.         15.         ]]
```



Thank-you for your Attention

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J. Tithof
B. Suri

Virginia Tech

M. Paul
M. Xu

General Homology Software
chomp.rutgers.edu

Persistent Homology Software
www.math.rutgers.edu/~vidit/perseus.html

