Beyond Vehicles and Other Flow Systems

Benedetto Piccoli

Rutgers University – Camden

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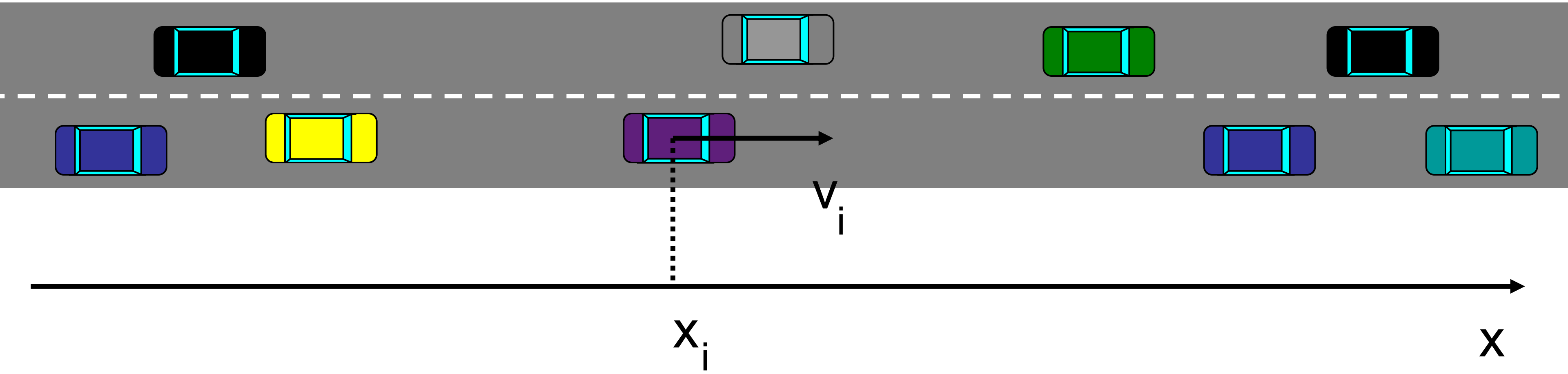
Part I: models and scales

Modelling of vehicular traffic

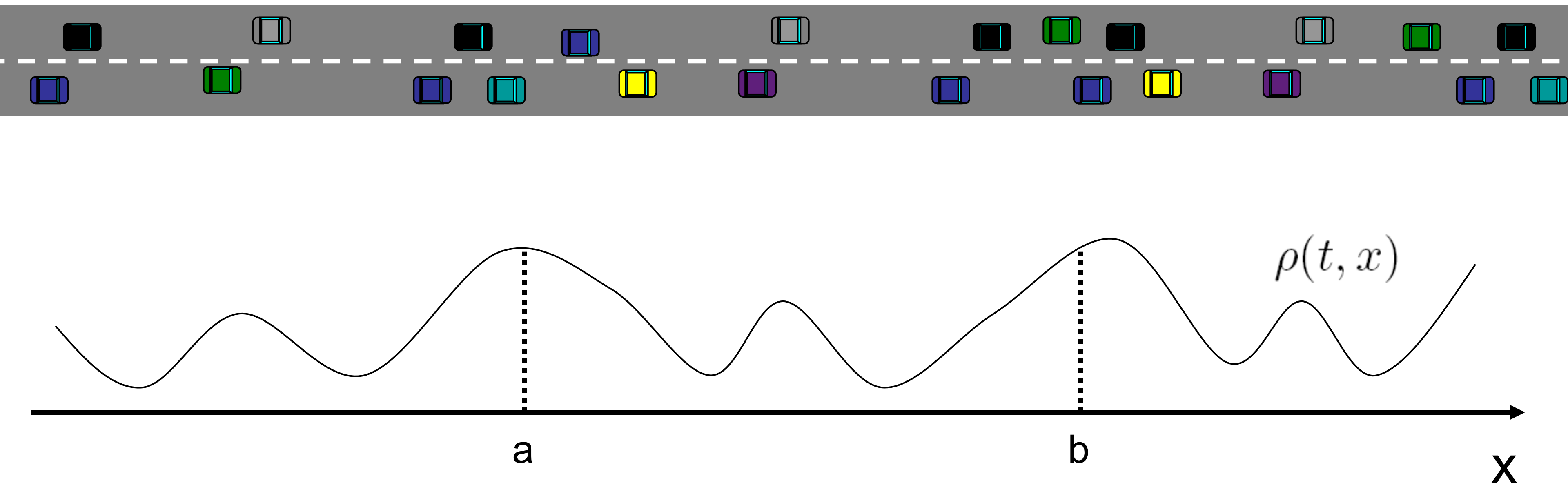


MICROSCOPIC

$$\dot{v}_i = C\,\frac{v_{i+1} - v_i}{x_{i+1} - x_i}$$

Modelling of vehicular traffic

MACROSCOPIC

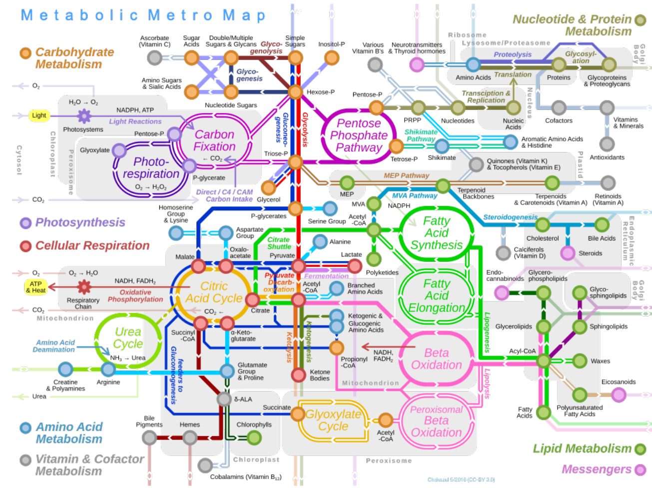


$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx \;=\; f(t,a) - f(t,b) \;=\; -\int_a^b \frac{d}{dx} f(t,x)\,dx$$

$$\int_a^b \left(\frac{d}{dt}\rho(t,x) + \frac{d}{dx} f(t,x) \right) dx = 0 \qquad\qquad \frac{d}{dt}\rho(t,x) + \frac{d}{dx} f(t,x) = 0$$

(Social) dynamics of multi-agent systems

Animal groups



Metabolic networks

Social networks

Crowd dynamics (pedestrian)

Example of social dynamics models

Hegselmann-Krause model $\dot{x}_i = \sum_{|x_j - x_i| < r} (x_j - x_i)$

R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence: models, analysis and simulation. J. Artificial Societies and Social Simulation, 5(3), 2002.

Cucker-Smale model $\dot{x}_i = v_i, \;\; \dot{v}_i = \frac{1}{N} \sum_j a(\|x_j - x_i\|)(v_j - v_i)$

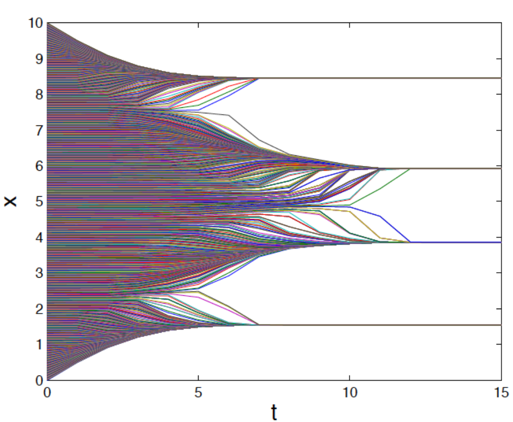
F. Cucker and S. Smale, Emergent behavior in flocks, IEEE Trans. Automat. Control, 52 (2007), pp. 852–862.

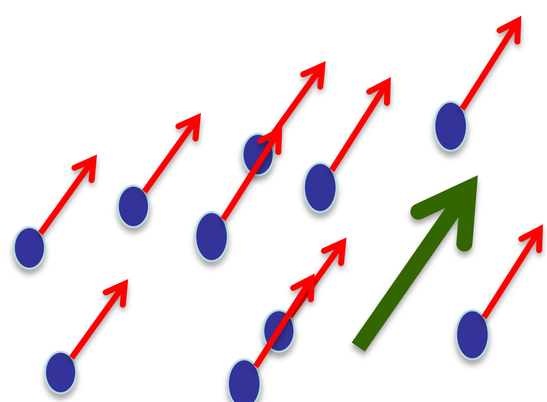
Migration model $\dot{v}_i = \alpha_i (V - v_i) + \frac{(1-\alpha_i)}{N} \sum_j a(\|x_j - x_i\|)(v_j - v_i)$

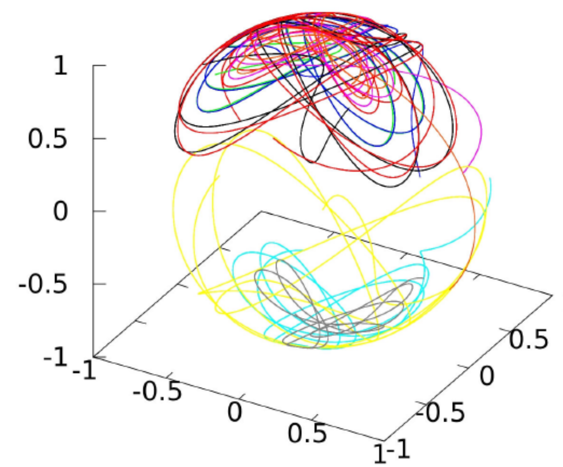
Darren Pais, Naomi E. Leonard, Adaptive network dynamics and evolution of leadership in collective migration, Physica D: Nonlinear Phenomena, Volume 267, 2014, Pages 81-93,

Consensus on sphere $$\dot{x}_i = \sum_{j=1}^{N} a_{ij} (x_j - \langle x_i, x_j \rangle x_i)$$

M. Caponigro, A. Lai, B. Piccoli: A nonlinear model of opinion formation on the sphere, *Dynamics of Continuous and Discrete Systems*, **35** (2015), 4241–4268.

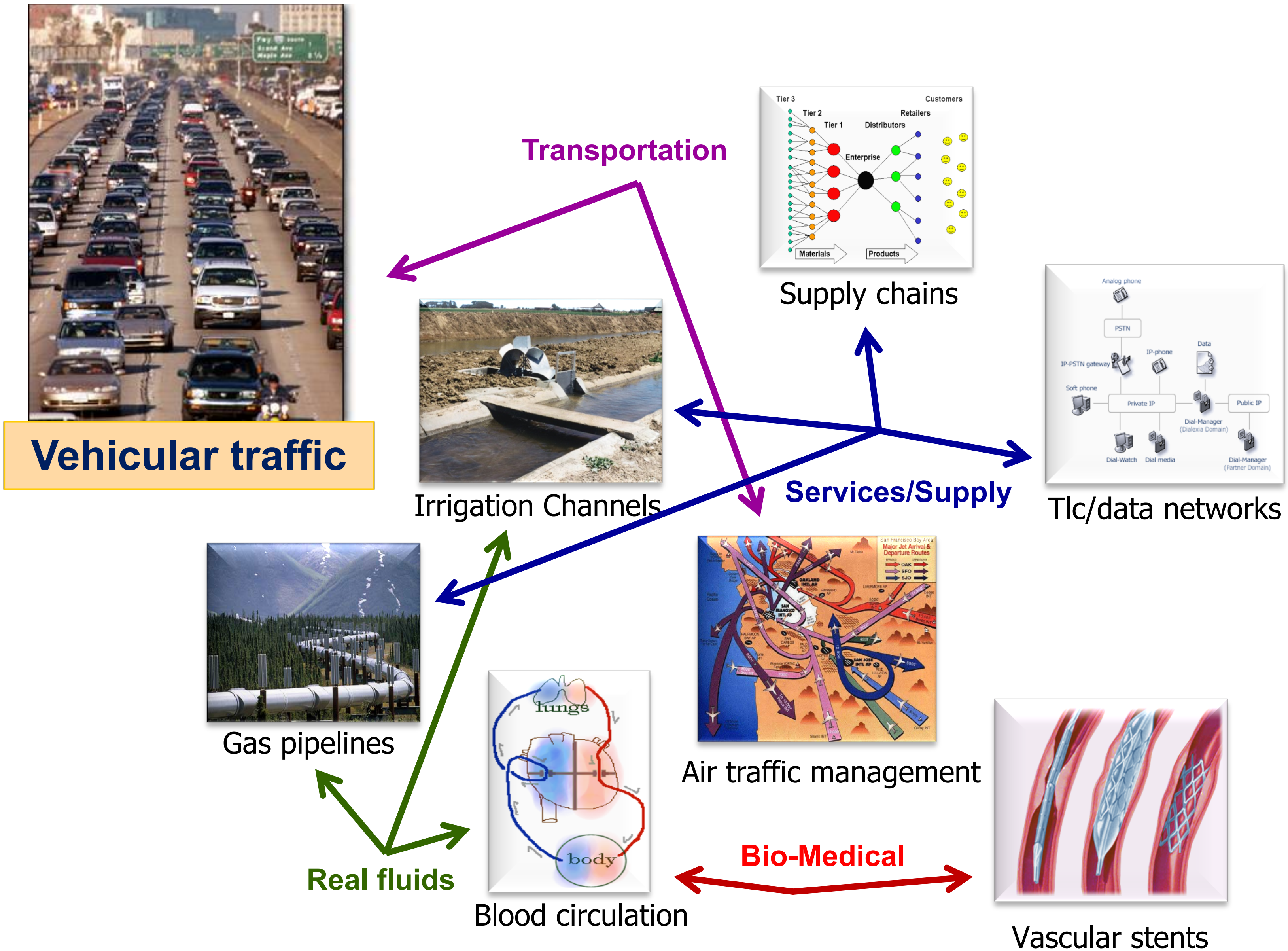




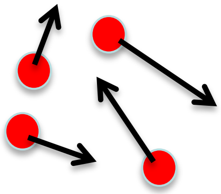



Social dynamics
Boids
Greedy
Clustering
Voter model
Self-propelled
Intelligent
Autonomous
Consensus
Self-driven
Panic
Bounded confidence
Multiscale
Crowd dynamics
Herding
Rational
Selfish
Opinion dynamics
Self-organization
Social forces
Interacting particles

Flow systems on networks

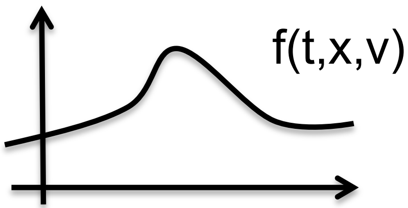


Modeling at different scales (gas dynamics)



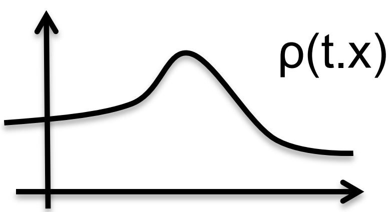
Microscopic

$$\begin{cases} \dot{x}_i &= v_i \\ \dot{v}_i &= \frac{1}{N}\sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) \end{cases}$$



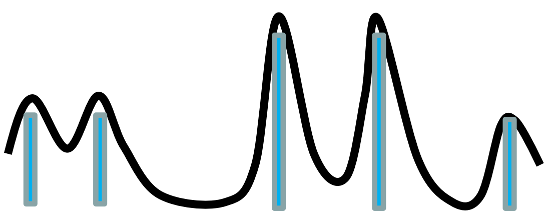
Mesoscopic

$$\partial_t f + v \cdot \nabla_x f + \lambda \nabla_v \cdot Q(f,f) = 0,$$



Macroscopic

$$\partial_t \rho + \partial_x q(\rho) = 0$$



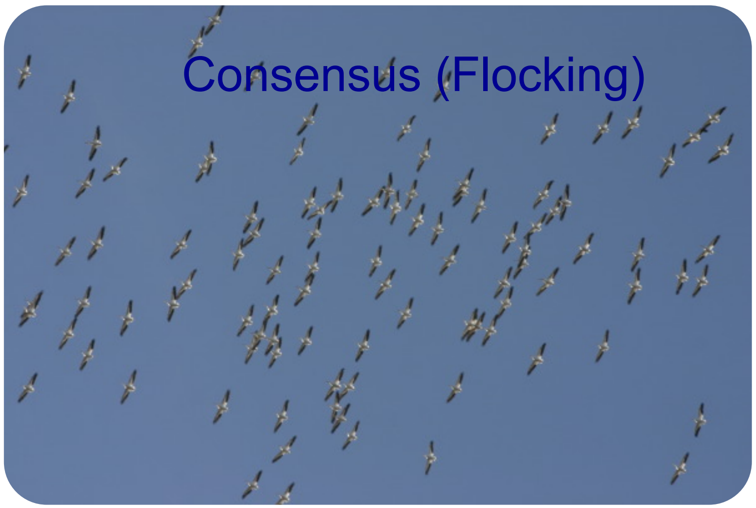
Multiscale

$$\begin{cases} \partial_t \mu + \nabla \cdot (v\mu) = 0 \\ \mu_{|t=0} = \mu_0 \end{cases}$$

Part II: microscopic controls

The Cucker and Smale model

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) + u_i \end{cases}$$



$$\dot{v}_i = \frac{1}{N}\sum_{j=1}^{N} \frac{v_j - v_i}{(1 + \|x_j - x_i\|^2)^\beta}$$

$$\beta \leqslant \frac{1}{2}$$

the system will converge to a consensus

Cucker-Smale : consensus (flocking) conditions for $\beta > 1/2$
Ha-Tadmor: kinetic and hydrodinamic limit of CS
Motsch-Tadmor: relative distance, asymmetric
Karper-Mellet-Triviza: hydrodinamic limit of MT model
Jabin-Motsch: first order clustering
Particle systems: Reynolds, Vicsek, Ben-Jacob et al, Krause, Couzin, Helbing, …
Degond, Carrillo, Hauray, Fornasier, Bertozzi, Toscani, Figalli, Vecil, Slepcev …

Sparse control of CS system (1)

$$\mathcal{V}_f = \{(v_1, \dots, v_N) \in \mathbb{R}^{dN} \mid v_1 = \cdots = v_N\},\ \mathcal{V}_\perp = \{v \mid \sum_i v_i = 0\}.$$

$$B(u,v) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

$$X := B(x,x) = \tfrac{1}{2N^2} \sum_{i,j} \|x_i - x_j\|^2,\ V := B(v,v) = \tfrac{1}{2N^2} \sum_{i,j} \|v_i - v_j\|^2$$

Proposition

Assume that $X_0 = B(x_0, x_0)$ *and* $V_0 = B(v_0, v_0)$ *satisfy*

$$\gamma(X_0) := \int_{\sqrt{X_0}}^{\infty} a(\sqrt{2N}r)dr \geq \sqrt{V_0}\,.$$

Then the solution of tends to consensus.

Proposition

The feedback control $u(t) = -\alpha v_\perp(t)$*, stabilizes the system to consensus in infinite time.*

Sparse control of CS system (2)

Definition

Let $U(x,v)$ be set of control solving

$$\min\left(B(v,u)+\gamma(B(x,x))\frac{1}{N}\sum_{i=1}^{N}\|u_i\|\right)\quad \text{subject to } \sum_{i=1}^{N}\|u_i\|\leq M\,.$$

Theorem

Set $F(x,v)=\{(v,-L_x v+u)\mid u\in U(x,v)\}$, then for every initial data the unique solution to

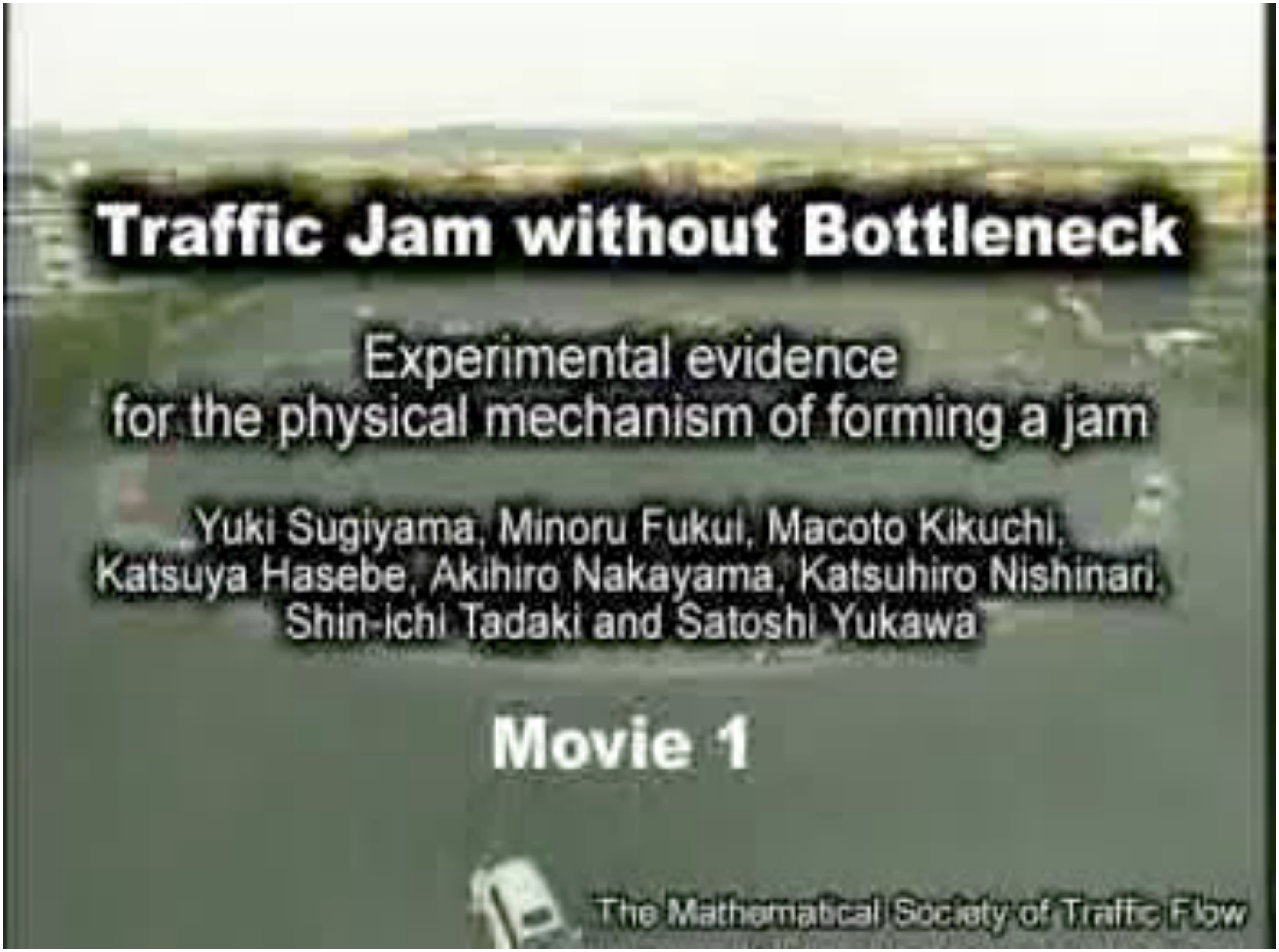
$$(\dot{x},\dot{v})\in F(x,v) \tag{1}$$

converges to consensus as t tends to $+\infty$.

Theorem

The single-valued lexicographic u_0 selection of F is sparse and admits sample-and-hold solutions (Clarke - Ledyaev - Sontag - Subbotin) tending to consensus for sufficiently small time steps. Moreover, u_0 minimizes $\frac{d}{dt}V$.

Traffic control



Part III: mean-field limits

Can we really control?
Mean-field limit: Vlasov-type equations

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = H \star \mu_N(x_i, v_i), \quad i = 1, \dots N, \end{cases}$$

$$H * \mu(x,v) = \int H(x-y, v-w) d\mu(y,w)$$

$$H(x,v) := a(|x|)v$$

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))}$$

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu)\, \mu]$$

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = H \star \mu_N(x_i, v_i) + u_i, \quad i = 1, \dots N, \end{cases}$$

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu)\, \mu + \nu]$$

Mean-field games
(not mean-field controlled equations)

Many agents with perfect knowledge of the system and strategy given by solving a game, then pass to the limit as in mean-field.

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu)\, \mu] \quad + \quad \sigma \Delta \mu$$

$$\partial_t v + \sigma \Delta v + \mathcal{H}(t, x, \nabla v, \mu) = 0$$

Huang M, Caines PE, Malhamé RP (2003) Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. In: Proceedings of the 42nd IEEE CDC, Maui, pp 98–103

Lasry J-M, Lions P-L (2007). Mean field games. Japan J Math 2:229–260

Control by leaders

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H \star \mu_N(y_k, w_k) + H \star \mu_m(y_k, w_k) + u_k & k = 1, \dots m, \\ \dot{x}_i = v_i, \\ \dot{v}_i = H \star \mu_N(x_i, v_i) + H \star \mu_m(x_i, v_i) & i = 1, \dots N, \end{cases} \quad (1)$$

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H \star (\mu + \mu_m)(y_k, w_k) + u_k, & k = 1, \dots m, \quad (2) \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot \left[(H \star (\mu + \mu_m)) \, \mu \right], \end{cases}$$

$$F_N(u) = \int_0^T \left\{ L(y_N(t), w_N(t), \mu_N(t)) + \tfrac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt,$$
$$F(u) = \int_0^T \left\{ L(y(t), w(t), \mu(t)) + \tfrac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt.$$

Theorem

F_N Γ-converges to F, thus optimal controls of (1) weakly converge to optimal controls of (2).

M. Fornasier, B. Piccoli, F. Rossi: Mean-field sparse optimal control, *Philosophical Transaction of Royal Society of London Series A*, **372** (2014), 20130400.

Mesoscopic scale: Boltzmann equation (not mean-field equation)

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f, f)$$

f if the probability distribution of particles,
Q is the interaction kernel

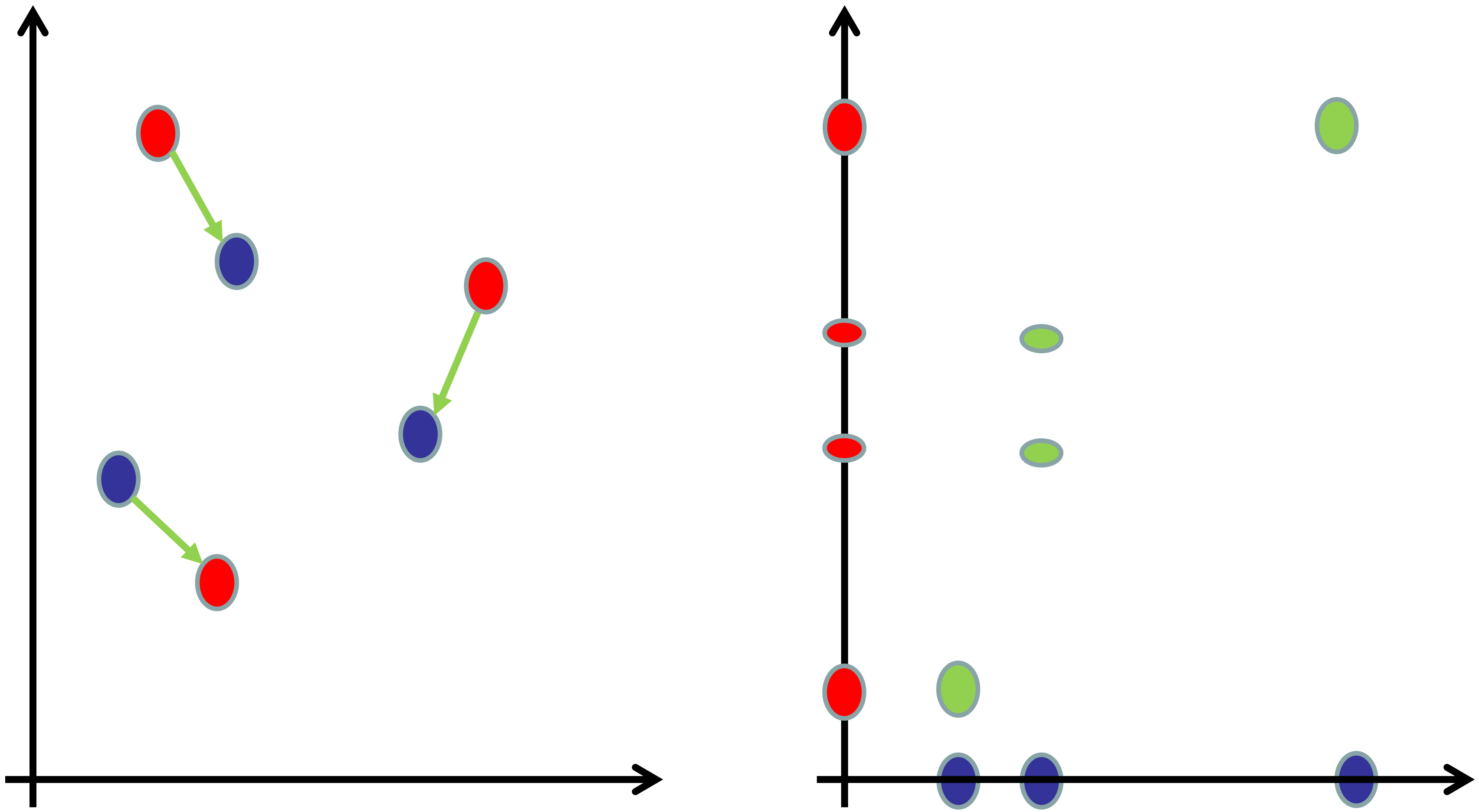
Idea for control:

1. Mimic micro-model interactions with Q
2. Insert optimized control in the microscopic dynamic for AV
3. Deduce Boltzmann equation (and hydrodynamic limit) for controlled traffic

Albi, Chiarello, Degond Herty, Pareschi, Toscani, Tosin, Zanella ...

Part IV:
measure differential equations

Wasserstein metric (Earth mover)



Wasserstein metric

Optimal transport first proposed by Monge in 1781. Particular case is given by the cost $c(x,y) = |x-y|^p$ with $p \geq 1$, defining the Wasserstein distance:

$$W_p(\mu,\nu) = \inf_{\gamma\#\mu=\nu} \left(\int_{\mathbb{R}^n} |\gamma(x) - x|^p \, d\mu \right)^{1/p}.$$

A probability measure π on $\mathbb{R}^d \times \mathbb{R}^d$, one can interpret it as a method to transfer a measure. μ is sent to ν if:

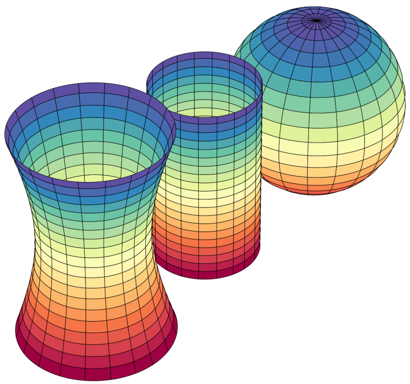
$$|\mu| \int_{\mathbb{R}^d} d\pi(x,\cdot) = d\mu(x), \qquad |\nu| \int_{\mathbb{R}^d} d\pi(\cdot,y) = d\nu(y).$$

Monge-Kantorovich problem. Wasserstein distance:

$$W_p(\mu,\nu) = \left(|\mu| \min_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \, d\pi(x,y) \right)^{1/p}.$$

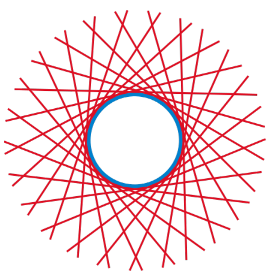
where $\Pi(\mu,\nu)$ is the set of transference plans from μ to ν.

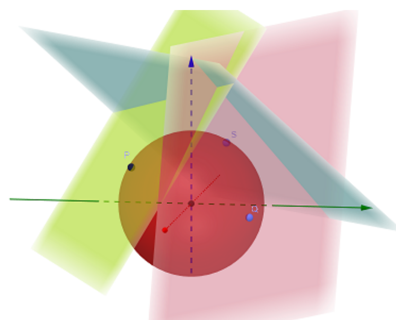
Language from differential geometry

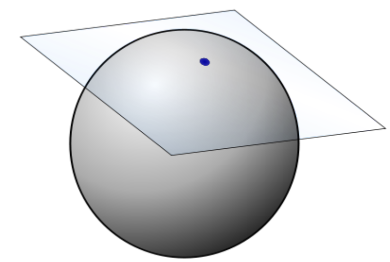


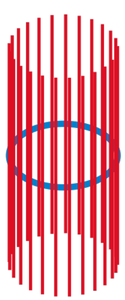
Manifold: space that locally is diffeomorphic to R^n

$T_x M$ tangent space at x or set of vector tangent to a point x
Each element is a couple $(x, v(x))$





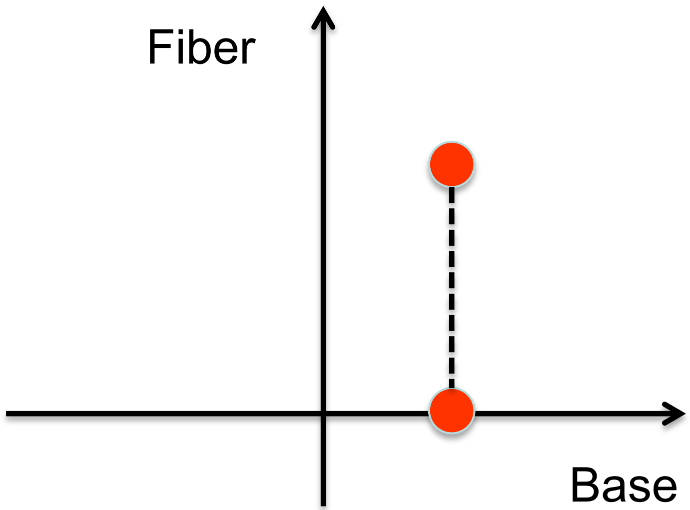




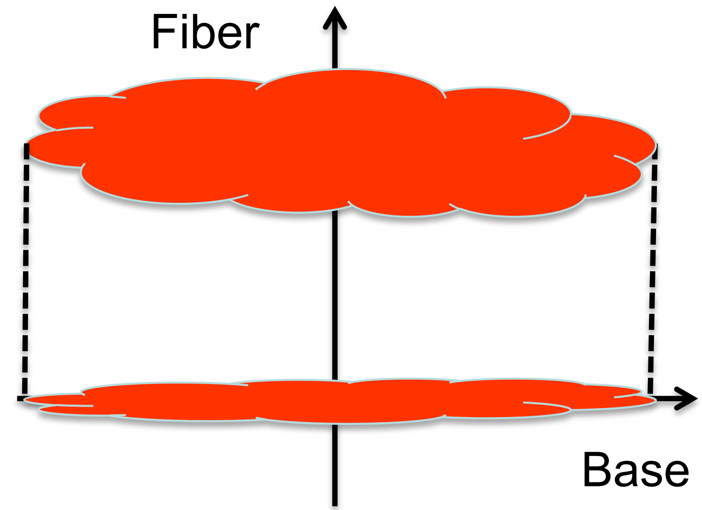
TM tangent bundle: union of all tangent spaces $\cup_x T_x M$

1. $TR^n = R^n \times R^n = R^{2n}$
2. Ordinary differential equation = vector field = section of tangent bundle
3. ODEs can be seen as maps $x \rightarrow (x, v(x))$

Measure differential equations



$x \to v(x)$ $\quad x \to (x, v(x))$



$\delta_x \to \delta_{(x,v(x))}$

Definition

A Probability Vector Field (briefly PVF) on $\mathcal{P}(\mathbb{R}^n)$ is a map $V : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(T\mathbb{R}^n)$ such that $\pi_1 \# V[\mu] = \mu$.

Measure Differential Equation: $\dot{\mu} = V[\mu]$

For every $f \in \mathbb{C}_c^\infty(\mathbb{R}^n)$,

$$\frac{d}{dt} \int_{\mathbb{R}^n} f(x)\, d\mu(t)(x) = \int_{T\mathbb{R}^n} (\nabla f(x) \cdot v)\, dV[\mu(t)](x, v). \qquad (1)$$

Solutions to MDEs

Assumptions for existence of solutions:

(H1) V is support sublinear, i.e. $\exists C > 0$ such that

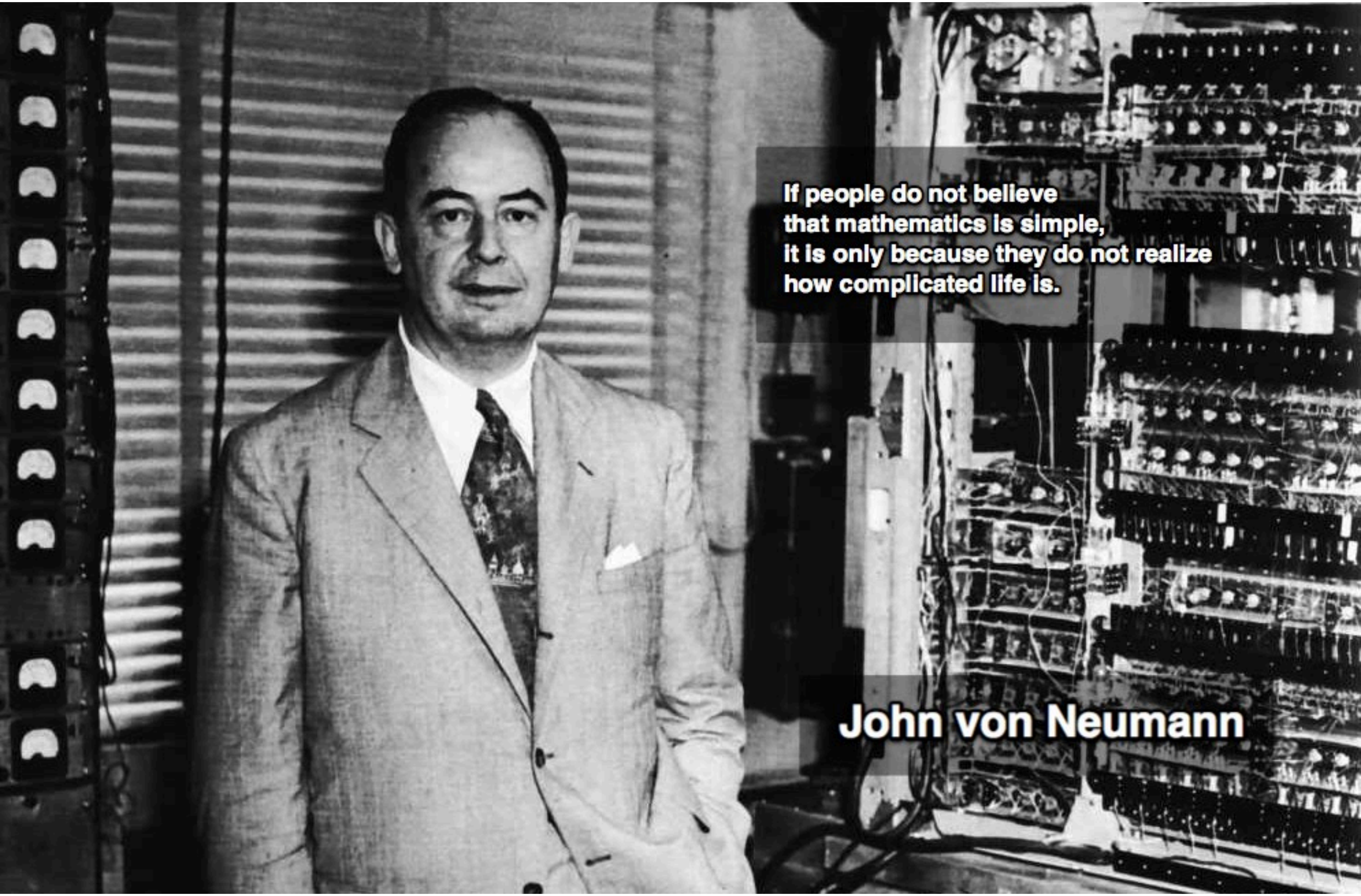
$$\sup_{(x,v)\in Supp(V[\mu])} |v| \leq C \left(1 + \sup_{x\in Supp(\mu)} |x| \right).$$

(H2) The map $V : \mathcal{P}_c(\mathbb{R}^n) \to \mathcal{P}_c(T\mathbb{R}^n)$ is continuous (for the Wasserstein metrics $W^{\mathbb{R}^n}$ and $W^{T\mathbb{R}^n}$.)

Lipschitz semigroup using:

$$\begin{aligned}\mathcal{W}(V_1, V_2) \; &= \inf \Big\{ \int_{T\mathbb{R}^n \times T\mathbb{R}^n} |v - w| \; dT(x, v, y, w) \;\; : \\ &\qquad T \in P(V_1, V_2), \; \pi_{13}\#T \in P^{opt}(\mu_1, \mu_2) \Big\}\end{aligned}$$

Part V: conclusions

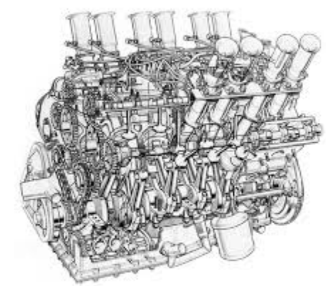

If people do not believe
that mathematics is simple,
it is only because they do not realize
how complicated life is.
John von Neumann

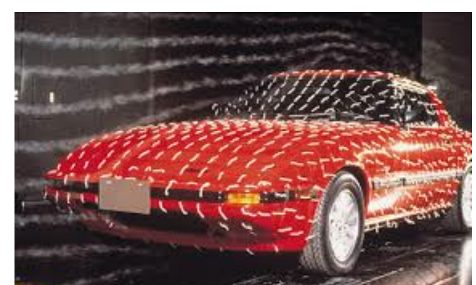
High complexity of traffic

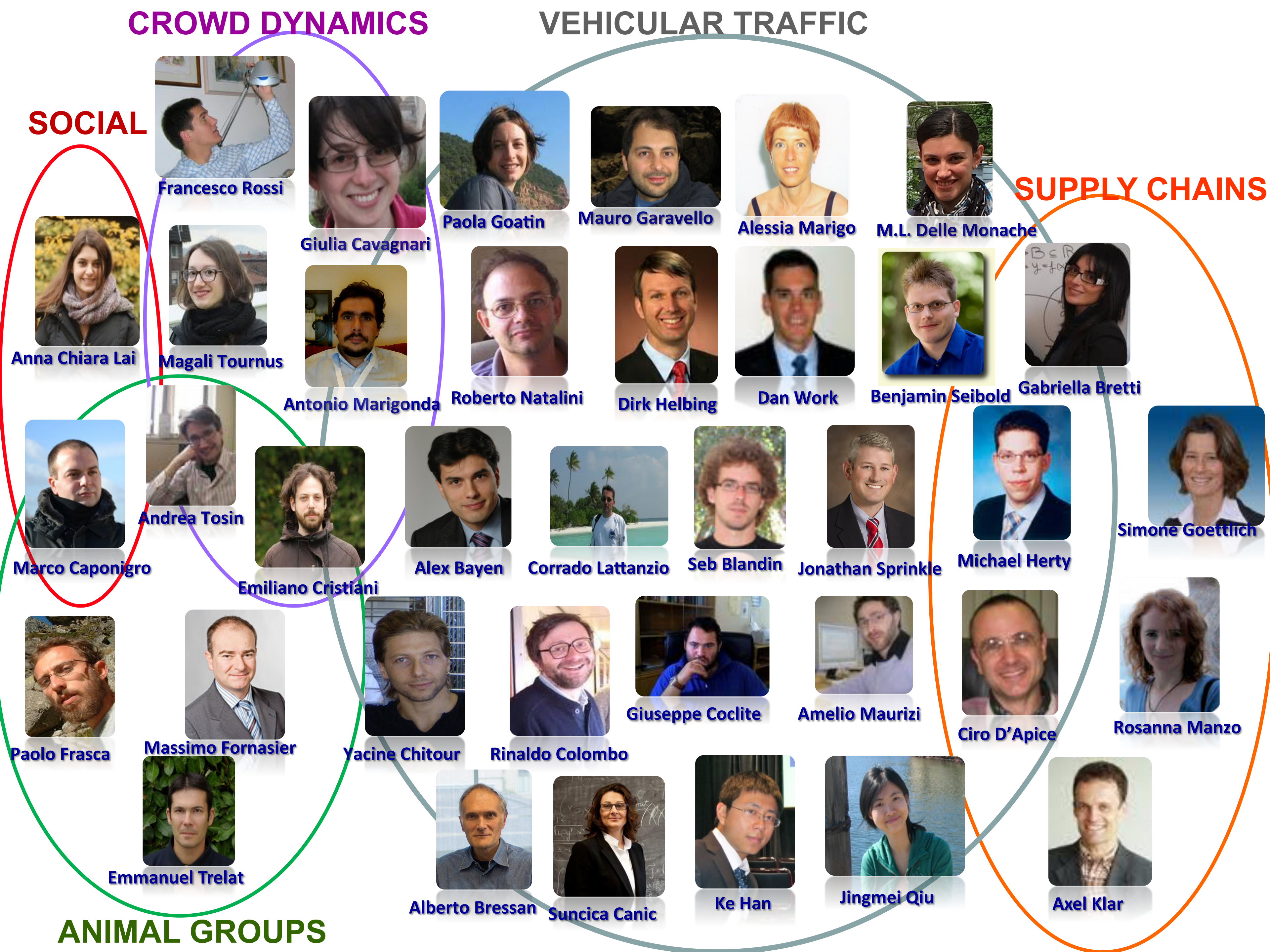
Infrastructure

Internal dynamics

Agents dynamics







CROWD DYNAMICS
VEHICULAR TRAFFIC
SOCIAL
SUPPLY CHAINS
ANIMAL GROUPS
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Giulia Cavagnari
Paola Goatin
Mauro Garavello
Alessia Marigo
M.L. Delle Monache
Anna Chiara Lai
Magali Tournus
Antonio Marigonda
Roberto Natalini
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Benjamin Seibold
Gabriella Bretti
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Jonathan Sprinkle
Michael Herty
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Paolo Frasca
Massimo Fornasier
Yacine Chitour
Rinaldo Colombo
Giuseppe Coclite
Amelio Maurizi
Ciro D'Apice
Rosanna Manzo
Emmanuel Trelat
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Ke Han
Jingmei Qiu
Axel Klar