

IPAM Workshop on “Autonomous Vehicles”

## Non-local macroscopic models accounting for the presence of autonomous vehicles

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joint works with

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The Inria logo is a stylized, cursive script in a reddish-orange color, with the letters 'i', 'n', 'r', and 'i' being more prominent and connected.

# Traffic flow modeling: approaches

## Microscopic

- individual agents
- ODE system
- many parameters
- low and high densities
- comp. cost  $\sim$  number of agents



@fotolia - MurielleB

## Macroscopic

- continuous flow
- PDEs
- few parameters
- high densities
- analytical theory
- comp. cost  $\sim$  domain size



@fotolia - VRD

## Macroscopic traffic flow models

Along a road (network) we need to measure aggregate quantities:

- the (mean) **traffic density**  $\rho$ : number of vehicles per unit space
- the (mean) **velocity**  $v$ : distance covered by vehicles per unit time
- the **traffic flow**  $q = \rho v$ : number of vehicles per unit time

Data sources:



@Wikipedia

magnetic loop detectors, video recordings, floating car data, etc



## Macroscopic traffic flow models

$$\left[ \text{number of vehicles in } [a, b] \text{ at time } t \right] = \int_a^b \rho(t, x) dx$$

must be conserved!

$$\int_a^b \rho(t_2, x) dx = \int_a^b \rho(t_1, x) dx$$

$$+ \int_{t_1}^{t_2} q(t, a) dt - \int_{t_1}^{t_2} q(t, b) dt$$

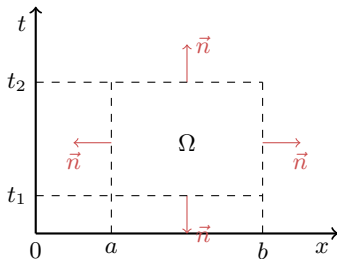
$\Downarrow$

divergence theorem

$$\int_{\Omega} \text{div}_{(t,x)}(\rho, q) = \int_{\partial\Omega} (\rho, q) \cdot \vec{n}$$

$\Downarrow$

$$\int_{t_1}^{t_2} \int_a^b (\partial_t \rho + \partial_x q) dx dt = 0$$

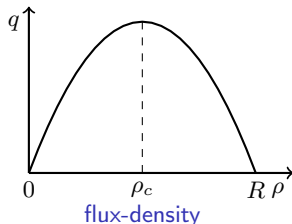
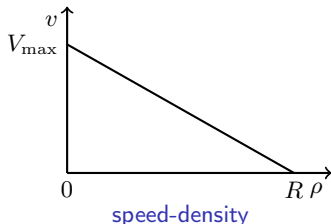


conservation law

## The Lighthill-Whitham-Richards (LWR) model (mid 50s)

- Mass conservation equation:  $\partial_t \rho + \partial_x(\rho v) = 0$
- Phenomenological speed-density relation:  $v(t, x) = v(\rho(t, x))$

fundamental diagram



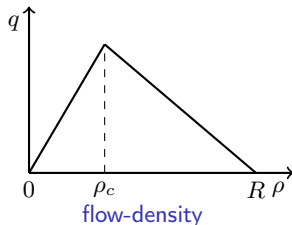
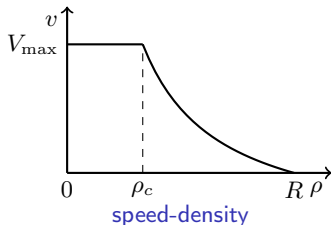
$R$  maximal or *jam* density,  $\rho_c$  critical density:

- flux is increasing for  $\rho \leq \rho_c$ : free-flow phase
- flux is decreasing for  $\rho \geq \rho_c$ : congestion phase

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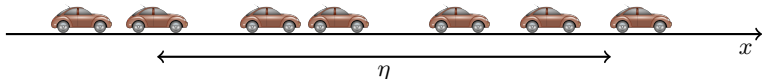
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## The idea

*Drivers adapt their speed to a weighted mean of the (downstream) traffic density*

$$v(t, x) = v \left( \int_x^{x+\eta} \rho(t, y) \omega(y - x) dy \right), \quad \omega' < 0$$



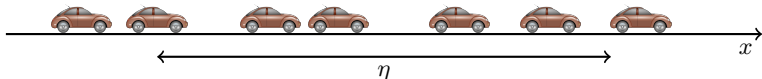
Features:

- bounded acceleration
- *modeling connected vehicles?*

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Features:

- bounded acceleration
- *modeling connected vehicles?*

**Question: Is such a model appropriate?**

## Non-local conservation laws

(Systems of) equations of the form

$$\partial_t U + \operatorname{div}_{\mathbf{x}} F(t, \mathbf{x}, U, w * U) = 0$$

with  $t \in \mathbb{R}^+$ ,  $\mathbf{x} \in \mathbb{R}^d$ ,  $U(t, \mathbf{x}) \in \mathbb{R}^N$ ,  $w(t, \mathbf{x}) \in \mathbb{R}^{m \times N}$

Applications:

- sedimentation [Betancourt&al, Nonlinearity 2011]
- granular flows [Amadori-Shen, JHDE 2012]
- crowd dynamics [Colombo&al, ESAIM COCV 2011; AMS 2011; M3AS 2012]
- supply chains [Colombo-Herty-Mercier, ESAIM COCV 2011]
- conveyor belts [Göttlich&al, Appl. Math. Modell., 2014]
- gradient constraint [Amorim, Bull. Braz. Math. Soc., 2012]

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General well posedness results:

- 1D scalar equations  
[Amorim-Colombo-Teixeira, ESAIM M2AN 2015; Keimer-Pflug, JDE 2017]
- multiD scalar equations  
[Colombo-Herty-Mercier, ESAIM COCV 2011; Keimer-Pflug-Spinola, JMAA 2018]
- multiD systems  
[Crippa-Mercier, NoDEA 2012; Aggarwal-Colombo-Goatin, SINUM 2015]

## A model with non-local velocity<sup>1</sup>

LWR model with downstream non-local velocity

$$\partial_t \rho(t, x) + \partial_x (\rho(t, x) V(t, x)) = 0$$

where

$$V(t, x) = v \left( \int_x^{x+\eta} \rho(t, y) w_\eta(y-x) dy \right), \quad \eta > 0$$

with  $w_\eta \in \mathbf{C}^1([0, \eta]; \mathbb{R}^+)$  **non-increasing** and  $\int_0^\eta w_\eta(x) dx = 1$

$v : [0, \rho_{\max}] \rightarrow \mathbb{R}^+$  s.t.  $-A \leq v' \leq 0$ ,  $v(0) = v_{\max}$ ,  $v(\rho_{\max}) = v_{\min}$

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<sup>1</sup>[Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016]

# A model with non-local velocity<sup>1</sup>

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Related works:

- sedimentation model:  $F(\rho, \rho * w) = \rho(1 - \rho)^\alpha V(\rho * w)$ ,  $\alpha = 0$  or  $\alpha \geq 1$   
[Betancourt&al, Nonlinearity 2011]
- Arrhenius look-ahead dynamics:  $F(\rho, \rho * w) = \rho(1 - \rho)e^{-(\rho * w)}$   
[Sopasakis-Katsoulakis, SIAM 2006; Kurganov-Polizzi, NHM 2009; Li-Li, NHM 2011]
- general flux:  $F(\rho, \rho * w) = f(\rho)V(\rho * w)$   
[Chiarello-Goatin, ESAIM M2AN 2018]
- weighted average velocity  $F(\rho, V * w) = \rho(V(\rho) * w)$   
[Friedrich-Kolb-Göttlich, NHM 2018]

<sup>1</sup>[Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016]

## Well-posedness

### Theorem

[Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016; Chiarello-Goatin, M2AN 2018]

Let  $\rho_0 \in \text{BV}(\mathbb{R}; [0, \rho_{\max}])$ . Then the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)V(t, x)) = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0, x) = \rho_0(x) & x \in \mathbb{R} \end{cases}$$

admits a unique weak (entropy) solution ( $\rho \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \text{BV}$ ), such that

$$\min_{\mathbb{R}} \{\rho_0\} \leq \rho(t, x) \leq \max_{\mathbb{R}} \{\rho_0\} \quad \text{for a.e. } x \in \mathbb{R}, t > 0$$

Here we consider the case  $f(\rho) = \rho$

## Uniqueness and stability<sup>2</sup>

### Theorem

Let  $\rho, \sigma$  be two entropy weak solutions of CP with initial data  $\rho_0, \sigma_0$  respectively. Then, for any  $T > 0$  there holds

$$\|\rho(t, \cdot) - \sigma(t, \cdot)\|_{\mathbf{L}^1} \leq e^{\mathcal{K}T} \|\rho_0 - \sigma_0\|_{\mathbf{L}^1} \quad \forall t \in (0, T].$$

where

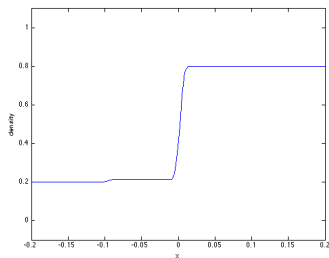
$$\begin{aligned} \mathcal{K} = & w_\eta(0) \|v'\|_\infty \left( \sup_{t \in [0, T]} \|\rho(t, \cdot)\|_{\mathbf{BV}(\mathbb{R})} + 2\|\rho_0\|_\infty \right) \\ & + \|\rho_0\|_1 \left( 2(w_\eta(0))^2 \|v''\|_\infty \|\rho_0\|_\infty + \|v'\|_\infty \|w'_\eta\|_{\mathbf{L}^\infty([0, \eta])} \right) \end{aligned}$$

*Proof.* Doubling of variables.

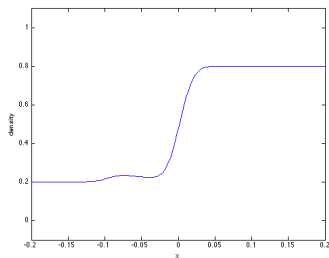
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<sup>2</sup>[Betancourt&al, Nonlinearity 2011; Keimer-Pflug, JDE 2017]

## Monotonicity preservation



(a)  $v(\rho) = 1 - \rho$



(b)  $v(\rho) = \ln\left(\frac{1}{\rho}\right)$

**Figure:** Density profiles at time  $t = 0.01$  corresponding to  $\rho_L = 0.2$ ,  $\rho_R = 0.8$  and kernel  $w_\eta(x) = 1/\eta$ ,  $\eta = 0.1$ .

## Dependence on the location of the kernel support

We set  $v(\rho) = 1 - \rho$  and

downstream: 
$$V_d(t, x) = 1 - \int_x^{x+\eta} \rho(t, y) w_\eta(y - x) dy$$

center: 
$$V_c(t, x) = 1 - \int_{x-\eta/2}^{x+\eta/2} \rho(t, y) w_\eta(y - x) dy$$

upstream : 
$$V_u(t, x) = 1 - \int_{x-\eta}^x \rho(t, y) w_\eta(y - x) dy$$

## Dependence on the location of the kernel support

### Rarefaction

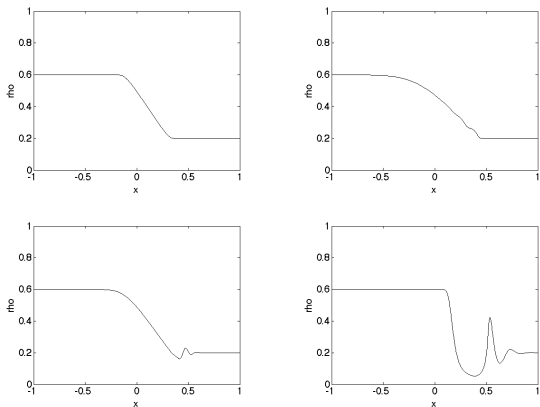


Figure:  $w_\eta(x) = 1/\eta$  with downstream, central and upstream supports respectively and initial data  $\rho_L = 0.6$ ,  $\rho_R = 0.2$

## Dependence on the kernel support

Shock

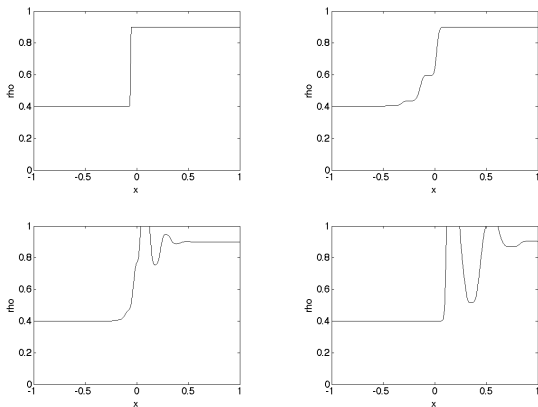


Figure:  $w_\eta(x) = 1/\eta$  with downstream, central and upstream supports respectively and initial data  $\rho_L = 0.4$ ,  $\rho_R = 0.9$

## Dependence on the kernel support

### Oscillating initial datum

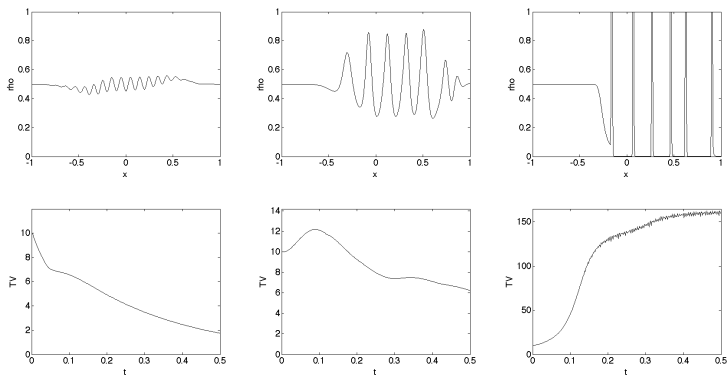
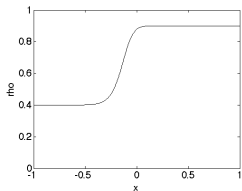
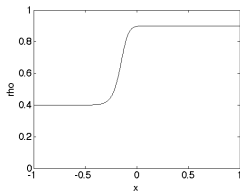


Figure:  $w_\eta(x) = 1/\eta$  with downstream, central and upstream supports respectively

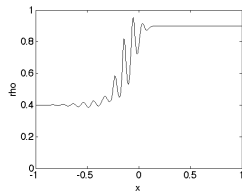
## Kernel monotonicity



(a)  $w_\eta(x) = 1/\eta$



(b)  $w_\eta(x) = 2(\eta - x)/\eta^2$



(c)  $w_\eta(x) = 2x/\eta^2$

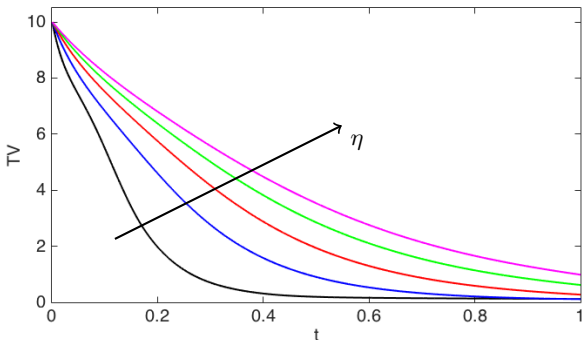
Figure:  $\rho(t = 0.5, \cdot)$  corresponding to  $\rho_L = 0.4$ ,  $\rho_R = 0.9$

## What this non-local model teaches us

- You should always look forward
- You should care more what is closer

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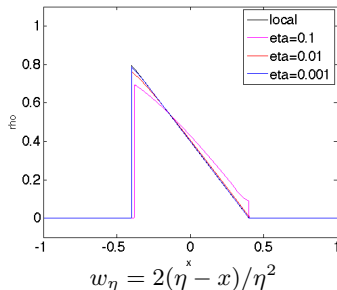
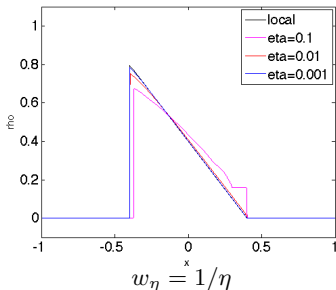
- You should always look forward
- You should care more what is closer
- The farther you see, the worse it is



Limit  $\eta \searrow 0^3$

$$\partial_t \rho + \partial_x (\rho v(\rho * w_\eta)) = 0 \quad \rightarrow \quad \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \quad ??$$

We consider  $v(\rho) = 1 - \rho$  and  $\rho_0(x) = \begin{cases} 0.8 & \text{if } -0.5 < x < -0.1 \\ 0 & \text{otherwise} \end{cases}$

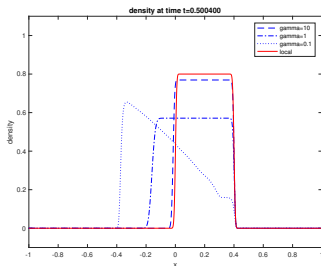


<sup>3</sup>[Colombo-Crippa-Spinolo, 2018]

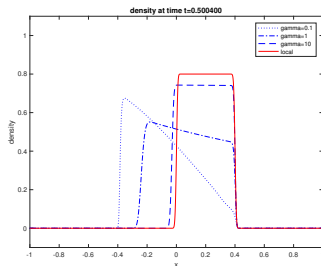
Limit  $\eta \rightarrow +\infty^4$

$$\partial_t \rho + \partial_x (\rho v(\rho * w_\eta)) = 0 \quad \rightarrow \quad \partial_t \rho + \partial_x (\rho v(0)) = 0$$

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$$w_\eta = 1/\eta$$



$$w_\eta = 2(\eta - x)/\eta^2$$

<sup>4</sup>[Chiarello-Goatin, M2AN 2018]

## A multi-class model with non-local velocity<sup>5</sup>

### Multi-class traffic model with downstream non-local velocity

$$\partial_t \rho_i(t, x) + \partial_x (\rho_i(t, x) v_i((r * \omega_i)(t, x))) = 0, \quad i = 1, \dots, M,$$

where

$$r(t, x) := \sum_{i=1}^M \rho_i(t, x), \quad v_i(\xi) := v_i^{\max} \psi(\xi),$$

$$(r * \omega_i)(t, x) := \int_x^{x+\eta_i} r(t, y) \omega_i(y-x) dy,$$

$$\begin{aligned} & \omega_i \in \mathbf{C}^1([0, \eta_i]; \mathbb{R}^+), & \omega_i' \leq 0, & \int_0^{\eta_i} \omega_i(y) dy = J_i. \\ \text{(H)} \quad & W_0 := \max_{i=1, \dots, M} \omega_i(0). & 0 < v_1^{\max} \leq v_2^{\max} \leq \dots \leq v_M^{\max}. \\ & \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ smooth} & \psi' \leq 0 \text{ s.t. } \psi(0) = 1 \text{ and } \psi(r) = 0 \text{ for } r \geq 1. \end{aligned}$$

<sup>5</sup>[Chiarello-Goatin, NHM 2019]

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- Local multi-class model: [Benzoni-Colombo, EJAM 2003]

<sup>5</sup>[Chiarello-Goatin, NHM 2019]

## Existence of weak solutions locally in time

Theorem [Chiarello-Goatin, NHM 2019]

Let  $\rho_i^0(x) \in (\text{BV} \cap \mathbf{L}^\infty)(\mathbb{R}; \mathbb{R}^+)$ , for  $i = 1, \dots, M$ , and assumptions **(H)** hold. Then the Cauchy problem

$$\begin{cases} \partial_t \rho_i(t, x) + \partial_x (\rho_i(t, x) v_i((r * \omega_i)(t, x))) = 0, & i = 1, \dots, M, \\ \rho_i(0, x) = \rho_i^0(x). \end{cases}$$

admits a weak solution on  $[0, T[ \times \mathbb{R}$ , for some  $T > 0$  **sufficiently small**.

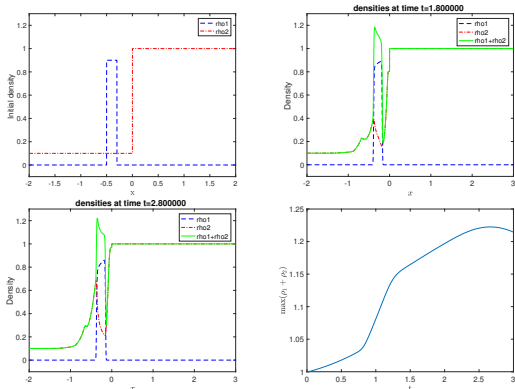
*Proof.* Helly's theorem + Lax-Wendroff type argument.

## Lack of uniform $L^\infty$ -bounds

Unlike the classical (local) multi-population model, the simplex

$$\mathcal{S} := \left\{ \rho \in \mathbb{R}^M : \sum_{i=1}^M \rho_i \leq 1, \rho_i \geq 0 \text{ for } i = 1, \dots, M \right\}$$

is not an invariant domain for the non-local multi-class model.



## Cars and trucks mixed traffic

$$\begin{cases} \partial_t \rho_1(t, x) + \partial_x (\rho_1(t, x) v_1^{\max} \psi((r * \omega_1)(t, x))) = 0, \\ \partial_t \rho_2(t, x) + \partial_x (\rho_2(t, x) v_2^{\max} \psi((r * \omega_2)(t, x))) = 0, \end{cases}$$

with

$$\omega_1(x) = \frac{2}{\eta_1} \left( 1 - \frac{x}{\eta_1} \right), \quad \eta_1 = 0.3,$$

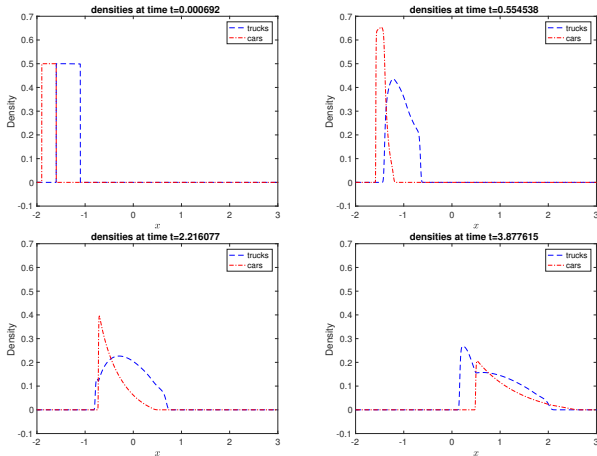
$$\omega_2(x) = \frac{2}{\eta_2} \left( 1 - \frac{x}{\eta_2} \right), \quad \eta_2 = 0.1,$$

$$\psi(\xi) = \max \{ 1 - \xi, 0 \}, \quad \xi \geq 0,$$

$$v_1^{\max} = 0.8, \quad v_2^{\max} = 1.3.$$

$$\begin{cases} \rho_1(0, x) = 0.5 \chi_{[-1.1, -1.6]}, & \text{trucks} \\ \rho_2(0, x) = 0.5 \chi_{[-1.6, -1.9]}, & \text{cars} \end{cases}$$

## Cars and trucks mixed traffic



**Figure:** Density profiles of cars and trucks at increasing times corresponding to the non-local model.

## Impact of connected autonomous vehicles

Let us consider a circular road modeled by the space interval  $[-1, 1]$  with periodic boundary conditions at  $x = \pm 1$ .

$$\begin{cases} \partial_t \rho_1(t, x) + \partial_x (\rho_1(t, x) v_1^{\max} \psi((r * \omega_1)(t, x))) = 0, & \text{autonomous,} \\ \partial_t \rho_2(t, x) + \partial_x (\rho_2(t, x) v_2^{\max} \psi((r * \omega_2)(t, x))) = 0, & \text{non - autonomous} \\ \rho_1(0, x) = \beta (0.5 + 0.3 \sin(5\pi x)), \\ \rho_2(0, x) = (1 - \beta) (0.5 + 0.3 \sin(5\pi x)), \end{cases}$$

with

$$\omega_1(x) = \frac{1}{\eta_1}, \quad \eta_1 = 1,$$

$$\omega_2(x) = \frac{2}{\eta_2} \left( 1 - \frac{x}{\eta_2} \right), \quad \eta_2 = 0.01,$$

$$\psi(\xi) = \max \{ 1 - \xi, 0 \}, \quad \xi \geq 0,$$

$$v_1^{\max} = v_2^{\max} = 1.$$

$\beta \in [0, 1]$  is the penetration rate of autonomous vehicles.

## Impact of connected autonomous vehicles

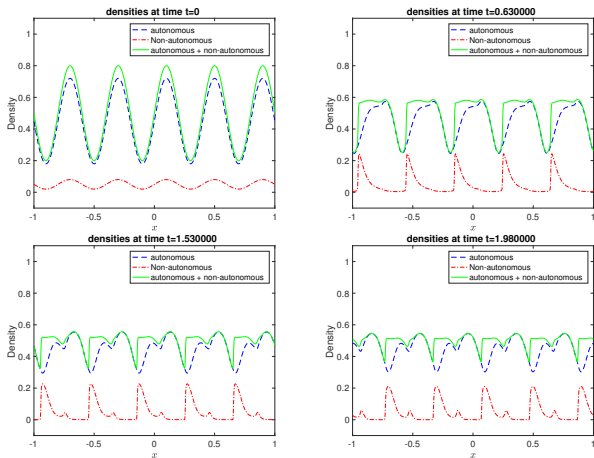


Figure: Density profiles corresponding to the non-local problem with  $\beta = 0.9$  at different times.

## Impact of connected autonomous vehicles

As a metric of traffic congestion, we consider the two following functionals<sup>6</sup>:

$$J(\beta) = \int_0^T d|\partial_x r| dt,$$

$$\Psi(\beta) = \int_0^T [\rho_1(t, \bar{x}) v_1^{\max} \psi((r * \omega_1)(t, \bar{x})) + \rho_2(t, \bar{x}) v_2^{\max} \psi((r * \omega_2)(t, \bar{x}))] dt,$$

where  $\bar{x} = x_0 \approx 0$ .

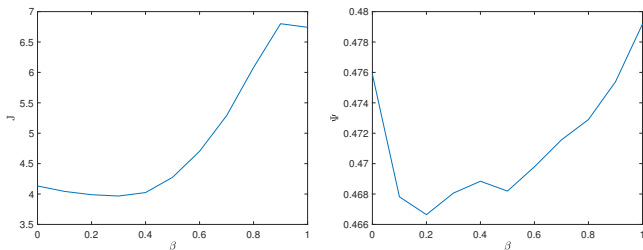


Figure: Functionals  $J$  (left) and  $\Psi$  (right).

<sup>6</sup> [Colombo-Groli, 2004], [Chiarello-Goatin, 2019]

## Impact of connected autonomous vehicles

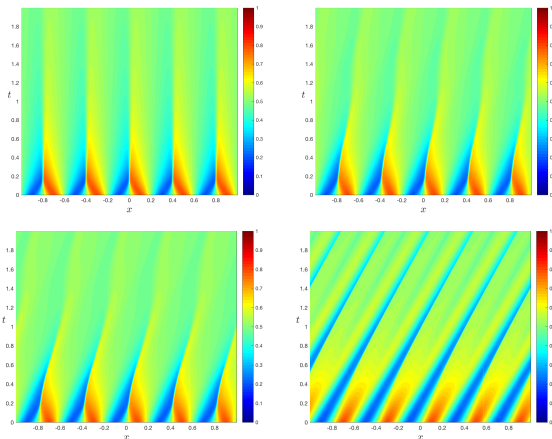


Figure:  $(t, x)$ -plots of the total traffic density  $r(t, x)$  corresponding to  $\beta = 0, 0.2, 0.3, 0.9$ .

# Numerical schemes

- First order
  - Lax-Friedrichs  
[Colombo-Amorim-Teixeira, ESAIM M2AS 2015; Blandin-Goatin, NumMath 2016; Aggarwal-Colombo-Goatin, SINUM 2015; ...]
  - Upwind  
[Friedrich-Kolb-Göttlich, NHM 2018; Chiarello-Goatin, NHM 2019]
  - Lagrangian-Remap  
[Chiarello-Goatin-Villada, submitted]
  
- High order
  - Discontinuous Galerkin and Finite Volume WENO  
[Chalons-Goatin-Villada, SISC 2018; Chiarello-Goatin-Villada, submitted]

# Perspectives

Some open problems:

- boundary conditions  
[DeFilippis-Goatin, *Nonlinear Anal.* 2017; Goatin-Rossi, submitted]
- extension to road networks (junctions)  
[Chiarello-Friedrichs-Goatin-Göttlich, ongoing]
- topological VS metric interactions  
[Costeseque-Goatin-Rossi, ongoing]

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Thank you!