The Shape of Inner Space

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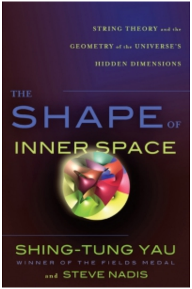
Introduction

I'd like to talk about how mathematics and physics can come together to the benefit of both fields, particularly in the case of Calabi-Yau spaces and string theory. This, not coincidentally, is the subject of a new book, THE SHAPE OF INNER SPACE, which I have written with Steve Nadis, a science writer.

Steve Nadis



Book Cover

This book tells the story of those spaces. It also tells some of my own story and a bit of the history of geometry as well. In that spirit, I'm going to back up and talk about my personal introduction to geometry and the evolution of the ideas that are discussed in this book.

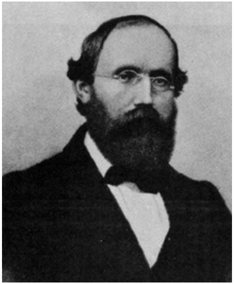
I wanted to write this book to give people a sense of how mathematicians think and approach the world. I also want people to realize that mathematics does not have to be a wholly abstract discipline, disconnected from everyday phenomena, but is instead crucial to our understanding of the physical world.

So we're now going to step back in time a bit. Or perhaps I should say step back in spacetime...

I. Riemannian Geometry

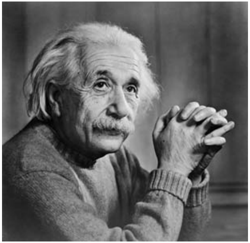
When I arrived in Berkeley in 1969 for graduate study, I learned that the concept of geometry had gone through a radical change in the 19th century, thanks to the contributions of Gauss and Riemann. Riemann revolutionized our notions of space. Objects no longer had to be confined to the flat, linear space of Euclidean geometry.

Gauss

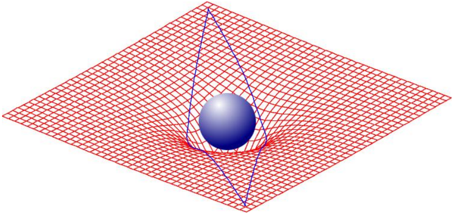


Riemann

Riemann instead proposed a much more abstract conception of space — of any possible dimension — in which we could describe distance and curvature. In fact, one can develop a form of calculus that is especially suited to such an abstract space. It took about fifty years until Einstein realized that this kind of geometry, which involved curved spaces, was exactly what he needed to unify Newtonian gravity with special relativity. This insight culminated in his famous theory of general relativity.



Einstein



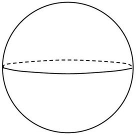
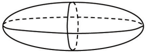
Curved Space-time

I learned about Riemannian geometry during my first year at Berkeley. After a couple of months, I started to toy around with some statements that related the curvature of a space — its exact shape or geometry — to a much cruder, more general way of characterizing shape, which we call topology.

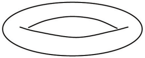
At Berkeley 1969

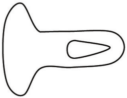
Topology is a concept of a space that is unrelated to the way that we measure distance in that space. In that sense, topology describes a space much less precisely than geometry. We need to know all the details of a space to measure the distance between any two points. The sum of all those details, which spell out the curvature at every point, is what we mean by geometry.

For example:

1. The sphere  and the ellipsoid 

   have the same topology, but they have a different shape (geometry).

2. The thin donut  has the same topology as

    , but they have a different shape (geometry).

I wrote down some of my thoughts concerning ways in which topology (or general shape) influenced the geometry (or exact shape) and vice versa. While I was photocopying those notes in the Xerox room, I ran into Arthur Fisher, a mathematician. He insisted on knowing what I had written. After reading through my notes, he told me that any principle that related curvature with topology would be useful in physics. His comments have stayed with me ever since.

II. General Relativity

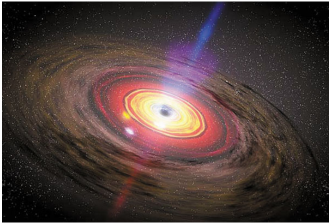
We learned through special relativity that space and time should not be treated separately but should instead be merged together to form spacetime. We learned that information should not, and indeed cannot, travel faster than the speed of light. The laws of gravity, moreover, should be independent of the coordinate system selected by observers trying to measure events.

Einstein struggled in his attempt to obtain a fundamental description of gravity. But he got some help from his friend Grossman, a mathematician, who told him of the work of other mathematicians, Riemann and Ricci.

Riemann provided the framework of abstract space, as well as the means for defining distance and curvature in such a space. Riemann thus provided the background space or setting in which gravity, as Einstein formulated it, plays out.

But Einstein also drew on the work of Ricci, who defined a special kind of curvature that could be used to describe the distribution of matter in spacetime. Through general relativity, Einstein offered a geometric picture of gravity. Rather than considering gravity as an attractive force between massive objects, it could instead be thought of as the consequence of the curvature of spacetime due to the presence of massive objects. The precise way in which spacetime is curved tells us how matter is distributed.

When I looked at the equations of Einstein, I was intrigued by the fact that matter only controls part of the curvature of spacetime. I wondered whether we could construct a spacetime that is a vacuum, and thus has no matter, yet its curvature is still pronounced. Well, the famous Schwarzschild solution to Einstein's equations is such an example. This solution applies to a non-spinning black hole — a vacuum that, curiously, has mass owing to its extreme gravity. But that solution admits a singular point, or singularity — a place where the laws of physics break down.



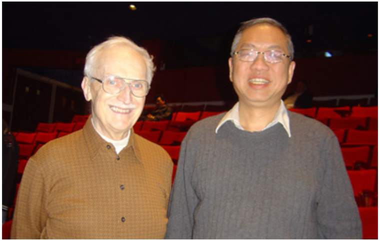
Black hole

I became interested in a different situation — a smooth space without a singularity that was compact and closed, unlike the open, extended space of the Schwarzschild solution. The question was: Could there be a compact space that contained no matter — a closed vacuum universe, in other words — whose force of gravity was nontrivial? I was obsessed with this question and believed that such a space could not exist. If I could prove that, I was sure that it would be an elegant theorem in geometry.

III. Calabi Conjecture

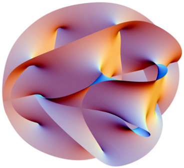
When I started thinking about this in the early 1970s, I did not realize that the geometer Eugenio Calabi had posed almost the exact same question. Calabi framed the problem in fairly complicated mathematical language — involving difficult concepts like Kahler manifolds, Ricci curvature, and Chern classes. Yet his abstract conjecture could also be framed in terms of Einstein's theory of general relativity. In that context his question translated to: Can there be gravity, or the curving of space, in a closed vacuum — a compact space that has no matter?

For about three years, my friends and I tried to prove that the class of spaces proposed by Calabi could not exist. We, along with many others, considered them to be "too good to be true." But try as we might, we could not prove that such spaces do not exist.



With Prof. Calabi, 2004

Until one day I thought I found a way to demonstrate that Calabi was wrong. I made this discovery at a big conference at Stanford, and I was asked to give a talk about it. However, a few months later, while trying to write up my proof in a rigorous fashion, I found that I could not complete my argument. I finally decided that the Calabi conjecture must be right after all, and I spent the next several years trying to prove it.



Calabi-Yau space

In 1975, I used my Sloan fellowship to visit Courant at the invitation of Professor Louis Nirenberg. My friend S.-Y. Cheng was there. Although a major motivation for me was to visit my girlfriend who worked in the Plasma lab in Princeton, I still got a lot of work done. I wrote several important papers with S.-Y. Cheng related to the real Monge-Ampère equation. They are very much related to the work on the complex Monge-Ampére equation, which appeared in the proof of Calabi conjecture. In fact, by this time, I had learned quite a bit about complex Monge-Ampére equations in terms of second-order estimates and was confident that I could solve the whole problem.

Looking back, this was rather amazing since I was spending so much time traveling back and forth between Princeton and New York, yet still managed to make progress with the research. My girlfriend's job was finishing by the summer of 1976 . She went to interview at TRW in Los Angeles in December of 1975. So I accompanied her to LA. She got an offer from TRW for a job in the following year.

So I decided to take a leave of absence from Stanford to come to UCLA for the 1976-1977 academic year. I wrote to Prof. Robert Greene and Barrett O'Neil. I have to say that I am grateful that my proposal was accepted immediately. I used my Sloan fellowship for the first quarter and for the rest of the academic year I had a teaching job at UCLA. Stanford allowed me to take the leave of absence. My girlfriend was very impressed by that. I was a new Ph.D., and the job situation in those years was tough.

My girlfriend and I got engaged in Princeton when I visited her in May 1976. In June, I drove cross country with her and her parents from Princeton to Los Angeles. It was a very enjoyable trip . Along the way, I was thinking about solving the Poincaré conjecture and the Calabi conjecture at the same time. For the Poincaré conjecture, I was hoping to use the theory of minimal surfaces. My idea did not quite work, but the potential was there.

As for the Calabi conjecture, I thought through the estimates that were needed to solve the equations, while I was enjoying the American countryside. (I did not tell my future wife what I was thinking about at the time.) When we arrived in LA, my friends at UCLA were very friendly. We found a temporary apartment, and I then went out to buy my first house with my future wife.

We got married in early September and moved to a house in the San Fernando Valley. I was given an office right next to Prof. Robert Greene. It was a small office but very nice. Best of all, I could talk with Robert and other faculty members. Marriage proved to be truly enjoyable, so much so that within a couple of weeks, I was able to put all my ideas together to find a proof of the Calabi conjecture.

Life was good, as they say. The proof of the Calabi conjecture looked beautiful to me, especially after such a long struggle. It was extremely satisfying to be the first person to understand such a proof, and I felt certain that it would eventually be important in physics. There is a poem that conveys some of what I was feeling:

> *In the spring, the flowers are falling while I was watching alone. The pair of birds (swallows) were flying together in the light rain.*

I felt that I was truly mingled with nature.

But then I got practical. I remembered all of my earlier efforts to disprove the Calabi conjecture. Each of the supposed counterexamples I had gathered turned out to be theorems for which I now had a proof, and many of these statements were important.

In September of 1976, David Mumford gave a seminar talk at UCLA on solitons. I attended that lecture and another lecture he gave at UC Irvine. There he discussed a conjecture related to the work of Bogomolov about some inequalities between topological numbers of algebraic surfaces. After staring at it, I realized it was exactly a consequence of the Calabi conjecture I mentioned above. I had used that same inequality about three years ago in my attempt to disprove the Calabi conjecture. (This idea was inspired by works of Hitchin and Grey.) So I told Mumford about it.

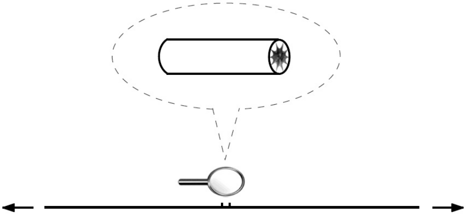
I double checked it at home and sent the details to Mumford a week later. I was gratified that the expected inequality turned out to be true. But I was also able to prove a further result that led to a solution of the famous Severi conjecture, which concerns the algebraic structure of the so-called “projective space”. This conjecture can be viewed as the Poincaré conjecture in an algebraic setting.

The math department at UCLA provided me with a comfortable space to develop my thinking. Within a month or so, I met Bill Meeks, and we immediately got involved in a major development on minimal surfaces, which related geometry with topology. It was used to solve the Smith conjecture later.

Hence, in the period of less than a year, I managed to solve several major mathematical problems. Needless to say, it was the most fruitful year in my career, both personally and professionally.

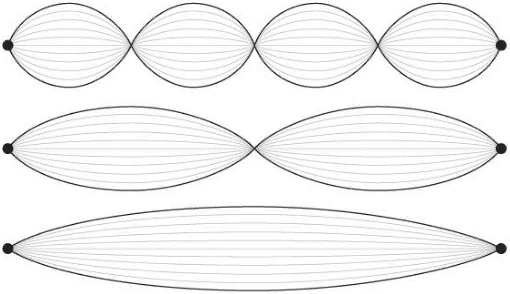
IV. String Theory

A couple of years later, when I visited my wife in San Diego, I got several phone calls. Horowitz and his colleague Andy Strominger said that they were very excited about a model for describing the vacuum state of the universe, based on a new theory called string theory.



Kaluza-Klein garden hose

String theory is built on the assumption that particles, at their most basic level, are made of vibrating bits of tiny strings. In order for the theory to be consistent with quantum theory, spacetime has a certain symmetry built into it called supersymmetry. Spacetime is also assumed to be ten dimensional.



Vibrating strings

Horowitz and Strominger were interested in the multidimensional spaces whose existence I proved, mathematically, in my confirmation of the Calabi conjecture. They believed that these spaces could play an important role in string theory, as they seemed to be endowed with the right kind of supersymmetry — a property deemed essential to their theory. They asked me if their assessment of the situation was correct and, to their delight, I told them that it was.

Ed Witten, whom I'd met in Princeton, was now collaborating with Philip Candelas, Horowitz, and Strominger, trying to figure out the shape, or geometry, of the six "extra" dimensions of string theory. The physicists believed these six dimensions were curled up in a tiny space, which they called Calabi-Yau space — the same family of spaces originally proposed by Calabi and later proved by me.

Witten

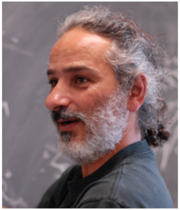
String theory, again, assumes that spacetime has 10 dimensions overall. The three large spatial dimensions that we're familiar with, plus time, make up the four-dimensional spacetime of Einstein's theory. But there are also six additional dimensions hidden away in Calabi-Yau space, and this invisible space exists at every point in "real space," according to the theory, even though we can't see it.

The existence of this extra-dimensional space is fantastic on its own, but string theory goes much farther. It says that the exact shape, or geometry, of Calabi-Yau space dictates the properties of our universe and the kind of physics we see. The shape of Calabi-Yau space — or the "shape of inner space," as we put it in our book — determines the kinds of particles that exist, their masses, the ways in which they interact, and maybe even the constants of nature.

While Einstein had said the phenomenon of gravity is really a manifestation of geometry, string theorists boldly proclaimed that the physics of our universe is a consequence of the geometry of Calabi-Yau space. That's why string theorists were so anxious to figure out the precise shape of this six — dimensional space — a problem we're still working on today.

The great excitement over Calabi-Yau spaces started in 1984, when physicists first found out about them. That enthusiasm kept up for a couple years, before waning. But the excitement picked up again in the late 1980s, when Brian Greene, Ronen Plesser, Candelas, and others began exploring the notion of "mirror symmetry."

Greene



Plesser

The basic idea here was that two different Calabi-Yau spaces, which had different topologies and seemed to have nothing in common, nevertheless gave rise to the same physics. This established a previously unknown kinship between so-called mirror pairs of Calabi-Yau's.

The connection uncovered through physics proved to be extremely powerful in the hands of mathematicians. When they were stumped trying to solve a problem involving one Calabi-Yau space, they could try solving the same problem on its mirror pair. On many occasions, this approach was successful. As a result, problems that had defied resolution, sometimes for as long as a century, were now being solved. And a branch of mathematics called enumerative geometry was suddenly rejuvenated. These advances gave mathematicians greater respect for physicists, as well as greater respect for string theory itself.

V. Conclusion

Before we get too carried away, we should bear in mind that string theory, as the name suggests, is just a theory. It has not been confirmed by physical experiments, nor have any experiments yet been designed that could put that theory to a definitive test. So the jury is still out on the question of whether string theory actually describes nature, which was the original intent.

On the positive side of the ledger, some extremely intriguing, as well as powerful, mathematics has been inspired by string theory. Mathematical formulae developed through this connection have proved to be correct independent of the scientific validity of string theory. So far it stands as the only consistent theory that unifies the different forces. And it is beautiful. Moreover, the effort to unify the different forces of nature has unexpectedly led to the unification of different areas mathematics that at one time seemed unrelated.

We still don't know what the final word will be. In the past two thousand years, the concept of geometry has evolved over several important stages to the current state of modern geometry. Each time geometry has been transformed in a major way, the new version has incorporated our improved understanding of nature arrived at through advances in theoretical physics. It seems likely that we shall witness another major development in the 21st century, the advent of quantum geometry — a geometry that can incorporate quantum physics in the small and general relativity in the large.

The fact that abstract mathematics can reveal so much about nature is something I find both mysterious and fascinating. This is one of the ideas that my coauthor and I have tried to get across in our book, The Shape of Inner Space. We also hope that the book gives you a description of how mathematicians work. They are not necessarily weird people, such as a janitor who solves centuries-old math problems on the side while mopping and dusting floors, as described in the movie “Good Will Hunting”. Nor does a brilliant mathematician have to be mentally ill, or exhibit otherwise bizarre behavior, as depicted in another popular movie and book.

Mathematicians are just scientists who look at nature from a different, more abstract point of view than the empiricists. But the work mathematicians do is still based on the truth and beauty of nature, the same as it is in physics. Our book tries to convey the thrill of working at the interface between mathematics and physics, showing how important ideas flow through different disciplines, with the result being the birth of new and important subjects.

In the case of string theory, geometry and physics have come together to produce some beautiful mathematics, as well as some very intriguing physics. The mathematics is so beautiful, in fact, and it has branched out into so many different areas, that it makes you wonder whether the physicists might be onto something after all.

The story is still unfolding, to be sure, and I consider myself lucky to have been part of it.