The polynomial rate of convergence of critical interfaces.

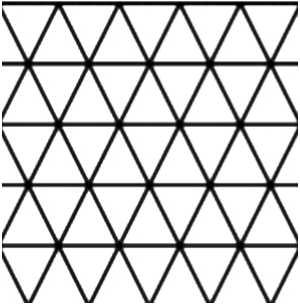
Ilia Binder
University of Toronto

based on joint works with Lincoln Chayes (UCLA), Helen Lei and Larissa Richards (Toronto)

Analysis and Geometry of Random Shapes
January 9, 2019

1. Percolation: the models.
2. Exploration process for percolation and crossing probability.
3. Polynomial rate of convergence for the Cardy-Smirnov observable: joint work with Lincoln Chayes and Helen Lei.
4. From convergence of observables to convergence of interfaces: joint work with Larissa Richards (Toronto).

Site percolation on triangular lattice



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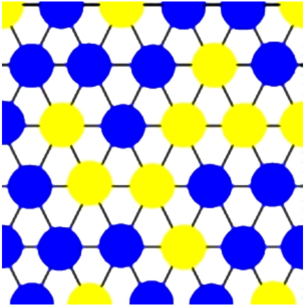

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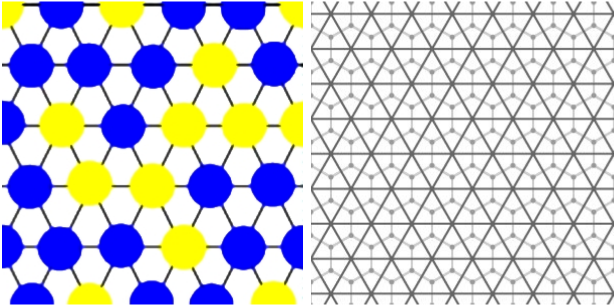

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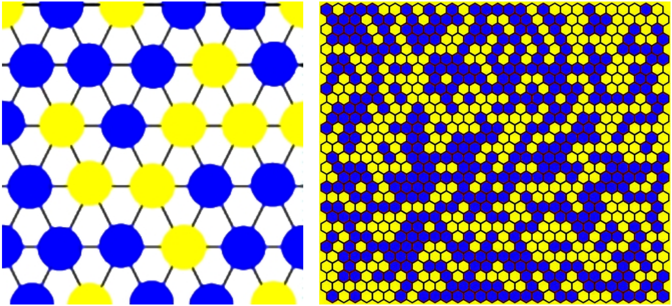
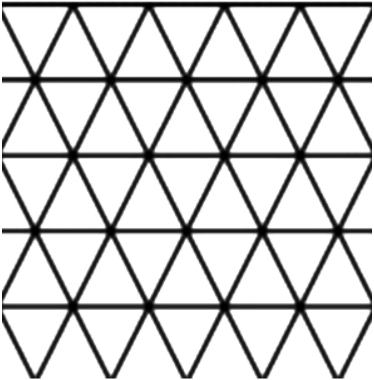


Bond percolation on triangular lattice



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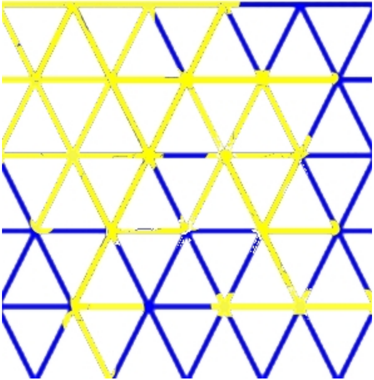

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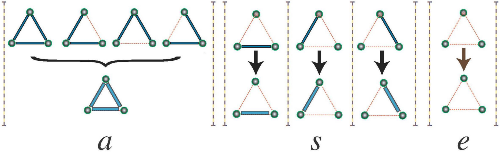
To hexagonal lattice: bond percolation

Interested only in connectivity properties, so can group triangles with the same connectivity:



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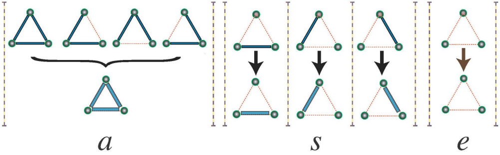

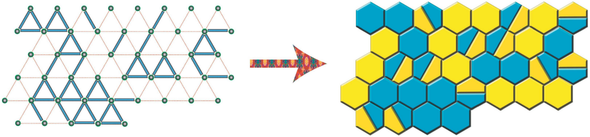

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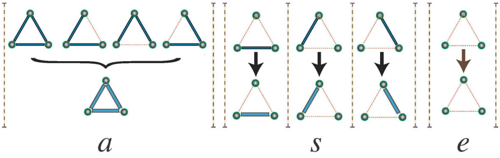
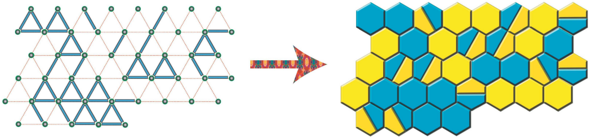
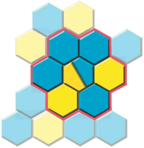




Not quite symmetric:

Modified bond percolation

Introduce flowers and irises.



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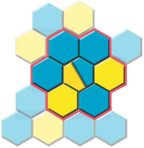

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Rules

1. Flowers are disjoint
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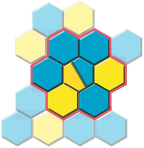

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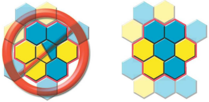

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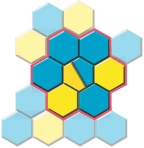

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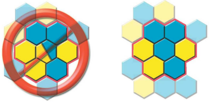

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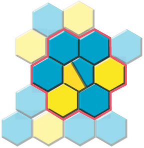

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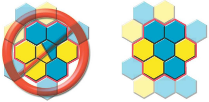

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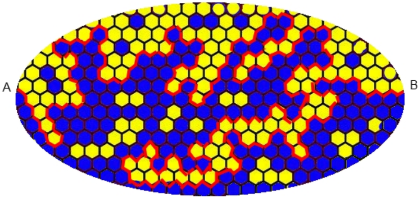
The last rule introduces local correlations. A generalization of site percolation.

Exploration process.

In a simply connected domain Ω with two points (or prime ends) A, B on the boundary, color all $1/n$-hexagons on $[AB]$ blue, on $[BA]$ yellow. The unique interface between yellow and blue, a random curve from A to B in Ω is called the *exploration process*.



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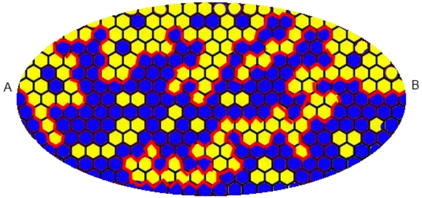

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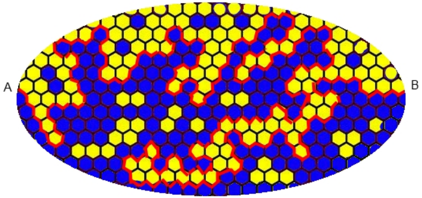

Converges to SLE_6 when $n \to \infty$. Proven for triangular site percolation (Smirnov, 2000) and the modified bond percolation, but only when $\mathrm{Mdim}(\partial\Omega) < 2$ (B-Chayes-Lei, 2010).

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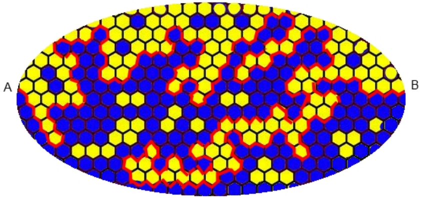

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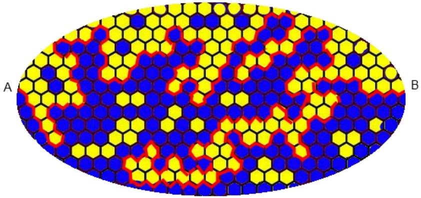

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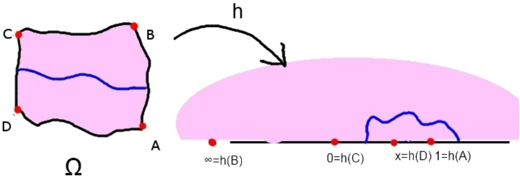
One should expect power rate of convergence ($n^{-\psi}$ for some $\psi > 0$).

Cardy's formula



Cardy (1992):

$$\lim_{n\to\infty} S_n(\Omega, A, B, C, D) = S_\infty(\Omega; A, B, C, D) = G(x) := \frac{\int_0^x (s(1-s))^{-2/3}\, ds}{\int_0^1 (s(1-s))^{-2/3}\, ds}$$

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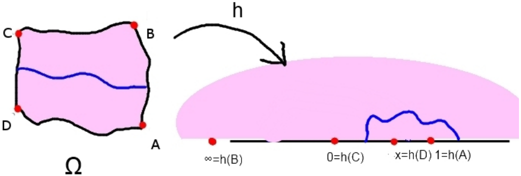

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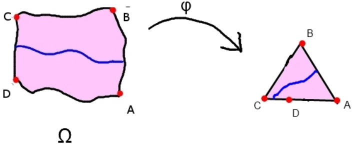

Carleson:

$$S_\infty(\Omega; A, B, C, D) := \frac{|CD|}{|AC|}$$

Cardy-Smirnov observable

Smirnov (2000): combinatorial description of the discrete approximation of the mapping φ_Ω to the standard equilateral triangle Δ with the vertices $\{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$.

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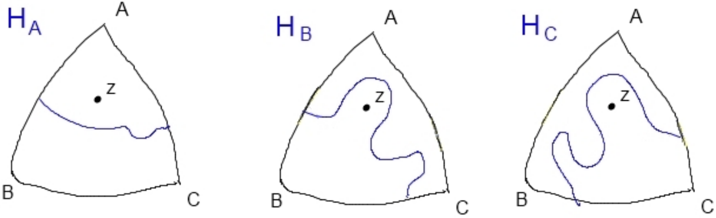

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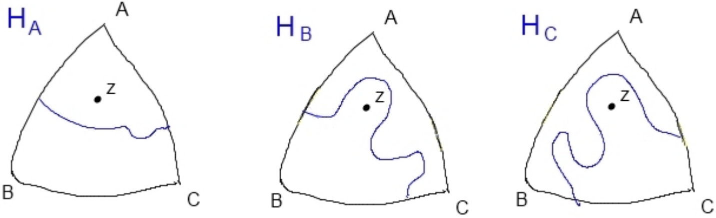

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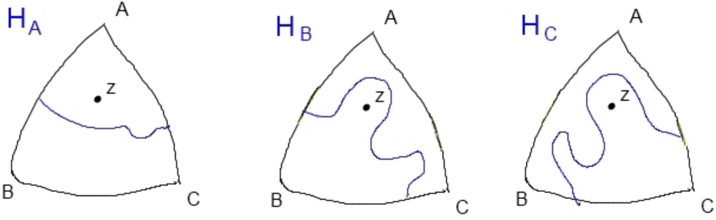

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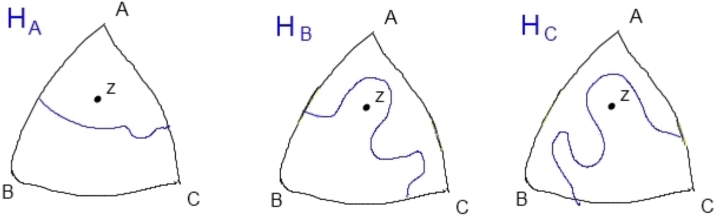

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Also works for the modified bond percolation when $\mathrm{Mdim}(\partial\Omega) < 2$ (B.-Chayes-Lei, 2010).

The rate of convergence for Cardy-Smirnov observable

Theorem (B.-Chayes-Lei, 2014).

Let Ω be a bounded simply connected domain, A, B, C, D be prime ends at the boundary. Then for some universal constant $\psi > 0$ and for large enough n,

$$|S_n(\Omega; A, B, C, D) - S_\infty(\Omega; A, B, C, D)| < n^{-\psi}.$$

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Mendelson, Nachmias, Watson (2014): the same result for the usual site percolation and for domains with piecewise-analytic boundary, but with any exponent $\psi < \frac{1}{6}$.

Sketch of a proof

Let Ω_n be a "good" $1/n$-hexagonal approximation of Ω, and the points A_n, B_n, C_n, and D_n are selected to be the conformally closest grid points to A, B, C, and D correspondingly. Let φ_n be the unique conformal map of Ω_n that sends A_n, B_n, C_n to the corresponding vertices of Δ.

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$$\oint_{\gamma_n} H_n(z)\,dz = \sum_{j=0}^{k} \frac{1}{2}\left(H_n(v_{j+1}) + H_n(v_j)\right)(v_{j+1} - v_j) = 0,$$

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and uniformly(in n) *Hölder continuous.*

Cauchy transform as a bridge between discrete and continuous worlds

In an ideal world, one can show that the Cauchy transform

$$F_n(z) := \frac{1}{2\pi i} \oint_{\partial\Omega_n} \frac{H_n(\xi)}{\xi - z}\, d\xi$$

is polynomially close to $H_n(z)$ *and* for z *inside* Ω_n, when

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Almost discrete analyticity and Hölder continuity

In reality:

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$$\oint_{\gamma_n} H_n(z)\,dz \leq Cn^{-\rho}\,\mathrm{length}(\gamma_n), \quad \rho > 0$$

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Cauchy transform

Step 1: Replacing a function H_n with its Cauchy integral extension

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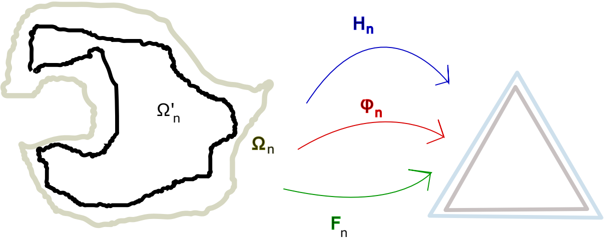

Analytic and percolation arguments

Step 2: Since in Ω'_n, F_n is power close to H_n, we can use a percolation argument (the so-called *Harris systems* on the level $N \gg n$) to show that Ω'_n is "close" to Ω in the conformal sense: $\partial\left(\varphi(\Omega'_n)\right)$ is n^{-b} close to $\partial\Delta$ in the supremum norm.

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Just use the lemma for g and $\Delta_1 = \varphi(\Omega_n')$ to show that uniformly

$$|F_n(z) - \varphi(z)| \le Cn^{-d}, \qquad z \in \Omega_n'.$$

The main statement for Percolation.

In the spirit of LERW result of Benes-Kozdron-Viklund(2013) and Viklund(2015):

Theorem (B-Richards, 2018).

There exists a $\rho > 0$ *such that for any domain* Ω *with two boundary prime ends* A, B *there exists a sequence of n-lattice approximations* (Ω_n, A_n, B_n) *and a coupling of the exploration process* γ_n *in* Ω_n *from* A_n *to* B_n *and* SLE_6*-curve* γ *in* Ω *from* A *to* B *which satisfy*

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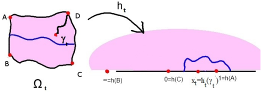
Where prime ends are a concern, a weighted sum of the distances within various regions defining the prime ends is considered.

Cardy-Smirnov observable determines the driving function.



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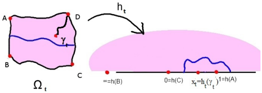

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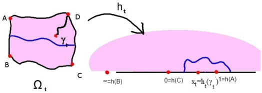

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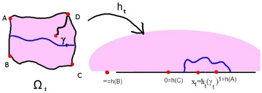

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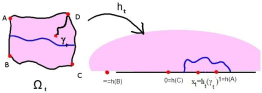

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This means that $\lambda_n(t)$ must have law of approximately $B(6t)$.

From the convergence of driving function to the convergence of interface

Tip structure modulus (Viklund) for a Löwner curve $\gamma(t)$, $t \leq T$, $\gamma(0) = D$ is defined as

$$\eta_{tip}(\delta) = \sup \operatorname{diam}(N_S),$$

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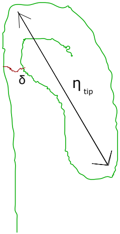

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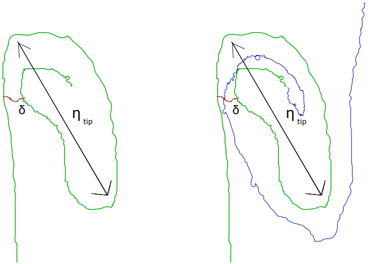

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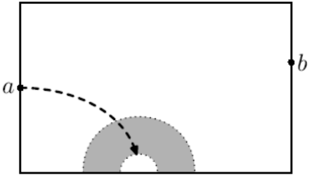
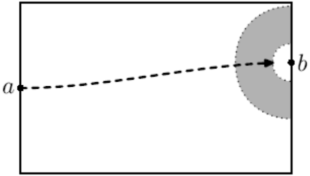
For percolation: condition 2 is essentially standard 6-arm exponent estimate, while condition 3 is needed only for SLE_6 and is established by Viklund (for any $\kappa < 8$).

Kemppainen-Smirnov Condition



Unforced crossing



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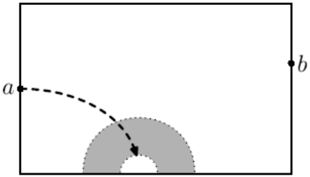

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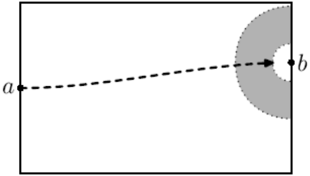


Forced crossing

Definition.

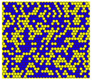
A family of random simple curves satisfy Kemppainen-Smirnov (KS) Condition if for some $C > 1$ *and for any random curve in the family and for any stopping time* τ*, for any annulus*

$$A(z_0, r, R) := \{z : r < |z - z_0| < R\}$$

with $r \geq Cr > 0$*, the probability of unforced crossing of* $A(z_0, r, R)$ *in* $\Omega \setminus \gamma[0, \tau]$ *is at most* $\frac{1}{2}$.

Models with KS condition.

KS condition is proved for:

- Critical site percolation on hexagonal lattice. 

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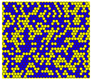
- Critical site percolation on hexagonal lattice. 
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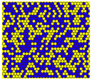
- Critical site percolation on hexagonal lattice. 
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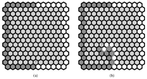
- Harmonic explorer. 

(a) (b)

KS condition and tip modulus.

Theorem. (Kemppainen-Smirnov, 2015)

Any family of random curves satisfying KS condition is tight, all the limit points lie on Hölder continuous Löwner curves and the corresponding driving functions have bounded exponential moments for some positive $\varepsilon > 0$.

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Suppose that a random family of curves satisfy Kemppainen-Smirnov Condition. Then, for some $\beta > 0$, $\varepsilon > 0$, $c > 0$, $C > 0$, *if* $r = cR^{1+\varepsilon}$ *then*

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Corollary.

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