A Proof of Alon's Second Eigenvalue Conjecture

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A Sampler...

Conj: $\forall \varepsilon > 0$, "most" d-regular graphs on n vertices have

$$\lambda_2 \leq 2\sqrt{d-1} + \varepsilon$$

for n large.

Thm: Model: $d/2$ permutations

"most" $= 1 - O_\varepsilon(n^{-\tau})$

$$\tau = \lceil (\sqrt{d-1} - 1)/2 \rceil$$

Thm 2: $d/2$ cyclic permutations (or d perf. match.)

$$\tau = \lceil \sqrt{d-1} \rceil - 1$$

N.B. τ is best, unless...

Trace Method [Wigner, Broder-Shamir, ...]

$$\sum_{i=1}^{n} \lambda_i^k = \mathrm{Tr}(A^k) = \#\text{ closed walks of length } k$$

Traces $\xrightarrow{[B\text{-}S]}$ Irreducible Traces

$\rightsquigarrow$ Selective Irred. Traces

$\rightsquigarrow$ Selective Strongly Irred. Traces

What do closed walks "look like"?

$\pi_1, \ldots, \pi_{d/2}$ random permutations

$$E[\#\text{fixed points } \pi_1 \pi_2 \pi_1^{-1} \pi_3] = 1$$

$$E[\text{'' '' '' } \pi_2 \pi_3^{-1} \pi_1 \pi_1] = 1$$

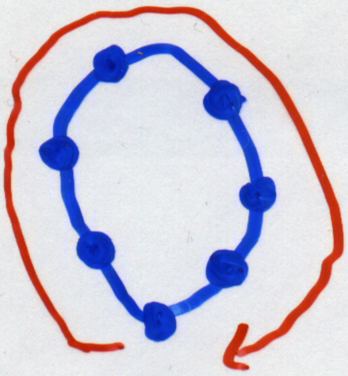
$$E[\text{'' '' '' } \pi_1 \pi_2 \pi_2^{-1} \pi_1^{-1}] = n$$

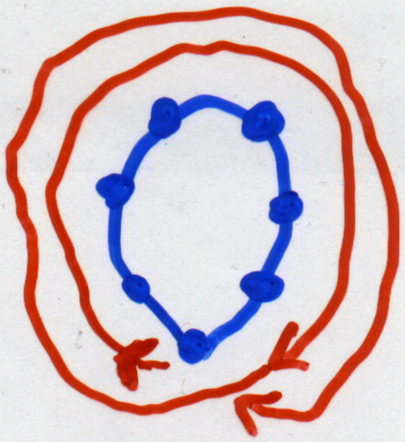
$$\pi_1 \pi_2 \pi_2^{-1} \pi_1^{-1} = \pi_1 \pi_1^{-1} = \text{id.}$$

$$E[\text{word}] = n\left[p_0(\text{word}) + \frac{p_1(\text{word})}{n} + \ldots\right]$$

$$E[\pi_1 \pi_1 \pi_1 \pi_1] = 3, \quad E[\pi_1^a \pi_2^b] = \ldots$$

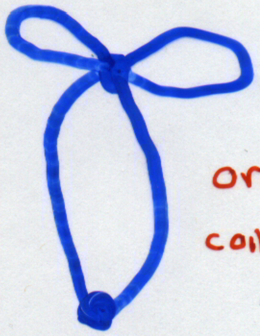
What do irreducible closed walks look like?





ord 0
coin 1

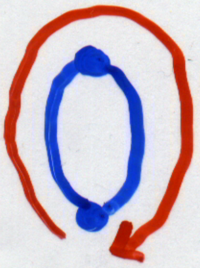
ord 1
coin 2

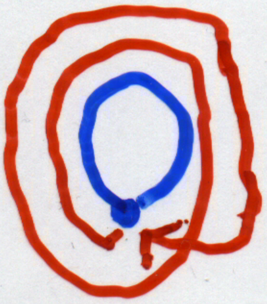


ord 2
coin 3

"Form," "type," "new type," ...

$\pi, \pi,$





Thm: $E\Big[\#$ irred closed walks, length k type  $\Big]$

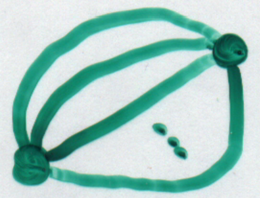
$$= f_0(k) + \frac{f_1(k)}{n} + \frac{f_2(k)}{n^2} + \cdots$$

$f_1, f_2, \ldots$ "d-Ramanujan"

$$f_i(k) = \text{poly}_i(k)\,(d-1)^k + \text{error}_i(k)$$

$$|\text{error}_i(k)| \le c\,k^c\,(d-1)^{k/2}$$

Problem: Not true for all types



Problem when type has too many closed, irreducible walks, i.e.

$$\lambda_{\text{Irred}} \gtrsim \sqrt{d-1}$$

e.g. $\lambda_{\text{Irred}}\Big($  $\Big) = 2$

① Such types key to τ s.t.

$\lambda_2 \leq 2\sqrt{d-1}+\varepsilon$ prob. $1 - O_\varepsilon(n^{-\tau})$

② Require "selective trace" for such types

(Almost) Matching Bounds:

 $\in G \implies$

$\lambda_2(G) \geq \lambda_1\Big($  $\Big) - o_n(1)$

d=7

Thm:

$$\lambda_1(\mathrm{Tree}_d(\Psi)) > 2\sqrt{d-1}$$

$$\Longleftrightarrow$$

$$\lambda_{\mathrm{Irred}}(\Psi) > \sqrt{d-1}$$

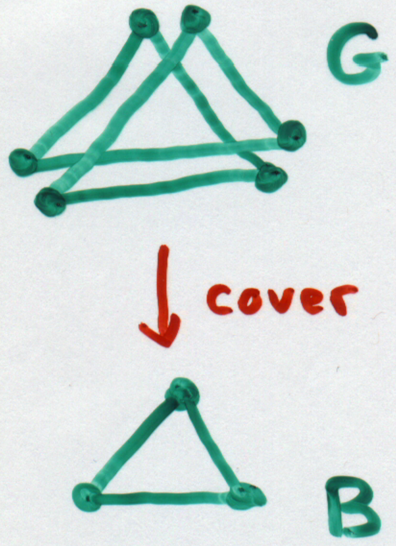
Further directions:

(0) Alternate method

(1) $\lambda_2 \leq 2\sqrt{d-1}$

(2) τ bound, critical tangles

(3) Relative Alon Conjecture:



$$\lambda_{\substack{\text{new,}\\ \text{max}}} \leq \rho + \varepsilon \ ?$$

$$\rho = \lambda_1 \left(\substack{\text{univ.}\\ \text{cover}} \ B \right)$$

Thm: $\lambda_{\substack{\text{new}\\ \text{max}}} \leq \sqrt{\lambda_1(B)\, \rho} + \varepsilon$