

A Proof of Alon's Second Eigenvalue Conjecture

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A Sampler...

Conj: $\forall \epsilon > 0$, "most" d -regular graphs on n vertices have

$$\lambda_2 \leq 2\sqrt{d-1} + \epsilon$$

for n large.

Thm: Model: $d/2$ permutations

$$\text{"most"} = 1 - O_\epsilon(n^{-\tau})$$

$$\tau = \lceil (\sqrt{d-1} - 1) / 2 \rceil$$

Thm 2: $d/2$ cyclic permutations

(or d perf. match.)

$$\tau = \lceil \sqrt{d-1} \rceil - 1$$

N.B. τ is best, unless...

Trace Method [Wigner, Broder-Shamir, ...]

$$\sum_{i=1}^n \lambda_i^k = \text{Tr}(A^k) = \text{\# closed walks of length } k$$

Traces $\xrightarrow{[B-S]}$ Irreducible Traces

$\xrightarrow{\quad}$ Selective Irred. Traces

$\xrightarrow{\quad}$ Selective Strongly Irred.

Traces

What do closed walks "look like"?

$\pi_1, \dots, \pi_{d/2}$ random permutations

$$E[\# \text{ fixed points } \pi_1 \pi_2 \pi_1^{-1} \pi_2^{-1}] = 1$$

$$E[" " " \pi_2 \pi_3 \pi_1 \pi_1^{-1}] = 1$$

$$E[" " " \pi_1 \pi_2 \pi_2^{-1} \pi_1^{-1}] = n$$

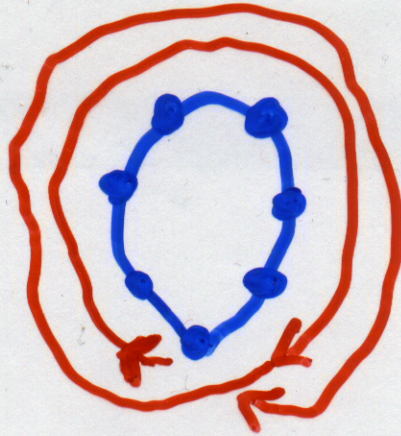
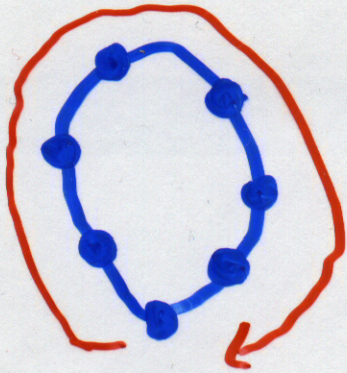
$$\pi_1 \pi_2 \pi_2^{-1} \pi_1^{-1} = \pi_1 \pi_1^{-1} = \text{id.}$$

$$E[\text{word}] = n \left[p_0(\text{word}) + \frac{p_1(\text{word})}{n} + \dots \right]$$

$$E[\pi_1 \pi_1 \pi_1 \pi_1] = 3, \quad E[\pi_1^a \pi_2^b] = \dots$$

closed

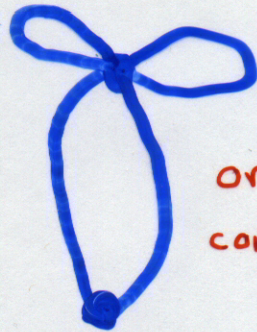
What do irreducible walks look like?



ord 0
coin 1



ord 1
coin 2



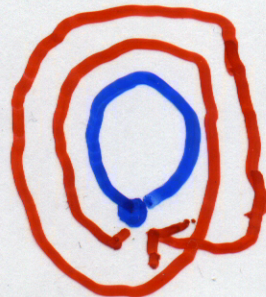
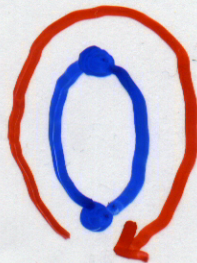
ord 2
coin 3

"Form,"

"type,"

"new type," ...

$\pi, \pi,$



Thm: $E[\# \text{ irred closed walks, length } k \text{ type } \textcircled{1}]$

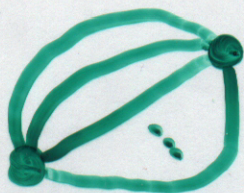
$$= f_0(k) + \frac{f_1(k)}{n} + \frac{f_2(k)}{n^2} + \dots$$

f_1, f_2, \dots "d-Ramanujan"

$$f_i(k) = \text{poly}_i(k) (d-1)^k + \text{error}_i(k)$$

$$|\text{error}_i(k)| \leq c k^c (d-1)^{k/2}$$

Problem: Not true for all types



Problem when type has too many closed, irreducible walks, i.e.

$$\lambda_{\text{Irred}} \geq \sqrt{d-1}$$

e.g. $\lambda_{\text{Irred}}(\text{circle}) = 2$

① Such types key to τ s.t.

$$\lambda_2 \leq 2\sqrt{d-1} + \epsilon \text{ prob. } 1 - O_\epsilon(n^{-\tau})$$

② Require "selective trace"

for such types

(Almost) Matching Bounds:



$$\lambda_2(G) \geq \lambda_1 \left(\begin{array}{c} \text{flower} \\ \vdots \\ \text{tree} \end{array} \right) - o_n(1)$$

$d=7 \rightarrow$

The diagram shows a tree structure with a root node at the top, connected to three loops. Below the root, there are several branches extending downwards, some ending in dots to indicate further structure. This is enclosed in large parentheses.

Thm:

$$\lambda_1(\text{Tree}_d(\Psi)) > 2\sqrt{d-1}$$



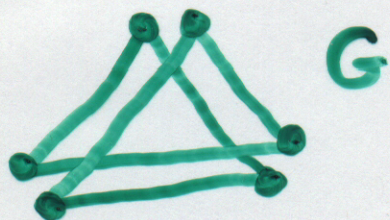
$$\lambda_{\text{Irred}}(\Psi) > \sqrt{d-1}$$

Further directions: (0) Alternate method

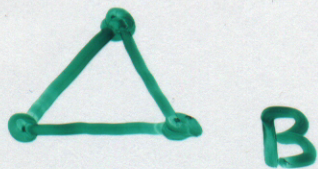
(1) $\lambda_2 \leq 2\sqrt{d-1}$

(2) τ bound, critical tangles

(3) Relative Alon Conjecture:



↓ cover



$$\lambda_{\text{new, max}} \leq \rho + \varepsilon ?$$

$$\rho = \lambda_1(\text{univ. cover } B)$$

Thm: $\lambda_{\text{new max}} \leq \sqrt{\lambda_1(B) \rho} + \varepsilon$