Construction of Irreducible Tempered Unitary Representations of the Free Group Using Vector-Valued Multiplicative Functions

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Notes available.

Throughout, let A_+ be a finite set of at least two elements and let Γ be the nonabelian free group generated by A_+. If $A_+ = \{\alpha, \beta\}$, a typical element of Γ would be:

$$\alpha\beta^{-1}\alpha\beta^{-1}\beta^{-1}\beta^{-1}\beta^{-1}\alpha^{-1}\beta$$

Throughout let $A \subset \Gamma$ be the set of generators and their inverses. If $A_+ = \{\alpha, \beta\}$, then $A = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$. The letters a and b will always stand for elements of A.

Let $e \in \Gamma$ denote the neutral element. Each element $x \in \Gamma$ has a unique expression as a **reduced word**:

$$x = a_1 a_2 \ldots a_n \qquad \text{with } a_j \in A,\ a_j a_{j+1} \neq e$$

For this x, let the **length** of x, denoted $|x|$, be n, the number of letters in its reduced word expression. Set $|e| = 0$.

The **Cayley graph** of Γ is defined as follows:

- There is a vertex for each $x \in \Gamma$.
- There is an edge for each pair (x, xa) with $x \in \Gamma$, $a \in A$.
- The pair of directed edges (x, xa), (xa, x) is considered the equivalent of a single undirected edge.

The Cayley graph of Γ is a **tree**.

The Cayley graph structure is defined using **right** multiplication by $a \in A$. Consequently, the action of Γ on itself by **left** translations preserves that structure.

Figure: The Cayley graph of Γ for the case $A = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$.

For $x \in \Gamma$, $d(e, x) = |x|$.

A **unitary representation** $(\pi_\Gamma, \mathcal{H})$ of Γ is a group homomorphism:

$$\pi_\Gamma : \Gamma \to \mathcal{U}(\mathcal{H})$$

into the group of unitary operators on $\mathcal{H}$.

A unitary representation $(\pi_\Gamma, \mathcal{H})$ of Γ is called **irreducible** if the only closed subspaces of $\mathcal{H}$ stable under the action of Γ are the zero subspace and $\mathcal{H}$ itself.

Two unitary representations π_Γ and $\pi_\Gamma^\sharp$ are called **equivalent** if there is a unitary Γ-map $J : \mathcal{H} \to \mathcal{H}^\sharp$.

A unitary Γ-representation π_Γ is called **tempered** if

$$\sum_{x\in\Gamma} e^{-\epsilon|x|}|\langle v, \pi_\Gamma(x)v\rangle|^2 < +\infty \qquad \text{for any } \epsilon > 0,\ v \in \mathcal{H}$$

Proposition: π_Γ is tempered $\iff$ π_Γ is weakly contained in the regular representation.

Observation: If $\sum_{x\in\Gamma} |\langle v, \pi_\Gamma(x)v\rangle|^2 < +\infty$, then π_Γ is actually contained in the regular representation, and so (see [Cecchini, Figà-Talamanca, 1974]) it cannot be irreducible.

Various examples of irreducible, tempered, unitary representations of Γ have been constructed over the years. The real difficulty is not in constructing the examples, but in proving them irreducibile. Here are some references:

[Yoshizawa, 1951], [Figà-Talamanca, Picardello, 1982, 1983], [Angelini, 1989], [Figà-Talamanca, Steger, 1994], [Kuhn, Steger, 1996], [Paschke, 2001], [Paschke, 2002].

We construct a family of representations which

- includes all of the examples in the above papers as special cases,
- allows a uniform proof of irreducibility,
- and a uniform proof of inequivalence between representations in the family.

Moreover

- although the construction makes use of the generating set A, the family of representation constructed doesn't depend on the choice of A.

It is impossible, **in principle**, to "list" the tempered unitary dual of Γ. (This is because $C^*_{\mathrm{red}}(\Gamma)$ is not a Type I algebra.)

Our construction covers all tempered unitary Γ-representations

- explictly constructed
- in the literature
- using "tree" methods
- which have been proved irreducible
- that I know of
- as of today, 10 February, 2004.

Notwithstanding all these caveats, the construction does cover an awful lot of representations, and does present points of interest quite apart from any sort of universality.

Definition: A **matrix system** (V_a, H_{ba}) consists of

- a finite-dimensional complex vector space V_a for each $a \in A$, and
- a linear map $H_{ba} : V_a \to V_b$ for each pair $a, b \in A$,
- where $H_{ba} = 0$ if $ab = e$.

An **invariant subsystem** of (V_a, H_{ba}) is a tuple (W_a) of linear subspaces $W_a \subseteq V_a$ such that $H_{ba} W_a \subseteq W_b$. A nonzero matrix system (V_a, H_{ba}) is **irreducible** if its only irreducible subsystems are the zero subsystem and the the full subsystem.

Equivalence of two matrix systems is defined as you would expect.

As input for the construction, start with an irreducible matrix system (V_a, H_{ba}). This will have to satisfy an additional condition ($\rho = 1$) to be explained later.

The space $\mathcal{H}^\infty$ of **(vector-valued) multiplicative functions** consists of all $f : \Gamma \to \coprod_a V_a$ satisfying, for some N,

- $f(xa) \in V_a$ whenever $|xa| = |x| + 1 \geq N + 1$, and
- $f(xab) = H_{ba} f(xa)$ whenever $|xab| = |x| + 2 \geq N + 2$.

We consider $f_1, f_2 \in \mathcal{H}^\infty$ to be equal if they agree except on some finite subset of Γ.

Γ acts on $\mathcal{H}^\infty$ by **left** translations. These preserve multiplicativity, which is defined in terms of **right** multiplication.

$$(\pi_\Gamma(y) f)(x) = f(y^{-1}x)$$

Figure: A schematic diagram of $f \in \mathcal{H}^\infty$

Actually, f is multiplicative only outside the ball of radius N.

Next, one wishes to put an **inner product** on $\mathcal{H}^\infty$. For this, one needs for each $a \in A$ a positive definite, sesquilinear inner product $B_a : V_a \times V_a \to \mathbf{C}$.

The desired definition of the inner product on $\mathcal{H}^\infty$ is:

$$\langle f_1, f_2 \rangle = \sum_{x,a,|x|=N,|xa|=N+1} B_a(f_1(xa), f_2(xa))$$

where we choose N large enough so that f_1 and f_2 are both multiplicative outside the ball of radius N.

In order that the definition be independent of N, it is necessary that for $a \in A$ and $v_1, v_2 \in V_a$:

$$B_a(v_1, v_2) = \sum_b B_b(H_{ba}v_1, H_{ba}v_2)$$

To express the same thing with some extra notation, define

$\mathcal{V} = \{B = (B_a)_a\,;\ B_a$ is a symmetric sesquilinear form on $V_a\}$

$\mathcal{P} = \{B \in \mathcal{V}\,;$ each B_a is positive semidefinite$\}$

and let $\mathcal{T} : \mathcal{V} \to \mathcal{V}$ be defined by

$$(\mathcal{T}B)_a(v_1, v_2) = \sum_b B_b(H_{ba}v_1, H_{ba}v_2)$$

$\mathcal{T}$ is a linear operator on the real vector space $\mathcal{V}$ and preserves the cone $\mathcal{P}$. Let $\rho = \rho(\mathcal{T})$ be the spectral radius of $\mathcal{T}$. By the Perron–Frobenius Theorem of [Vandergraft, 1968] there exists a tuple $B = (B_a)_a \in \mathcal{P}$, unique up to multiplication by a positive scalar, such that $\mathcal{T}B = \rho B$.

What we want is a $\mathcal{T}$-fixed tuple $(B_a)_a \in \mathcal{P}$. So we impose the condition $\rho = 1$. Then the tuple $B = (B_a)_a$ exists and is essentially unique.

Let $\mathcal{H}$ be the completion of $\mathcal{H}^\infty$ with respect to our inner product. For $y \in \Gamma$, $\pi_\Gamma(y)$ (and likewise $\pi_\Gamma(y^{-1})$) preserve the inner product on $\mathcal{H}^\infty$. Hence $\pi_\Gamma(y)$ can be extended to a unitary operator on $\mathcal{H}$. This completes the construction of $(\pi_\Gamma, \mathcal{H})$.

Results:

- π_Γ is always tempered.
- Generically, π_Γ is irreducible. In certain well-understood cases it is the direct sum of exactly **two** inequivalent irreducible representations.
- Generically, inequivalent matrix systems give rise to inequivalent unitary Γ-representations. In certain well-understood cases, exactly **two** (equivalence classes of) irreducible matrix systems give rise to the same Γ-representation.

A **semiinfinite geodesic** in (the Cayley graph of) Γ is a sequence of the form:

$$(x, xa_1, xa_1a_2, xa_1a_2a_3, \ldots)$$

for $a_j \in A$, $a_ja_{j+1} \neq e$.

One constructs a **boundary** of (the Cayley graph of) Γ by assigning an ideal limit point to each semiinfinite geodesic. By definition, two geodesics will have the same limit point if and only if they have a common final subgeodesic.

Figure: Two geodesics with the same ideal limit point

The boundary is the set of all such ideal limit points, and will be denoted Ω. Individual boundary points will be denoted ω.

Figure: The boundary, Ω, of (the Cayley graph of) Γ

The action of Γ on its Cayley graph by left translations induces an action of Γ on Ω.

For $y \in \Gamma$, we define

$$\Gamma(y) = \{x \in \Gamma\,;\ \text{the reduced word for } x \\ \text{starts with the reduced word for } y\}$$

$$\Omega(y) = \{\omega \in \Omega\,;\ \omega \text{ is the limit point} \\ \text{of some geodesic lying in } \Gamma(y)\}$$

Figure: $\Gamma(y)$ and $\Omega(y)$

Use the sets $\Omega(y)$ as a basis for the topology on Ω. For any N, $\Omega = \coprod_{|y|=N} \Omega(y)$. Consequently, each $\Omega(y)$ is open/closed.

This definition makes Ω a compact, perfect, totally disconnected, metrizable space — Ω is homeomorphic to the Cantor set. The action of Γ on Ω is by homeomorphisms.

By $C(\Omega)$ is meant the space of continuous functions from Ω to $\mathbf{C}$. This is a commutative C*-algebra under pointwise addition, multiplication, and conjugation.

The left action of Γ on $C(\Omega)$ is $\lambda : \Gamma \to \mathrm{Aut}(C(\Omega))$ defined by:

$$(\lambda(x)F)(\omega) = F(x^{-1}\omega) \qquad \text{for } x \in \Gamma,\ F \in C(\Omega)$$

Recall that $\mathcal{H}$ denotes the representation space of π_Γ. Define a $*$-algebra homomorphism $\pi_\Omega : C(\Omega) \to \mathcal{L}(\mathcal{H})$ starting with

$$(\pi_\Omega(\mathbf{1}_{\Omega(y)})f)(x) = \mathbf{1}_{\Gamma(y)}(x)f(x)$$

and then using linearity and continuity.

The spectral theorem says that this (or any) $*$-action of $C(\Omega)$ on $\mathcal{H}$ can be obtained by identifying $\mathcal{H}$ with an L^2-space on Ω (or a direct sum of L^2-spaces) and having $C(\Omega)$ act by pointwise multiplication.

The triple $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ satisfies the following

Definition: A **boundary representation** (i.e., representation of $\Gamma \ltimes_\lambda C(\Omega)$) is a triple $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ satisfying:

- $\mathcal{H}$ is a Hilbert space,
- $\pi_\Gamma : \Gamma \to \mathcal{U}(\mathcal{H})$ is a group homomorphism.
- $\pi_\Omega : C(\Omega) \to \mathcal{L}(\mathcal{H})$ is a $*$-algebra homomorphism.
- For $x \in \Gamma$ and $F \in C(\Omega)$

$$\pi_\Gamma(x)\pi_\Omega(F)\pi_\Gamma(x)^{-1} = \pi_\Omega(\lambda(x)F)$$

A boundary representation is **irreducible** if the only closed subspaces of $\mathcal{H}$ invariant under both π_Γ and π_Ω are 0 and all of $\mathcal{H}$. Two boundary representations $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ and $(\pi^\sharp_\Gamma, \pi^\sharp_\Omega, \mathcal{H}^\sharp)$ are **equivalent** if there exists a unitary map $J : \mathcal{H} \to \mathcal{H}^\sharp$ which is both a Γ-map and a $C(\Omega)$-map.

Proposition: For any boundary representation $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ the Γ-representation π_Γ is tempered. Conversely, given any tempered π_Γ, there exists a boundary representation $(\pi_\Gamma^\sharp, \pi_\Omega^\sharp, \mathcal{H}^\sharp)$ such that π_Γ occurs as a subrepresentation of $\pi_\Gamma^\sharp$.

$\Longrightarrow$: See [Quigg, Spielberg, 1992] or [Kuhn, Steger, 1996, Section 2].

$\Longleftarrow$: As observed by **Eliot Gootman**, this is a consequence of the fact that the map

$$C^*_{\mathsf{red}}(\Gamma) \to \Gamma \ltimes C(\Omega)$$

exists and is an inclusion. That it exists follows from the first implication. It must then be an inclusion since $C^*_{\mathsf{red}}(\Gamma)$ is simple (see [Powers, 1975]).

Theorem:

- If (V_a, H_{ba}) is an irreducible matrix system with $\rho = 1$, then $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ is an irreducible boundary representation.

- If (V_a, H_{ba}) and $(V_a^\sharp, H_{ba}^\sharp)$ are irreducible and inequivalent with $\rho = \rho^\sharp = 1$, then $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ and $(\pi_\Gamma^\sharp, \pi_\Omega^\sharp, \mathcal{H}^\sharp)$ are inequivalent as boundary representations.

This theorem is weaker (and easier to prove) than the analogous results for the Γ-representations π_Γ and $\pi_\Gamma^\sharp$. But it is a crucial ingredient in the proofs of the stronger results.

Idea of **Figà-Talamanca** ($\approx$ 1980): it's hard to prove irreducibility for Γ-representations because Γ lacks big compact subgroups K which you can average over. One should use Ω, out at infinity, as a replacement for K.

Starting with the matrix system (V_a, H_{ba}), there is a recipe (a formula) for calculating a certain "transition matrix". What's really important is the generalized 1-eigenspace of the transition matrix, which is of dimension 2 or 4. Based on the dimension and structure of that eigenspace, you can assign the matrix system to one of 3 cases. The properties of π_Γ (and the proofs of those properties) depend on which case you're in.

Case 1 (generic) — **Monotony**

In this case

- π_Γ is irreducible.
- Suppose $(V_a^\sharp, H_{ba}^\sharp)$ is an irreducible matrix system, not equivalent to (V_a, H_{ba}), with $\rho^\sharp = 1$. Then π_Γ and $\pi_\Gamma^\sharp$ are not equivalent as Γ-representations.

Moreover, π_Γ satisfies monotony, defined as follows:

Definition: An irreducible, tempered, unitary Γ-representation π_Γ is called **monotonous** when

- π_Γ can be extended to a boundary representation $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$, and
- if $(\pi^\sharp_\Gamma, \pi^\sharp_\Omega, \mathcal{H}^\sharp)$ is a boundary representation, and if $J : \mathcal{H} \to \mathcal{H}^\sharp$ is a Γ-map, then J is also a $C(\Omega)$-map.

A monotonous representation π_Γ determines its own π_Ω.

Moreover, for some constant $C_0 > 0$ and for $\alpha = 2$ or $\alpha = 3$ the following holds for all $f_1, f_3 \in \mathcal{H}$ and $f_2, f_4 \in \mathcal{H}^\infty$:

$$\lim_{\epsilon \to 0+} \epsilon^\alpha \sum_{x \in \Gamma} e^{-\epsilon|x|} \langle f_1, \pi_\Gamma(x) f_2 \rangle \overline{\langle f_3, \pi_\Gamma(x) f_4 \rangle} = C_0 \langle f_1, f_3 \rangle \overline{\langle f_2, f_4 \rangle}$$

The value of α depends only on the structure of the generalized 1-eigenspace of the transition matrix. $\alpha = 2$ is generic.

Moreover, for any $y \in \Gamma$, and for all $f_1, f_3 \in \mathcal{H}$ and $f_2, f_4 \in \mathcal{H}^\infty$

$$\lim_{\epsilon \to 0+} \epsilon^\alpha \sum_{x \in \Gamma(y)} e^{-\epsilon|x|} \langle f_1, \pi_\Gamma(x) f_2 \rangle \overline{\langle f_3, \pi_\Gamma(x) f_4 \rangle}$$

$$= C_0 \langle \pi_\Omega(\mathbf{1}_{\Omega(y)}) f_1, f_3 \rangle \overline{\langle f_2, f_4 \rangle}$$

Using this formula one can calculate π_Ω in terms of π_Γ.

Further, for any $z \in \Gamma$

$$\lim_{\epsilon \to 0+} \epsilon^\alpha \sum_{x \in \Gamma(y), x^{-1} \in \Gamma(z)} e^{-\epsilon|x|} \langle f_1, \pi_\Gamma(x) f_2 \rangle \overline{\langle f_3, \pi_\Gamma(x) f_4 \rangle}$$

$$= C_0 \langle \pi_\Omega(\mathbf{1}_{\Omega(y)}) f_1, f_3 \rangle \overline{\langle \pi_\Omega(\mathbf{1}_{\Omega(z)}) f_2, f_4 \rangle}$$

Case 2 — **Duplicity**

Define the matrix system $(\hat{V}_a, \hat{H}_{ba})$ by

$$\hat{V}_a = V^*_{a^{-1}} \qquad \text{the space of antilinear maps } V_{a^{-1}} \to \mathbf{C}$$

$$\hat{H}_{ba} = H^*_{a^{-1}b^{-1}}$$

From $\rho = 1$ it follows easily that $\hat{\rho} = 1$.

In this case (V_a, H_{ba}) and $(\hat{V}_a, \hat{H}_{ba})$ are inequivalent matrix systems and

- π_Γ is irreducible.
- π_Γ and $\hat{\pi}_\Gamma$ are equivalent Γ-representations.
- Suppose $(V^\sharp_a, H^\sharp_{ba})$ is an irreducible matrix system, not equivalent to either (V_a, H_{ba}) or $(\hat{V}_a, \hat{H}_{ba})$, with $\rho^\sharp = 1$. Then π_Γ and $\pi^\sharp_\Gamma$ are not equivalent as Γ-representations.

Moreover, π_Γ satisfies duplicity, defined as follows:

Definition: An irreducible, tempered, unitary Γ-representation π_Γ is called **duplicitous** when

- there are two different extensions of π_Γ to a boundary representation, $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ and $(\pi_\Gamma, \hat{\pi}_\Omega, \mathcal{H})$,
- if $(\pi_\Gamma^\sharp, \pi_\Omega^\sharp, \mathcal{H}^\sharp)$ is a boundary representation, and if $J : \mathcal{H} \to \mathcal{H}^\sharp$ is a Γ-map, then J factors as follows:

$$\mathcal{H} \xrightarrow{\quad \Delta \quad} \mathcal{H} \oplus \mathcal{H} \xrightarrow{\quad \tilde{J} \quad} \mathcal{H}^\sharp$$

  where Δ is the diagonal map, $f \mapsto (f, f)$, and where $\tilde{J}$ is both a Γ-map and a $C(\Omega)$-map from $(\pi_\Gamma, \pi_\Omega, \mathcal{H}) \oplus (\pi_\Gamma, \hat{\pi}_\Omega, \mathcal{H})$ to $(\pi_\Gamma^\sharp, \pi_\Omega^\sharp, \mathcal{H}^\sharp)$.

A duplicitous representation π_Γ admits **exactly two** possibilities for π_Ω.

Moreover, for some constant $C_0 > 0$ the following holds for all $f_1, f_3 \in \mathcal{H}$ and $f_2, f_4 \in \mathcal{H}^\infty$:

$$\lim_{\epsilon \to 0+} \epsilon \sum_{x \in \Gamma} e^{-\epsilon |x|} \langle f_1, \pi_\Gamma(x) f_2 \rangle \overline{\langle f_3, \pi_\Gamma(x) f_4 \rangle} = C_0 \langle f_1, f_3 \rangle \overline{\langle f_2, f_4 \rangle}$$

Note that the factor in front is ϵ, while in the previous case it was ϵ^2 or ϵ^3. So the matrix coefficients in this case are "smaller" at infinity than the matrix coefficients in the monotonous case.

Case 3 — **Oddity**

In this case (V_a, H_{ba}) and $(\hat{V}_a, \hat{H}_{ba})$ are equivalent matrix systems and

- π_Γ is the direct sum of **exactly two** irreducible, inequivalent Γ-representations, π'_Γ and π''_Γ, where $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$.

- Suppose $(V_a^\sharp, H_{ba}^\sharp)$ is an irreducible matrix system, not equivalent to (V_a, H_{ba}). Then neither π'_Γ nor π''_Γ is equivalent to any direct summand of $\pi_\Gamma^\sharp$.

Moreover, π'_Γ satisfies oddity, defined as follows:

Definition: An irreducible, tempered, unitary Γ-representation π'_Γ is called **odd** when

- there is a second, inequivalent, irreducible, tempered Γ-representation π''_Γ and an extension of $\pi'_\Gamma \oplus \pi''_\Gamma$ to a boundary representation, $(\pi'_\Gamma \oplus \pi''_\Gamma, \pi_\Omega, \mathcal{H}' \oplus \mathcal{H}'')$,
- if $(\pi^\sharp_\Gamma, \pi^\sharp_\Omega, \mathcal{H}^\sharp)$ is a boundary representation, and if $J : \mathcal{H}' \to \mathcal{H}^\sharp$ is a Γ-map, then J factors as follows:

  $$\mathcal{H}' \xrightarrow{\ \iota_1\ } \mathcal{H}' \oplus \mathcal{H}'' \xrightarrow{\ \tilde{J}\ } \mathcal{H}^\sharp$$

  where ι_1 is the inclusion, $f \mapsto (f, 0)$, and where $\tilde{J}$ is both a Γ-map and a $C(\Omega)$-map from $(\pi'_\Gamma \oplus \pi''_\Gamma, \pi_\Omega, \mathcal{H}' \oplus \mathcal{H}'')$ to $(\pi^\sharp_\Gamma, \pi^\sharp_\Omega, \mathcal{H}^\sharp)$,
- and similarly for any Γ-map $J : \mathcal{H}'' \to \mathcal{H}^\sharp$.

Of course π''_Γ is also odd.

An odd representation π'_Γ can be extended to a boundary representation only after one puts it together with its "twin" π''_Γ. The extension is then unique.

Duplicity Conjecture: If π_Γ is any irreducible, unitary, tempered Γ-representation then π_Γ is either monotonous, or duplicitous, or odd.

The evidence is strictly heuristic. There are many examples where one can prove monotony, duplicity, or oddity. This includes, but is not restricted to, the representations constructed here. There are certain other examples where one suspects monotony, duplicity, or oddity, but cannot prove it. No plausible counterexample is known.

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