# $\tilde{A}_n$ groups & their applications

Partly: joint work with Patrick Solé & Andrzej Żuk

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# $\tilde{\mathrm{A}}_{\mathrm{n}}$ buildings.

Let $F$ be a local field with valuation ring $\mathcal{O}$ and residue field $\mathcal{O}/\varpi\mathcal{O} \cong \mathbb{F}_q$. Let $\mathrm{ord} : \mathbb{F}^{\times} \to \mathbb{Z}$ be the valuation on $F$. For example, $F = \mathbb{F}_q((X))$ or $F = \mathbb{Q}_p$.

For each integer $n \geq 1$ there is a building of type $\tilde{A}_n$ associated with $F$.

Its vertices are the lattice classes $[L]$, where $L \subset F^{n+1}$ is an $\mathcal{O}$-lattice, and where $L \sim L'$ if $L' = tL$ for some $t \in F^{\times}$.

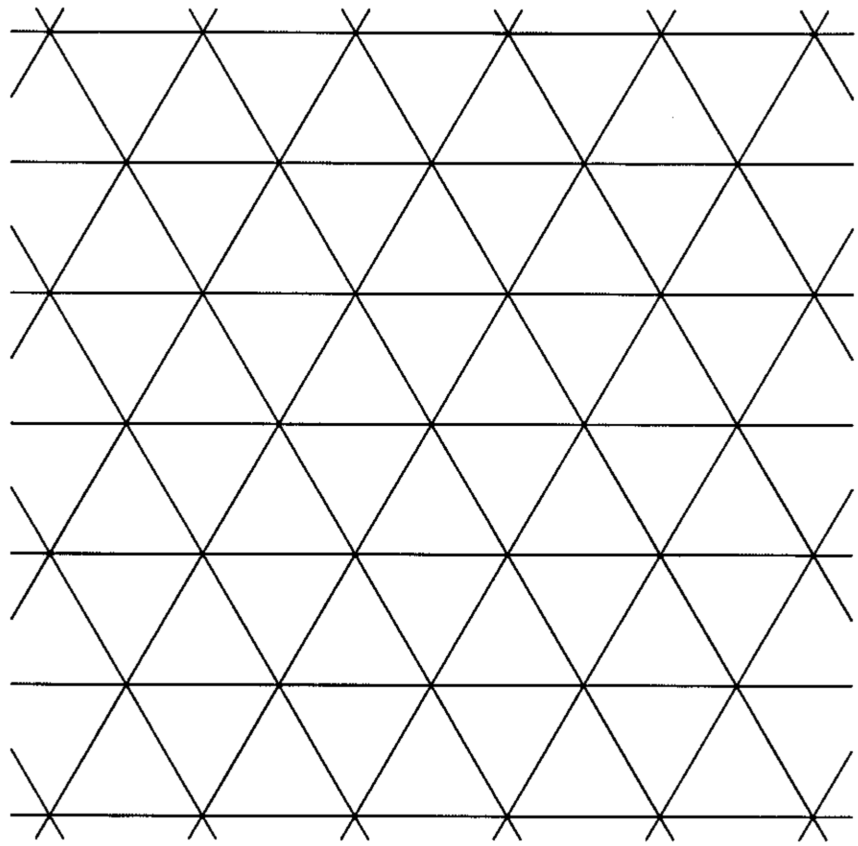
- $[L]$ is **adjacent** to $[L']$ if $\varpi L \subsetneq L' \subsetneq L$.
- **A simplex** consists of a set of pairwise adjacent vertices.
- If $n = 1$, the building is a homogeneous tree; each vertex has valency $q + 1$.

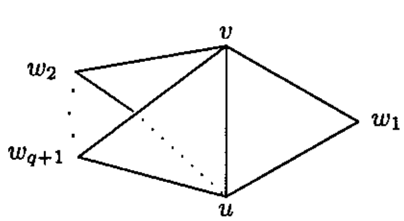
**Notation.** Throughout this talk, let $d = n + 1$. It's easiest to forget about $n$ altogether. The building we are talking about is denoted $\mathcal{B}_{d,F}$, but it is of type $\tilde{A}_{d-1}$.

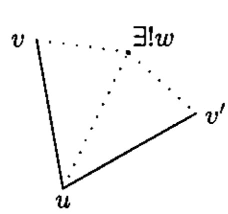
# $\tilde{\mathrm{A}}_{\mathrm{d}-1}$ buildings (cont).

- The neighbours of $[L]$ correspond to nonzero proper subspaces of $\mathbb{F}_q^d$. A neighbour $[L']$ corresponds to $L'/\varpi L \subset L/\varpi L \cong \mathbb{F}_q^d$.
- The simplexes containing $[L]$ correspond to flags of subspaces of $\mathbb{F}_q^d$: $V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{F}_q^d$.
- Maximal simplexes (chambers) have $d$ vertices.
- Each lattice can be written $g(L_0)$ for some $g \in GL(d, F)$, where $L_0 = \mathcal{O}^d$.
- Each vertex $v$ has a **type** $\tau(v) \in \mathbb{Z}/d\mathbb{Z}$, so that each chamber has one vertex of each type. The type of $[g(L_0)]$ is $\mathrm{ord}(\det(g))$ (mod $d$).
- $G = PGL(d, F)$ acts transitively on the vertex set of $\mathcal{B}_{d,F}$.
- The stabilizer of $[L_0]$ is $K = PGL(d, \mathcal{O})$.

**Some features of an $\tilde{A}_2$ building.**





## Adjacency operators on an $\tilde{\mathrm{A}}_{\mathrm{d}-1}$ building.

For functions $f$ defined on the vertex set of the building $\mathcal{B} = \mathcal{B}_{d,F}$, we can define $d-1$ adjacency operators $A_1, \dots, A_{d-1}$ by

$$(A_k f)(v) = \sum_w f(w)$$

where the sum is over the neighbours $w$ of $v$ such that $\tau(w) = \tau(v) + k$ (mod $d$).

The number of terms in the sum is $q$ binomial coefficient

$$\begin{bmatrix} d \\ k \end{bmatrix}_q = \frac{(q^d-1)(q^{d-1}-1)\cdots(q-1)}{(q^k-1)\cdots(q-1)\cdot(q^{d-k}-1)\cdots(q-1)}$$

(the number of subspaces of $\mathbb{F}_q^d$ of dimension $k$).

## Properties of the $\mathrm{A_k}$'s.

$\mathcal{A} \to \mathbb{C}$

- The $A_k$'s commute, and generate an algebra $\mathcal{A}$ isomorphic to the convolution algebra $\mathcal{C}_c(K\backslash G/K)$.

- Acting on $\ell^2(\mathcal{B}^0)$ (where $\mathcal{B}^0$ is the vertex set), the $A_k$'s are bounded operators, and the adjoint of $A_k$ is $A_{d-k}$. So $\mathcal{A} \subset \mathcal{L}(\ell^2(\mathcal{B}^0))$.

- The closure of $\mathcal{A}$ in $\mathcal{L}(\ell^2(\mathcal{B}^0))$ is a commutative $C^*$-algebra, with spectrum

$$\{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_j|\text{'s} = 1 \text{ and } z_1 z_2 \cdots z_d = 1\}/S_d.$$

- So $\bar{A}$ is isomorphic to the space of symmetric continuous functions on

$$\{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_j|\text{'s} = 1 \text{ and } z_1 z_2 \cdots z_d = 1\}.$$

# Spectrum of the $\mathrm{A_k}$'s.

Under this isomorphism, $A_k$ corresponds to

$$q^{k(d-k)/2}\sigma_k(z_1,\ldots,z_d),$$

where $\sigma_k$ is the $k$-th elementary symmetric polynomial in $z_1,\ldots,z_d$.

The spectrum $\mathrm{Spec}(A_k)$ of $A_k$ is the range of this function.

**Example.** If $d=2$, $A_1$ has spectrum equal to

$$\begin{aligned}&\{q^{1/2}(z_1+z_2) : |z_1|=|z_2|=1 \text{ and } z_1z_2=1\}\\ &=\{q^{1/2}(e^{i\theta}+e^{-i\theta}) : \theta\in\mathbb{R}\}\\ &=[-2\sqrt{q},2\sqrt{q}].\end{aligned}$$

# Spectrum of the $\mathrm{A_k}$'s (cont).

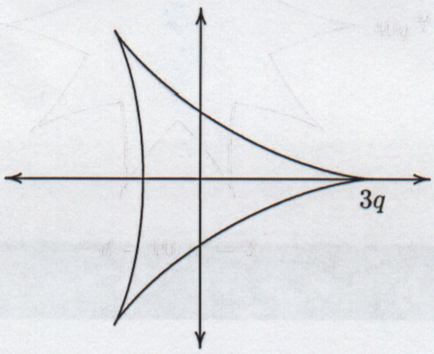
If $d = 3$, then $A_1$ has spectrum

$$\{q\big(e^{i\theta_1} + e^{i\theta_2} + e^{-i(\theta_1+\theta_2)}\big) : \theta_1, \theta_2 \in \mathbb{R}\}.$$

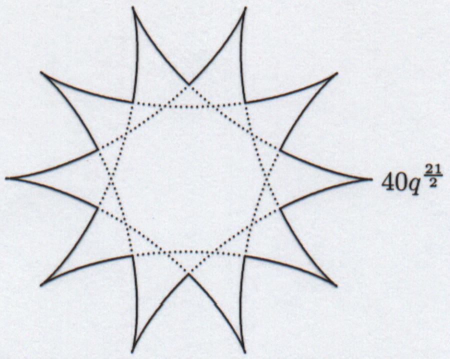
This is the region bounded by the hypocycloid $\{q(2e^{i\theta} + e^{-2\theta}) : \theta \in \mathbb{R}\}$.

The spectrum of $A_2 = A_1^*$ is the same.

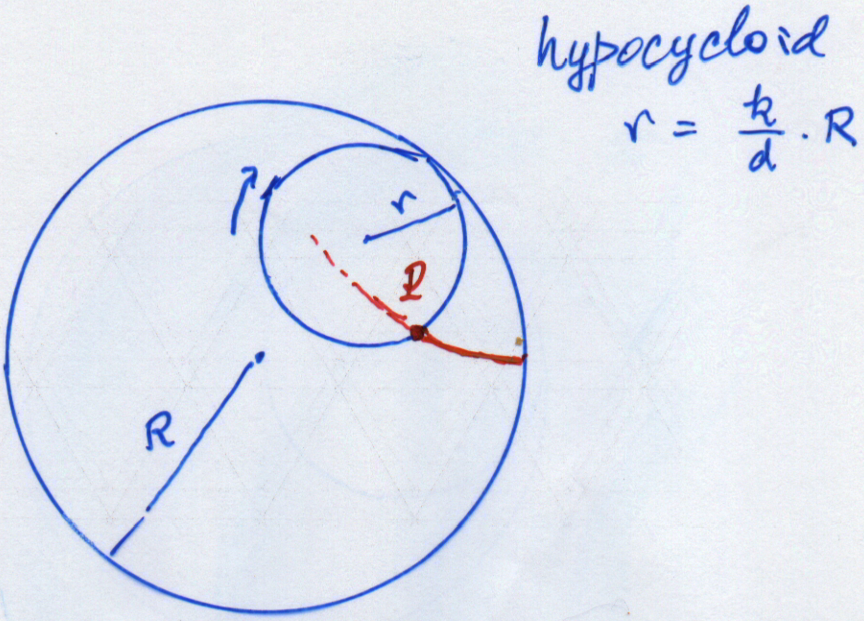
For larger $d$ and $k$, $\mathrm{Spec}(A_k)$ has a more complicated appearance (see Cartwright/Steger. Canadian J. Math 2002).




The case $d = 3$, $k = 1$ or $k = 2$.




$d = 10$, $k = 3$



## Ramanujan complexes of type $\tilde{\mathrm{A}}_{\mathrm{d-1}}$.

Let $\mathcal{B} = \mathcal{B}_{d,F}$ and let $\Gamma$ be a discrete co-compact subgroup of $G$. Then $\Gamma\backslash\mathcal{B}$ is finite, and we can define adjacency operators $A'_k$ for functions on $\Gamma\backslash\mathcal{B}^0$:

$$\big(A'_k f\big)(\Gamma v) = \sum f(\Gamma w),$$

the sum over the set of neighbours of $v$ such that $\tau(w) = \tau(v) + k$ (mod $d$).

The constant function $f \equiv 1$ is an eigenfunction for the eigenvalue

$$\begin{bmatrix} d \\ k \end{bmatrix}_q$$

of $A'_k$. If $\tau'(\Gamma v) = \tau(v)$ well-defines a type map on $\Gamma\backslash\mathcal{B}^0$, then the numbers

$$e^{2\pi ijk/d}\begin{bmatrix} d \\ k \end{bmatrix}_q$$

($j = 0, \ldots, d-1$) are also eigenvalues for $A'_k$ — the **trivial eigenvalues**.

Following Lubotzky, Samuels and Vishne, call $\Gamma\backslash\mathcal{B}$ a **pseudo-Ramanujan** complex if for each $k$, the eigenvalues of $A'_k$ are either trivial or in $\mathrm{Spec}(A_k)$.

Again following [LSV], call $\Gamma\backslash\mathcal{B}$ a **Ramanujan** complex if for any non-trivial **joint eigenfunction** $f$ of $A'_1, \ldots, A'_{d-1}$, with $A'_1 f = \lambda_1 f, \ldots, A'_{d-1} f = \lambda_{d-1} f$, there exist $z_1, \ldots, z_d \in \mathbb{C}$ of modulus 1 so that $z_1 z_2 \cdots z_d = 1$ and

$$\lambda_k = q^{k(d-k)/2} \sigma_k(z_1, \ldots, z_d) \quad \text{for } k = 1, \ldots, n.$$

For $d = 2$, the trivial eigenvalues are $\pm(q+1)$, and "pseudo-Ramanujan" $=$ "Ramanujan".

For $d = 3$, the trivial eigenvalues are $(q^2 + q + 1)e^{2\pi j/3}$, $j = 0, 1, 2$. Again "pseudo-Ramanujan" $=$ "Ramanujan" because $A'_2$ is the adjoint of $A'_1$.

# Motivation from representation theory.

If $z_1, \ldots, z_d \in \mathbb{C}$ have modulus 1 and $z_1 z_2 \cdots z_d = 1$, then we get a unitary character on the group of upper triangular matrices in $GL(d, F)$

$$\begin{pmatrix} a_1 & * & \cdots & * \\ 0 & a_2 & \cdots & * \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & a_d \end{pmatrix} \mapsto z_1^{\mathrm{ord}(a_1)} z_2^{\mathrm{ord}(a_2)} \cdots z_d^{\mathrm{ord}(a_d)}.$$

This is trivial on the scalar matrices $\lambda I$, and so gives a unitary character $\chi$ on the image $B$ in $G = PGL(d, F)$ of the upper triangulars. Then

$$\pi = \mathrm{Ind}_B^G \chi \qquad \text{(unitary induction)}$$

is an irreducible unitary representation of $G$ having a non-zero $K$-fixed vector. These representations are called the **principal series spherical representations**.

We also have $d$ characters of $G$ which factor through the map

$$gZ \mapsto \mathrm{ord}(\det(g)) \pmod d$$

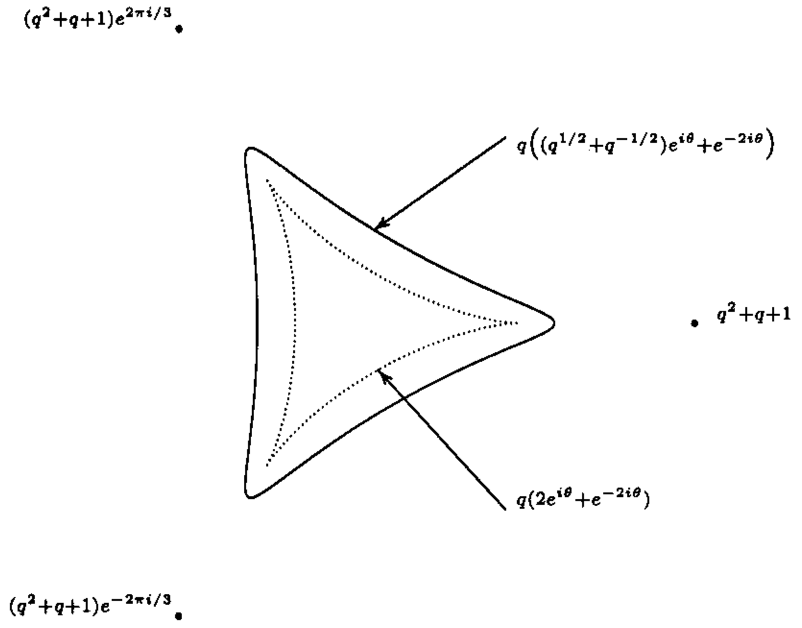
from $G$ to $\mathbb{Z}/d\mathbb{Z}$. All are trivial on $K$.

If $f$ is a simultaneous eigenfunction for the $A'_k$'s on $\Gamma\backslash\mathcal{B}$, then $f$ gives rise to an irreducible subrepresentation $\pi$ of $L^2(\Gamma\backslash G)$ which is spherical (i.e., having a non-zero $K$-fixed vector). So $\Gamma\backslash\mathcal{B}$ being Ramanujan just means that $\pi$ must be one of these characters or a principal series representation.

There are other spherical representations of $G$ — the complementary series representations.

If $d = 2$, these are indexed by $(-(q+1), (q+1)) \setminus [-2\sqrt{q}, 2\sqrt{q}]$.

If $d = 3$, they are indexed by the $z \in \mathbb{C}$ which are on or inside the curve $q\big((q^{1/2} + q^{-1/2})e^{i\theta} + e^{-2i\theta}\big)$ and outside the hypocycloid $q(2e^{i\theta} + e^{-2i\theta})$.




$d = 3$. Parameter sets for the principal and complementary series representations.

# $\tilde{\mathbf{A}}_{\mathbf{d-1}}$ groups

An $\tilde{\mathbf{A}}_{\mathbf{d-1}}$ **group** is a subgroup $\Gamma_0$ of $G = PGL(d, F)$ which acts **simply transitively** on the set of vertices of $\mathcal{B}_{d,F}$. This means that $\Gamma_0 \cap K = \{1\}$ and $\Gamma_0 K = G$. So $\Gamma_0$ is a co-compact lattice subgroup of $G$.

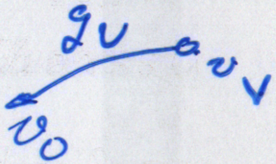
**Theorem 1 (Cartwright - Steger)**
*Let $F = \mathbb{F}_q((X))$, where $q$ is any prime power. For any $d \geq 2$, $G$ contains an $\tilde{A}_{d-1}$ group.*

# Other known examples of $\tilde{\mathbf{A}}_{\mathbf{d}-1}$ groups

- The tree $\mathcal{B}_{2,F}$ depends only on $q$. The $\tilde{A}_1$ groups are free products of $r$ copies of $\mathbb{Z}/2\mathbb{Z}$ and $s$ copies of $\mathbb{Z}$, where $r + 2s = q + 1$. (Tits)
- When $d = 2$, and $q$ is odd, the CS group is $\mathbb{Z}^{*(q+1)/2}$, the free group on $(q+1)/2$ generators; if $q$ is even, it is $(\mathbb{Z}/2\mathbb{Z})^{*(q+1)}$.
- For $q = 2$ and 3, the $\tilde{A}_2$ groups have all been found, and exist only for $F = \mathbb{Q}_q$ and $F = \mathbb{F}_q((X))$. (Cartwright, Mantero, Steger, Zappa)
- For $q = 5$ and 7, Voskuil has found $\tilde{A}_2$ groups for $F = \mathbb{Q}_q$.
- For $q = 2$, the $\tilde{A}_3$ groups have all been found, and exist only for $F = \mathbb{F}_2((X))$. (Cartwright, Steger)
- There is an $\tilde{A}_3$ group for $F = \mathbb{Q}_5$. (Svenson)

## General properties of an $\tilde{\mathrm{A}}_{\mathrm{d-1}}$ group $\Gamma_0$.

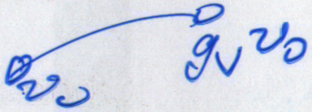
- $\Gamma_0$ has a natural set of generators $g_V$, indexed by the nonzero proper vector subspaces $V$ of $\mathbb{F}_q^d$.



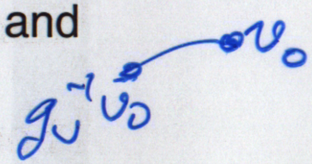
  - In fact, $\{g_V : \dim(V) = 1\}$ generates $\Gamma_0$.

- $\Gamma_0$ has a presentation involving relations in the $g_V$'s of two types:

  - $g_V^{-1} = g_{\lambda(V)}$, where $\dim(\lambda(V)) = d - \dim(V)$.

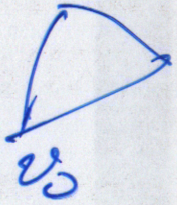


  - $g_U g_V g_W = 1$, where $\dim(U) + \dim(V) + \dim(W) = d$ and $V \subsetneqq \lambda(U)$.



- Each $g \in \Gamma_0$ has a unique normal form

$$g = g_{V_1} g_{V_2} \cdots g_{V_r},$$



where $\lambda(V_i) + V_{i+1} = \mathbb{F}_q^d$ for all $i < r$.

## An $\tilde{\mathbf{A}}_{\mathbf{d-1}}$ group for $\mathbf{F} = \mathbb{F}_{\mathbf{q}}((\mathbf{X}))$.

Let $Y$ be an indeterminate. Then $\mathbb{F}_{q^d}(Y)$ is a Galois extension of $\mathbb{F}_q(Y)$. Its Galois group is generated by $\varphi$, where

$$\varphi(a) = a^q \text{ (for } a \in \mathbb{F}_{q^d}) \quad \text{and} \quad \varphi(Y) = Y.$$

Form the algebra $\mathcal{D}$ of formal sums

$$a_0 + a_1\sigma + \cdots + a_{d-1}\sigma^{d-1},$$

where the $a_j$'s are in $\mathbb{F}_{q^d}(Y)$, and where

$$\sigma a = \varphi(a)\sigma \quad \text{and} \quad \sigma^d = Y + 1.$$

It is a cyclic simple algebra of dimension $d^2$ over $\mathbb{F}_q(Y)$. It is a division algebra, but $\mathcal{D} \otimes \mathbb{F}_q((Y))$ is isomorphic to the matrix algebra $M_d(\mathbb{F}_q((Y)))$.

## The group $\Gamma_0$.

Let

$$b_1 = Y + 1 + \sigma + \cdots + \sigma^{d-1} \in \mathcal{D}.$$

Then $b_1^{-1} = \frac{1}{Y}(1 - \sigma^{-1})$. Form the conjugates

$$b_x = x b_1 x^{-1} \quad (x \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times).$$

We map these elements into $PGL(d, \mathbb{F}_q((Y)))$ via

$$\mathcal{D}^\times \hookrightarrow \big(\mathcal{D} \otimes \mathbb{F}_q((Y))\big)^\times \cong \big(M_d(\mathbb{F}_q((Y)))\big)^\times = GL(d, \mathbb{F}_q((Y))) \\ \to PGL(d, \mathbb{F}_q((Y))).$$

Let $\Gamma_0$ be the subgroup of $G = PGL(d, \mathbb{F}_q((Y)))$ by the images of the $b_x$'s.

The elements $g_x$, $x \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$, map $v_0 = [L_0]$ onto the $(q^d - 1)/(q - 1)$ distinct neighbours of $v_0$ of type 1. It follows easily that $\Gamma_0$ acts transitively on the vertex set of the building.

The Skolem-Noether theorem says that any automorphism of the algebra $\mathcal{D}$ has the form $\xi \mapsto g\xi g^{-1}$, where $g \in \mathcal{D}^\times$. Thus $\mathrm{Aut}(\mathcal{D}) \cong \mathcal{D}^\times / Z(\mathcal{D}^\times)$. It turns out that with respect to a natural basis of $\mathcal{D}$ over $\mathbb{F}_q(Y)$, each automorphism

$$\xi \mapsto b_x \xi b_x^{-1}$$

has a matrix with entries in $\mathbb{F}_q[\frac{1}{Y}]$. So if $\mathbf{A}$ is the corresponding algebraic group, then

$$\Gamma_0 \subset \mathbf{A}\Big(\mathbb{F}_q[\frac{1}{Y}]\Big),$$

which is a lattice subgroup in $\mathbf{A}(\mathbb{F}_q((Y))) \cong PGL(d, \mathbb{F}_q((Y)))$ by a general theorem of Borel and Harder.

This implies that $\Gamma_0 \cap K$ is finite. A direct calculation shows that it is in fact $\{1\}$.

$\mathcal{D}$ is not a matrix algebra because $Y+1$ is not the norm $N_{\mathbb{F}_{q^d}(Y)/\mathbb{F}_q(Y)}(\xi)$ of any element $\xi$ of $\mathbb{F}_{q^d}(Y)$. Here is a neat way to realize the elements of $\mathcal{D}$ as matrices:

Consider another indeterminate $X$, and choose any $\beta \in \mathbb{F}_{q^d}$ such that $\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\beta) = t_1 \neq 0$. Then the norm $N_{\mathbb{F}_{q^d}(X)/\mathbb{F}_q(X)}(1+\beta X)$ equals $1 + t_1 X + t_2 X^2 + \cdots + t_d X^d$ for certain other elements $t_2, \ldots, t_d$ of $\mathbb{F}_q$. We embed $\mathbb{F}_q(Y)$ into $\mathbb{F}_q(X)$ and $\mathbb{F}_{q^d}(Y)$ into $\mathbb{F}_{q^d}(X)$ by mapping $Y$ to $t_1 X + t_2 X^2 + \cdots + t_d X^d$. First define $\Psi : \mathcal{D} \otimes \mathbb{F}_q(X) \to M_d(\mathbb{F}_{q^d}(X))$ via

$$x \mapsto \Psi(x) = \begin{pmatrix} x & 0 & \ldots & 0 \\ 0 & \varphi(x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \varphi^{d-1}(x) \end{pmatrix} \qquad \text{for } x \in \mathbb{F}_{q^d}(X),$$

and $\sigma$ to $\Psi(\sigma) =$

$$\begin{pmatrix} 0 & 1+\beta X & 0 & \dots & 0 \\ 0 & 0 & 1+\varphi(\beta)X & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1+\varphi^{d-2}(\beta)X \\ 1+\varphi^{d-1}(\beta)X & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Let $\xi_0, \dots, \xi_{d-1}$ be a basis for $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$, and let $Q \in GL(d, \mathbb{F}_{q^d})$ have $(i,j)$-th entry $\varphi^j(\xi_i)$ for $i, j = 0, \dots, d-1$. The conjugation $M \mapsto QMQ^{-1}$ maps the image of $\Psi$ into $M_d(\mathbb{F}_q(X))$, because the $(i,j)$-th entry of $Q^{-1}$ is $\varphi^i(\eta_j)$, where $\mathrm{Tr}(\xi_i \eta_j) = \delta_{i,j}$. The map $\xi \mapsto Q\Psi(\xi)Q^{-1}$ is an isomorphism $\mathcal{D} \otimes \mathbb{F}_q(X) \to M_d(\mathbb{F}_q(X))$.

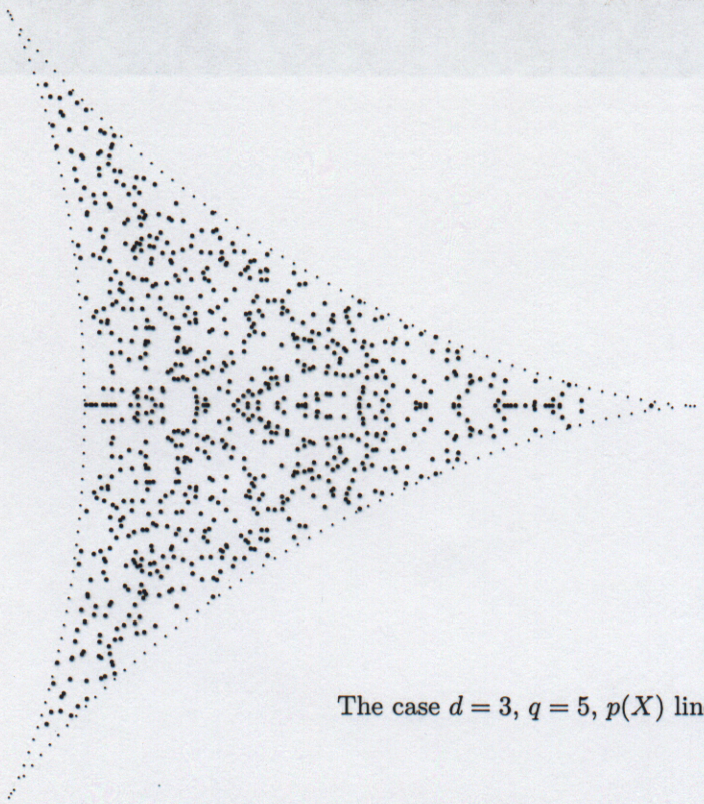
It is clear that the elements $b_x$ are mapped in this way to matrices $g_x$ with entries in $\mathbb{F}_q[X]$, and all have determinant

$$Y^{d-1}(Y+1) = (t_1X + \cdots + t_dX^d)^{d-1}(1 + t_1X + \cdots + t_dX^d).$$

So if we choose an irreducible degree $r$ polynomial $p(X)$ not dividing this determinant, we can form the ideal $I = p(x)\mathbb{F}_q[X]$ of $\mathbb{F}_q[X]$, and map the $g_x$'s to matrices in $GL(d, \mathbb{F}_q[X]/I) \cong GL(d, \mathbb{F}_{q^r})$. So we get a group homomorphism

$$\Gamma_0 \to PGL(d, \mathbb{F}_{q^r}).$$

Let $\Gamma$ be the kernel of this homomorphism. Solé and Żuk and I conjectured that $\Gamma \backslash \mathcal{B} = \Gamma \backslash \Gamma_0$ was a Ramanujan complex.



The case $d = 3$, $q = 5$, $p(X)$ linear.



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