Asymptotics of principal evaluations of Schubert polynomials

Alejandro Morales

UMASS, Amherst

IPAM Workshop Asymptotic Algebraic Combinatorics

February 5, 2020

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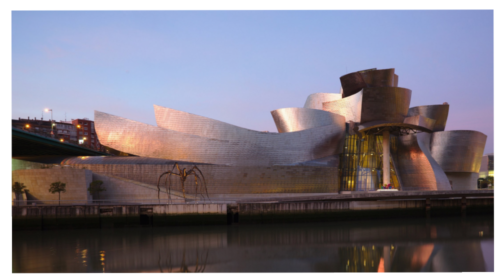
2

$G_w(1)$



$$\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

joint work with Igor Pak, Greta Panova

Outline

* Schubert polynomials $\mathfrak{S}_w(x_1, x_2, \ldots, x_{n-1})$
* conjectures on max $\mathfrak{S}_w(1, 1, \ldots, 1)$
* special cases of $\mathfrak{S}_w(1, 1, \ldots, 1)$
* progress on conjectures

Schubert polynomials

$w = w_1 w_2 \cdots w_n$ permutation

Schubert polynomials $\mathfrak{S}_w(x_1, \ldots, x_{n-1}) \in \mathbb{N}[x_1, x_2, \ldots, x_{n-1}]$

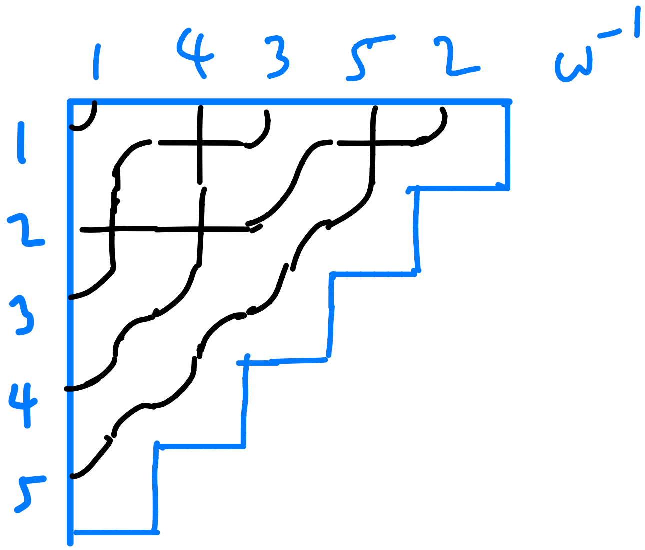
- defined recursively by Lascoux – Schützenberger (1982) to study Schubert varieties

- equivalent explicit formulas for $\mathfrak{S}_w(\underline{x}) = \sum_D \underline{x}^D$ in terms of combinatorial objects

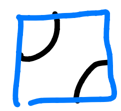
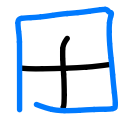
  * rc-graphs / pipe dreams (Bergeron-Billey 93)
  * "bumpless pipe dreams" (Lam-Lee-Shimozono 18)

  ⋮

RC graphs (reduced pipe dreams)

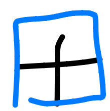


$w = 1\,5\,3\,2\,4$

- tiling D $\mathbb{Z}_{>} \times \mathbb{Z}_{>}$ with , 

- label wires left side 1, 2, 3, ...

- wires of tiling create a history of a permutation w^{-1} on the top

- D is reduced if #  $= \ell(w)$

  $\Rightarrow$ no two wires cross more than once

RC graphs (reduced pipe dreams)

Theorem (Bergeron–Billey 93)

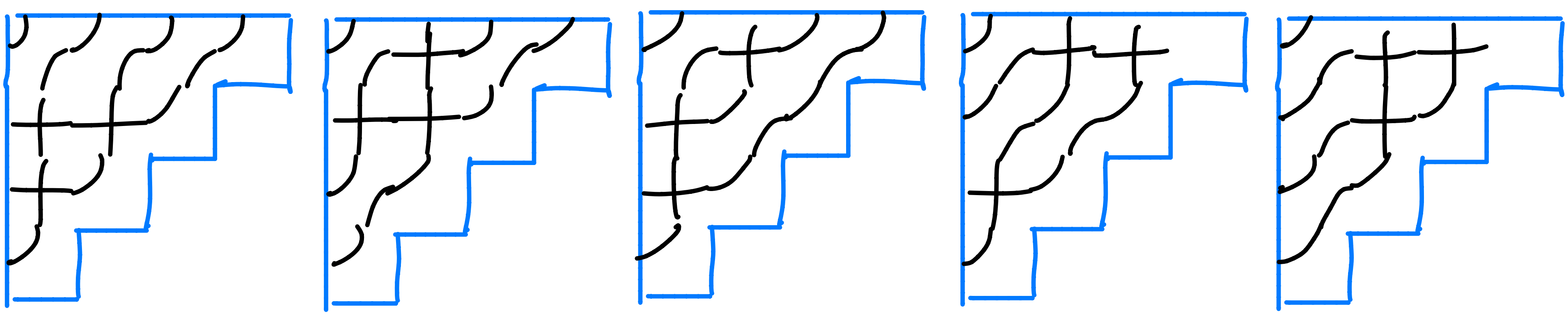
$$\mathfrak{S}_w(\underline{x}) = \sum_{D} \prod_{\substack{\boxed{+} \in D \\ \text{row } i}} x_i$$

sum over reduced pipe dreams of w

Example $w = 1432$

1 4 3 2

1
2
3
4



$$x_2^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1^2 x_2$$

Why study principal specialisation: $\mathfrak{S}_w(1,1,\dots,1)$

① $\mathfrak{S}_w(1,1,\dots,1) = \#$ reduced pipe dreams of w

<u>Example</u> $\mathfrak{S}_{1432}(1,1,1) = 5$

$$\mathfrak{S}_{1\,n+1\,n\cdots 2\,1}(1,\dots,1) = \frac{1}{n+1}\binom{2n}{n}$$

②

Theorem (Macdonald 1991)

$$\mathfrak{S}_w(1,1,\dots,1) = \frac{1}{\ell(w)!}\sum_{\underline{a}} a_{i_1} a_{i_2} \cdots a_{i_\ell}$$

sum is over <u>reduced words</u> $\underline{a} = a_{i_1} \cdots a_{i_\ell}$ of w

→ Bijective proof by Billey–Holroyd–Young 2017

Why study principal specialisation: $\mathfrak{S}_w(1,1,\ldots,1)$

② Theorem (Macdonald 1991)

$$\mathfrak{S}_w(1,1,\ldots,1) = \frac{1}{\ell(w)!} \sum_{\underline{a}} a_{i_1} a_{i_2} \cdots a_{i_\ell}$$

sum is over reduced words $\underline{a} = a_{i_1} \cdots a_{i_\ell}$ of w

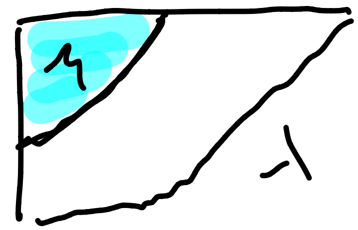
* $\underline{a} = a_1 a_2 \cdots a_\ell$ is a reduced word of w if $s_{a_1} \cdots s_{a_\ell}$ is a reduced decomposition of w into $s_i = (i, i+1)$

Example $w = 1432 = s_2 s_3 s_2 = s_3 s_2 s_3$

$$5 = \mathfrak{S}_{1432}(1,1,1) = \frac{1}{3!}\left(2\cdot 3\cdot 2 + 3\cdot 2\cdot 3\right)$$

Why study principal specialisation: $\mathfrak{S}_w(1,1,\dots,1)$

* recall, a skew shape λ/μ

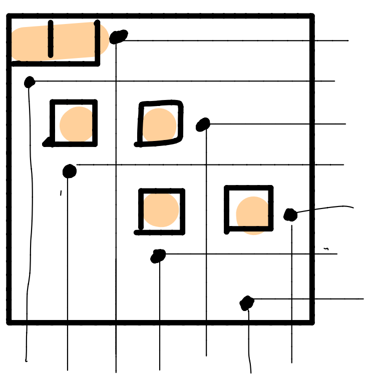


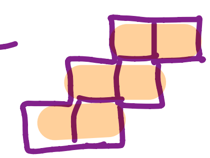
Corollary (Billey-Jockusch-Stanley 93)

If w avoids 321 then its (Rothe) diagram gives a skew shape λ/μ. Conversely, every skew shape λ/μ comes from the diagram of some 321-avoiding w.

$$\mathfrak{S}_w(1,1,\dots,1) = \frac{1}{\ell!} \, r_1\cdots r_\ell \, \#SYT(\lambda/\mu)$$

$\ell = |\lambda/\mu|$, $r_1 \cdots r_\ell$ is a reduced word of w

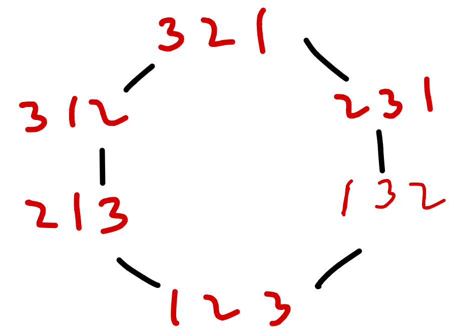


$\mathfrak{S}_w(1,1,\dots,1) = \#SYT$  $= 61$

Why study principal specialisation: $\mathfrak{S}_w(1,1,\ldots,1)$

Schubert Shenanigans (Stanley 2017)

$*$ $W_n = (S_n, \leq)$ <u>weak order</u> S_n



$*$ Björner studied whether W_n has certain property called <u>Sperner</u>

Stanley's conjecture

> <u>Theorem</u> (Gaetz-Gao 18, Hamacher-Pechenik-Speyer-Weigandt 18)
>
> Certain matrices of principal evaluations $\mathfrak{S}_w(1 \ldots 1)$ are invertible.

$*$ This implies W_n is Sperner. (Stanley)

How small can $\gamma_w := \mathfrak{S}_w(1,1,\ldots,1)$ be ?

* $\gamma_w = 1$ if and only if w is 132-avoiding

Theorem (Weigandt 2017, conj. Stanley 2017)
$\gamma_w = 2$ if and only if w contains pattern 132 <u>once</u>

* $\gamma_w \geq 1 + \underbrace{n_{132}(w)}_{\#\text{ patterns } 132 \text{ in } w}$ (Weigandt 2017)

* $\gamma_w \geq 1 + n_{132}(w) + n_{1432}(w)$ (Gao 2019)

How large can $\Upsilon_w := \mathfrak{S}_w(1,1,\ldots,1)$ be ?

Let $u(n) := \max_{w \in S_n} (\Upsilon_w : w \in S_n)$

Proposition (Stanley 2017)

$$\frac{1}{4} \leq \liminf \frac{\log_2 u(n)}{n^2} \leq \limsup \frac{\log_2 u(n)}{n^2} \leq \frac{1}{2}$$

<u>Proof</u> Use Cauchy identity:

$$\sum_{v^{-1}u = n\,n-1\cdots 1} \Upsilon_v \cdot \Upsilon_u = 2^{\binom{n}{2}}$$

$\boxtimes$

Conjecture I (Stanley 2017)

There is a limit $c := \lim_{n \to \infty} \frac{\log_2 u(n)}{n}$

Progress on conjectures $u(n) = \max_{w \in S_n} \mathfrak{S}_w(1,1,\ldots,1)$

Conjecture I (Stanley 2017)

There is a limit $c := \lim_{n\to\infty} \frac{\log_2 u(n)}{n}$

Known bounds

$0.25 \le c \le 0.5$ (Stanley 17)

$0.25162 \le c$ (M-Pak-Panova 17)

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- Some Schubert shenanigans

  A conjectured determinant evaluation related to Schubert polynomials, which implies that the weak order on the symmetric group S_n is strongly Sperner.

  Update. Conjecture 4.1 was first proved by Anna Weigandt and later by Greta Panova. The lower bound 1/4 near the bottom of page 7 was improved to about .251

$(\overset{?}{=})$

$0.29323672\ldots = \alpha \le c$ (M-Pak-Panova 18)

$c \le 0.46$ (Pak-Yeliussizov 18+)

$c \le 0.37$ (???)

From $c \geq 0.25$ to $c \geq 0.2516\ldots$

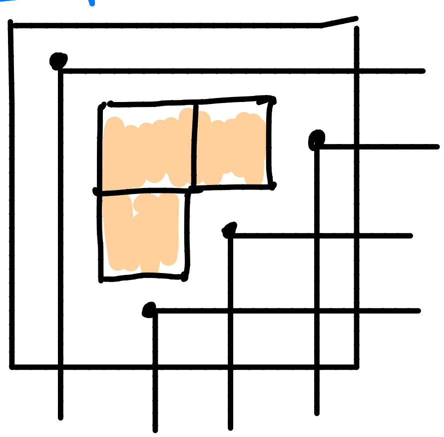
Corollary (Wachs 85, Knutson-Miller-Yong 09)

If w is 2143-avoiding then there is a shape λ such that

$$\mathfrak{G}_w(1,1,\ldots,1) = \#\ \underline{\text{flagged}}\ \text{SSYT shape } \lambda$$

↑ entries in row $i \leq c_i$

<u>example</u> $w = 1432$



Rothe diagram

$$\mathfrak{G}_{1432}(1,1,1) = \#\ \text{SSYT} \quad \begin{array}{|c|c|l}\hline & & \leq 2 \\ \hline & \multicolumn{2}{|l}{\leq 3} \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$\mathfrak{S}_w(1,1,\ldots,1)$ for vexillary permutations

Corollary (Wachs 85, Knutson-Miller-Yong 09)

If w is 2143-avoiding then there is a shape λ such that

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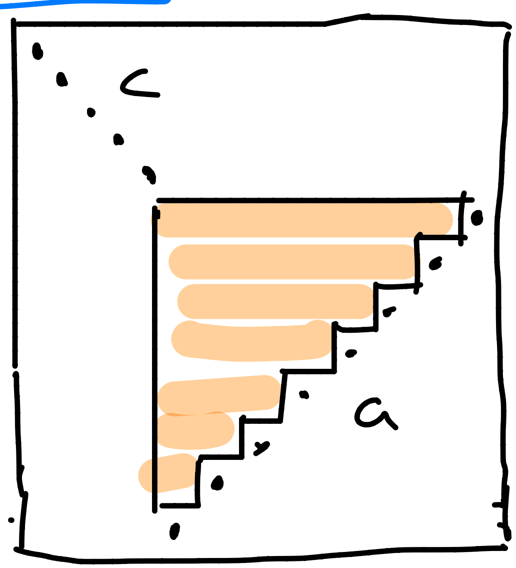
↑ entries in row $i \le c_i$

Corollary



$$\mathfrak{S}_{w_0(a,c)}(1,1,\ldots,1) = \#\ \text{SSYT}$$

$$= \prod_{1 \le i < j \le a} \frac{2c+i+j-1}{i+j-1}$$

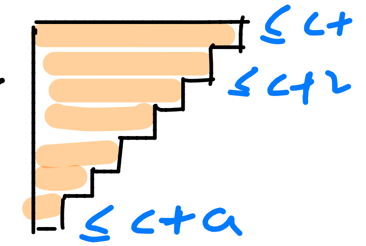
(Proctor 90)

$$w(a,c) := 1^c \times \underbrace{w_0(a)}_{a\ a-1 \cdots 1}$$

From $c \geq 0.25$ to $c \geq 0.2516\ldots$

Corollary

$$\mathfrak{S}_{w_0(a,c)}(1,1,\ldots,1) = \#\,\mathrm{SSYT}$$



$$= \prod_{1 \le i < j \le a} \frac{2c+i+j-1}{i+j-1} \qquad \text{(Proctor 90)}$$

$$w(a,c) := 1^c \times \underbrace{w_0(a)}_{a\ a-1 \cdots 1}$$

Question For which a,c with $a+c=n$ is $\mathfrak{S}_{w_0(a,c)}(1,1,\ldots,1)$ maximal?

Proposition (M-Pak-Panova 17)

For $a = \frac{n}{3}$: $\quad \frac{1}{n^2}\log_2 \mathfrak{S}_{w_0(a,n-a)}(1,1,\ldots,1) \to C \approx 0.25162$

This is largest limit among all ratios a/n.

Progress on conjectures $u(n) := \max_{w \in S_n} \mathfrak{S}_w(1,1,\ldots,1)$

Conjecture I (Stanley 2017)

There is a limit $c := \lim_{n\to\infty} \frac{\log_2 u(n)}{n^2}$

Known bounds

$0.25 \leq c \leq 0.5$ (Stanley 17)

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$0.29323672\ldots = \alpha \leq c$ (M-Pak-Panova 18)

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$c \leq 0.37$ (???)

What w maximize $\Upsilon_w := \mathfrak{S}_w(1,1,\ldots,1)$?

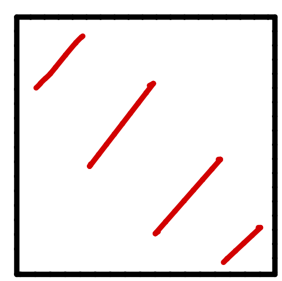
(Merzon-Smirnov 16
Stanley 17)

Data

n	$u(n)$	w
3	2	132
4	5	1432
5	14	12543, 15432, 21543
6	84	126543, 216543
7	660	1327654
8	9438	13287654
9	163592	132987654
10	4424420	1,4,3,2,10,9,8,7,6,5

Conjecture 2 (Merzon-Smirnov 16, Stanley 17)

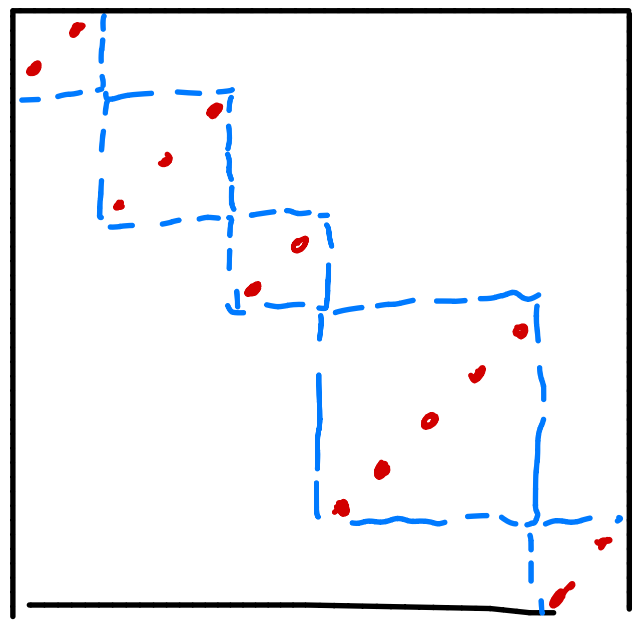
Υ_w is maximal at layered permutations



Layered permutations

* given composition $b_1 + \cdots + b_{k-1} + b_k = n$
  let $w(b_k, b_{k-1}, \ldots, b_1)$ be the layered permutation with runs $b_k, b_{k-1}, \ldots, b_1$.

(Richardson perms, pop-stack sortable, Claesson 17, ...)



$w(2,3,2,5,2)$

* let L_n be set of layered permutations in S_n

Max Υ_w on layered permutations

$*$ let $\quad v(n) := \max\limits_{w \in \mathcal{L}_n} \Upsilon_w$ $\leftarrow$ layered permutations in S_n

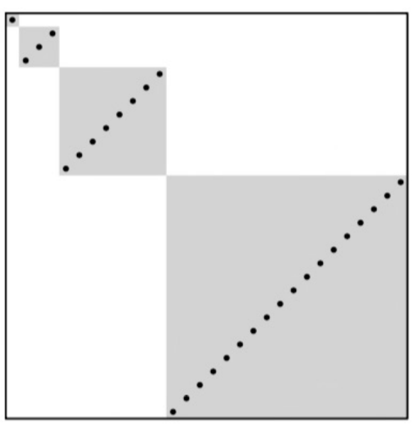
Theorem (M-Pak-Panova 18)

$*$ There is a limit $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} \log_2 v(n) = \frac{\beta}{\log 2} \approx 0.2932\ldots$

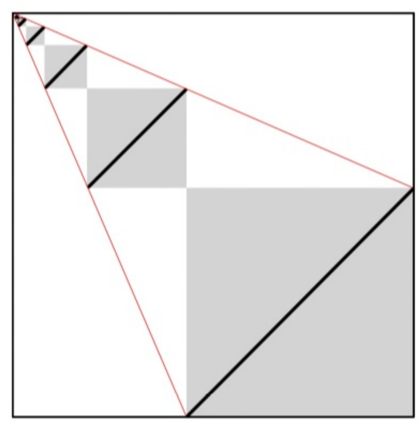
$*$ The max value $v(n)$ is achieved at

$w(\ldots, b_2, b_1)$ with $b_i \sim \alpha^{i-1}(1-\beta)n$

for some $\alpha \approx 0.43318\ldots$



$w(1,3,8,18)$



$w(2,4,9,20,46,106,246,567)$

Progress on conjectures $u(n) = \max_{w \in S_n} \mathfrak{S}_w(1,1,\ldots,1)$

Conjecture I (Stanley 2017)
There is a limit $c := \lim_{n \to \infty} \frac{\log_2 u(n)}{n^2}$

Known bounds

$0.25 \leq c \leq 0.5$ (Stanley 17)

$0.25162 \leq c$ (M-Pak-Panova 17)

$0.29323672\ldots = \alpha \leq c$ (M-Pak-Panova 18)

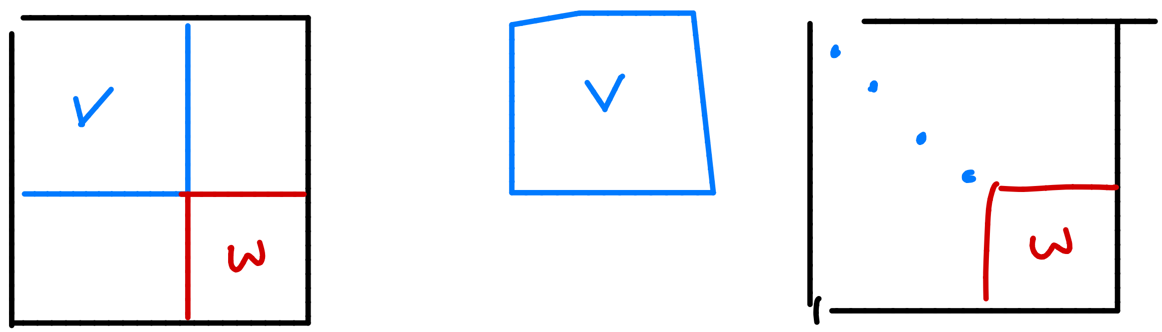
(conjectured value of c)

$c \leq 0.46$ (Pak-Yeliussizov 18?)

$c \leq 0.37$ (???)

Υ_w on layered permutations

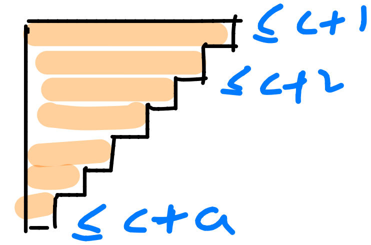
$*$ $\mathfrak{S}_{v\times w}(\underline{x}) = \mathfrak{S}_v(\underline{x})\cdot\mathfrak{S}_{1^m\times w}(\underline{x})$



$P(c,a)$

$\|$

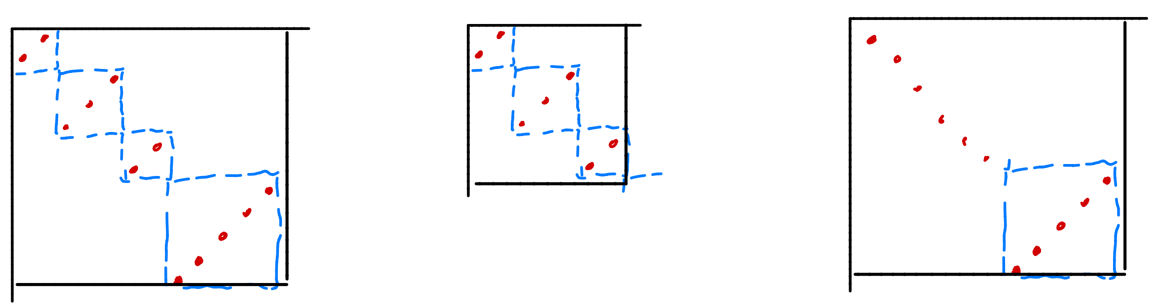
Recall $\mathfrak{S}_{w_0(a,c)}(1,1,\dots,1) = \#\,SSYT$



$= \prod_{1\le i<j\le a}\frac{2c+i+j-1}{i+j-1}$

Proposition $\Upsilon_{w(b_k,\dots,b_1)} = \Upsilon_{w(b_k,\dots,b_2)}\cdot P(|b|-b_1,b_1)$

Proof $w(b_k,\dots,b_1) = w(b_k,\dots,b_2)\times w_0(b_k)$



Data max $\Upsilon_w(1,1,\ldots,1)$ for layered permutations

* let $P(m,b) := \prod_{1 \le i < j \le b} \frac{2m+i+j-1}{i+j-1}$

Proposition $\Upsilon_{w(b_k,\ldots,b_1)} = \Upsilon_{w(b_k,\ldots,b_2)} \cdot P(|b|-b_1, b_1)$

* running a dynamic program

Data max $\gamma_w(1,1,\dots,1)$ for layered permutations

* let $P(m,b) := \prod_{1\le i<j\le b} \frac{2m+i+j-1}{i+j-1}$

Proposition $\gamma_{w(b_k,\dots,b_1)} = \gamma_{w(b_k,\dots,b_2)} \cdot P(|b|-b_1, b_1)$

* running a dynamic program

max $\qquad$ $\log \frac{1}{n^2} \log_2 \gamma(n)$

n	$(\dots, b_2, b_1)$	$f(n)$
1	(1)	0.000000
2	(1, 1)	0.000000
3	(1, 2)	0.111111
4	(1, 3)	0.145121
5	(1, 1, 3)	0.152294
6	(1, 1, 4)	0.177564
7	(1, 2, 4)	0.191149
8	(1, 2, 5)	0.206317
9	(1, 2, 6)	0.213824
10	(1, 3, 6)	0.220771
11	(1, 3, 7)	0.227005
12	(1, 3, 8)	0.229879
13	(1, 1, 3, 8)	0.233769
14	(1, 1, 4, 8)	0.237048
15	(1, 1, 4, 9)	0.241677
16	(1, 1, 4, 10)	0.244446
17	(1, 2, 4, 10)	0.246954
18	(1, 2, 4, 11)	0.249509
19	(1, 2, 5, 11)	0.251966
20	(1, 2, 5, 12)	0.254240
21	(1, 2, 5, 13)	0.255575
22	(1, 2, 6, 13)	0.257354
23	(1, 2, 6, 14)	0.258685
24	(1, 3, 6, 14)	0.260063
25	(1, 3, 6, 15)	0.261360
26	(1, 3, 7, 15)	0.262425
27	(1, 3, 7, 16)	0.263673
28	(1, 3, 7, 17)	0.264435
29	(1, 3, 8, 17)	0.265233
30	(1, 3, 8, 18)	0.266034
31	(1, 1, 3, 8, 18)	0.266811

n	$(\dots, b_2, b_1)$	$f(n)$
51	(1, 2, 5, 13, 30)	0.276896
52	(1, 2, 6, 13, 30)	0.277275
53	(1, 2, 6, 13, 31)	0.277550
54	(1, 2, 6, 14, 31)	0.277807
55	(1, 2, 6, 14, 32)	0.278094
56	(1, 3, 6, 14, 32)	0.278322
57	(1, 3, 6, 14, 33)	0.278618
58	(1, 3, 6, 14, 34)	0.278815
59	(1, 3, 6, 15, 34)	0.279103
60	(1, 3, 6, 15, 35)	0.279313
61	(1, 3, 7, 15, 35)	0.279525
62	(1, 3, 7, 15, 36)	0.279747
63	(1, 3, 7, 16, 36)	0.279962
64	(1, 3, 7, 16, 37)	0.280192
65	(1, 3, 7, 16, 38)	0.280344
66	(1, 3, 7, 17, 38)	0.280532
67	(1, 3, 7, 17, 39)	0.280698
68	(1, 3, 8, 17, 39)	0.280862
69	(1, 3, 8, 17, 40)	0.281038
70	(1, 3, 8, 18, 40)	0.281178
71	(1, 3, 8, 18, 41)	0.281363
72	(1, 3, 8, 18, 42)	0.281486
73	(1, 1, 3, 8, 18, 42)	0.281670
74	(1, 1, 3, 8, 18, 43)	0.281803
75	(1, 1, 3, 8, 19, 43)	0.281969
76	(1, 1, 3, 8, 19, 44)	0.282112
77	(1, 1, 4, 8, 19, 44)	0.282210
78	(1, 1, 4, 8, 19, 45)	0.282361
79	(1, 1, 4, 8, 20, 45)	0.282488
80	(1, 1, 4, 8, 20, 46)	0.282646
81	(1, 1, 4, 8, 20, 47)	0.282755

Data max $\Upsilon_{w(1,1,\ldots,1)}$ for layered permutations

* let $P(m,b) := \prod_{1 \le i < j \le b} \frac{2m+i+j-1}{i+j-1}$

Proposition $\Upsilon_{w(b_k,\ldots,b_1)} = \Upsilon_{w(b_k,\ldots,b_2)} \cdot P(|b|-b_1, b_1)$

* running a dynamic program

max $\qquad \log \frac{1}{n^2} \log_2 \gamma(n)$

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17	(1, 2, 4, 10)	0.246954
18	(1, 2, 4, 11)	0.249509
19	(1, 2, 5, 11)	0.251966
20	(1, 2, 5, 12)	0.254240
21	(1, 2, 5, 13)	0.255575
22	(1, 2, 6, 13)	0.257354
23	(1, 2, 6, 14)	0.258685
24	(1, 3, 6, 14)	0.260063
25	(1, 3, 6, 15)	0.261360
26	(1, 3, 7, 15)	0.262425
27	(1, 3, 7, 16)	0.263673
28	(1, 3, 7, 17)	0.264435
29	(1, 3, 8, 17)	0.265233
30	(1, 3, 8, 18)	0.266034
31	(1, 1, 3, 8, 18)	0.266811

n	$(\ldots,b_2,b_1)$	$f(n)$
51	(1, 2, 5, 13, 30)	0.276896
52	(1, 2, 6, 13, 30)	0.277275
53	(1, 2, 6, 13, 31)	0.277550
54	(1, 2, 6, 14, 31)	0.277807
55	(1, 2, 6, 14, 32)	0.278094
56	(1, 3, 6, 14, 32)	0.278322
57	(1, 3, 6, 14, 33)	0.278618
58	(1, 3, 6, 14, 34)	0.278815
59	(1, 3, 6, 15, 34)	0.279103
60	(1, 3, 6, 15, 35)	0.279313
61	(1, 3, 7, 15, 35)	0.279525
62	(1, 3, 7, 15, 36)	0.279747
63	(1, 3, 7, 16, 36)	0.279962
64	(1, 3, 7, 16, 37)	0.280192
65	(1, 3, 7, 16, 38)	0.280344
66	(1, 3, 7, 17, 38)	0.280532
67	(1, 3, 7, 17, 39)	0.280698
68	(1, 3, 8, 17, 39)	0.280862
69	(1, 3, 8, 17, 40)	0.281038
70	(1, 3, 8, 18, 40)	0.281178
71	(1, 3, 8, 18, 41)	0.281363
72	(1, 3, 8, 18, 42)	0.281486
73	(1, 1, 3, 8, 18, 42)	0.281670
74	(1, 1, 3, 8, 18, 43)	0.281803
75	(1, 1, 3, 8, 19, 43)	0.281969
76	(1, 1, 3, 8, 19, 44)	0.282112
77	(1, 1, 4, 8, 19, 44)	0.282210
78	(1, 1, 4, 8, 19, 45)	0.282361
79	(1, 1, 4, 8, 20, 45)	0.282488
80	(1, 1, 4, 8, 20, 46)	0.282646
81	(1, 1, 4, 8, 20, 47)	0.282755

max

n	$(\ldots,b_2,b_1)$	$f(n)$
101	(1, 2, 4, 11, 25, 58)	0.284828
102	(1, 2, 4, 11, 25, 59)	0.284891
103	(1, 2, 4, 11, 26, 59)	0.284978
104	(1, 2, 4, 11, 26, 60)	0.285046
105	(1, 2, 5, 11, 26, 60)	0.285148
106	(1, 2, 5, 11, 26, 61)	0.285222
107	(1, 2, 5, 11, 27, 61)	0.285289
108	(1, 2, 5, 11, 27, 62)	0.285368
109	(1, 2, 5, 12, 27, 62)	0.285434
110	(1, 2, 5, 12, 27, 63)	0.285518
111	(1, 2, 5, 12, 27, 64)	0.285577
112	(1, 2, 5, 12, 28, 64)	0.285657
113	(1, 2, 5, 12, 28, 65)	0.285720
114	(1, 2, 5, 12, 29, 65)	0.285766
115	(1, 2, 5, 12, 29, 66)	0.285834
116	(1, 2, 5, 13, 29, 66)	0.285892
117	(1, 2, 5, 13, 29, 67)	0.285965
118	(1, 2, 5, 13, 29, 68)	0.286015
119	(1, 2, 5, 13, 30, 68)	0.286074
120	(1, 2, 5, 13, 30, 69)	0.286129
121	(1, 2, 6, 13, 30, 69)	0.286201
122	(1, 2, 6, 13, 30, 70)	0.286261
123	(1, 2, 6, 13, 31, 70)	0.286306
124	(1, 2, 6, 13, 31, 71)	0.286369
125	(1, 2, 6, 13, 31, 72)	0.286413
126	(1, 2, 6, 14, 31, 72)	0.286472
127	(1, 2, 6, 14, 31, 73)	0.286519
128	(1, 2, 6, 14, 32, 73)	0.286576
129	(1, 2, 6, 14, 32, 74)	0.286628
130	(1, 3, 6, 14, 32, 74)	0.286667
131	(1, 3, 6, 14, 32, 75)	0.286723
132	(1, 3, 6, 14, 33, 75)	0.286768
133	(1, 3, 6, 14, 33, 76)	0.286827
134	(1, 3, 6, 14, 33, 77)	0.286868

max

n	$(\ldots,b_2,b_1)$	$f(n)$
151	(1, 3, 7, 16, 38, 86)	0.287573
152	(1, 3, 7, 16, 38, 87)	0.287612
153	(1, 3, 7, 17, 38, 87)	0.287643
154	(1, 3, 7, 17, 38, 88)	0.287684
155	(1, 3, 7, 17, 38, 89)	0.287713
156	(1, 3, 7, 17, 39, 89)	0.287750
157	(1, 3, 7, 17, 39, 90)	0.287782
158	(1, 3, 8, 17, 39, 90)	0.287814
159	(1, 3, 8, 17, 39, 91)	0.287849
160	(1, 3, 8, 17, 40, 91)	0.287879
161	(1, 3, 8, 17, 40, 92)	0.287916
162	(1, 3, 8, 17, 40, 93)	0.287942
163	(1, 3, 8, 18, 40, 93)	0.287975
164	(1, 3, 8, 18, 40, 94)	0.288003
165	(1, 3, 8, 18, 41, 94)	0.288040
166	(1, 3, 8, 18, 41, 95)	0.288071
167	(1, 3, 8, 18, 42, 95)	0.288093
168	(1, 3, 8, 18, 42, 96)	0.288126
169	(1, 1, 3, 8, 18, 42, 96)	0.288155
170	(1, 1, 3, 8, 18, 42, 97)	0.288191
171	(1, 1, 3, 8, 18, 42, 98)	0.288216
172	(1, 1, 3, 8, 18, 43, 98)	0.288244
173	(1, 1, 3, 8, 18, 43, 99)	0.288272
174	(1, 1, 3, 8, 19, 43, 99)	0.288302
175	(1, 1, 3, 8, 19, 43, 100)	0.288332
176	(1, 1, 3, 8, 19, 44, 100)	0.288355
177	(1, 1, 3, 8, 19, 44, 101)	0.288387
178	(1, 1, 3, 8, 19, 44, 102)	0.288410
179	(1, 1, 4, 8, 19, 44, 102)	0.288432
180	(1, 1, 4, 8, 19, 44, 103)	0.288457
181	(1, 1, 4, 8, 19, 45, 103)	0.288486
182	(1, 1, 4, 8, 19, 45, 104)	0.288513
183	(1, 1, 4, 8, 20, 45, 104)	0.288533
184	(1, 1, 4, 8, 20, 45, 105)	0.288563

Proof max Υ_w on layered permutations

<u>Proposition</u> $\Upsilon_{w(b_k,\dots,b_1)} = \Upsilon_{w(b_k,\dots,b_2)} \cdot P(|b|-b_1, b_1)$ $(\dagger)$

* $$V(n) = \max_{b,\ |b|=n} \Upsilon_{w(\underline{b})}$$

* By $(\dagger)$ $$V(n) = \max_{1 \le b_1 \le n} \{ V(n-b_1) \cdot P(n-b_1, b_1) \}$$

* We use very precise estimates for

$$\log P(m,b) = \log \prod_{1 \le i < j \le b} \frac{2m+i+j-1}{i+j-1}$$

$$= \sum_{1 \le i \le j' \le b-1} \left(\log(2m+i+j') - \log(i+j') \right)$$

Estimates for $\log P(m, n-m)$

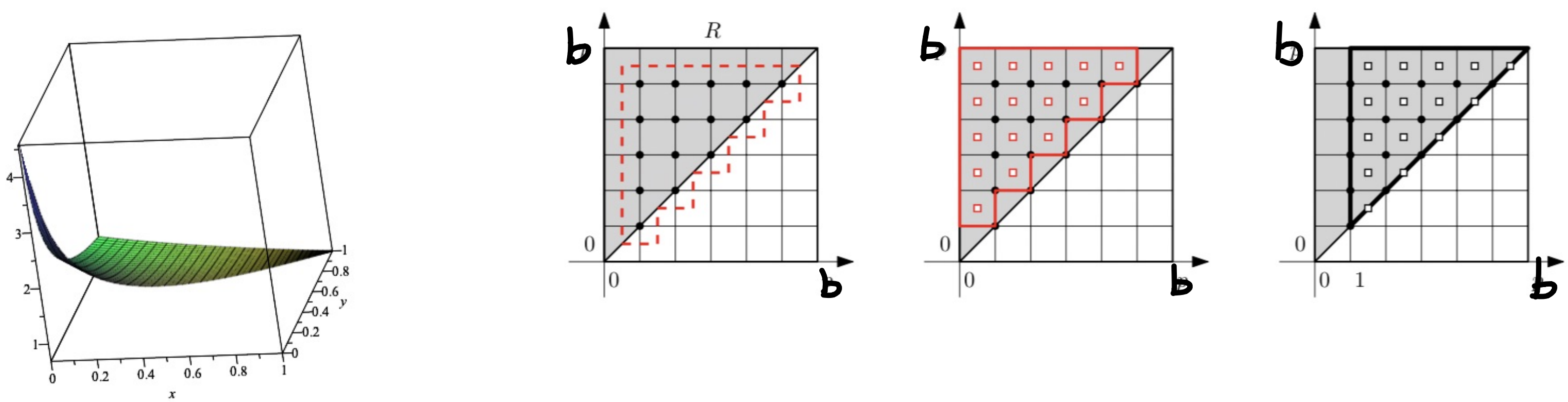
Lemma 1 for $n \geq m \geq 0$

$$-2n \leq \log P(m, n-m) - n^2 f(m/n) \leq 0$$

where $f(x) := x^2 \log x - \frac{1}{2}(1-x)^2 \log(1-x) - \frac{1}{2}(1+x)^2 \log(1+x) + 2x \log 2$

* use double integral that approximates $\log P(m, n-m)$

$$\int_0^b \int_y^b \left(\log(2m+x+y) - \log(x+y) \right) dx\, dy = (m+b)^2 f\left(\frac{m}{m+b}\right)$$



Estimates for $\log P(m, n-m)$

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$- \frac{1}{2}(1+x)^2 \log(1+x) + 2x \log 2$

* so we approximate $\log P(m, n-m)$ by $n^2 f(x)$
$x = m/n \in [0,1]$

* Next, find unique constant β so that
$f(x) + \beta x^2$ has unique max in $[0,1)$

Optimizing constants

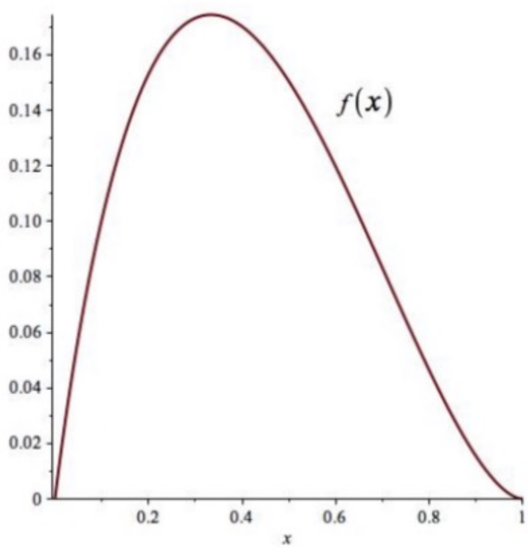
* we approximate $\log P(m, n-m)$ by $n^2 f(x)$

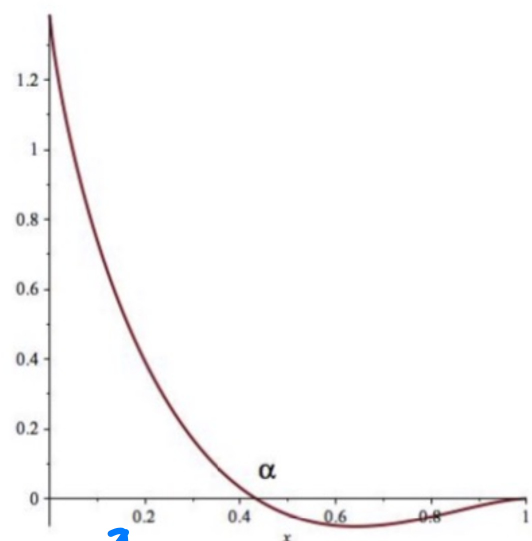
  $x = m/n \in [0,1]$

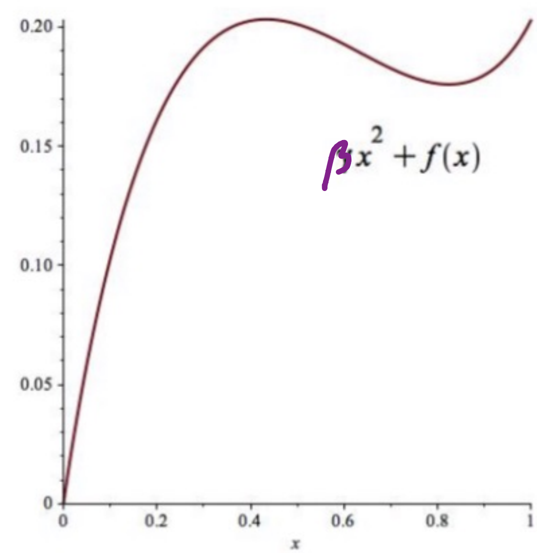
* Next, find unique constant β so that

  $f(x) + \beta x^2$ has unique max α in $[0,1)$

* max value is β : $f(\alpha) + \beta \alpha^2 = \beta$







$q(x) = f(x)2x + f'(x)(1-x)$

Proof max Υ_w on layered permutations

Lemma 2 $\left| \log v(n) - \beta n^2 \right| \leq 4n$

and if $\left| \log \Upsilon_{w(b)} - \beta n^2 \right| \leq 4n$

then $b = (\ldots, b_2, b_1)$ with $b_i \sim (1-\alpha)\alpha^{i-1} n$

Pf induction + Lemma 1

UB
$$\log v(n) = \max_{m<n} \left(\log P(m, n-m) + \log v(m) \right)$$
$$\leq \max_{m<n} \left(\log P(m, n-m) + \beta m^2 + 4m \right)$$
$$\leq n^2 \cdot \max_{x \in [0,1)} \left(f(x) + \beta x^2 \right) + 4n$$
$$= n^2 \beta + 2n$$

$\left(\text{at } x = \frac{m}{n} = \alpha \quad b_1 = m - n = (1-\alpha) n \right)$

Proof max Υ_w on layered permutations

$$\boxed{\begin{array}{l} \textbf{Lemma 2} \quad \left| \log v(\lambda) - \beta n^2 \right| \le 4n \\ \text{and if} \quad \left| \log \Upsilon_{w(b)} - \beta n^2 \right| \le 4n \\ \text{then} \quad b = (\ldots, b_2, b_1) \quad \text{with} \quad b_i \sim (1-\alpha)\alpha^{i-1} n \end{array}}$$

Pf induction + Lemma 1

LB $\log v(\lambda) \geq \log v(n\alpha) + \log P(n\alpha, n - n\alpha)$

$$\geq \left(\beta n^2 \alpha^2 - 4n\alpha\right) + \left(n^2 f(\alpha) - 2n\right)$$

$$\geq \beta n^2 - 2(1+2\alpha) n$$

$$\geq \beta n^2 - 4n$$

Proof max Υ_w on layered permutations

Lemma 2 $\left| \log v(n) - \beta n^2 \right| \le 4n$

and if $\left| \log \Upsilon_{w(b)} - \beta n^2 \right| \le 4n$

then $b = (\ldots, b_2, b_1)$ with $b_i \sim (1-\alpha)\alpha^{i-1} n$

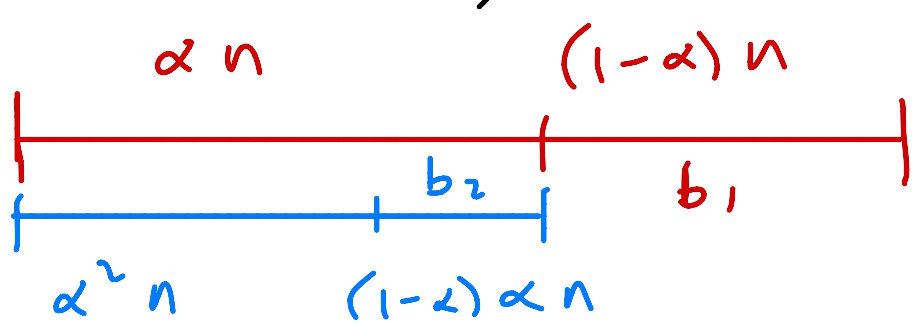
Pf UB + LB $\Longrightarrow$ $\left| \log v(n) - \beta n^2 \right| \le 4n$

bound attained when $b_1 \sim (1-\alpha) n$

$b_2 \sim (1-\alpha)\alpha n$

$\vdots$

recursively $b_i \sim (1-\alpha)\alpha^{i-1} n$



Progress on conjectures $u(n) = \max_{w \in S_n} \mathfrak{S}_w(1,1,\ldots,1)$

Conjecture I (Stanley 2017)

There is a limit $c := \lim_{n \to \infty} \frac{\log_2 u(n)}{n^2}$

Known bounds

$0.25 \leq c \leq 0.5$ (Stanley 17)

$0.25162 \leq c$ (M-Pak-Panova 17)

$\underline{0.29323672\ldots} = \alpha \leq c$ (M-Pak-Panova 18)

(conjectured value of c)

$c \leq 0.46$ (Pak-Yeliussizov 18+)

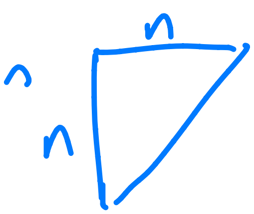
$c \leq 0.37$ (???)

Upper bound from 0.5 to 0.37

Recall $u(n) := \max_{w \in S_n} \Upsilon_w$

Let $a(n) := \sum_{w \in S_n} \Upsilon_w$

total number of reduced pipe dreams in



Proposition

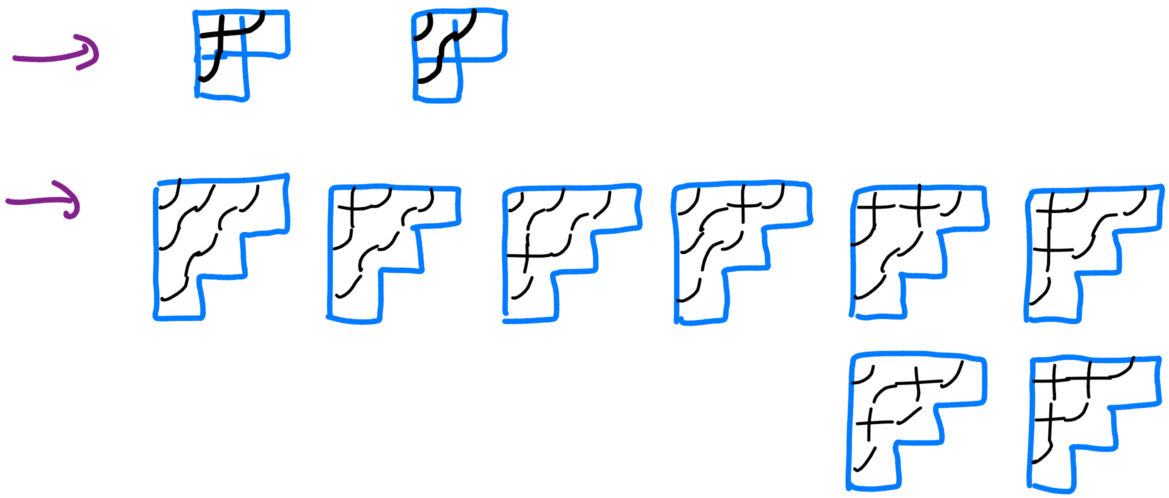
$$\lim_{n\to\infty} \frac{\log_2 u(n)}{n^2} = c \quad \text{iff} \quad \lim_{n\to\infty} \log_2 \frac{a(n)}{n^2} = c$$

Proof

$$u(n) \le a(n) \le n!\, u(n) \qquad \boxtimes$$

Data for $a(n) = \sum_{w \in S_n} \gamma_w$

n	$a(n)$
1	1
2	2
3	7
4	41
5	393



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1	1
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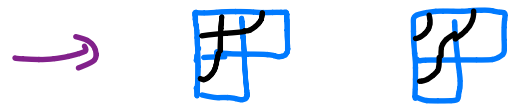

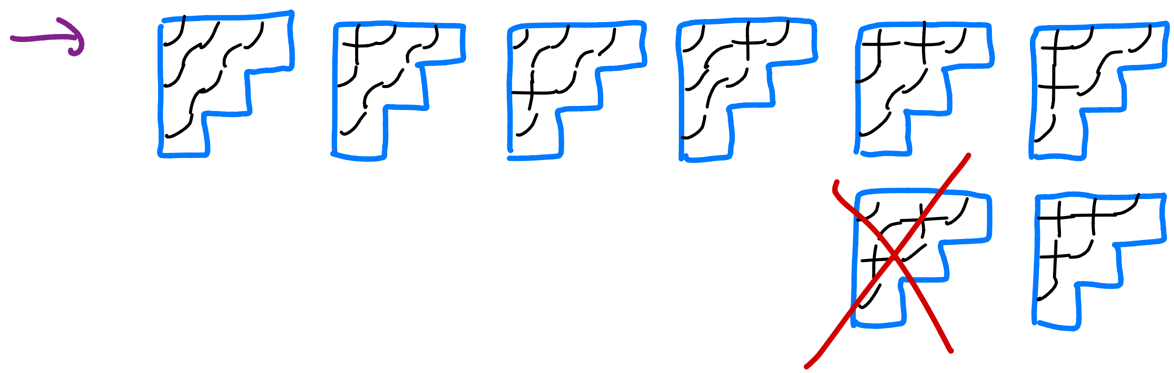

Data for $a(n) = \sum_{w \in S_n} \gamma_w$

n	$a(n)$		?
1	1	1	1
2	2	2	2
3	7	3	7
4	41	9	42
5	393	21	429

Data for $a(n) = \sum_{w \in S_n} \gamma_w$

n	$a(n)$	$ASM(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$
1	1	1
2	2	2
3	7	7
4	41	42
5	393	429

$n \times n$ alternating sign matric

Theorem (Weigandt 2018+) $a(n) \leq ASM(n)$

Data for $a(n) = \sum_{w \in S_n} \gamma_w$

$$ASM(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

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1	1	1
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Theorem (Weigandt 2018+) $a(n) \leq ASM(n)$

Corollary $\lim_{n \to \infty} \frac{\log_2 a(n)}{n^2} \leq 0.37$

<u>Proof</u> $\frac{\log_2 ASM(n)}{n^2} \longrightarrow 0.37$ $\boxtimes$

Bumpless pipe dreams

A bumpless pipe dream is a tiling of the $n \times n$ board with

so that there are n wires

- start right edge
- end bottom edge

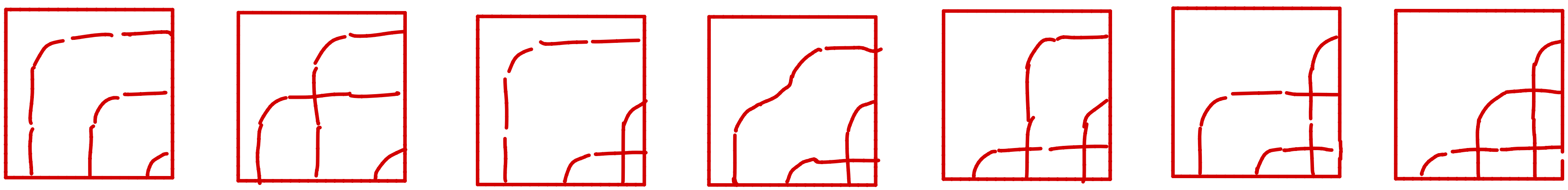
it is reduced if

- pair of wires cross at most once

Bumpless pipe dreams

Theorem (Lam-Lee-Shimozono 18)

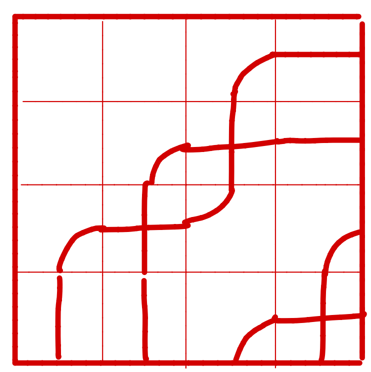
$$\sum_{w \in S_n} \mathfrak{S}_w = \#\ n \times n \text{ reduced bumpless pipe dreams}$$



Theorem (Weigandt 18+)

$$ASM(n) = \#\ n \times n \text{ bumpless pipe dreams}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



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There is a limit $c := \lim_{n\to\infty} \frac{\log_2 u(n)}{n^2}$

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$0.29323672\ldots = \alpha \leq c$ (M-Pak-Panova 18)

(conjectured value of c)

$c \leq 0.46$ (Pak-Yeliussizov 18+)

$c \leq 0.37$ (Cor. Weigandt 18+)

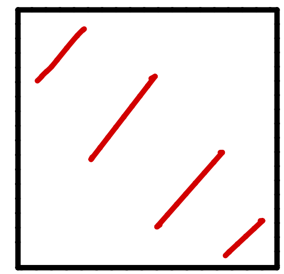
Summary: $u(n) := \max_{w \in S_n} \Upsilon_w \qquad v(n) := \max_{w \in S_n,\, \text{layered}} \Upsilon_w$

Conjecture 1 (Stanley 2017)

There is a limit $c := \lim_{n \to \infty} \dfrac{\log_2 u(n)}{n^2}$

Conjecture 2 (Merzon-Smirnov 16, Stanley 17)

Υ_w is maximal at layered permutations



Theorem (M-Pak-Panova 2018)

$$\lim_{n \to \infty} \frac{\log_2 v(n)}{n^2} = 0.29323672\ldots$$

Best bounds

$$0.29323672\ldots \leq c \leq 0.37$$

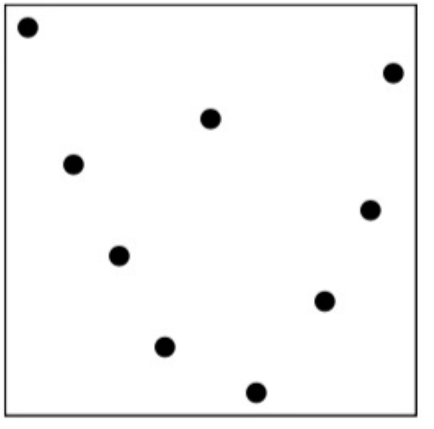
Maximum terms in the Cauchy identity

Recall first bounds for $\displaystyle\lim_{n\to\infty} \frac{\log_2 u(n)}{n^2}$ came from

$$\sum_{w^{-1}u = w_0} \Upsilon_w \cdot \Upsilon_u = 2^{\binom{n}{2}} \qquad \text{(Cauchy identity)}$$

$*$ let $\displaystyle u'(n) := \max_{w \in S_n} \left(\Upsilon_w \cdot \Upsilon_{w w_0^{-1}}\right)$

n	$u'(n)$	w
3	2	132
4	6	1423
5	33	15243
6	286	162534
7	4620	1736254
8	162360	18527364
9	9057090	195283746



$w = 195283746$

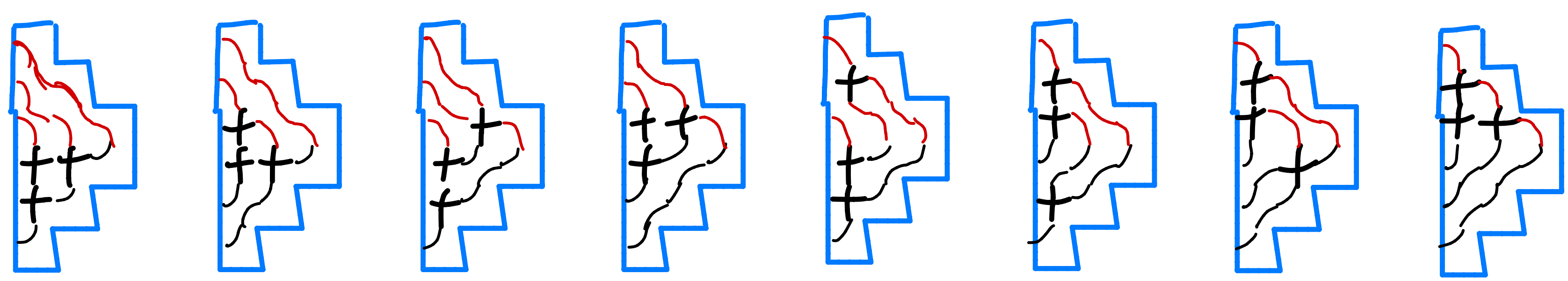
Double rc graphs

$$\sum_{w^{-1}u = w_0} \Upsilon_w \cdot \Upsilon_u = 2^{\binom{n}{2}} \qquad \text{(Cauchy identity)}$$

Comes from Cauchy identity of double Schubert polynomials

$$\sum_{w^{-1}u=w_0} \mathfrak{S}_w(\underline{x})\,\mathfrak{S}_u(\underline{y}) = \mathfrak{S}_{w_0}(\underline{x}, \underline{y})$$

$*$ Bergeron-Billey 93 have a proof of $(*)$ with double rc-graphs



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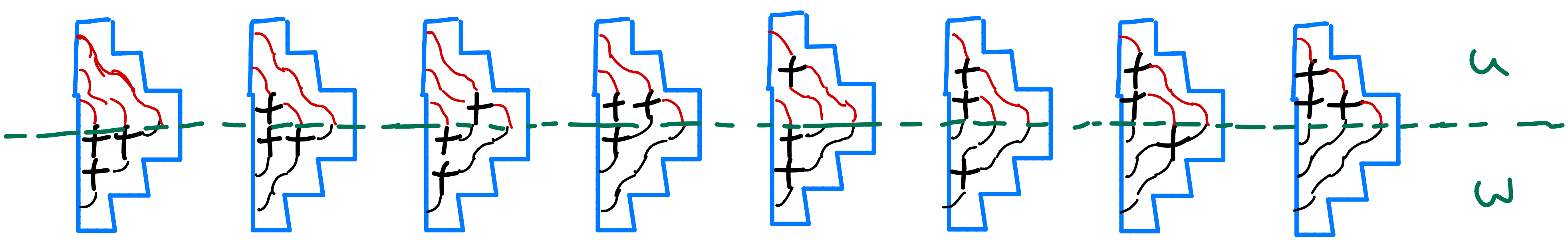

Double rc graphs

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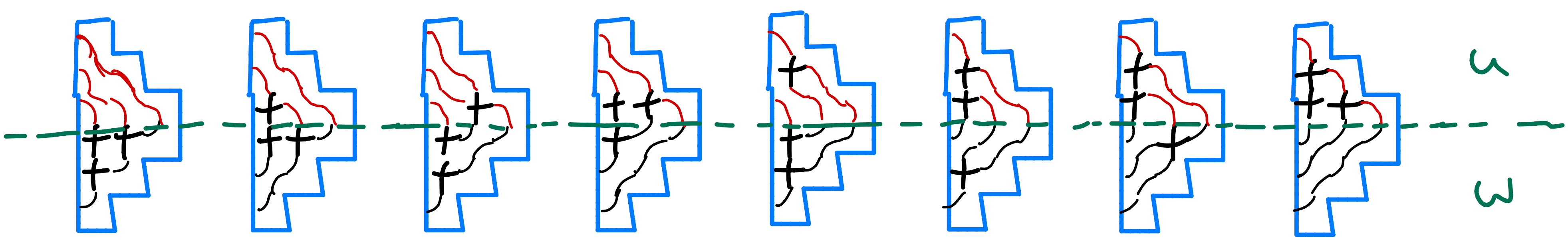
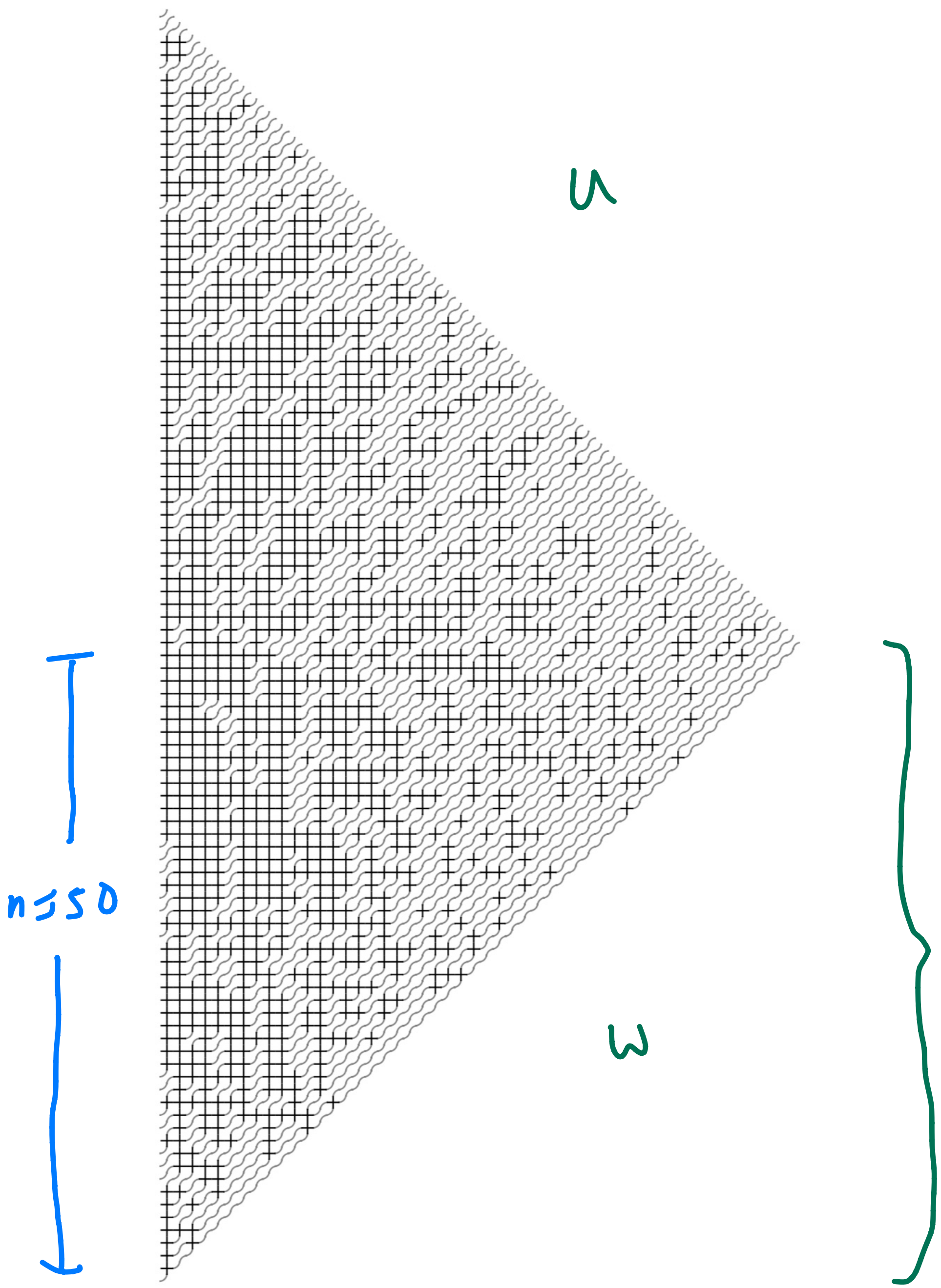
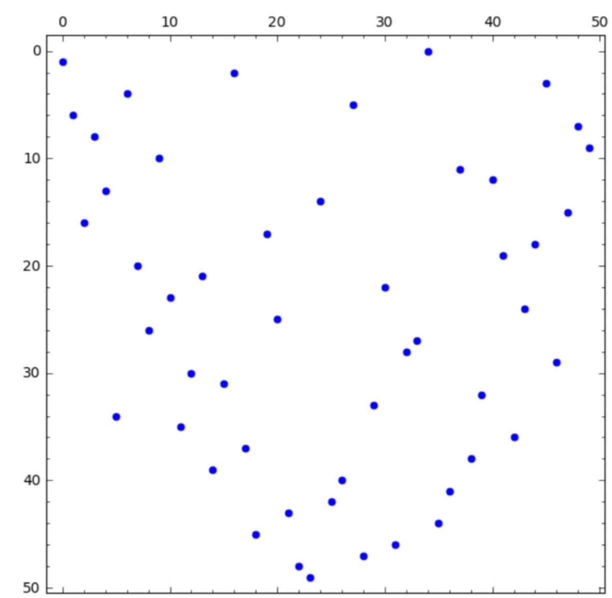


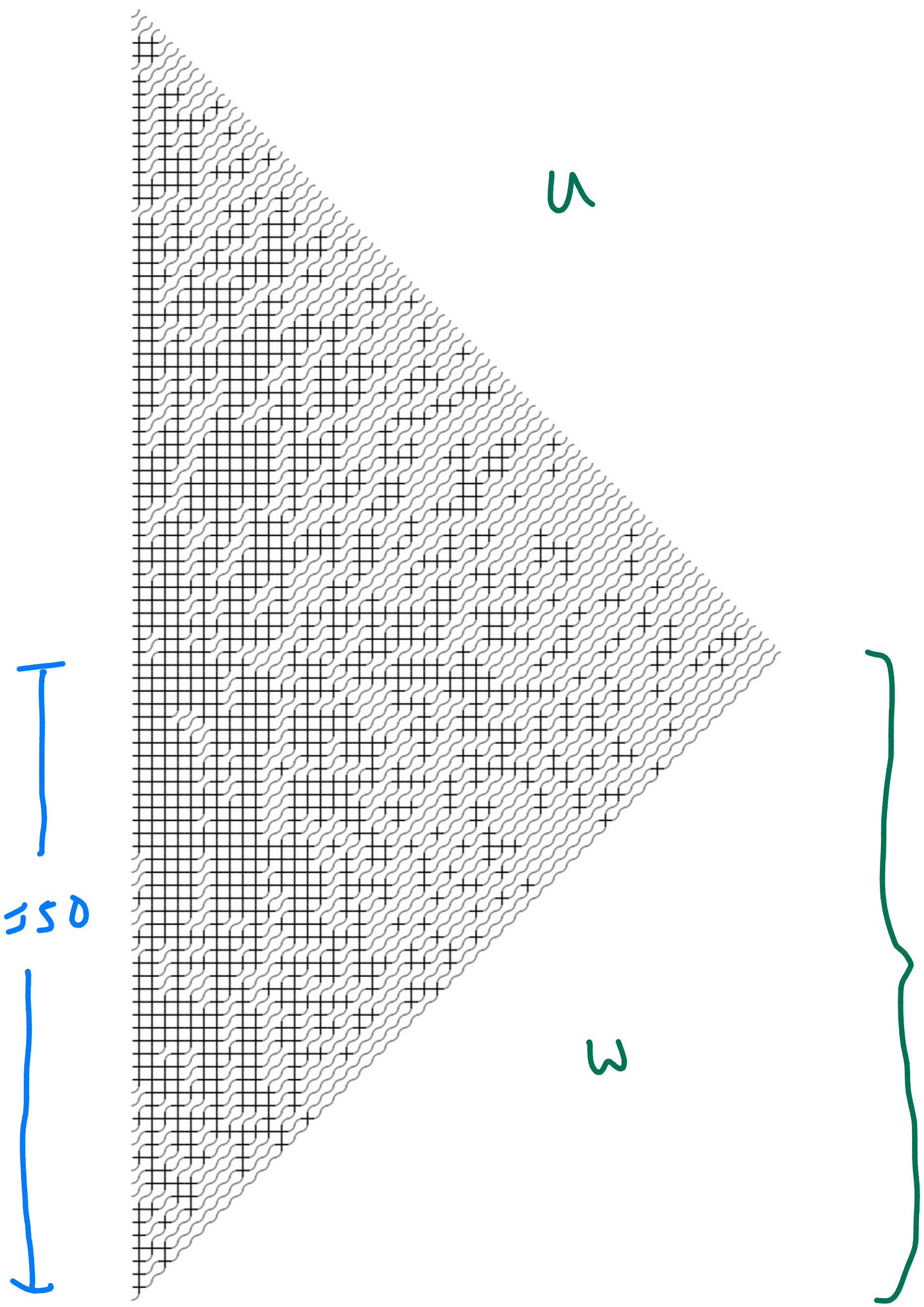
* double rc-graphs can be generated with local moves

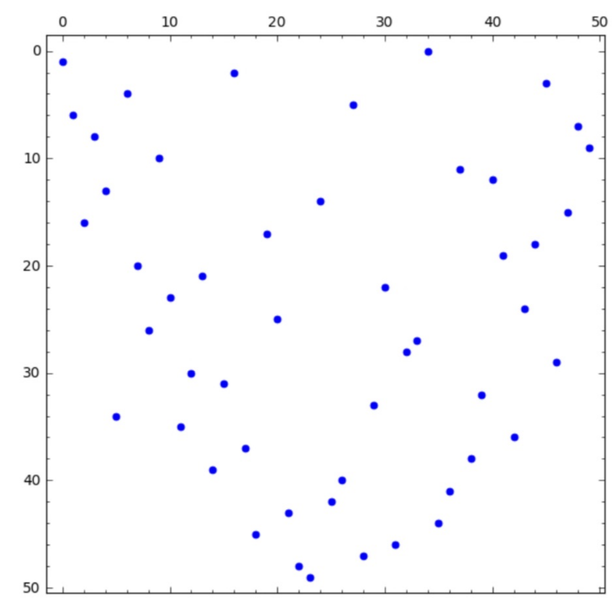
Simulation of a Markov chain of double rc-graphs

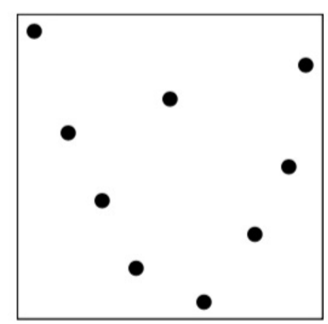




Simulation of a Markov chain of double rc-graphs







$w = 195283746$

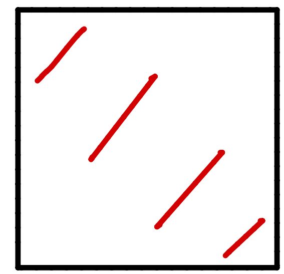
Summary: $u(n) := \max_{w \in S_n} \Upsilon_w \qquad v(n) := \max_{w \in S_n,\ \text{layered}} \Upsilon_w$

Conjecture 1 (Stanley 2017)

There is a limit $c := \lim_{n \to \infty} \frac{\log_2 u(n)}{n^2}$

Conjecture 2 (Merzon-Smirnov 16, Stanley 17)

Υ_w is maximal at layered permutations



Theorem (M-Pak-Panova 2018)

$$\lim_{n \to \infty} \frac{\log_2 v(n)}{n^2} = 0.2932372\ldots$$

Best bounds

$$0.2932372\ldots \leq c \leq 0.37$$

Thank you!

¡Gracias!