Plane partitions with periodic weights

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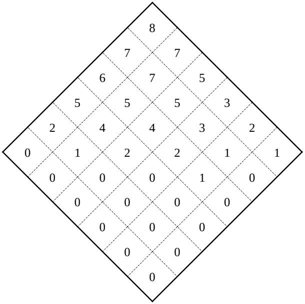
(parts joint with L. Petrov)

Plane partitions

Model: Fill an $aN \times bN$ rectangular portion of the square lattice with non-negative integers weakly decreasing in the directions of the axes.



Plane partitions

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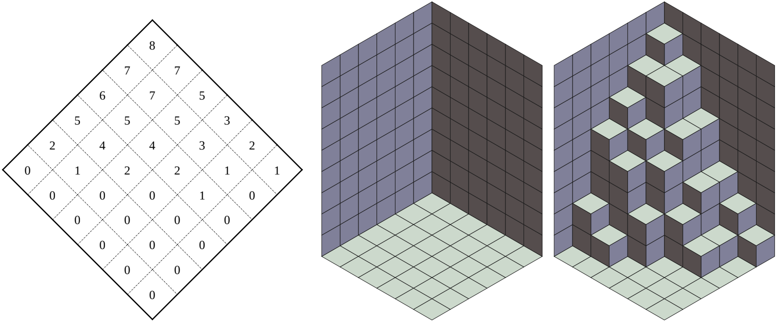

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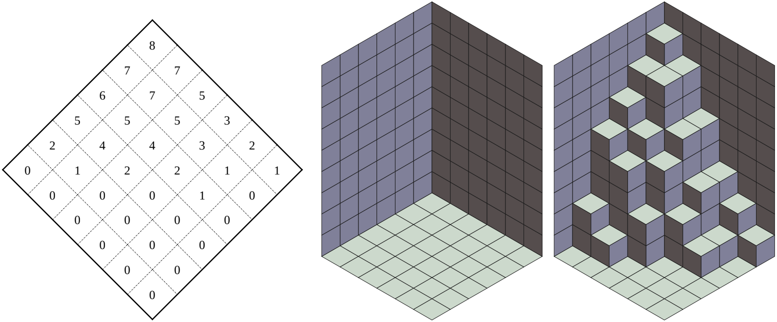
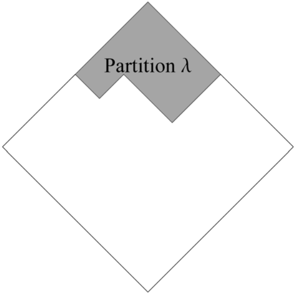


If put a restriction on the max of the integers (i.e. the height), get lozenge tilings of the hexagon.

Skew plane partitions

Remove a Young diagram λ_N from the corner of the $aN \times bN$ rectangle.



Skew plane partitions

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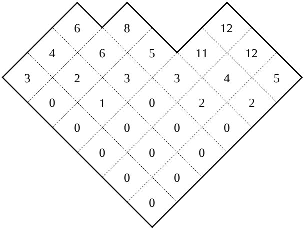
Fill the rest with non-negative integers, weakly decreasing as before.



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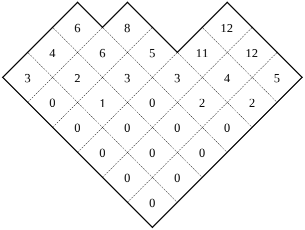
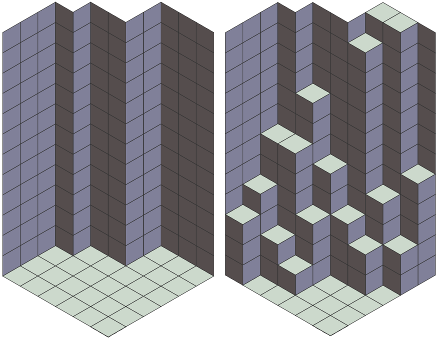
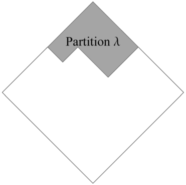
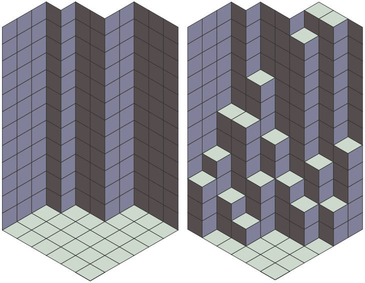




The volume measure





- Consider the stacks of boxes with boundary λ, confined to the $aN \times bN$ box, with the distribution

$$Prob(\pi) \propto q^{|\pi|} = q^{volume} = \prod_{\text{all boxes}} q_{\text{box}},$$

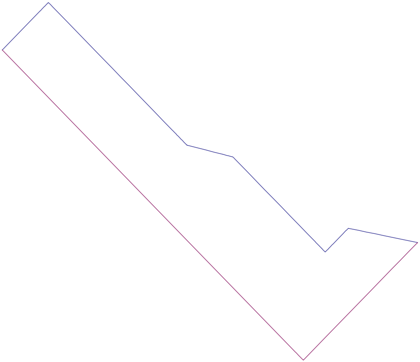
  for some $q \in (0,1)$, where $|\pi|$ is the total volume (i.e. the number of boxes). More precisely, $Prob(\pi) = \frac{q^{|\pi|}}{Z}$, where $Z = \sum_{\pi} q^{|\pi|}$ is the normalizing constant/partition function, which needs to be finite for the model to make sense.
- In the case when the height is restricted (i.e. lozenge tilings of a hexagonal region), $q = 1$ gives the uniform measure.

The scaling limit

Study the scaling limit of the system (Nienhuis-Hilhorst-Blöte 1984, Kenyon, Okounkov-Reshetikhin 2003, 2007, Boutillier-M.-Reshetikhin-Tingley 2012, M. 2011.)

- Should have a, b fixed, $N \to \infty$
- Have $q \in (0, 1)$. If q is fixed, the expected volume of the system is finite, so should have $q \to 1^-$. The right rate is $q = e^{-\gamma/N}$ for some $\gamma > 0$.
- The sequence of cutouts λ_N can converge to a 1-Lipschitz function.
- Most generally, the case when the limit is a piecewise linear function of slopes in $[-1, 1]$ has been studied.

Limit shapes - infinite regions



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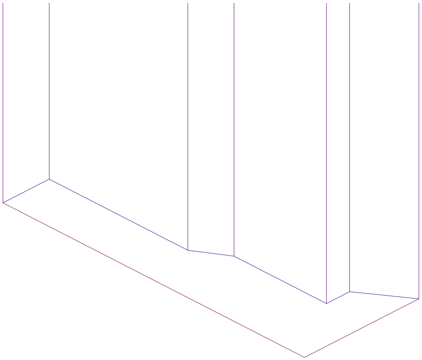

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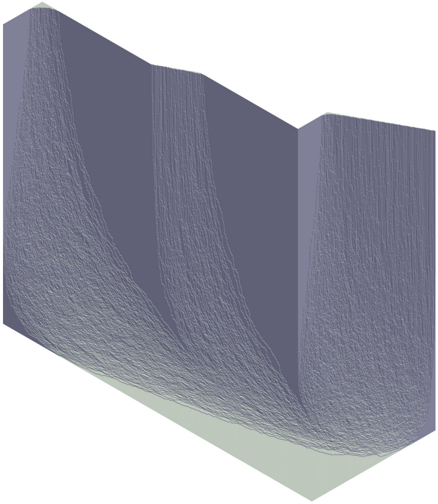

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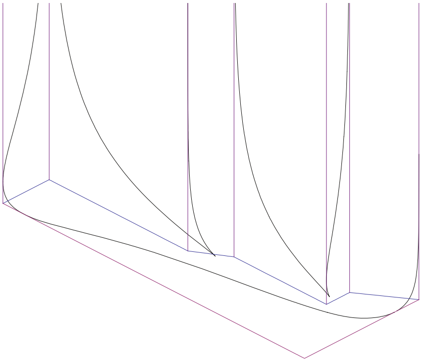
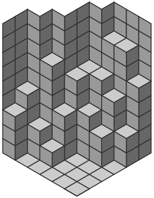


The correlation functions



- The positions of the horizontal tiles completely determine the (skew) plane partition.
- To understand the fluctuations, study the local correlation functions of the positions of the horizontal tiles.
- Denote by $\rho((t_1, h_1), \dots, (t_n, h_n))$ the probability that there are horizontal tiles at positions $(t_i, h_i), i = 1, \dots, n$.

The discrete correlation kernel

Theorem (Okounkov-Reshetikhin 2003)

- $\rho((t_1, h_1), \ldots, (t_n, h_n)) = \det(K((t_i, h_i), (t_j, h_j))_{i,j=1}^n$.
- *The correlation kernel* $K((t_1, h_1), (t_2, h_2))$ *is given by:*

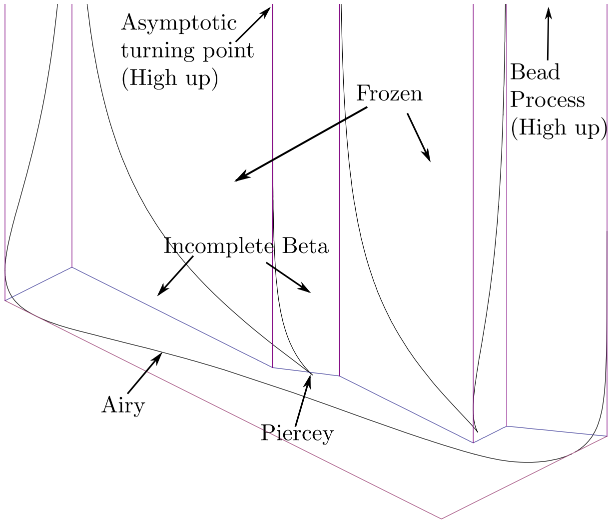
$$\frac{1}{(2\pi i)^2} \iint \frac{\Phi_-(z, t_1)\Phi_+(w, t_2)}{\Phi_+(z, t_1)\Phi_-(w, t_2)} \frac{\sqrt{zw}}{z - w} z^{-h_1 + b_{\lambda_q}(t_1) - 1/2} w^{h_2 - b_{\lambda_q}(t_2) - 1/2} \frac{dzdw}{zw},$$

where

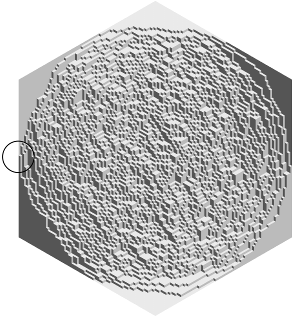
$$\Phi_\pm(z, t) = \prod_{\substack{m \gtrless t,\, m \in \mathbb{Z} + \frac{1}{2} \\ \text{slope at } m \text{ is } \mp 1}} (1 \mp z^{\pm 1} q^{\pm m}),$$

and b_{λ_q} *encodes the "back wall".*

Scaling limit - limit shape



The birth of a random matrix



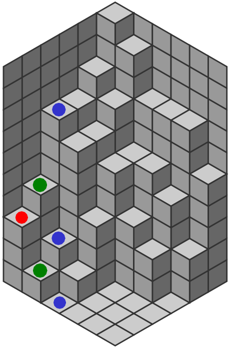
Conjecture (Okounkov-Reshetikhin 2006)

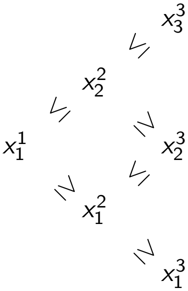
The point process at turning points converges to the GUE corners process.

Turning points

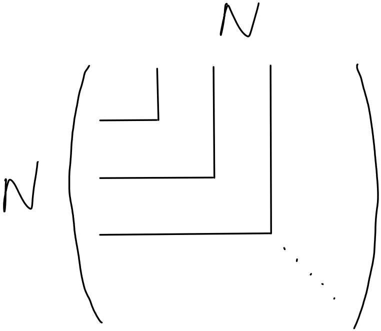
Consider the process near turning points:

- On a slice at distance k from the edge we have k horizontal lozenges. Let the heights be
$$x_1^k \leq x_2^k \leq \cdots \leq x_k^k.$$
- Slices interlace:





The GUE corners process



Let $\lambda_1^k \le \lambda_2^k \le \cdots \le \lambda_k^k$ be the eigenvalues of the $k \times k$ corner of H.
Two neighboring rows interlace:

$$\begin{array}{ccccccccc}
\lambda_1^k & & \lambda_2^k & & \cdots & & \lambda_{k-1}^k & & \lambda_k^k. \\
& \rotatebox{-45}{$\le$}\ \lambda_1^{k-1}\ \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$}\ \lambda_2^{k-1}\ \rotatebox{45}{$\le$} & \cdots & & & \rotatebox{-45}{$\le$}\ \lambda_{k-1}^{k-1}.\ \rotatebox{45}{$\le$} &
\end{array}$$

The GUE corners process

- The joint distribution of

$$\begin{array}{c} \lambda_1^k, \lambda_2^k, \quad \ldots, \quad \lambda_{k-1}^k, \lambda_k^k \\ \lambda_1^{k-1}, \lambda_2^{k-1}, \ldots, \lambda_{k-1}^{k-1} \\ \ldots \\ \lambda_1^3, \lambda_2^3, \lambda_3^3 \\ \lambda_1^2, \lambda_2^2 \\ \lambda_1^1 \end{array}$$

  is called the GUE corners process. Denote by $\mathbb{GUE}_k$.
- Conditioned on the top row and the interlacing condition, the lower entries are unifromly distributed.

The Okounkov-Reshetikhin conjecture

Conjecture (Okounkov-Reshetikhin 2006)

After appropriate centering and scaling the joint law of the heights x_i^k of the horizontal lozenges in the first k slices converges to the joint law of the eigenvalues λ_i^k of the first k corners of a GUE random matrix.

The Okounkov-Reshetikhin conjecture

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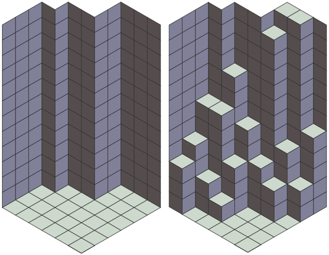


$$Prob(\pi) \propto q^{|\pi|} = q^{volume}$$

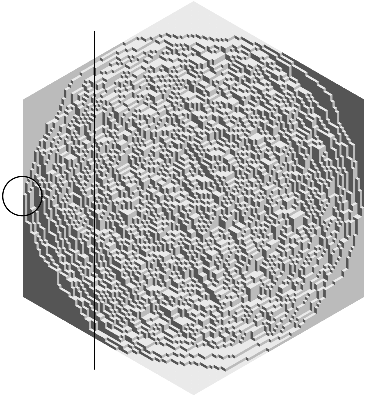
$$q \in (0,1)$$

Let $x_i^k, i \leq k$ be the heights of the first k slices as defined above.

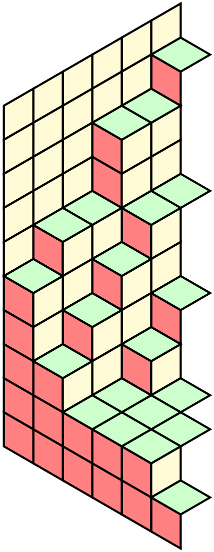
Theorem (Okounkov-Reshetikhin 2006)

Let $q = e^{-1/N}$. There exist constants C_0 and C_1 such that in the limit $q \to 1$ we have $\frac{x_i^k - NC_0}{\sqrt{NC_1}} \to \mathbb{GUE}_k$ in distribution.

Specifying a slice

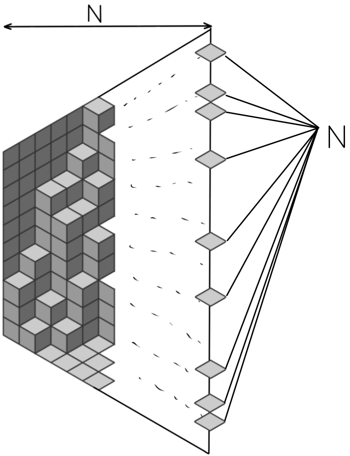


Stacks of boxes



CLT for the turning point

Let $x(1), x(2), \ldots$ be a sequence of signatures, where $x(N)$ has N parts. Consider the q^{volume} distribution on lozenge tilings whose N'th slice is $x(N)$. Let $x^k(N)$ be the k'th slice.



CLT for the turning point

Theorem (M.,Petrov 2017)

Suppose there exists a nonconstant weakly decreasing function $f(t)$ such that $x_i(N)/N$ converges pointwise and uniformly to f. E.g.

$$\left|\frac{x_i(N)}{N} - f(i/N)\right| = o(1/\sqrt{N})$$

as $N \to \infty$ will suffice. Then for every k, as $N \to \infty$ and $q \to 1$ as $q = e^{-\gamma/N}$ for some constant $\gamma \geq 0$, we have

$$\frac{x^k(N) - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k,$$

in the sense of weak convergence, for some explicit constants $E(f)$ and $S(f)$.

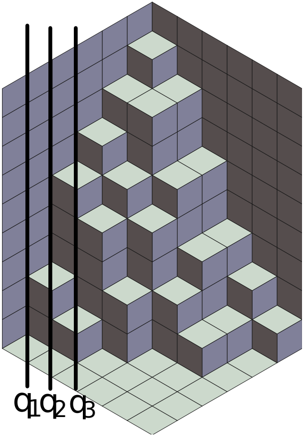
- $\gamma = 0$ corresponds to the uniform measure. Obtained earlier by Gorin,Panova. A different proof by Novak.
- Piecewise constant f corresponds to some polygonal regions, incuding the hexagon. Hexagon with the uniform measure was done first by Johansson, Nordenstam

Breaking away from the GUE corners process

Given $\{q_i\}_{i \in \mathbb{Z}}$, $q_i > 0$, consider plane partitions with the distribution

$$Prob(\pi) \propto \prod_{i \in \mathbb{Z}} q_i^{|\pi^i|},$$

where $|\pi^i|$ is the total volume of the i-th slice of π.



Periodic weights

- Consider periodic weights with period k:

$$\begin{aligned} q_0 &= q_{\pm k} = q_{\pm 2k} = \dots \\ q_1 &= q_{\pm k+1} = q_{\pm 2k+1} = \dots \\ q_2 &= q_{\pm k+2} = q_{\pm 2k+2} = \dots \\ &\dots \\ q_{k-1} &= q_{\pm 2k-1} = q_{\pm 3k-1} = \dots \end{aligned}$$

- What scaling limit should we study?
- Taking $q_i \to 1^-$ is natural, but doesn't produce new phenomena.
- Consider weights

$$q_i = \alpha_i q,$$

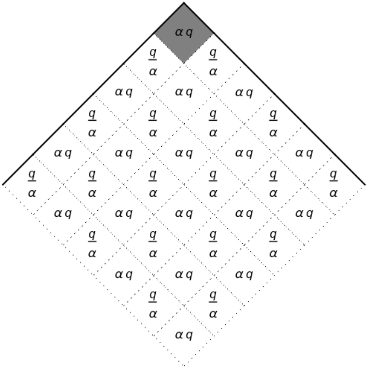
  $i = 0, \dots, k-1$ and consider the limit $q \to 1^-$.
- Must have

$$\prod_{i=0}^{k-1} \alpha_i = 1,$$

  however if we have $\alpha_i > 1$ for some i, the partition function will be infinite.

2-periodic weights - cut corner out

Consider weights αq, $\frac{1}{\alpha} q$ with $\alpha > 1$.



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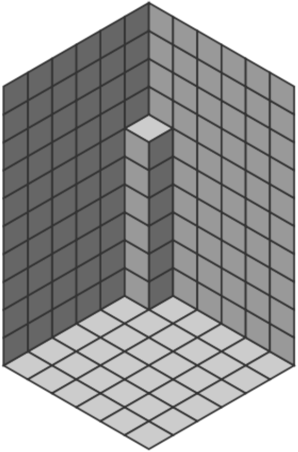

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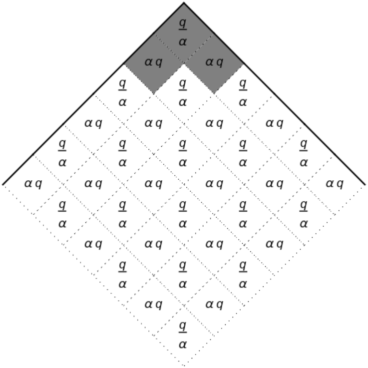

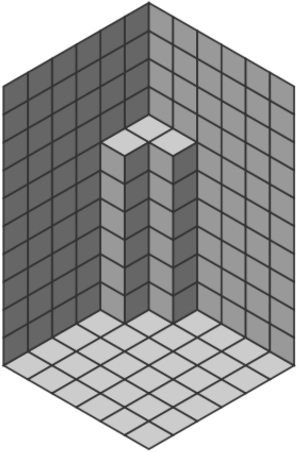

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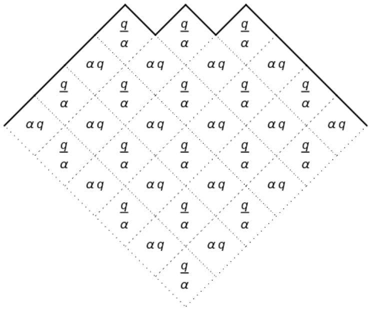

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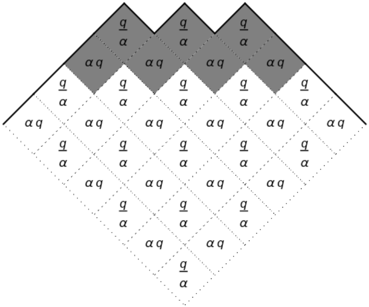

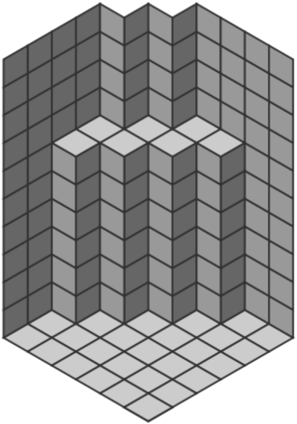

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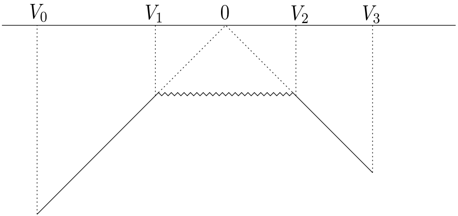
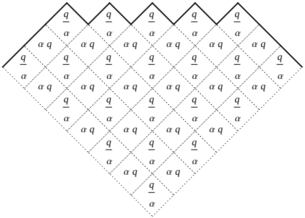
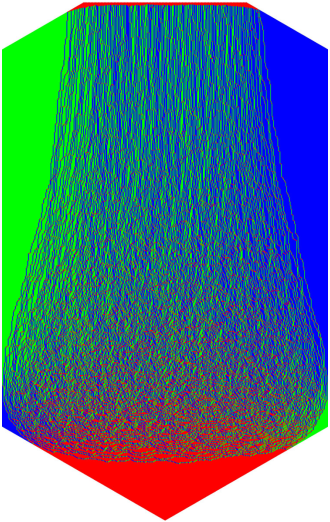
2-periodic weights - cut corner out

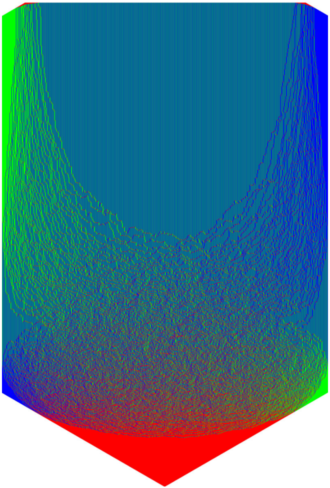
Consider weights $\alpha q, \frac{1}{\alpha} q$ with $\alpha > 1$.
If the cut is of width fN, then the weight of a strip is $\alpha q^{fN} = \alpha e^{-\frac{\gamma}{N} fN} = \alpha e^{-\gamma f}$, so if f is large enough, this is < 1, and the partition function ends up being finite.



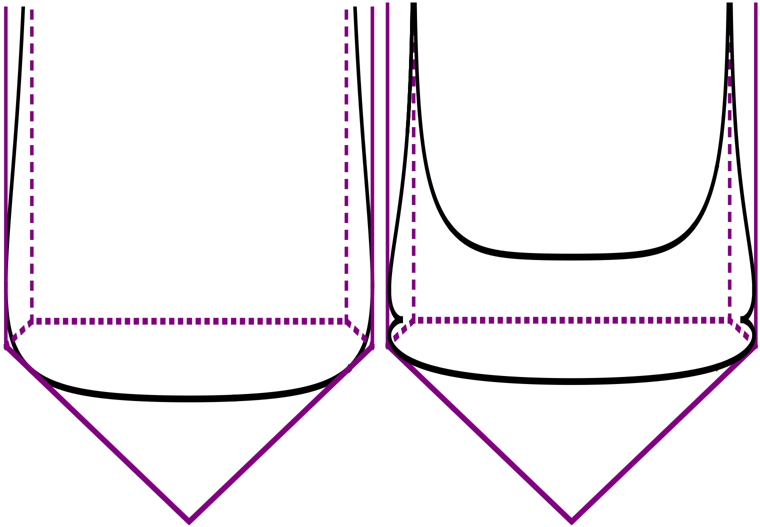


2-periodic weights - cut corner: a sample



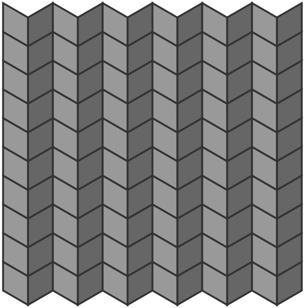


2-periodic weights - cut corner: frozen boundary



Turning points

- There are two turning points near each vertical boundary section.
- The distance between the turning points converges to zero when α converges to 1.
- Turning points are separated by a deterministic region of two types of tiles:



GUE corners at the turning points

The fact that there are two turning points implies that locally you do not have the GUE corners process - the number of particles is not correct.

- To have GUE corners, the number of particles on the vertical slices should be $1, 2, 3, 4, 5, 6, 7, 8, \ldots$
- Now, the number of particles on vertical slices near each turning point is $1, 1, 2, 2, 3, 3, 4, 4, \ldots$
- However, if you pick only the particles on the odd-number slices you get the right number of particles and indeed the GUE corners process. The same goes for the even-number slices.
- In fact, you have two copies of the GUE corners process, non-trivially correlated.

Periodic weights with one defect

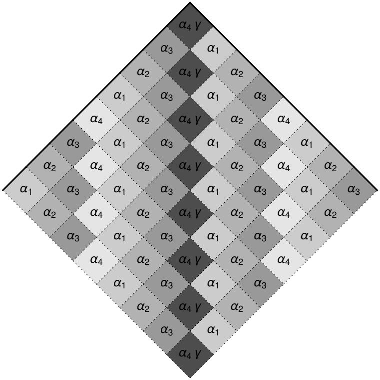
- For 2-periodic weights the issue of ∞ partition function can be resolved by taking the right boundary conditions, but that doesn't work for higher periods.
- Alternate solution: periodic weights with one "defect": let

$$\gamma = \prod_{\alpha_i < 1} \alpha_i$$

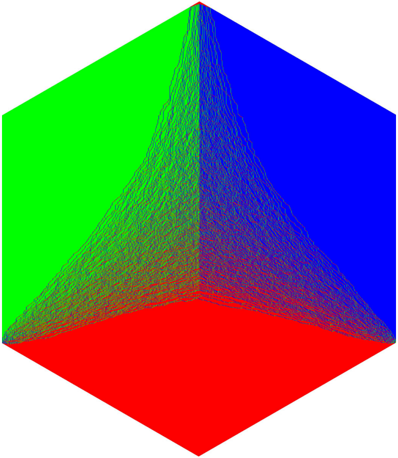
and consider plane partitions with weights

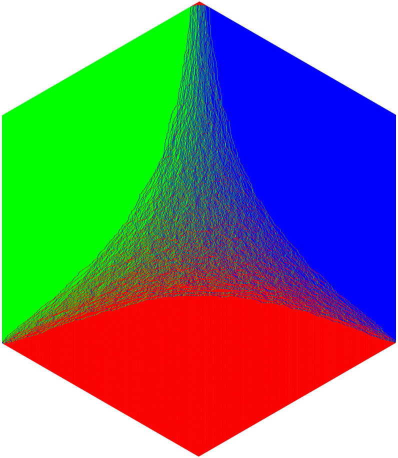
$$q_i = \begin{cases} q\alpha_{i \bmod k}, & i \neq 0 \\ q\gamma\alpha_0, & i = 0. \end{cases} .$$

- This way, for all $m < 0 < n$ we have $\prod_{i=m}^{n} q_i < 1$, so the partition function doesn't blow up.



Samples





Kernel at the transition - 2-periodic weights

Given integers t_1, t_2, let $N(t_1, t_2)$ be the number of half integers m between t_1 and t_2 such that $y_m \neq 1$, taken with a positive sign if $t_1 \geq t_2$ and with a negative sign otherwise, with y_m as in the figure.

y_m	$\frac{1}{\alpha}$	1	$\frac{1}{\alpha}$	1	$\frac{1}{\alpha}$	1	1	$\frac{1}{\alpha}$	1	$\frac{1}{\alpha}$	1	$\frac{1}{\alpha}$
m	$-\frac{11}{2}$	$-\frac{9}{2}$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$

Theorem (M. 2019)

Let $r > 0$ and $q = e^{-r}$. Consider plane partitions with 2-periodic weights with a defect in the middle slice. In the limit $r \to 0$, along the middle slice, the correlation functions of the system are given by determinants of the kernel

$$\lim_{r\to 0} K_{\bar{q}}((t_1, h_1), (t_2, h_2)) = \frac{1}{2\pi \mathfrak{i}} \int_C (1-z)^{\Delta t - N(t_1,t_2)} (1 - z/\alpha)^{N(t_1,t_2)} z^{-\Delta h - \frac{1}{2}\Delta t - 1} dz,$$

where $\Delta t = t_1 - t_2$, $\Delta h = h_1 - h_2$, t_1, t_2 are fixed, $rh_i \to \chi$ with Δh fixed.

The bulk process - k-periodic weights

Theorem (M. 2019)

Let $\beta_{i+\frac{1}{2}} = \alpha_0\alpha_1 \cdot \ldots \cdot \alpha_i$ and let l be the number of distinct β's. Given a β_i, let ΔN_i count the number of half-integers m between t_1 and t_2 such that $\beta_m = \beta_{i+\frac{1}{2}}$. The correlation functions of the system near a point (τ, χ) in the bulk are given by determinants of the kernel

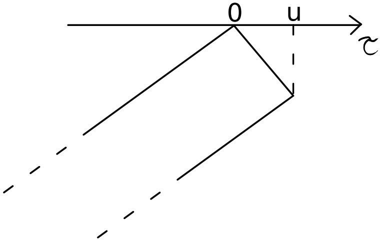
$$\lim_{r\to 0} K_{\bar{q}}((t_1, h_1), (t_2, h_2)) = \frac{1}{2\pi\mathfrak{i}} \int_C \prod_{i=1}^{l} \left(1 - ze^{-\tau}\beta_{i+\frac{1}{2}}^{-1}\right)^{\Delta N_i} z^{-\Delta h - \frac{1}{2}\Delta t - 1} dz,$$

where $\Delta t = t_1 - t_2$, $\Delta h = h_1 - h_2$, $rt_i \to \tau$, and $rh_i \to \chi$.

Notice, that $\sum_{i=1}^{l} \Delta N_i = \Delta t$. In particular, in the homogeneous case $\alpha_1 = \cdots = \alpha_k = 1$, we have $k = 1$ and $\beta_1 = 1$, so we recover the incomplete beta kernel. Otherwise, the process we observe is translation invariant under translations by $k\mathbb{Z} \times \mathbb{Z}$ only and is a special case of processes studied previously by Borodin and Shlosman (2010).

Turning points with k-periodic weights

- For simplicity consider a semi-infinite floor.



- Let $\beta_i = \prod_{j=1}^{i} \alpha_j$ for $i = 1, \ldots, k$.
- Let

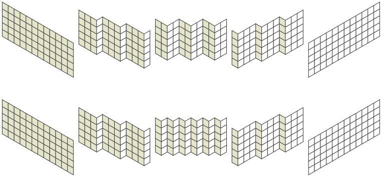
$$\tilde{\beta}_1 < \cdots < \tilde{\beta}_m$$

be distinct such that

$$\{\beta_1, \ldots, \beta_k\} = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_m\}.$$

Turning points and the GUE Corners process

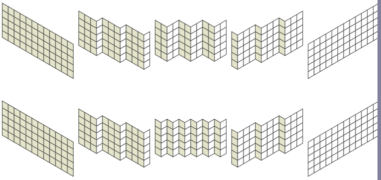
- m turning points.
- Separated by new types of frozen regions/ facets.



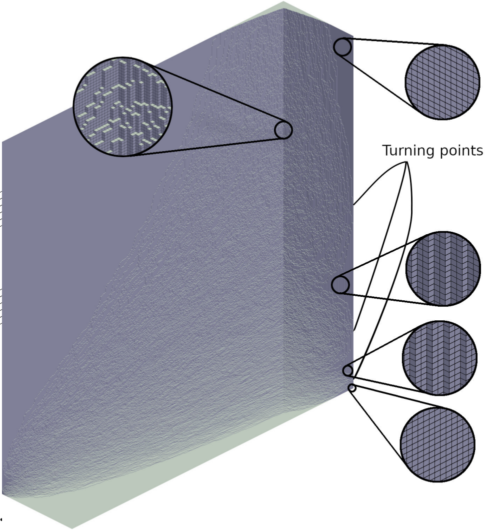
- Number of particles on the vertical slices near a turning point are $1, 1, \ldots, 1, 2, 2, \ldots, 2, 3, 3, \ldots, 3, \ldots$.

Turning points and the GUE Corners process

- m turning points.
- Separated by new types of frozen regions/ facets.



- Number of particles on the vertical slices near a turning point are $1, 1, \ldots, 1, 2, 2, \ldots, 2, 3, 3, \ldots$



Turning points for k-periodic weights

Theorem (M. 2019)

Let $r > 0$. Consider plane partitions with periodic weights as above. Near the vertical boundary the system develops m turning points, one for each $\tilde{\beta}_i$. The correlation functions of the system near it are given by determinants of the kernel

$$K_{\bar{q}}^{j}((\hat{t}_1, \hat{h}_1), (\hat{t}_2, \hat{h}_2)) = \frac{\sqrt{r}}{(2\pi \mathrm{i})^2} \int_{z \in C_z'} \int_{w \in C_w'} e^{C_j(\xi^2 - \omega^2)} \frac{\omega^{N_{\hat{t}_2, j}}}{\xi^{N_{\hat{t}_1, j}}} \frac{e^{\hat{h}_2 \omega}}{e^{\hat{h}_1 \xi}} \frac{d\xi \; d\omega}{(\xi - \omega)},$$

where $\hat{t}_i = u - t_i$ is the horizontal distance from the boundary, $\frac{\hat{h}_i}{r^{1/2}}$ is the vertical distance from χ_j and $N_{t,j} = \#\{m \in \mathbb{Z} + \frac{1}{2} : u - t < m < u, \beta_{d-m-\frac{1}{2}} = \tilde{\beta}_j\}$.

Corollary

For any sequence of slices $t_1 > t_2 > \dots$ such that $N_{t_i, j} = i$ (i.e. the i'th chosen slice has i horizontal lozenges), the point process of the horizontal lozenges on these slices is the GUE corner process.

Remark: Interlacing is not a geometric constraint anymore.

Intermediate regime

- Consider two-periodic weights αq, $\frac{1}{\alpha} q$.
- Question: What happens when $\alpha \to 1$?
- Consider two-periodic weights q_t given by

$$q_t = \begin{cases} e^{-r+\gamma r^{1/2}}, & t \text{ is even} \\ e^{-r-\gamma r^{1/2}}, & t \text{ is odd} \end{cases}, \tag{1}$$

  where $\gamma > 0$ is an arbitrary constant. This is an intermediate regime between the homogeneous weights and the inhomogeneous weights considered earlier.
- The macroscopic limit shape and correlations in the bulk are the same as in the homogeneous case.
- Periodicity disappears in the limit and we have a $\mathbb{Z} \times \mathbb{Z}$ translation invariant ergodic Gibbs measure in the bulk. However, the local point process at turning points is different from the homogeneous one. In particular, while we only have one turning point near each edge, we still do not have the GUE corners process, but rather a one-parameter deformation of it.

Turning point correlations in the intermediate regime

Theorem (M. 2014)

Let χ be the expected hight of a turning point and let $h_i = \lfloor \frac{\chi}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$, where $q = e^{-r}$. The correlation functions near a turning point of the system with periodic weights(1) are given by

$$\lim_{r \to 0} r^{-\frac{1}{2}} K_{\lambda, \bar{q}}((t_1, h_1), (t_2, h_2))$$
$$= \frac{1}{(2\pi \mathrm{i})^2} \iint e^{C_{cr}(\zeta^2 - \omega^2)} \frac{e^{\tilde{h}_2 \omega}}{e^{\tilde{h}_1 \zeta}} \frac{\omega^{\lfloor \frac{t_2+1}{2} \rfloor}}{\zeta^{\lfloor \frac{t_1+1}{2} \rfloor}} \frac{(\omega - \gamma)^{\lfloor \frac{t_2+2}{2} \rfloor}}{(\zeta - \gamma)^{\lfloor \frac{t_1+2}{2} \rfloor}} \frac{d\zeta \, d\omega}{\zeta - \omega}.$$

Thank you