Understanding the asymptotics of the number of tableaux of skew shape through a variational principle

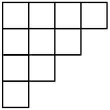
Martin Tassy, joint work with Alejandro Morales and Igor Pak

Dartmouth College

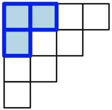
February 6, 2020

Notations

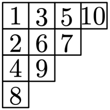
λ: partition shape



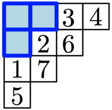
λ/μ: skew shape



$f^{\lambda} = \#$ SYT of shape λ



$f^{\lambda/\mu} = \#$ SYT of shape λ/μ

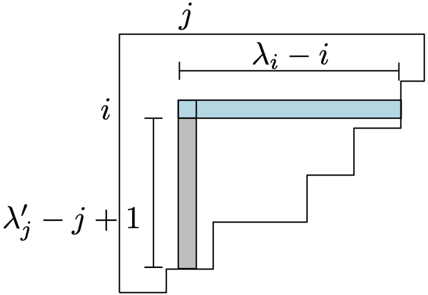


Hook-length formula

Theorem (Frame-Robinson-Thrall 1954)

$$f^\lambda = N! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)},$$

$h(i,j) = \lambda_i - i - \lambda'_j - j + 1$ *is the* ***hook-length*** *of* (i,j).

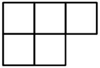


Hook-length formula

Theorem (Frame-Robinson-Thrall 1954)

$$f^{\lambda} = N! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)},$$

$h(i,j) = \lambda_i - i - \lambda'_j - j + 1$ *is the* ***hook-length*** *of* (i,j).



4	3	1
2	1	

$$f^{\yng(3,2)} = \frac{5!}{1^2 \cdot 2 \cdot 3 \cdot 4} = 5$$

Outline

- $f^\lambda = \dfrac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$
- asymptotics

- $f^{\lambda/\mu} = ?$

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

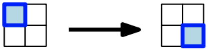
Theorem (Naruse 2014)

$$f^{\lambda/\mu} = N! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

where $\mathcal{E}(\lambda/\mu)$ is the set of ***excited diagrams*** *of λ/μ.*

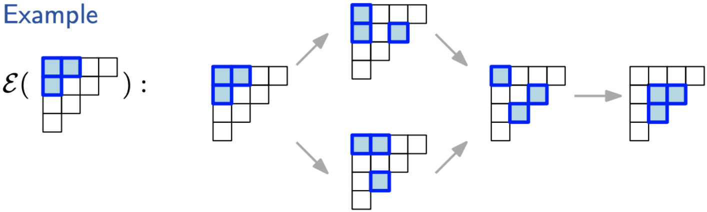
Excited diagrams of λ/μ

An **excited move** on an excited cell (i, j) in $S \subset \lambda$ replaces (i, j) in S by $(i+1, j+1)$.



Definition

Excited diagrams $\mathcal{E}(\lambda/\mu)$: diagrams obtained from μ by applying iteratively excited moves on excited cells.

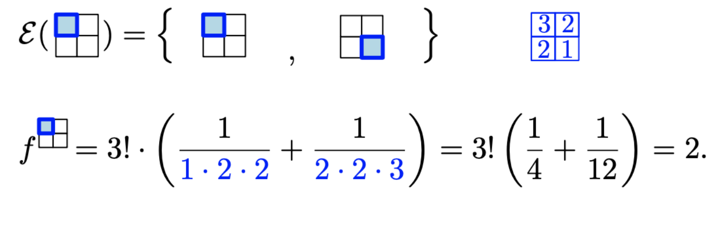


Theorem (Naruse 2014)

$$f^{\lambda/\mu} = N! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

where $\mathcal{E}(\lambda/\mu)$ *is the set of* ***excited diagrams*** *of* λ/μ.

Example



Outline

- $f^\lambda = \dfrac{N!}{\prod_{u\in\lambda} h(u)}$
- asymptotics

- $f^{\lambda/\mu} = N! \sum_{D\in\mathcal{E}(\lambda/\mu)} \cdots$
- asymptotics?

Conjecture on the asymptotics of tableaux of skew shape

A sequence of partitions $\{\lambda^{(N)}\}$ is strongly stable if it satisfies the following property:

$$(\sqrt{N} - L)\psi < [\lambda^{(N)}] < (\sqrt{N} + L)\psi$$

Conjecture (Morales, Panova and Pak)

For a strong stable shape $\nu_N = \lambda^{(N)}/\mu^{(N)}$ converging to ψ/ϕ and such that $area(\psi/\phi) = 1$:

$$\frac{1}{N}\left(\log f^{\nu_N} - \frac{1}{2}N\log N\right) \to c(\psi/\phi)$$

Rewriting Naruse's formula

$$f^{\lambda/\mu} = N! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)}$$

$$= N! \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)} \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} h(i,j)$$

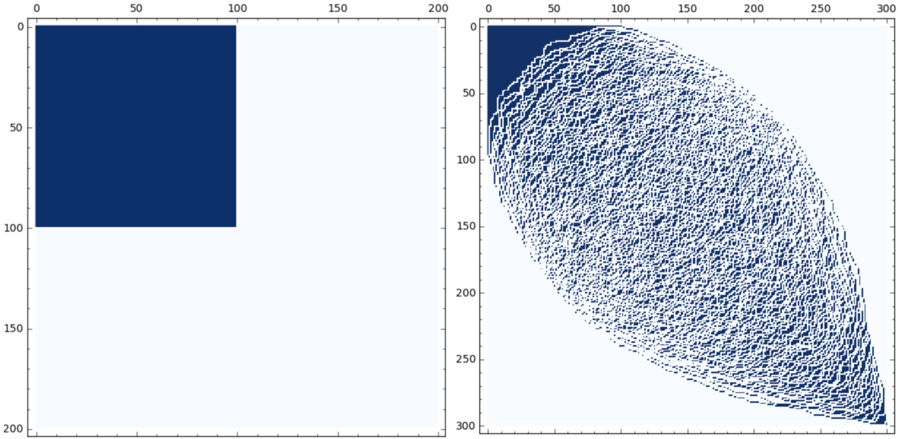
Taking the logarithm we obtain:

$$\log f^{\lambda/\mu} = \log \left(N! \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)} \right) + \log \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} h(i,j)$$

The asymptotics of the first term in the LHS are obtainable through the hook formula. Hence we are interested in

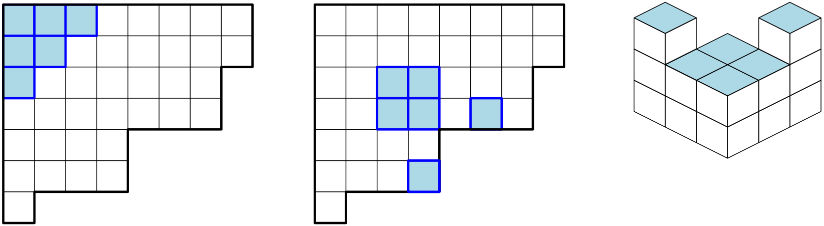
$$\log \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} h(i,j).$$

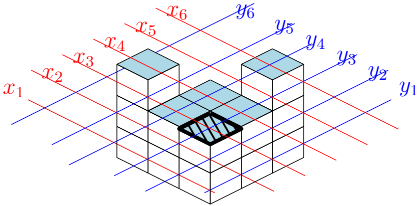
Simulations of the excited measure



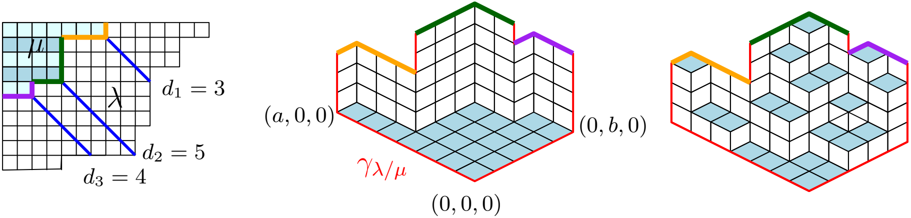
The existence of a limit shape suggests that the "excited measure" concentrates towards an asymptotic profile.

From excited diagrams to weighted lozenge tilings





The weight of a blue lozenge with coordinates (i, j) is given by $\lambda_i - i + \lambda'_j - j + 1$.



The boundary curve γ_n of the corresponding region can be described in the following way:

1. The bottom and sides of the region are obtained by interpolating linearly between the points $(a, 0, d_1)$, $(a, 0, 0)$,$(0, 0, 0)$,$(0, b, 0)$ and $(0, b, d_k)$.
2. Starting from the pont $(0, a, d_1)$ moves:
   - In the xy-plane, according to μ between any two inner corners of μ.
   - Vertically by $d_i - d_{i+1}$ at each inner corner.

The variational principle for lozenge tilings

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of curves such that $\frac{1}{n}\gamma_n$ converges to a closed curve γ in $\mathbb{R}^3$ in the ℓ_∞ norm.

Theorem (Kenyon 2009)

The number N_{γ_n} of lozenge tlings with boundary γ_n satisfies

$$\lim_{n\to\infty}\frac{1}{n^2}\log N_{\gamma_n}\to\Phi(g_{\max})$$

where $g_{\max}$ is the only extension of γ which maximizes the inegral

$$\Phi(g):=\int_U\sigma(\nabla g)dx_1dx_2$$

and U is the region enclosed by the projection of γ in the xy-plane.

Important observations

1. The VP gives the first term of the asymptotics for the logarithm of the partition function.

Important observations

1. The VP gives the first term of the asymptotics for the logarithm of the partition function.
2. The VP implies that all the tilings look asymptotically the same with overwhelming probability.

Important observations

1. The VP gives the first term of the asymptotics for the logarithm of the partition function.
2. The VP implies that all the tilings look asymptotically the same with overwhelming probability.
3. Requires that all the discrete quantities converge to continuous objects after rescaling.

Rescaling the hook function

At this stage, the local hamiltonian function representing the hook

$$\log h(x, y) = \log \left(\lambda_x - x + \lambda'_y - y + 1\right)$$

does not rescale with the system.

Rescaling the hook function

At this stage, the local hamiltonian function representing the hook

$$\log h(x, y) = \log \left(\lambda_x - x + \lambda'_y - y + 1\right)$$

does not rescale with the system.

We rewrite it as:

$$\log h(x, y) = \log n + w_n(x, y)$$

with $w_n(x, y) = \log \left(\frac{1}{n}\left(\lambda_x - x + \lambda'_y - y + 1\right)\right)$. Now $w_n(nx, ny)$ has a well defined limit $\hbar_\psi$ when $n \to \infty$.

Rescaling the hook function

We can now rewrite our asymptotics as:

$$\begin{aligned}\log \sum_{T\in\mathcal{T}(\lambda/\mu)} \prod_{(x,y)\in\blacklozenge} h(x,y) &= \log \sum_{T\in\mathcal{T}(\lambda/\mu)} \prod_{(x,y)\in\blacklozenge} h(x,y) \\ &= \log \sum_{T\in\mathcal{T}(\lambda/\mu)} \prod_{(x,y)\in\blacklozenge} e^{\log n + w_n(x,y)} \\ &= |\mu_n| \log n + \log \sum_{T\in\mathcal{T}(\lambda/\mu)} \prod_{(x,y)\in\blacklozenge} e^{w_n(x,y)}\end{aligned}$$

Under the hyptothesis of strong strability the term $|\mu_n| \log n$ converges nicely. Hence we are solely interested in

$$\log \sum_{T\in\mathcal{T}(\lambda/\mu)} \prod_{(x,y)\in\blacklozenge} e^{w_n(x,y)}$$

Proof of the weighted variational principle

The idea of the proof is similar to the original VP:

1. Undercount and overcount the weight of configurations which stays close to a piecewise affine asymptotic profile f.
2. Let the mesh of the affine interpolation go to 0 to obtain bounds for all asymptotic profiles.
3. Show that both bounds are aymptotically the same.

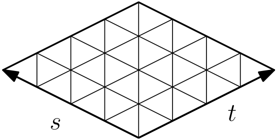
Proof of the weighted variational principle

The idea of the proof is similar to the original VP:

1. Undercount and overcount the weight of configurations which stays close to a piecewise affine asymptotic profile f.
2. Let the mesh of the affine interpolation go to 0 to obtain bounds for all asymptotic profiles.
3. Show that both bounds are aymptotically the same.

The expression of the functionnal in the integral arises whene evaluating the weight of configurations with approximately linear slope.

Tiling weight of a macroscopic lozenge



The weight of tilings with slope (s,t) on a lozenge of size m centered at (na, nb) can be written as:

$$
\begin{aligned}
\log \sum_{T \in \mathcal{T}_m(s,t)} \prod_{(x,y)\in \blacklozenge} e^{w_n(x,y)} &= \log \sum_{T \in \mathcal{T}_m(s,t)} \prod_{(x,y)\in \blacklozenge} e^{\hbar_\psi(a,b)+o(n)} \\
&= \log \sum_{T \in \mathcal{T}_m(s,t)} e^{\left(\hbar_\psi(a,b)+o(n)\right)|\blacklozenge|} \\
&= e^{m^2\sigma(s,t)+O(m)} e^{\left(\hbar_\psi(a,b)+o(n)\right)(1-s-t)m^2+o(m^2)} \\
&= e^{m^2\left(\sigma(s,t)+\hbar_\psi(a,b)(1-s-t)\right)+o(m^2)}
\end{aligned}
$$

Variational principle for lozenge tlings with local weights

Let $\{\gamma_n\}_{n\in N}$ be a sequence of curves such that $\frac{1}{n}\gamma_n$ converges to a closed curve γ in $\mathbb{R}^3$ in the ℓ_∞ norm and let $\{w_n\}_{n\in N}$ be a sequence of weight functions such that
$\lim_{n\to\infty} \sup_{(x,y)\in U} \|w_n(nx, ny) - \rho(x,y)\|_\infty = 0.$

Theorem (Morales, Pak, T.)

The number N_{γ_n} of lozenge tlings with boundary γ_n satisfies

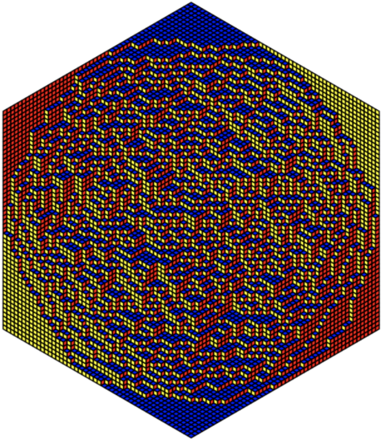
$$\lim_{n\to\infty} \frac{1}{n^2} \log W_{\gamma_n} \to \Phi(g_{\max})$$

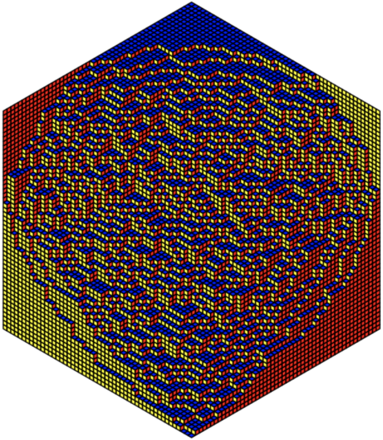
where $g_{\max}$ is the only extension of γ in $P_{1,1,1}$ which maximizes the integral

$$\Phi(g) := \int_U \sigma(\nabla g) + L_\rho(x_1, x_2, \nabla g) dx_1 dx_2$$

and $L_\rho(x_1, x_2, \nabla g) = (\rho_1, \rho_2, \rho_3) \cdot (\nabla g_1, \nabla g_2, 1 - \nabla g_1 - \nabla g_2)$

Excited measure vs uniform measure for lozenge tilings





Consequence for SYT of skew shape

1. The conjecture about strongly stable shapes is true. The constant $c(\psi/\phi)$ is given by

$$c(\psi/\phi) := k(\psi) + \Psi(f_{\max}),$$

where

$$k(\psi) = \int_{\mathcal{C}(\psi)} \hbar_\psi(x, y) dx dy,$$

$$\Psi(f_{\max}) = \max_{f \in P_{1,1,1}} \int_{\mathcal{C}(\phi)} \Big(\sigma(\nabla f) + (1 - \partial_x f - \partial_y f) \hbar_\psi(x, y) \Big) dx dy,$$

Consequence for SYT of skew shape

1. The conjecture about strongly stable shapes is true. The constant $c(\psi/\phi)$ is given by

$$c(\psi/\phi) := k(\psi) + \Psi(f_{\max}),$$

where

$$k(\psi) = \int_{\mathcal{C}(\psi)} \hbar_\psi(x,y)dxdy,$$

$$\Psi(f_{\max}) = \max_{f \in P_{1,1,1}} \int_{\mathcal{C}(\phi)} \Big(\sigma(\nabla f) + (1 - \partial_x f - \partial_y f)\hbar_\psi(x,y)\Big)dxdy,$$

2. Under the "hook measure", the excited lozenge concentrate around the unique asymptotic profile $f_{\max}$.

References

- Morales, A., Pak, I. and Panova, G. ”Hook formulas for skew shapes III. Multivariate and product formulas.” *arXiv:1707.00931 (2017).*

- Cohn, H., Kenyon, R., and Propp, J. (2001). A variational principle for domino tilings. *Journal of the American Mathematical Society, 14(2), 297-346.*

- Morales, A., Pak, I., and T. (2018). Asymptotics for the number of standard tableaux of skew shape and for weighted lozenge tilings. *arXiv:1805.00992.*

- Krieger, A., Menz, G. and T. ”Deducing a variational principle with minimal a priori assumptions.” *arXiv:1910.01042 (2019).*

- Sun, W. ”Dimer model, bead model and standard young tableaux: finite cases and limit shapes.” *arXiv:1804.03414 (2018)*