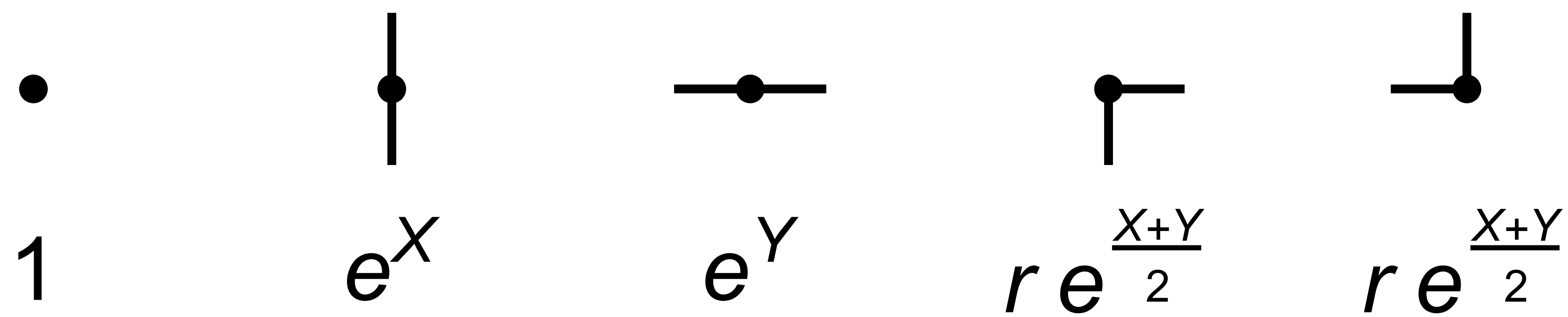
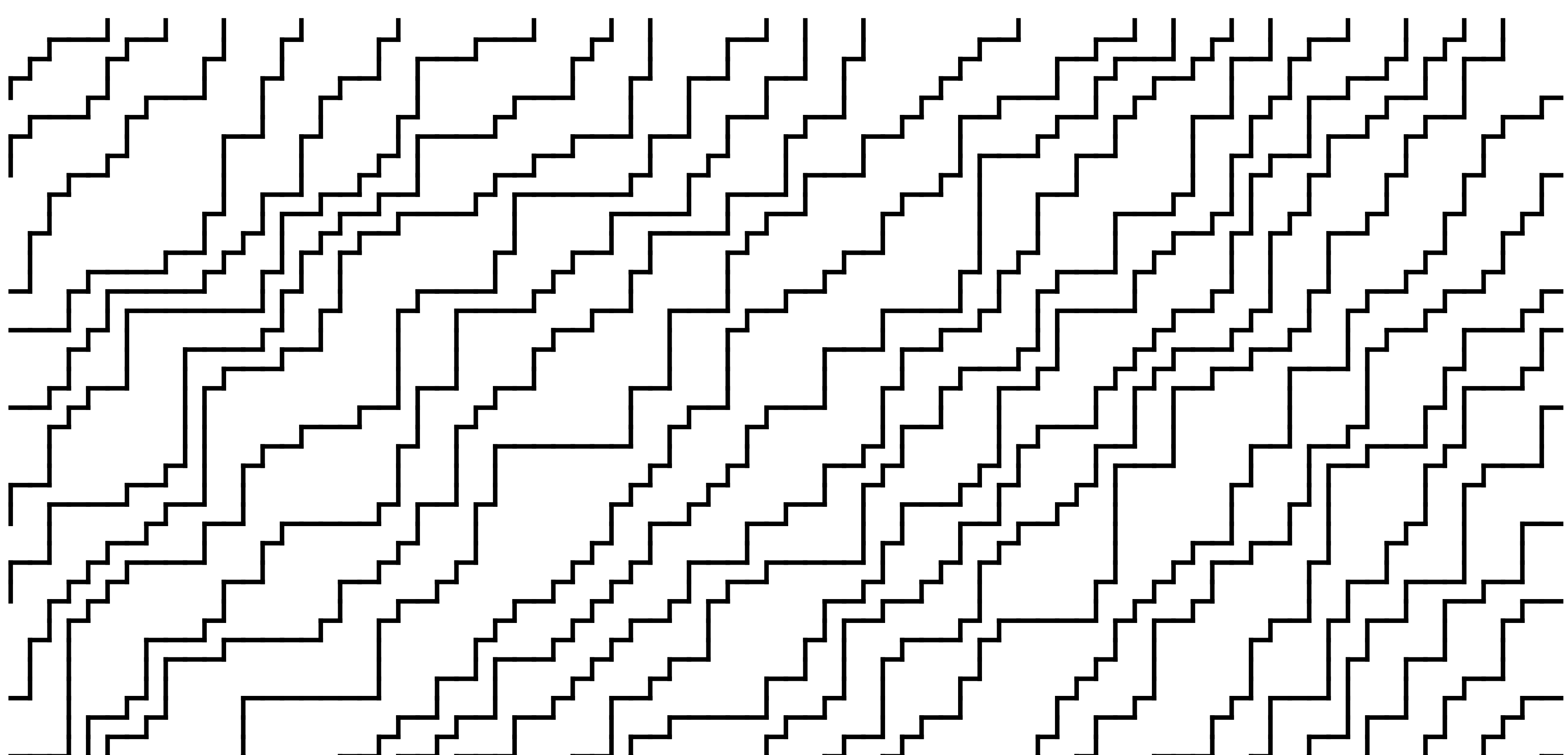
ON THE 5-VERTEX MODEL

Richard Kenyon (Yale)

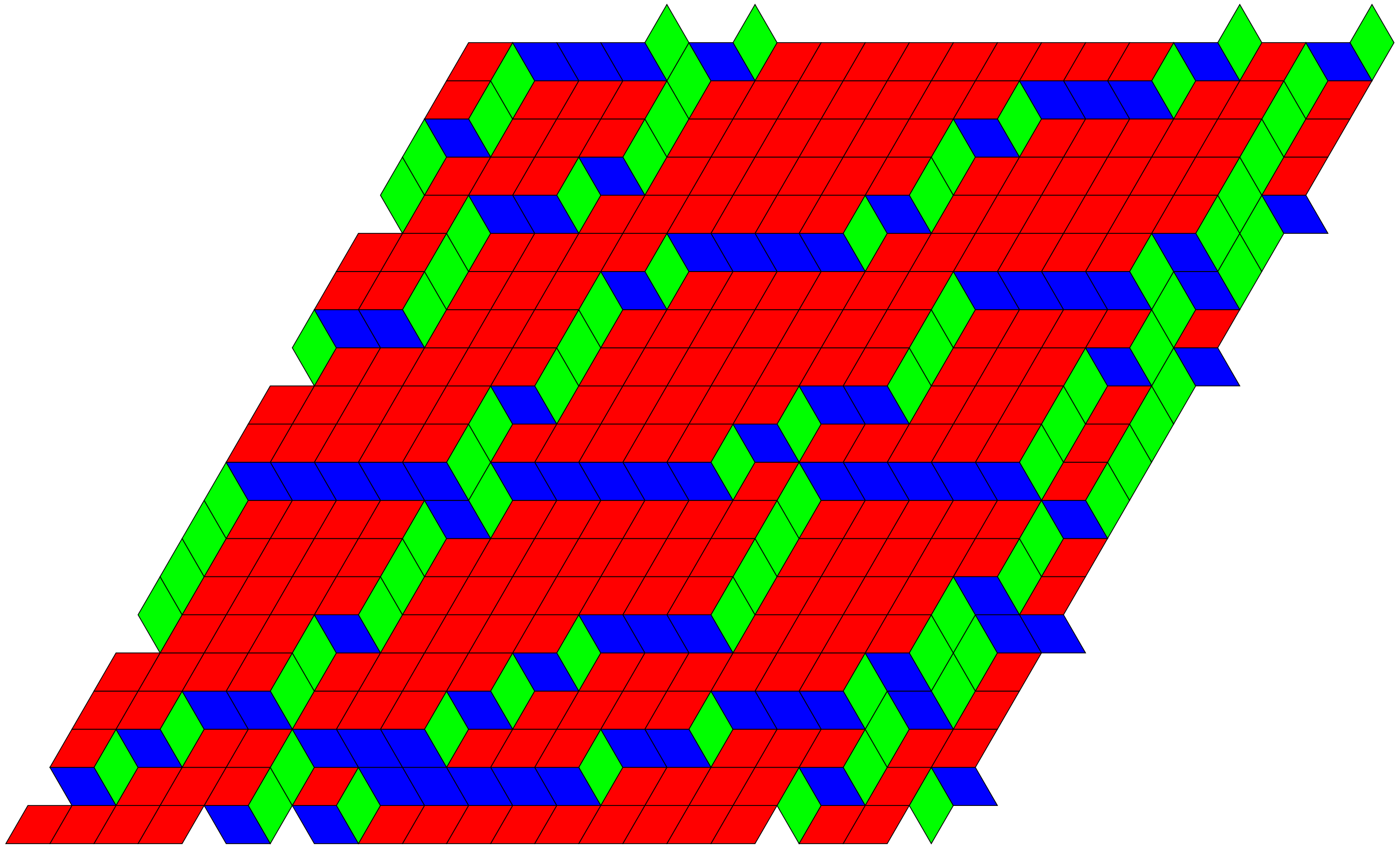
The five vertex model: a generalization of the lozenge tiling model
a special case of the six-vertex model $(\Delta \to \infty)$

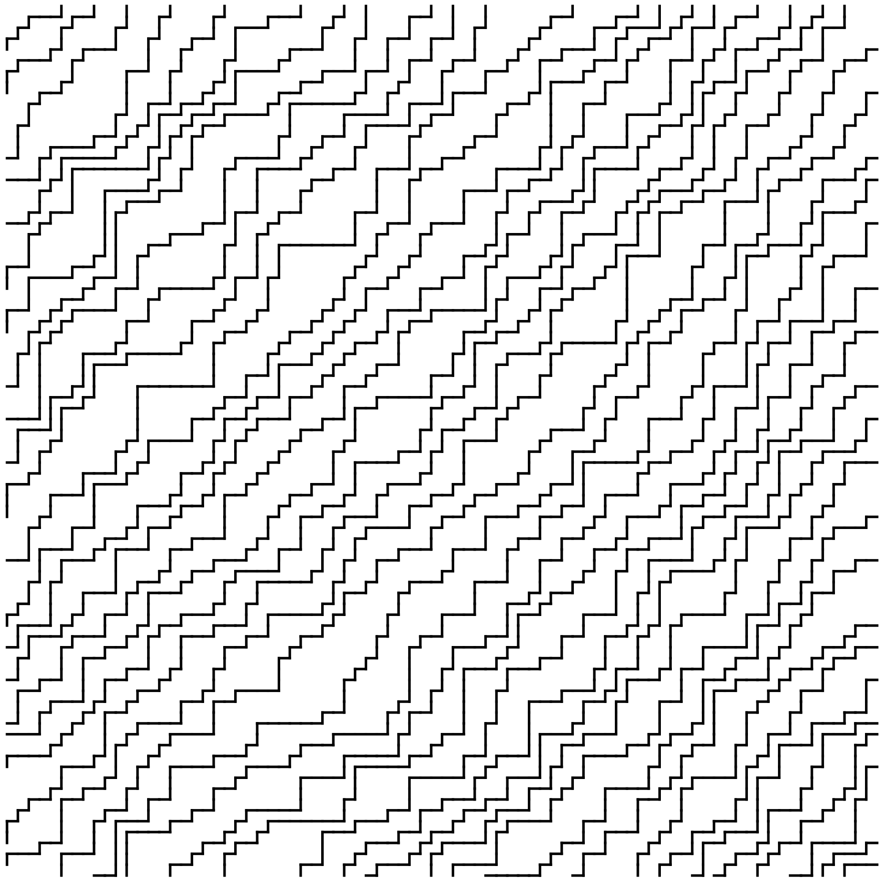


A configuration has probability $\frac{1}{Z} e^{vX+hY} r^c$ where r is the number of corners, v is the number of vertical edges, h is the number of horizontal edges.



lozenge tilings and the 5-vertex model

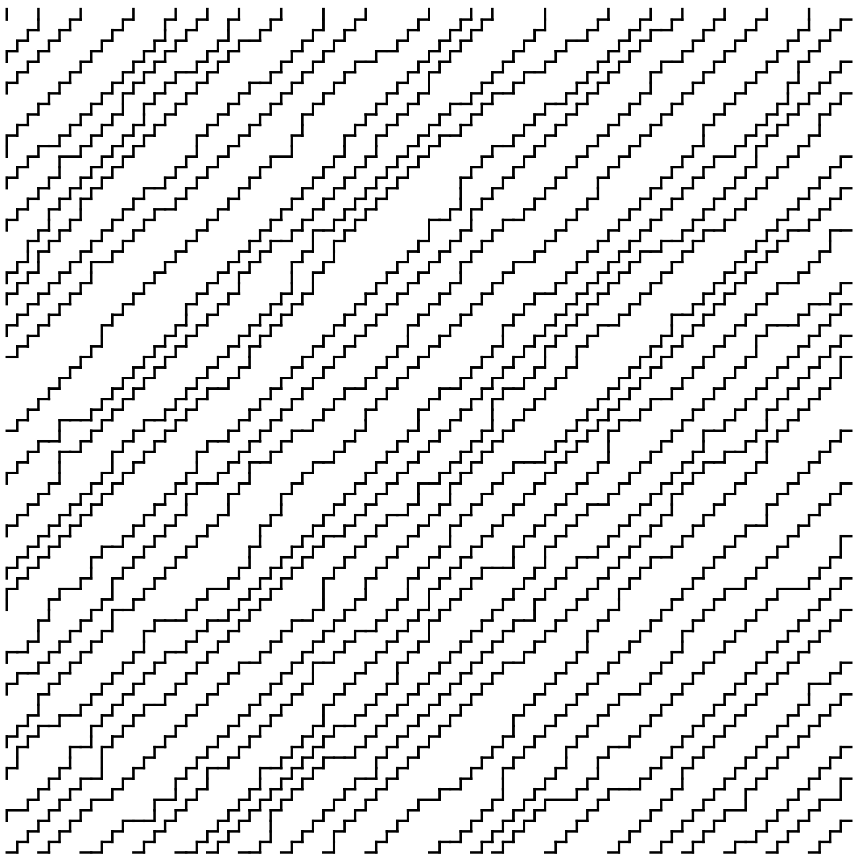




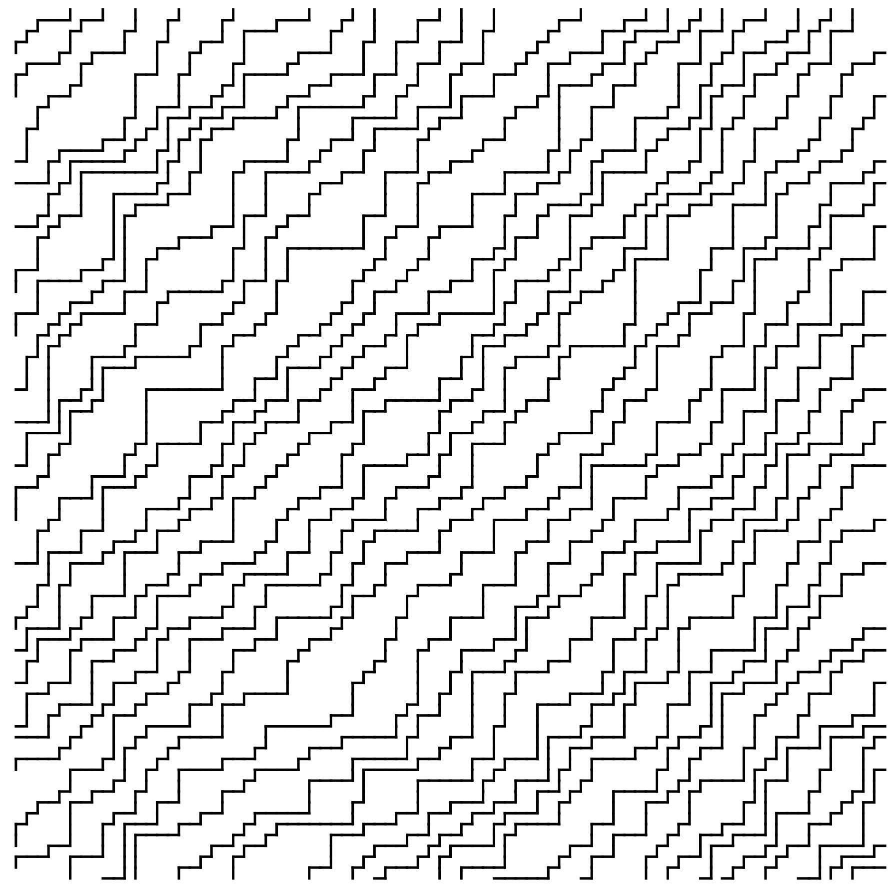
The 5 vertex model with $r = 1$ is the lozenge tiling model.

$r \neq 1$ means blue and green lozenges "interact".

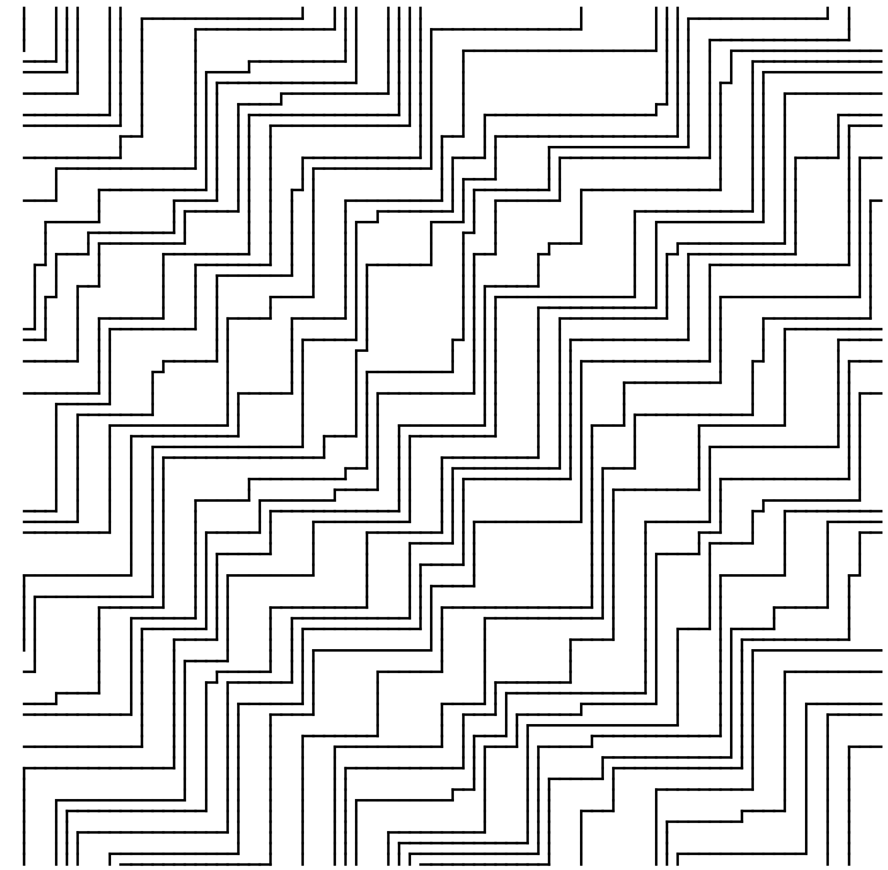
Simulations



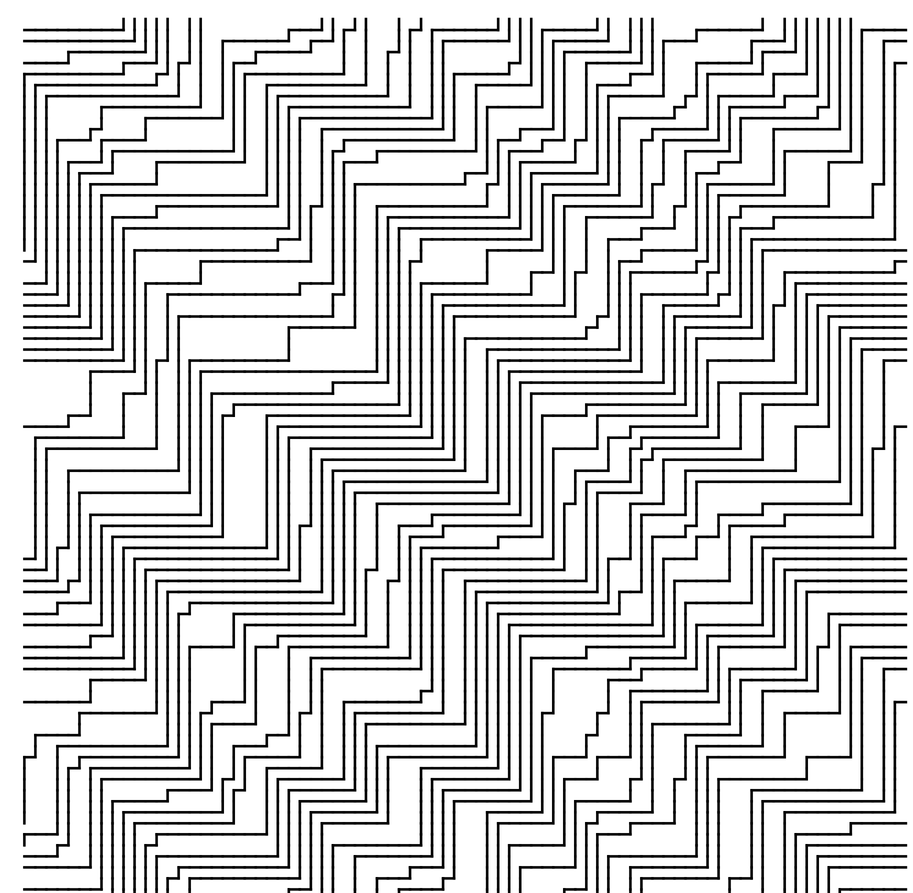
$r = 10$

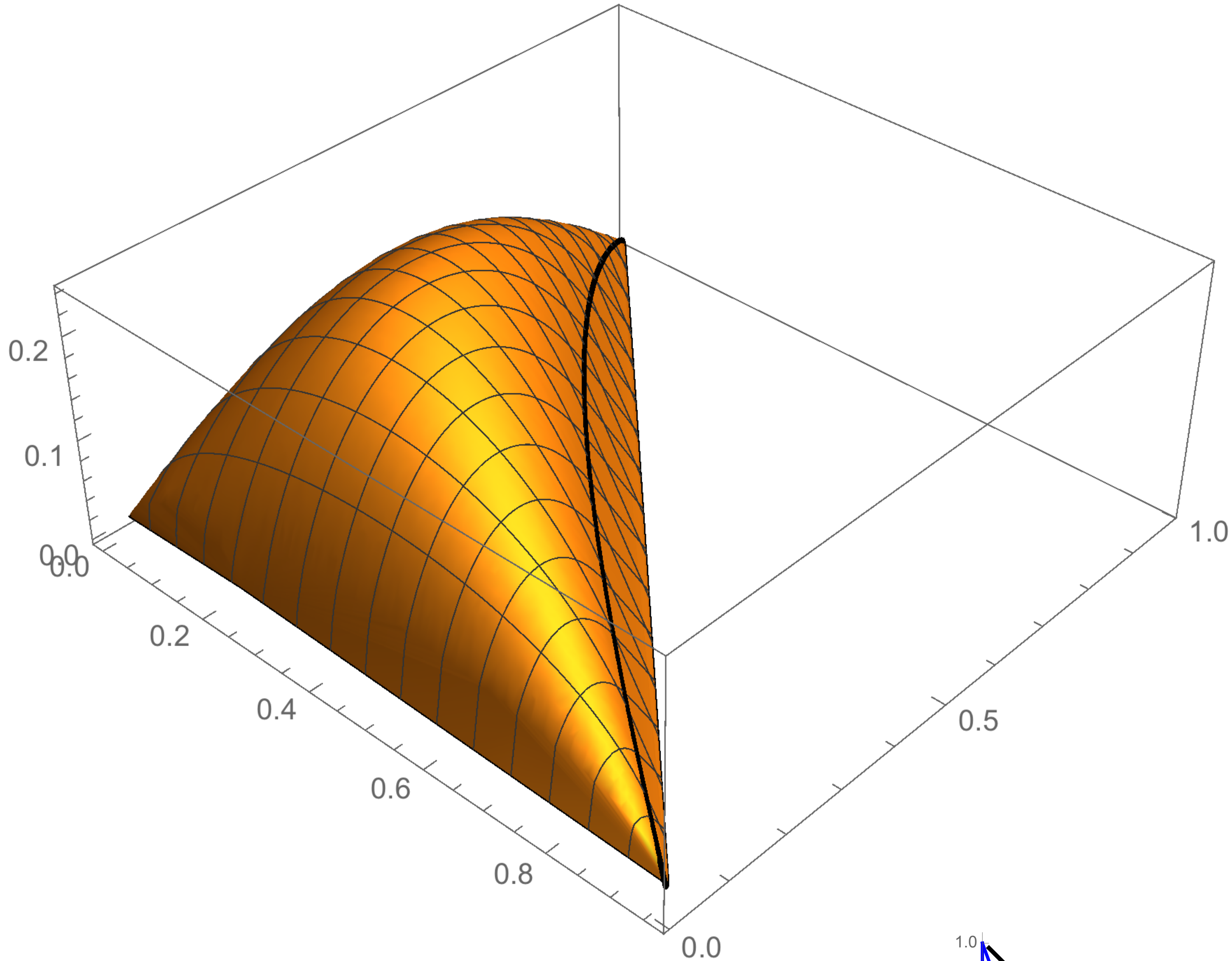


$r = 1$



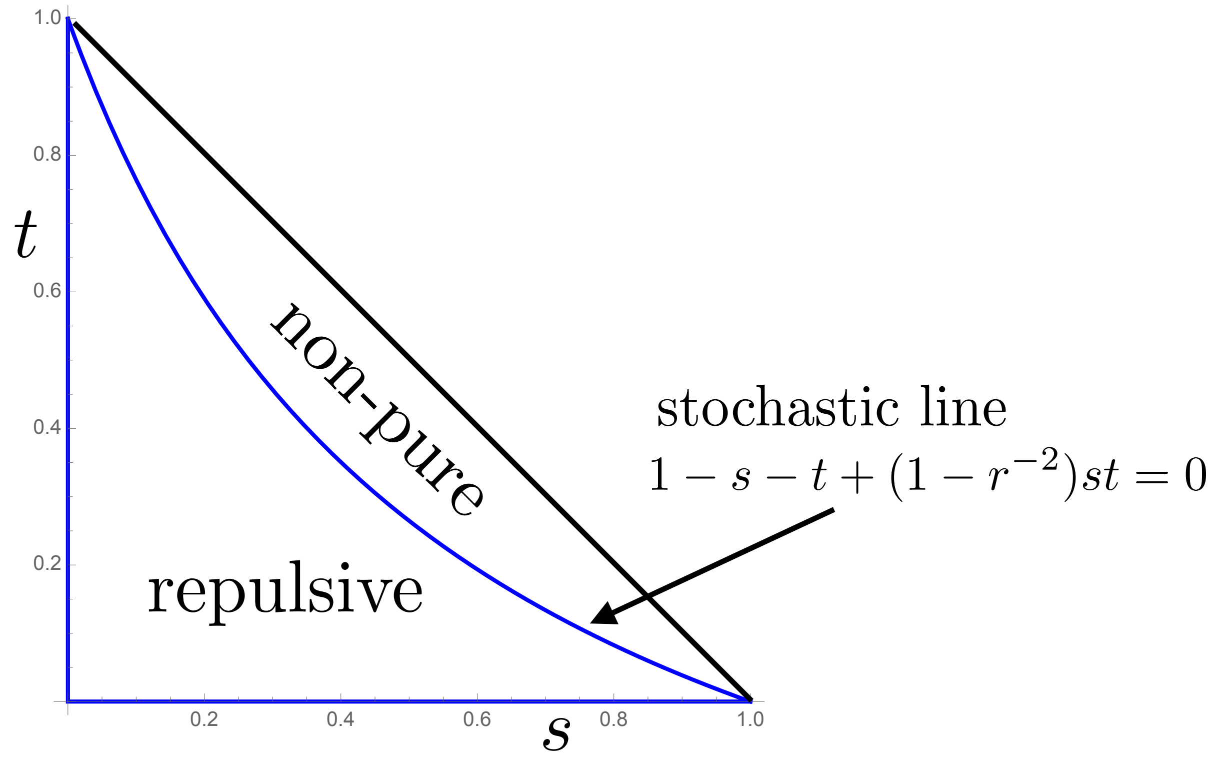
$r = .1$

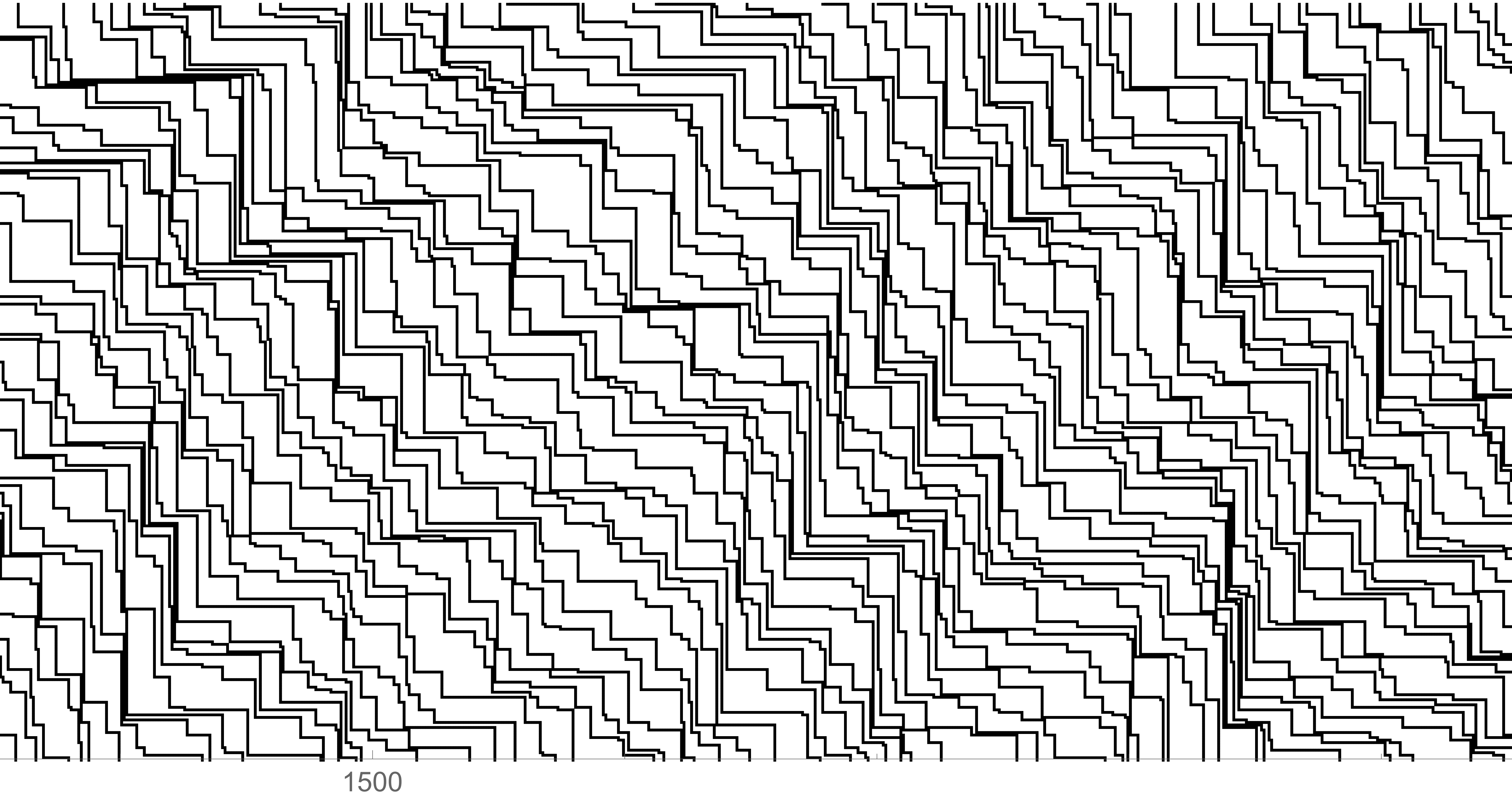




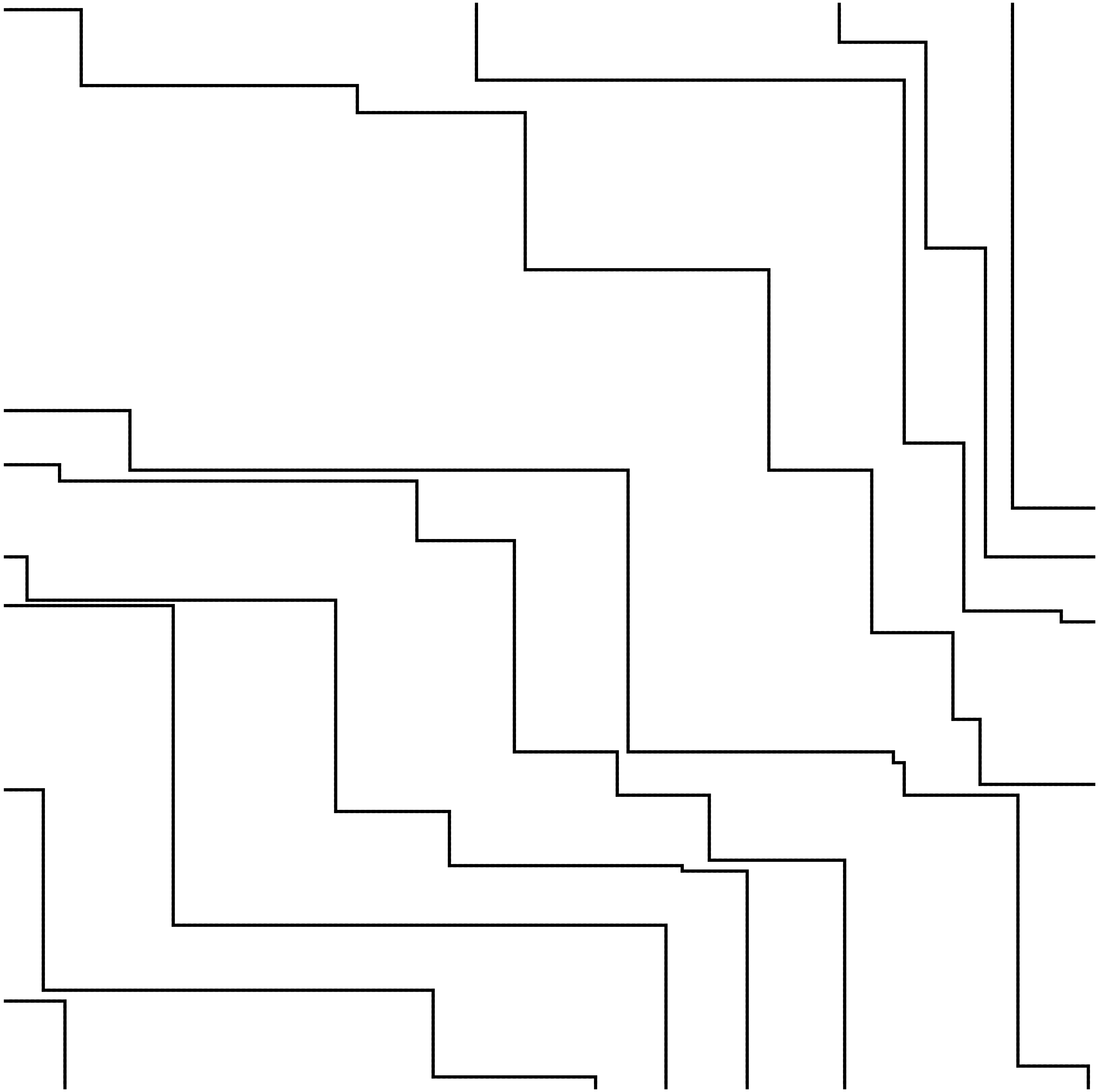
plot of $-\sigma$

when $r < 1$, $\sigma(s,t)$ is piecewise analytic:





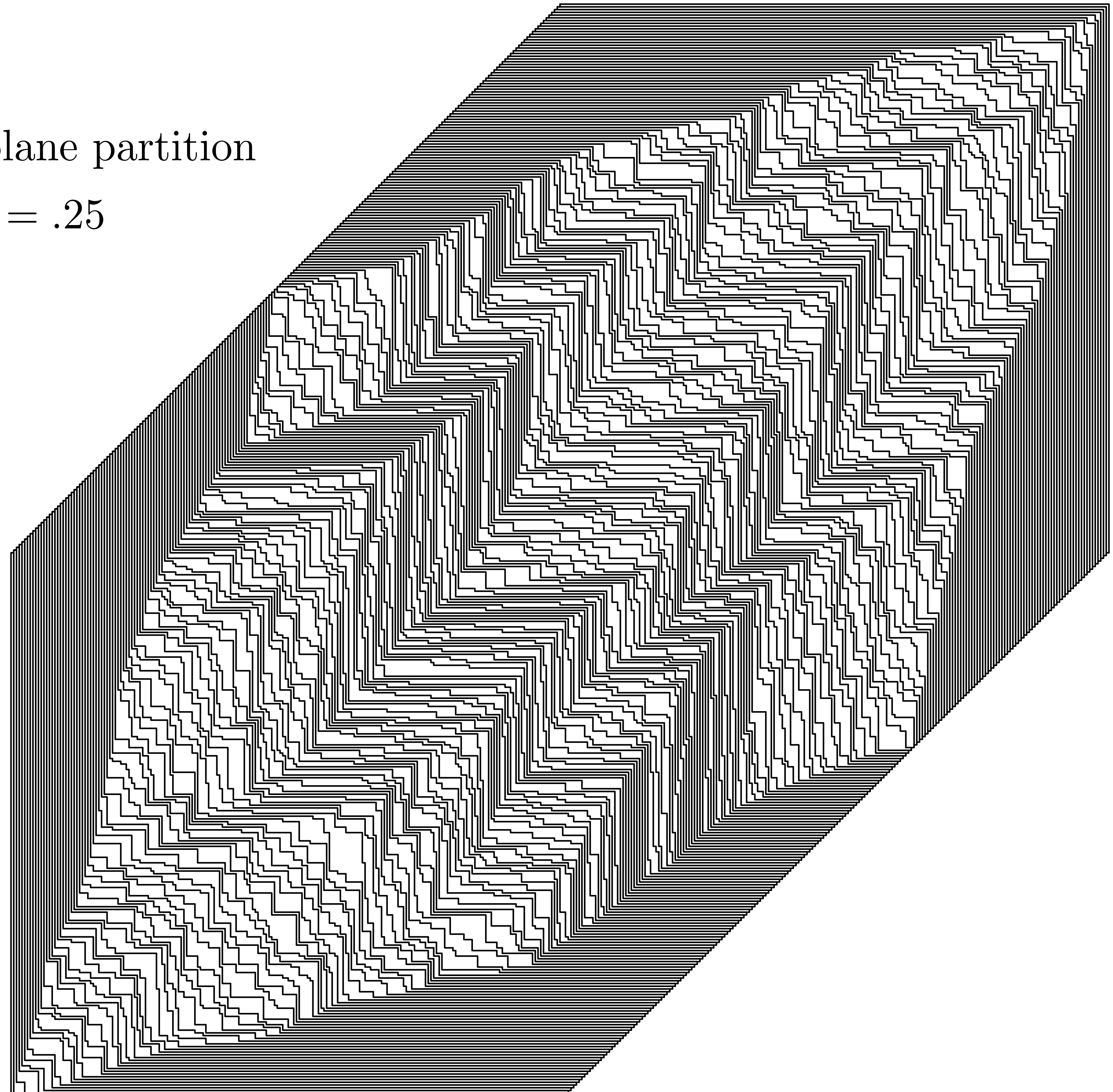
sample from a stochastic state

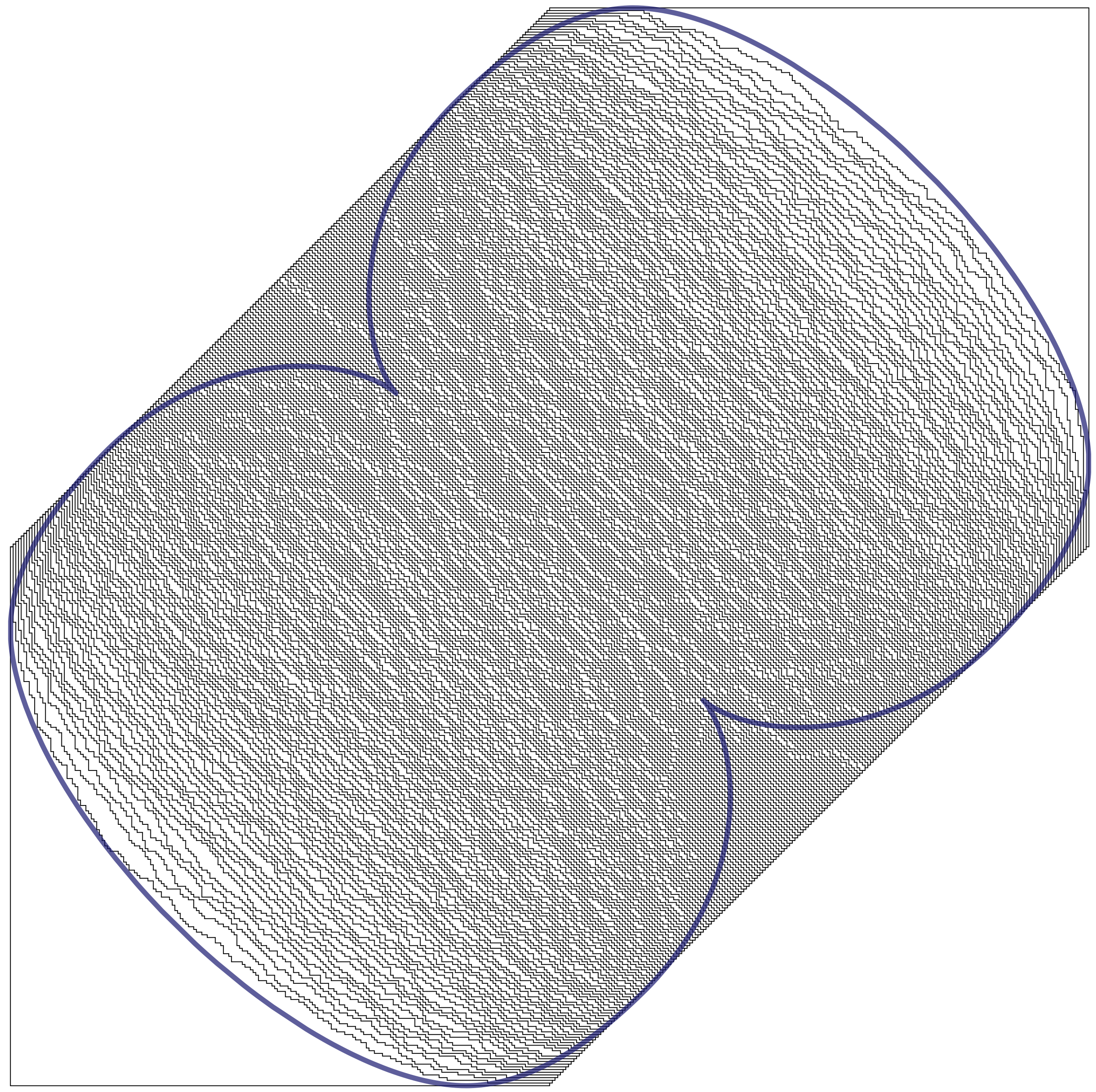


$r = .03$

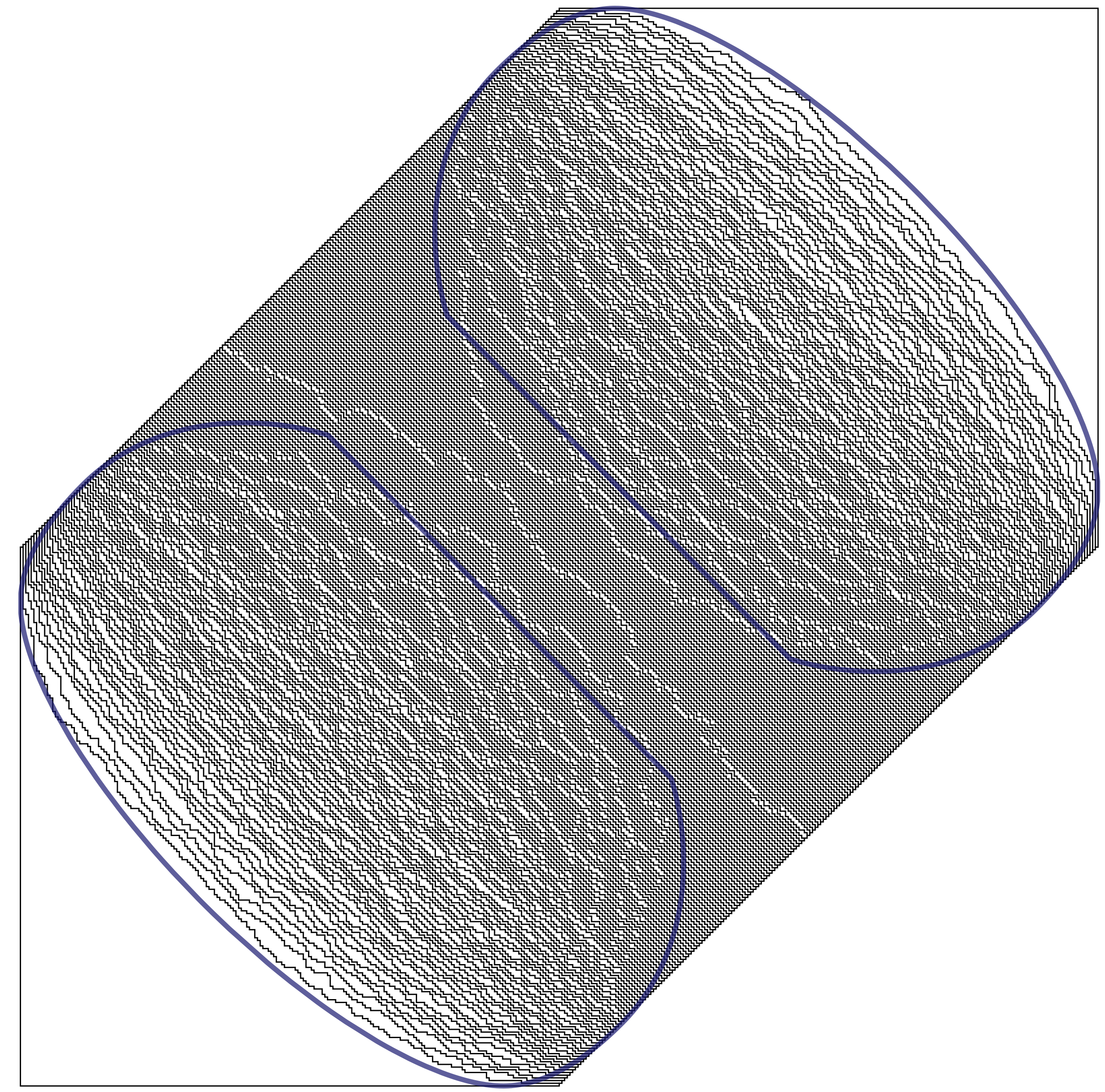
Boxed plane partition

$r = .25$





$r = 2.5$

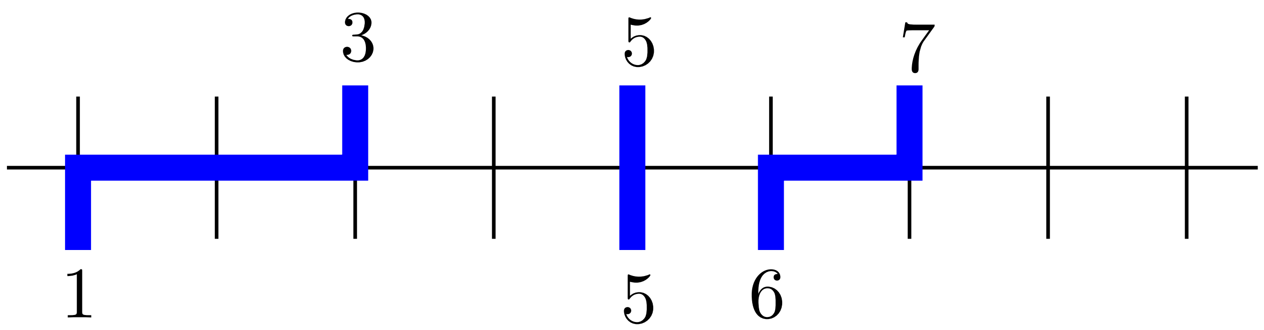


$r = 3.1$

General r case: how to find the free energy $F(X,Y)$?

No determinant formula...need to use Bethe Ansatz

that is, find an explicit diagonalization of the $2^N \times 2^N$ transfer matrix T



$$T(\{1,5,6\},\{3,5,7\}) = r^4 e^{3Y}$$

$T^k(A,B)$ is the total weight of configurations starting in state A and ending k steps later in state B.

The Free energy $F_r(X,Y)$ is determined from the leading eigenvalue of T.

$$F_r(X,Y) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda.$$

Eigenvectors are determinants

For $n = 3$ particles, the eigenvectors are of the form

$$F(x_1, x_2, x_3) = \begin{vmatrix} z_1^{x_1} & z_2^{x_1} & z_3^{x_3} \\ A(z_1) z_1^{x_2} & A(z_2) z_2^{x_2} & A(z_3) z_3^{x_2} \\ A(z_1)^2 z_1^{x_3} & A(z_2)^2 z_2^{x_3} & A(z_3)^2 z_3^{x_3} \end{vmatrix}$$

where $A(z) = (1 - r^2)\frac{y}{z} - 1$. Here the z_i are distinct roots of

$$\boxed{z^N A(z)^n = (-1)^{n+1} \prod_{i=1}^{n} A(z_i).}$$

(and $y = e^Y$)

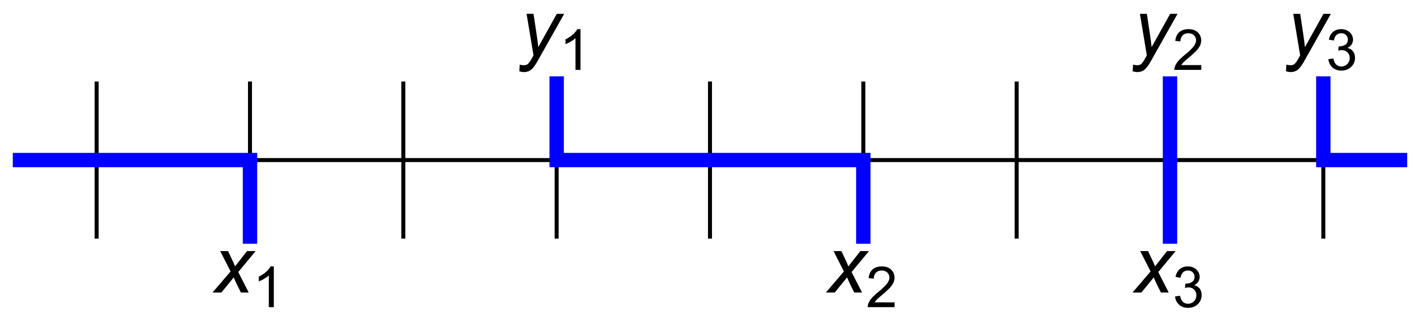
$$z^N A(z)^n = (-1)^{n+1} \prod_{i=1}^{n} A(z_i).$$

Lemma $F(x_1, x_2, x_0 + N) = F(x_0, x_1, x_2)$.

Proof:

$$F(x_1, x_2, x_0 + N) =$$

$$= \left| \begin{matrix} z_i^{x_1} \\ A(z_i) z_i^{x_2} \\ A(z_i)^2 z_i^N z_i^{x_0} \end{matrix} \right|_{i=1..3} = \left(\prod A(z_i)^{-1} \right) \left| \begin{matrix} A(z_i) z_i^{x_1} \\ A(z_i)^2 z_i^{x_2} \\ A(z_i)^3 z_i^N z_i^{x_0} \end{matrix} \right|_{i=1..3} = \left| \begin{matrix} z_i^{x_0} \\ A(z_i) z_i^{x_1} \\ A(z_i)^2 z_i^{x_2} \end{matrix} \right|_{i=1..3} . \quad \square$$



To check that F is an eigenvector,

$$TF(x_1,x_2,x_3) = \sum_{y_1=x_1+1}^{x_2} \sum_{y_2=x_2+1}^{x_3} \sum_{y_3=x_3+1}^{x_1+N} T_{x_1,x_2,x_3}^{y_1,y_2,y_3} f(y_1,y_2,y_3).$$

Applying the sum to the determinant (we just give the first column),

$$TF = \left| \begin{matrix} z_1^{x_2}(1+r^2(\frac{y}{z_1}+\cdots+(\frac{y}{z_1})^{x_2-x_1-1})) \\ A(z_1)z_1^{x_3}(1+r^2(\frac{y}{z_1}+\cdots+(\frac{y}{z_1})^{x_3-x_2-1})) \\ A(z_1)^2 z_1^{x_1+N}(1+r^2(\frac{y}{z_1}+\cdots+(\frac{y}{z_1})^{N+x_1-x_3-1})) \end{matrix} \right|$$

$$= \left| \begin{matrix} z_1^{x_2}(1+r^2\frac{(\frac{y}{z_1})^{x_2-x_1}-\frac{y}{z_1}}{\frac{y}{z_1}-1}) \\ A(z_1)z_1^{x_3}(1+r^2\frac{(\frac{y}{z_1})^{x_3-x_2}-\frac{y}{z_1}}{\frac{y}{z_1}-1}) \\ A(z_1)^2 z_1^{N+x_1}(1+r^2\frac{(\frac{y}{z_1})^{N+x_1-x_3}-\frac{y}{z_1}}{\frac{y}{z_1}-1}) \end{matrix} \right|$$

$$= \begin{vmatrix} z_1^{x_2} B(z_1) + z_1^{x_1} y^{x_2-x_1} D(z_1) \\ A(z_1)(z_1^{x_3} B(z_1) + z_1^{x_2} y^{x_3-x_2} D(z_1)) \\ A(z_1)^2 (z_1^{N+x_1} B(z_1) + y^{N+x_1-x_3} z^{x_3} D(z_1)) \end{vmatrix}$$

where

$$B(z) = 1 - \frac{r^2 y/z}{y/z - 1} = \frac{(1-r^2)y/z - 1}{y/z - 1}$$

and $D(z) = \frac{r^2}{y/z-1}$.

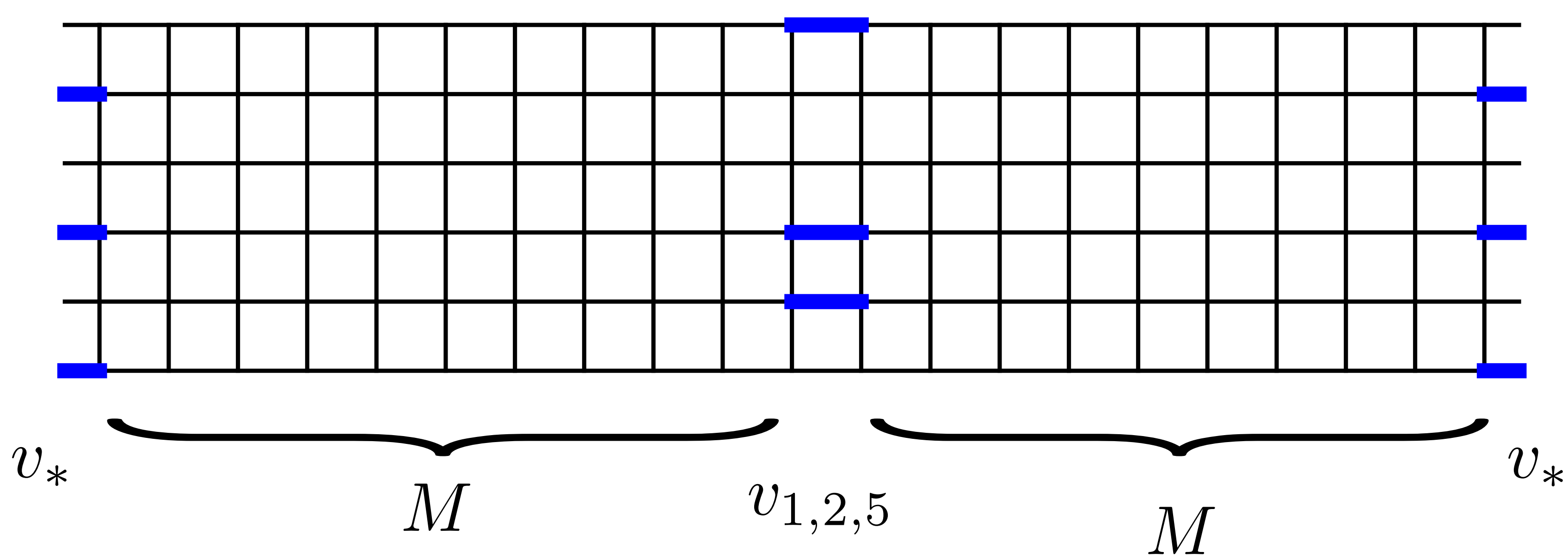
$$= \prod (y/z_i - 1)^{-1} \begin{vmatrix} z_1^{x_2} A(z_1) + z_1^{x_1} y^{x_2-x_1} r^2 \\ A(z_1)^2 z_1^{x_3} + A(z_1) z_1^{x_2} y^{x_3-x_2} r^2 \\ A(z_1)^3 z_1^N z_1^{x_1} + A(z_1)^2 z^{x_3} y^{N+x_1-x_3} r^2 \end{vmatrix}$$

$$= \prod (y/z_i - 1)^{-1} \begin{vmatrix} r^2 y^{x_2-x_1} & 1 & 0 \\ 0 & r^2 y^{x_3-x_2} & 1 \\ z^N A(z_1)^3 & 0 & r^2 y^{N+x_1-x_3} \end{vmatrix} F. \qquad \square$$

Probabilities

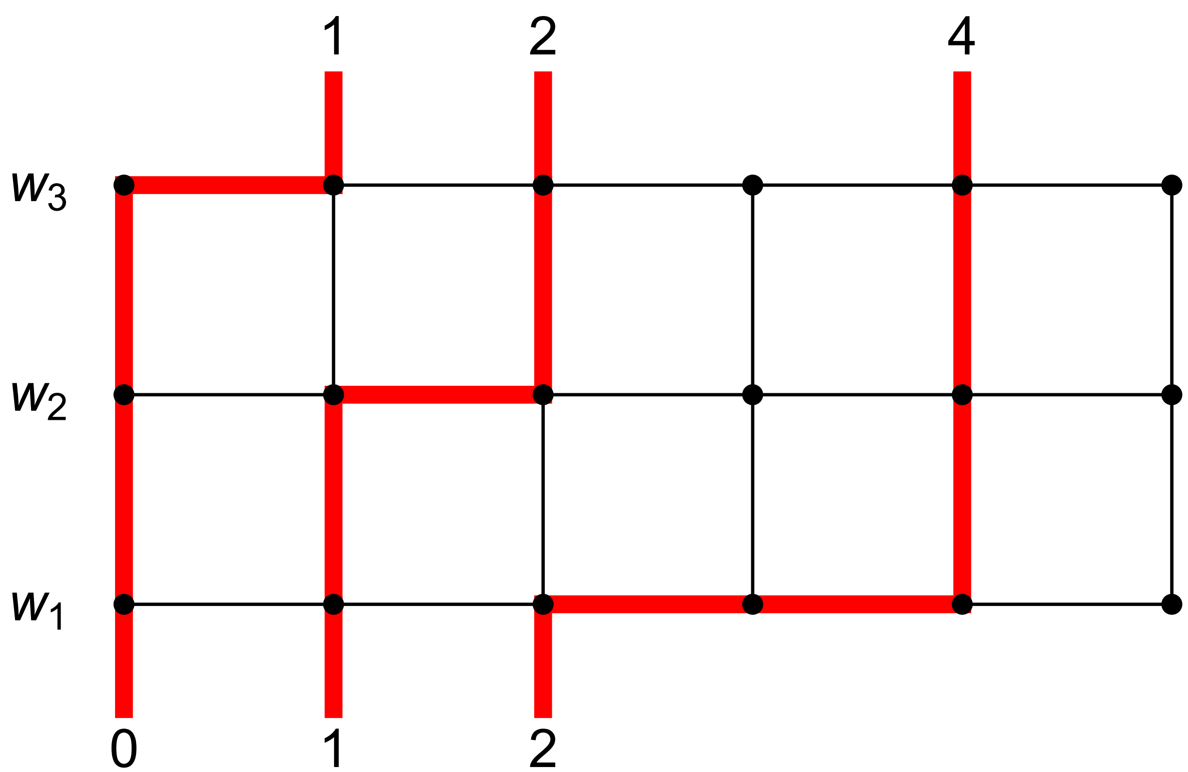
$$Pr(x_1, x_2, x_5) = \frac{\langle v_* | T^M | v_{1,2,5} \rangle \langle v_{1,2,5} | T^M | v_* \rangle}{\langle v_* | T^{2M} | v_* \rangle}$$

$$Pr(x_1, x_2, x_5) = \frac{1}{Z} F(x_1, x_2, x_5) F(N - x_5, N - x_2, N - x_1)$$



Schur function

$n = 3, N = 6$



$$s_{112} = \cdots + w_1^2 w_2 w_3 + \ldots$$

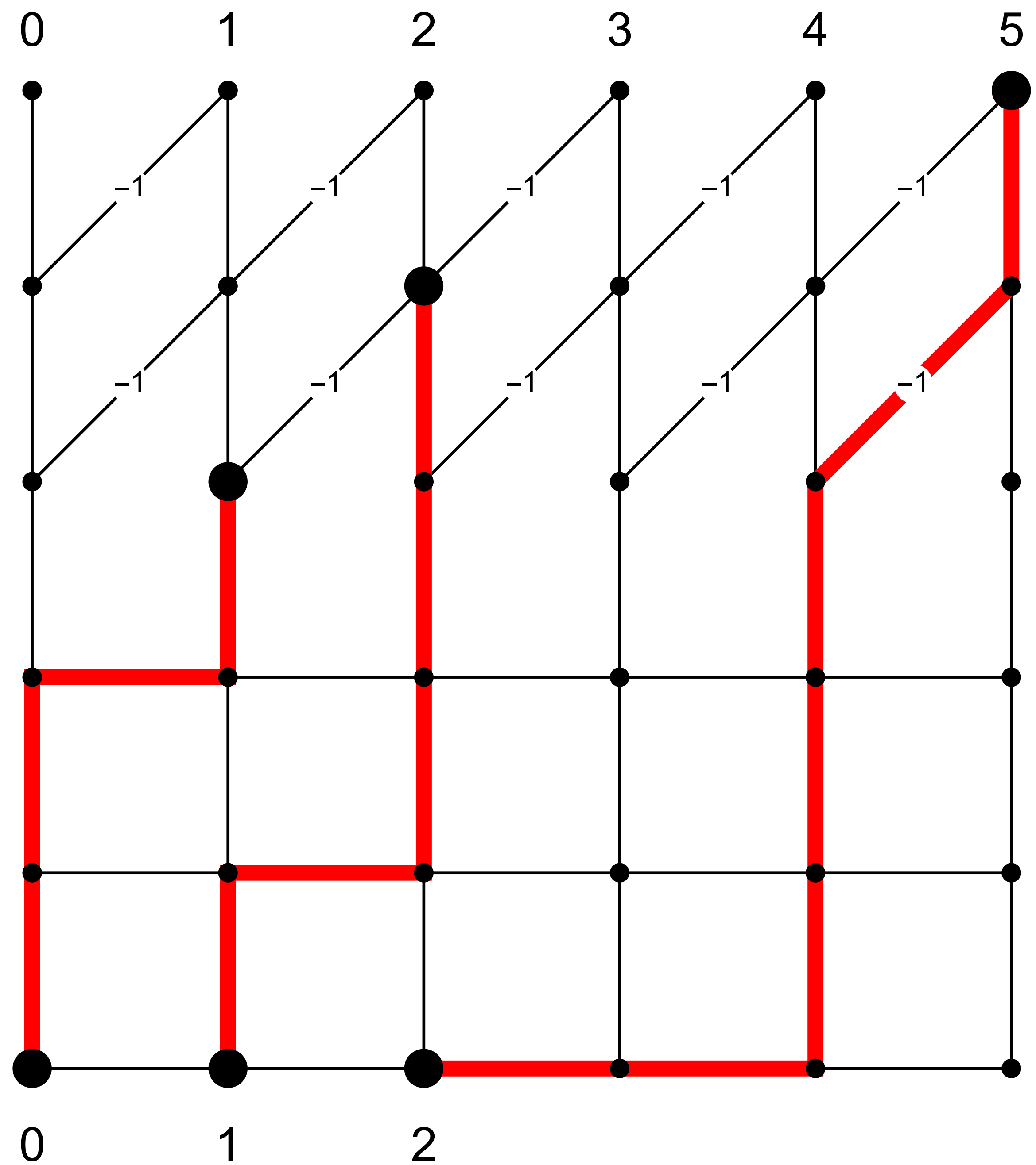
$$s_{112} = \frac{\begin{vmatrix} w_1^{1+0} & w_2^{1+0} & w_3^{1+0} \\ w_1^{1+1} & w_2^{1+1} & w_3^{1+1} \\ w_1^{2+2} & w_2^{2+2} & w_3^{2+2} \end{vmatrix}}{\begin{vmatrix} w_1^{0+0} & w_2^{0+0} & w_3^{0+0} \\ w_1^{0+1} & w_2^{0+1} & w_3^{0+1} \\ w_1^{0+2} & w_2^{0+2} & w_3^{0+2} \end{vmatrix}}$$

← vandermonde Δ

Let $w_i = \frac{z_i}{(1-r^2)y}$. Then $A(w_i) = \frac{1}{w_i} - 1$

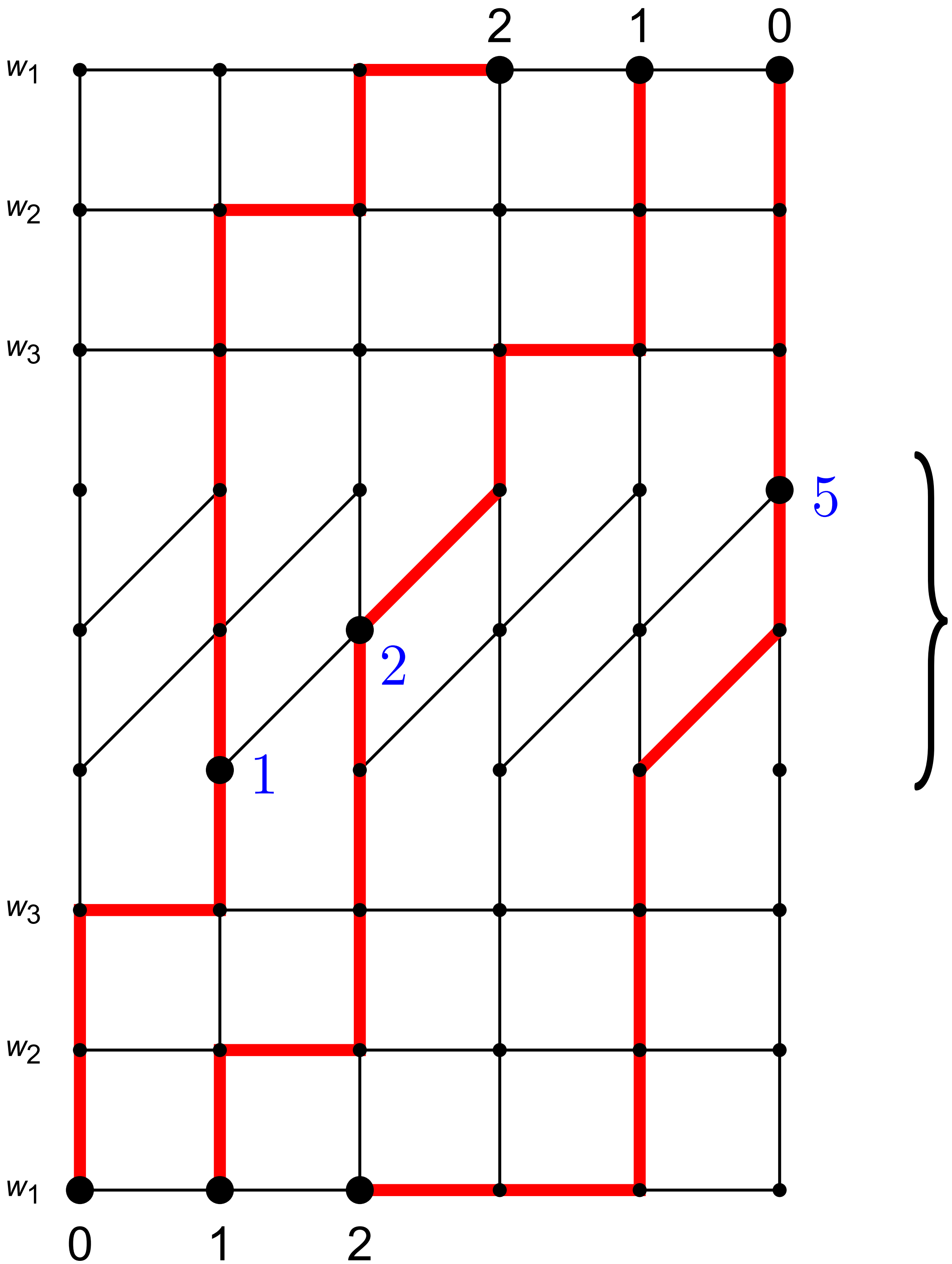
$$F(x_1, x_2, x_5) = D_2^2 D_1 s_{113} = \frac{\begin{vmatrix} w_1^1 & w_2^1 & w_3^1 \\ A(w_1)w_1^2 & A(w_2)w_2^2 & A(w_3)w_3^2 \\ A(w_1)^2 w_1^5 & A(w_2)^2 w_2^5 & A(w_3)^2 w_3^5 \end{vmatrix}}{\Delta}$$

discrete derivatives



where $A(w) = \frac{1}{w} - 1$.

$$Pr(x_1, x_2, x_5) = \frac{1}{Z} F(x_1, x_2, x_5) F(N - x_5, N - x_2, N - x_1)$$



$$Pr(x_1, x_2, x_5)$$

The red MNLP process is a
$\mathbb{C}$-valued determinantal process

$$h_k = [t^k] \prod_{i=1}^{n} \frac{1}{1 - tw_i}$$

By Cauchy-Binet and Jacobi-Trudi,

$$Z = \det \begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 & h_5 \\ 0 & 1 & h_1 & h_2 & h_3 & h_4 \\ 0 & 0 & 1 & h_1 & h_2 & h_3 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & 1 & -1 & & \\ & & & 1 & -1 & \\ & & & & 1 & -1 \\ & & & & & 1 \end{pmatrix}^2 \begin{pmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a Toeplitz matrix determinant...

$$Z = \det \begin{pmatrix} \tilde{h}_3 & \tilde{h}_4 & \tilde{h}_5 \\ \tilde{h}_2 & \tilde{h}_3 & \tilde{h}_4 \\ \tilde{h}_1 & \tilde{h}_2 & \tilde{h}_3 \end{pmatrix} \qquad \text{where } \tilde{H}(z) = (1-z)^{n-1} H(z)^2.$$

Local probabilities can be obtained from the associated projection kernel.

THANK YOU